Hidden strong symmetries in a range 3 deformation of the Hubbard model

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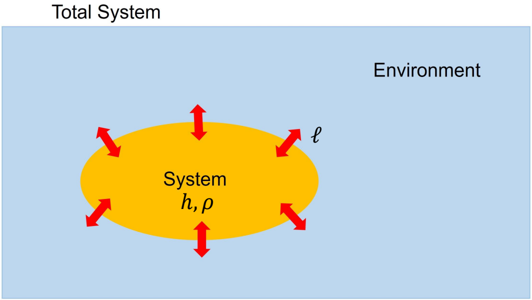
Context: Open quantum system

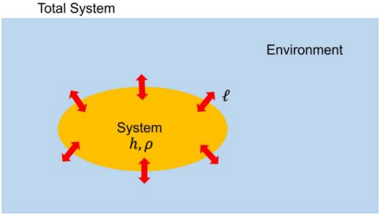


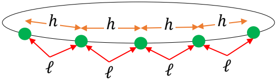
Total dynamic is hard: huge number of environment degrees of freedom

$$\downarrow$$

Focus: dynamic of the system and consider the effective action of the environment on it

Context: Open quantum system





Spin ½ chain

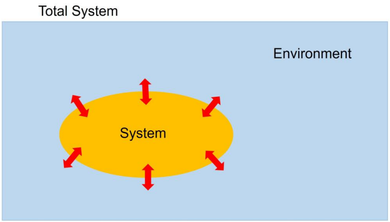
approximations

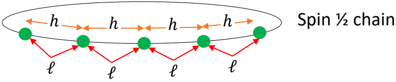
Lindblad equation

$$\dot{\rho} = \mathrm{i}\left[\rho, \sum_{i=1}^{L} h_{i,i+1}\right] - U \sum_{i=1}^{L}\left(\ell_{i,i+1}\rho\ell^{\dagger}_{i,i+1} - \frac{1}{2}\{\ell^{\dagger}_{i,i+1}\ell_{i,i+1}, \rho\}\right) \quad \rightarrow \quad \dot{\rho} \equiv \mathcal{L}\,\rho$$

Meaning:
Integrable Open quantum systems

[Medvedyeva, Essler, Prosen, Ziolkowska]





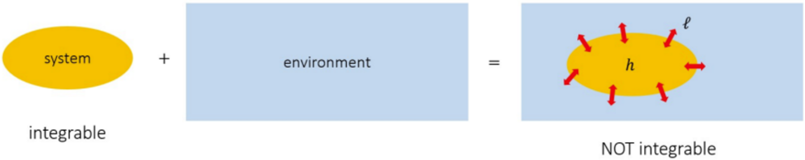
Lindblad equation

$$\dot{\rho} = \mathrm{i}\left[\rho, \sum_{i=1}^{L} h_{i,i+1}\right] - U \sum_{i=1}^{L}\left(\ell_{i,i+1}\rho\ell_{i,i+1}^{\dagger} - \frac{1}{2}\{\ell_{i,i+1}^{\dagger}\ell_{i,i+1}, \rho\}\right) \rightarrow \dot{\rho} \equiv \mathcal{L}\,\rho$$

Integrable open quantum system

$$\mathcal{L} \equiv \mathbb{Q}_2$$

Focus: Integrable Open quantum systems



Environment usually breaks integrability!

Question:
Are there cases where the "evolution remains integrable"?

Best option:
Models with a tunable coupling constant between the system and the environment

Hope:
Use integrability techniques (e.g. Bethe ansatz) to "solve" Open quantum systems

Find new integrable models: Boost approach

[de Leeuw, CP, Pribytok, Retore, Ryan]

$$\mathbb{Q}_2 \equiv \mathcal{L} \quad \xrightarrow{\text{Boost}} \quad \{\mathbb{Q}_3(\mathbb{Q}_2)\} \quad \xrightarrow{[\mathbb{Q}_2,\mathbb{Q}_3]=0} \quad R\text{ - matrix}$$

$$\mathcal{L} = -i\, h \otimes 1 + i\, 1 \otimes h^* + u \left(\ell \otimes \ell^* - \frac{1}{2} \left(\ell^\dagger \ell\right) \otimes 1 - \frac{1}{2} 1 \otimes \left(\ell^T \ell^*\right) \right)$$

Question:

For which h and ℓ we have an

Integrable open quantum system?

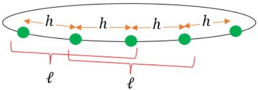
Steps:

New integrable model

$$h_{j,j+1} = \frac{1}{2}(X_j Y_{j+1} - Y_j X_{j+1})$$

$$\ell_{j,j+1,j+2} = Z_{j+1} + \kappa\,(X_j + X_{j+2})X_{j+1} - \kappa^2 X_j Z_{j+1} X_{j+2}$$

$$\mathcal{L} = -i\,h \otimes 1 + i\,1 \otimes h^* + u\,\left(\ell \otimes \ell^* - \frac{1}{2}\left(\ell^\dagger \ell\right) \otimes 1 - \frac{1}{2} 1 \otimes \left(\ell^T \ell^*\right)\right)$$



What is this model?!?

Integrable deformation of the Hubbard model

Our new model

$$\mathcal{L} = -i\, h \otimes 1 + i\, 1 \otimes h^* + u \left(\ell \otimes \ell^* - \frac{1}{2} \left(\ell^\dagger \ell\right) \otimes 1 - \frac{1}{2} 1 \otimes \left(\ell^T \ell^*\right) \right)$$

complexify u

$$H_3 = h \otimes 1 + 1 \otimes h + u \;\; \ell \otimes \ell$$

$$h = i(\sigma_i^+ \sigma_{i+1}^- - \sigma_i^- \sigma_{i+1}^+) \quad , \qquad \ell = Z_{j+1} + \kappa \big(X_j + X_{j+2}\big) X_{j+1} - \kappa^2 X_j Z_{j+1} X_{j+2}$$

$\kappa = \tanh \theta/2$

Hubbard model

$$H_{Hub} = \sum_j \underbrace{(c_j^\uparrow)^\dagger c_{j+1}^\uparrow + (c_{j+1}^\uparrow)^\dagger c_j^\uparrow}_{\to\, h \otimes 1} + \underbrace{(c_j^\downarrow)^\dagger c_{j+1}^\downarrow + (c_{j+1}^\downarrow)^\dagger c_j^\downarrow}_{\to\, 1 \otimes h} + \underbrace{U n_j^\uparrow n_j^\downarrow}_{\to\, \ell \otimes \ell}$$

Similarity + Jordan-Wigner transformation

$h \otimes 1$ $\quad$ $1 \otimes h$ $\quad$ $\ell \otimes \ell$

$\ell_{j,j+1} = Z_{j+1}$

$\kappa = 0$ Hubbard model

$\kappa \neq 0$ Integrable deformation of Hubbard

Main differences with the Hubbard model

- Interaction spans 3 sites
- Particle number not conserved

Is the range 3 model integrable?

YES!

It is a bond-site transformation of an integrable model

We find the R-matrix!

Some of the entries:

$$r_1 = -\frac{2\,i\,k\,\sin\left(\frac{1}{2}(\mathrm{Am}_u - \mathrm{Am}_v)\right)}{\mathrm{dn}_u + \mathrm{dn}_v} \qquad r_4 = \sec\left(\frac{1}{2}(\mathrm{Am}_u - \mathrm{Am}_v)\right)$$

Is the range 3 model new?

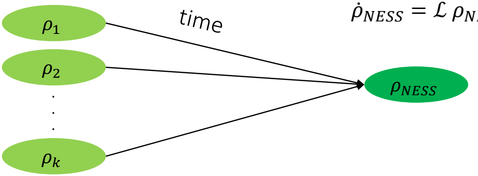
- Un-usual funcional dependence
- All the previous deformations of Hubbard:
  - Nearest-Neighbour
  - (At least) two local $U(1)$ charges → First range 3 deformation of the Hubbard model

Physical properties of this model

Environment in thermal equilibrium

Small system thermalizes:
details of the initial state are washed away

Typically



$$\dot{\rho}_{NESS} = \mathcal{L}\, \rho_{NESS} = 0$$

We find: L+1 NESS

Models with <u>non unique</u> NESS are exceptional: extra info about the initial state

What are the possible ways to have multiple NESS?

Presence of **extra** symmetry

Symmetry and conservation law

Hermitian quantum mechanics

$$[\mathbb{H}, Q] = 0 \quad \leftrightarrow \quad \dot{Q} = 0$$

Lindblad (not Hermitian) system

$$[h, Q] = [\ell, Q] = 0 \ \textit{strong} \text{ symmetry} \ \rightarrow \quad \dot{Q} = 0$$

[Albert, Jiang '14]

Hubbard model

$$\text{NESS} : L + 1$$

$$Q_0 = \sum_j Z_j \quad \text{total magnetization: strong symmetry}$$

$$\rho_0 = e^{\alpha Q_0}$$

Deformation of the Hubbard model

$$\text{NESS} : L + 1$$

Motivation?!

Reason of multiple NESS

$$Q_\kappa = T(\kappa)^\dagger Q_0 T(\kappa) \quad \kappa \text{ deformation parameter}$$

$$T(\kappa) = \mathrm{Tr}_{\mathcal{A}}(M_L(\kappa)\dots M_1(\kappa))$$

$$[T(\kappa), T(\kappa')] = 0 \qquad T(\kappa)^\dagger T(\kappa) = 1 + \kappa^L \prod_j Z_j$$

Hidden Strong Symmetry

$$[Q_\kappa, h] = [Q_\kappa, \ell] = 0\,, \quad [Q_\kappa, \mathcal{L}] = 0\,, \quad \dot{Q}_\kappa = 0$$

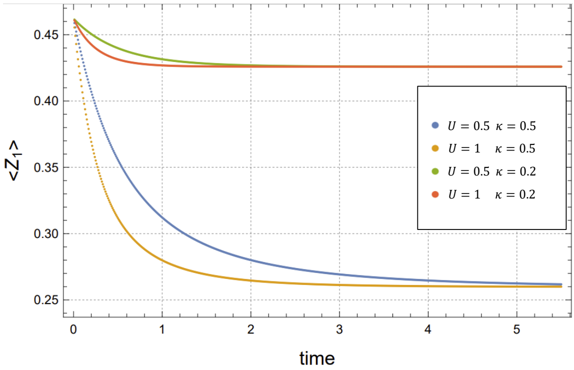
$$\text{NESS:} \quad \rho_\kappa = T(\kappa)^\dagger e^{\alpha Q_0} T(\kappa)$$

What is the role of the conserved charge Q_κ?
What is the role of the other conserved charges?

Evolution of the system

$$\text{initial state :} \quad \rho_\beta(t=0) = \frac{e^{\beta Q_0}}{(2\cosh\beta)^L}$$

Ansatz final state: Gibbs ensamble $\quad \rho_G \sim e^{-\lambda Q_\kappa}$



$$\lim_{t\to\infty} < Z_1 > = \frac{(\kappa^2-1)^2}{(1-\kappa^{2L})^2}\tanh\beta \, (1 - 2\,\kappa^L \tanh^{L-2}\beta + \kappa^{2L})$$

$$\lim_{t,L\to\infty} < Z_1 > = (1-\kappa^2)^2 \tanh\beta$$

Gibbs ensamble: Confirmed!

Evolution of the system

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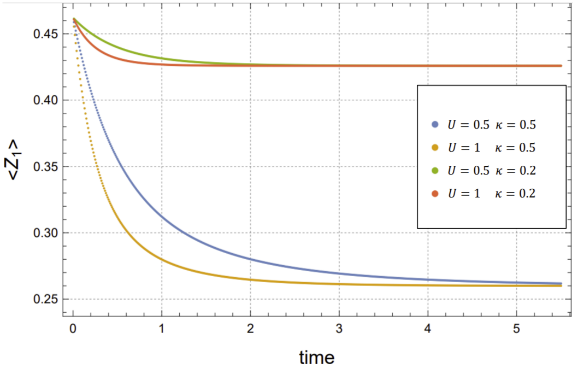

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Gibbs ensamble: Confirmed!

??? Role of the other charges

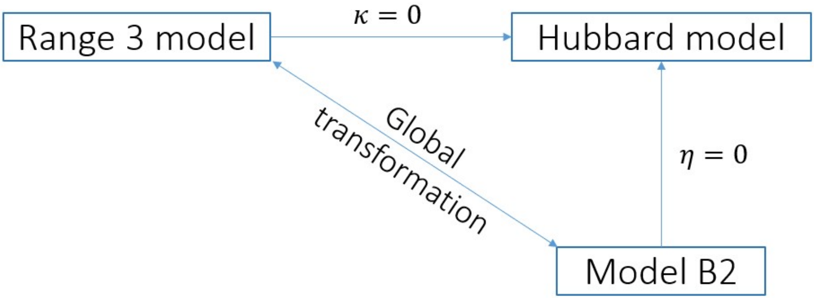
Connections between different integrable models

Action of T on the h and ℓ:

$$T(\kappa) h T(\kappa)^\dagger \to h$$

$$T(\kappa)\ell_{i,i+1,i+2}T(\kappa)^\dagger = Z_{i+1} + \eta(X_i X_{i+1} + Y_i Y_{i+1}) + \eta^2 Z_{i+1}$$

model B2 [Murakami, 98]



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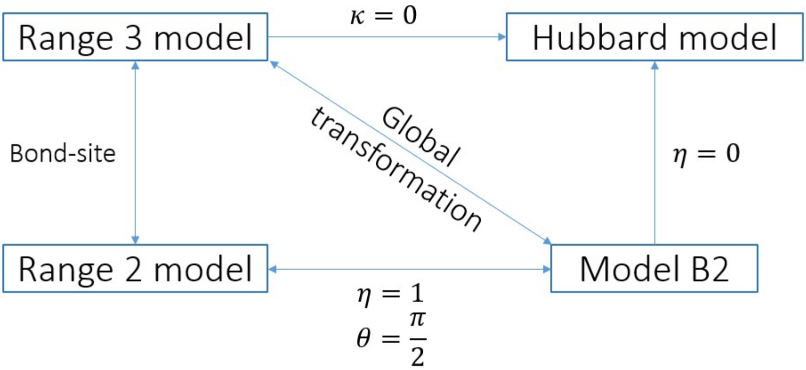


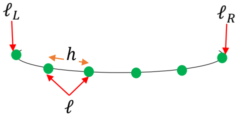
Conclusions and future work

What we saw:

- Integrabile model of range 3 → Deformation of the Hubbard model
- L+1 NESS → Hidden strong symmetry
- Role of the conserved charge → Gibbs Ensamble

Future work:

- Solution of the range 3 model: Liouville gap
- Can we classify integrable open quantum systems in the case of open boundary conditions?
- Existence of a bigger family of integrable Lindbladians containing all of them



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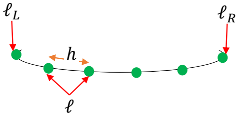


Thank you!

Back up slide:

Boost automorphism mechanism

1. Start from an ansatz for the nearest-neighbour Hamiltonian $\mathbb{Q}_2 = \sum \mathcal{H}_{j,j+1}$:

$$\mathcal{H}_{12}(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_1(\theta) & h_3(\theta) & 0 \\ 0 & h_4(\theta) & h_2(\theta) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

2. Use the boost operator to construct $\mathbb{Q}_3 \sim [\mathbb{B}[\mathbb{Q}_2], \mathbb{Q}_2]$

$$\mathbb{B}[\mathbb{Q}_2] := \partial_\theta + \sum_{n=-\infty}^{\infty} n\, \mathcal{H}_{n,n+1}$$

3. Impose $[\mathbb{Q}_2, \mathbb{Q}_3] = 0$

$$\dot{h}_3(h_1 + h_2) = (\dot{h}_1 + \dot{h}_2)h_3, \qquad \dot{h}_4(h_1 + h_2) = (\dot{h}_1 + \dot{h}_2)h_4$$

Potentially integrable $\mathbb{Q}_2$

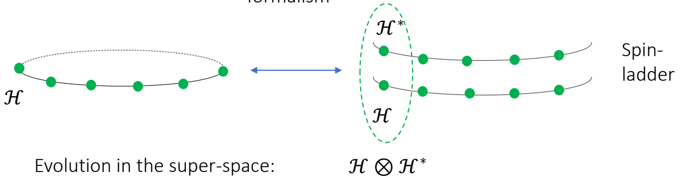
4. Compute the R-matrix from $(\partial_u \mathrm{YBE}|_{u\to v}) = 0$, $\quad [R_{13}R_{23}, \mathscr{H}_{12}(u)] = \dot{R}_{13}R_{23} - R_{13}\dot{R}_{23}$
$R(u,u) = P$, $P\dot{R}|_{u\to v} = \mathcal{H}$

Back up slide:

Meaning of integrability

$$\dot{\rho} = i[\rho, h] + u\left(\ell\rho\ell^\dagger - \frac{1}{2}\{\ell^\dagger\ell, \rho\}\right)$$

$\rho \sim |\ \rangle\langle\ |$ matrix ⟷ Super-operator formalism ⟷ $\rho \sim |\ \rangle|\ \rangle\rangle$ vector



Evolution in the super-space:

$$\dot{\rho} = \mathcal{L}\,\rho$$

$$\mathcal{L} = i\,1 \otimes h^T - i\,h \otimes 1 + u\left(\ell \otimes \ell^* - \frac{1}{2}\ell^\dagger\ell \otimes 1 - \frac{1}{2}1 \otimes \ell^T\ell^*\right)$$

Integrability:

$\mathcal{L} = \mathbb{Q}$ one of the conserved charges of an integrable model!

Back up slide: Symmetry Vs Conserved charges

Example 1

[Albert, Jiang '14]

$$h = 0\,, \qquad \ell = \sigma^- \qquad J = n\,,$$

$$[\mathcal{J}, \mathcal{L}] = 0\,, \qquad \dot{J} = \mathcal{L}(J) \neq 0$$