Elliptic Quantum Toroidal Algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ and Affine Quiver Gauge Theories

Hitoshi Konno

Tokyo University of Marine Science & Technology

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Review of EQG $U_{q,p}(\widehat{\mathfrak{g}})$ ($\widehat{\mathfrak{g}}$: affine Lie alg.)

$U_{q,p}(\widehat{\mathfrak{g}})$: (K '98, Jimbo-K-Odake-Shiraishi '99, $\cdots$)

- elliptic (p: elliptic nome) and dynamical analogue of $U_q(\widehat{\mathfrak{g}})$ in the Drinfeld realization
  - elliptic currents $E_j(z), F_j(z), \psi_j^{\pm}(z)$
- Hopf algebroid str. as a co-alg. str.,

  $\Delta : U_{q,p} \to U_{q,p} \widetilde{\otimes} U_{q,p}$ algebra hom.

$\rightsquigarrow$ Vertex operator $\Phi(z) : \mathcal{F} \to V_z \widetilde{\otimes} \mathcal{F}'$ s.t $\Phi(z)x = \Delta(x)\Phi(z)$, $\forall x \in U_{q,p}$.

$\rightsquigarrow$ $\langle \Phi(z_1)\Phi(z_2)\cdots\Phi(z_n)\rangle$ deformation of conformal block !

Characterizations of $U_{q,p}(\widehat{\mathfrak{g}})$

- p-deformation of the quantum aff. alg. $U_q(\widehat{\mathfrak{g}})$ + dynamical param.

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- q-deformation of the Feigin-Fuchs constr. of the W-algebra $W(\mathfrak{g})$ of the coset type

$U_q(\widehat{\mathfrak{g}})$

q-deform. $\nearrow$ $\searrow$ $p = q^{2r}$-deform. + dynamical param.

$\widehat{\mathfrak{g}}$ $U_{q,p}(\widehat{\mathfrak{g}})$

r-deform. + dynamical param. (Feigin-Fuchs constr.) $\searrow$ $\nearrow$ q-deform.

$W(\mathfrak{g})$

$\|$

$$(\widehat{\mathfrak{g}})_{r-h^\vee-k} \oplus (\widehat{\mathfrak{g}})_k \supset (\widehat{\mathfrak{g}})_{r-h^\vee}$$

$U_{q,p}(\widehat{\mathfrak{g}})$ is a q-deformation of the Feigin-Fuchs construction of $W(\mathfrak{g}) = (\widehat{\mathfrak{g}})_{r-h^\vee-k} \oplus (\widehat{\mathfrak{g}})_k \supset (\widehat{\mathfrak{g}})_{r-h^\vee}$. The Feigin-Fuchs construction is nothing but a r-deformation of $\widehat{\mathfrak{g}}$ to $W(\mathfrak{g})$. (K '98)

e.g. $\widehat{\mathfrak{sl}}_2 \quad (c=1) \quad \overset{r\text{-deform.}}{\longrightarrow} \quad \text{Vir.} \cong (\widehat{\mathfrak{sl}}_2)_{r-3} \oplus (\widehat{\mathfrak{sl}}_2)_1 \supset (\widehat{\mathfrak{sl}}_2)_{r-2}$

$$\left(c_{Vir} = 1 - 24\alpha_0^2, \quad \alpha_0 = \frac{1}{2\sqrt{r(r-1)}}\right)$$

$$T(z) = \frac{1}{2}(\partial\phi(z))^2 \quad \rightsquigarrow \quad T_{FF}(z) = \frac{1}{2}(\partial\phi(z))^2 + \sqrt{2}\alpha_0\partial^2\phi(z)$$

$$e(z) = e^{\alpha} z^h : e^{\sqrt{2}\widetilde{\phi(z)}} : \quad \rightsquigarrow \quad S_+(z) = e^{\sqrt{\frac{2r}{r-1}}Q} z^{\sqrt{\frac{2r}{r-1}}P} : e^{\sqrt{\frac{2r}{r-1}}\widetilde{\phi(z)}} :$$

$$f(z) = e^{-\alpha} z^{-h} : e^{-\sqrt{2}\widetilde{\phi(z)}} : \quad \rightsquigarrow \quad S_-(z) = e^{-\sqrt{\frac{2(r-1)}{r}}Q} z^{-\sqrt{\frac{2(r-1)}{r}}P} : e^{-\sqrt{\frac{2(r-1)}{r}}\widetilde{\phi(z)}} :$$

$$\left[\sqrt{2}a_m, \sqrt{2}a_n\right] = 2m\delta_{m+n,0} \quad \rightsquigarrow \quad \left[\sqrt{\frac{2r}{r-1}}a_m, \sqrt{\frac{2r}{r-1}}a_n\right] = 2m\frac{r}{r-1}\delta_{m+n,0}$$

q-deform $\downarrow$ $\qquad$ q-deform $\downarrow$

$$[\alpha_m, \alpha_n] = \frac{[2m]_q[m]_q}{m}\delta_{m+n,0} \quad \overset{p=q^{2r}\text{-deform.}}{\longrightarrow} \quad [\alpha_m, \alpha_n] = \frac{[2m]_q[m]_q}{m}\frac{1-p^m}{1-p^{*m}}\delta_{m+n,0}$$

$U_q(\widehat{\mathfrak{sl}}_2)$ $\qquad$ $U_{q,p}(\widehat{\mathfrak{sl}}_2) \quad p^* = q^{2(r-1)}$

Review of the Quantum Toroidal Algebra $U_{q,t}(\mathfrak{gl}_{1,tor})$

- $\mathfrak{gl}_{1,tor} = \oplus_{(m,n)\neq(0,0)}\mathbb{C}x^m y^n \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2, \quad xy = qyx$
  (Kac-Radul '93, Berman-Gao-Krylyuk '96)
- $U_{q,t}(\mathfrak{gl}_{1,tor})$ is a t-deform. of $\mathfrak{gl}_{1,tor} \cong (q,t)$-deform. of $W_{1+\infty}$ (Miki '07)
  $\cong$ the elliptic Hall algebra (Burban-Schiffmann '05, Schiffmann-Vasserot '11)
- $U_{q,t}(\mathfrak{gl}_{1,tor})$ is a relevant QG str. to discuss the instanton calculus and the AGT corresp. for the 5d & 6d lifts of the 4d $\mathcal{N}=2$ SUSY gauge theories (the linear quiver gauge theories).
  - $W_{q,t}(\mathfrak{g})$ is realized in $U_{q,t}(\mathfrak{gl}_{1,tor}) \otimes \cdots \otimes U_{q,t}(\mathfrak{g}_{1,tor})$

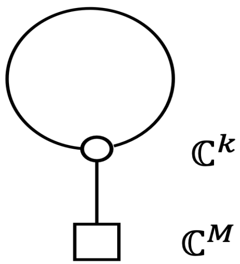
(Miki '07, Feigin-Hashizume-Hoshimo-Shiraishi-Yanagida '09, Berstein-Feigin-Merzon '15)

  - Intertwiners of $U_{q,t}(\mathfrak{gl}_{1,tor})$ w.r.t. the Drinfeld coproduct realize the refiened topological vertices (Awata-Feigin-Shiraishi '12)
  - A certain block of composition of such intertwiners realizes the intertwiner of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ ($\sim$ the vertex operator of $W_{q,t}(\mathfrak{sl}_N)$)
    (Zenkevich '18, Fukuda-Okubo-Shiraishi '19, '20)

This talk : Elliptic Quantum Toroidal Algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$

Main points:

- It has a nice co-alg. str. w.r.t. the Drinfeld copro., which yields the intertwiner $\Phi(u)$ and its "sifted inverse" $\Phi^*(u)$
- It realizes the Jordan quiver W-alg.$W_{p,p^*}(\Gamma(\widehat{A}_0))$(Kimura-Pestun '15), possessing the $SU(4)$ Ω-deformation parameters : $q,\ t,\ p,\ p^*$ s.t. $q/t = p^*/p$ (Nekrasov '16)
- It is a relevant QG str. to study instanton PFs of the 5d & 6d lifts of the 4d $\mathcal{N}=2^*$ gauge theories (i.e. the 4d $\mathcal{N}=2$ theories coupled with the adjoint matter) known also as the Jordan quiver gauge theories (the ADHM gauge theories)



Elliptic Quantum Toroidal Algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$

Let $q, t, p \in \mathbb{C}^*$, $|q|, |t|, |p| < 1$.
We define $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ to be a $\mathbb{C}[[p]]$-algebra generated by

$$\alpha_m, \quad x_n^{\pm}, \quad K^{\pm 1}, \quad \gamma^{\pm 1/2} \quad (m \in \mathbb{Z} \setminus \{0\}, n \in \mathbb{Z}).$$

Let us set

$$\psi^+(z) = K \exp\left(-\sum_{m>0} \frac{p^m}{1-p^m} \alpha_{-m} (\gamma^{-1/2} z)^m\right) \exp\left(\sum_{m>0} \frac{1}{1-p^m} \alpha_{-m} (\gamma^{-1/2} z)^{-m}\right),$$

$$\psi^-(z) = K^{-1} \exp\left(-\sum_{m>0} \frac{1}{1-p^m} \alpha_{-m} (\gamma^{1/2} z)^m\right) \exp\left(\sum_{m>0} \frac{p^m}{1-p^m} \alpha_{-m} (\gamma^{1/2} z)^{-m}\right),$$

$$x^{\pm}(z) = \sum_{n \in \mathbb{Z}} x_n^{\pm} z^{-n}$$

: elliptic currents

$$\gamma^{1/2}, \quad K : \text{central},$$

$$[\alpha_m, \alpha_n] = -\frac{\kappa_m}{m}(\gamma^m - \gamma^{-m})\gamma^{-m}\frac{1-p^m}{1-p^{*m}}\delta_{m+n,0}, \qquad p^* = p\gamma^{-2}$$

$$[\alpha_n, x^+(z)] = -\frac{\kappa_n}{n}\frac{1-p^n}{1-p^{*n}}(\gamma^{-3n/2}z)^n x^+(z),$$

$$[\alpha_n, x^-(z)] = \frac{\kappa_n}{n}(\gamma^{-1/2}z)^n x^-(z),$$

$$[x^+(z), x^-(w)] = \frac{(1-q)(1-1/t)}{(1-q/t)}\left(\delta(\gamma^{-1}z/w)\psi^+(w) - \delta(\gamma z/w)\psi^-(\gamma^{-1}w)\right),$$

$$z^3 G^+(w/z)g(w/z;p^*)x^+(z)x^+(w) = -w^3 G^+(z/w)g(z/w;p^*)x^+(w)x^+(z),$$

$$z^3 G^-(w/z)g(w/z;p)^{-1}x^-(z)x^-(w) = -w^3 G^-(z/w)g(z/w;p)^{-1}x^-(w)x^-(z),$$

$$g(\frac{w}{z};p^*)g(\frac{u}{w};p^*)g(\frac{u}{z};p^*)\left(\frac{w}{u}+\frac{w}{z}-\frac{z}{w}-\frac{u}{w}\right)x^+(z)x^+(w)x^+(u)$$
$$+\text{permutations in } z,w,u = 0,$$

$$g(\frac{w}{z};p)^{-1}g(\frac{u}{w};p)^{-1}g(\frac{u}{z};p)^{-1}\left(\frac{w}{u}+\frac{w}{z}-\frac{z}{w}-\frac{u}{w}\right)x^-(z)x^-(w)x^-(u)$$
$$+\text{permutations in } z,w,u = 0,$$

where

$$\kappa_m = (1-q^m)(1-t^{-m})(1-(t/q)^m), \quad G^\pm(z) = (1-q^{\pm1}z)(1-t^{\mp1}z)(1-(t/q)^{\pm1}z)$$

$$g(z;p) = \exp\left(\sum_{m>0}\frac{\kappa_m}{m}\frac{p^m}{1-p^m}z^m\right) \in \mathbb{C}[[p]][[z]] \quad etc.$$

Hopf Algebroid Structure via Δ^D

Let $\widetilde{\otimes}$ denote the ordinary tensor product with an extra condition

$$F(z,p^*)a\widetilde{\otimes}b = a\widetilde{\otimes}F(z,p)b, \qquad p^* = p\gamma^{-2}$$

The following gives the Drinfeld coproduct for $\mathcal{U}_{q,t,p} = U_{q,t,p}(\mathfrak{gl}_{1,tor})$.

$$\begin{aligned}
\Delta^D(\gamma^{\pm 1/2}) &= \gamma^{\pm 1/2}\widetilde{\otimes}\gamma^{\pm 1/2},\\
\Delta^D(\psi^{\pm}(z)) &= \psi^{\pm}(\gamma_{(2)}^{\mp 1/2}z)\widetilde{\otimes}\psi^{\pm}(\gamma_{(1)}^{\pm 1/2}z)\\
\Delta^D(x^+(z)) &= 1\widetilde{\otimes}x^+(\gamma_{(1)}^{-1/2}z) + x^+(\gamma_{(2)}^{1/2}z)\widetilde{\otimes}\psi^-(\gamma_{(1)}^{-1/2}z),\\
\Delta^D(x^-(z)) &= x^-(\gamma_{(2)}^{-1/2}z)\widetilde{\otimes}1 + \psi^+(\gamma_{(2)}^{-1/2}z)\widetilde{\otimes}x^-(\gamma_{(1)}^{1/2}z).
\end{aligned}$$

Here $\gamma_{(1)} = \gamma\widetilde{\otimes}1, \gamma_{(2)} = 1\widetilde{\otimes}\gamma$.

Definition 3.1

We say that an $\mathcal{U}_{q,t,p}$-module has level $(k,l)\in\mathbb{C}^2$ if $\gamma^{1/2}$ acts by $(t/q)^{k/4}$ and K acts by $(q/t)^{l/2}$.

Theorem 3.2 (q-Fock rep.)

Let $\lambda=(\lambda_1,\lambda_2,\cdots)$ be a partition. The following gives a level-(0,1) action of $\mathcal{U}_{q,t,p}$ on $\mathcal{F}_u^{(0,1)}$.

$$x^+(z)|\lambda\rangle_u = a^+(p)\sum_{i=1}^{\ell(\lambda)+1}\delta(u_i/z)\prod_{j=1}^{i-1}\frac{\theta_p(tu_i/u_j)\theta_p(qu_i/tu_j)}{\theta_p(qu_i/u_j)\theta_p(u_i/u_j)}|\lambda+\mathbf{1}_i\rangle_u,$$

$$x^-(z)|\lambda\rangle_u = a^-(p)(q/t)^{1/2}\sum_{i=1}^{\ell(\lambda)}\delta(q^{-1}u_i/z)\prod_{j=i+1}^{\ell(\lambda)}\frac{\theta_p(qu_j/tu_i)}{\theta_p(u_j/tu_i)}\prod_{j=i+1}^{\ell(\lambda)+1}\frac{\theta_p(tu_j/u_i)}{\theta_p(qu_j/u_i)}|\lambda-\mathbf{1}_i\rangle_u,$$

$$\psi^+(z)|\lambda\rangle_u = (q/t)^{1/2}\prod_{j=1}^{\ell(\lambda)}\frac{\theta_p(t^{-1}u_j/z)}{\theta_p(q^{-1}u_j/z)}\prod_{j=1}^{\ell(\lambda)+1}\frac{\theta_p(tu_j/qz)}{\theta_p(u_j/z)}|\lambda\rangle_u,$$

$$\psi^-(z)|\lambda\rangle_u = (q/t)^{-1/2}\prod_{j=1}^{\ell(\lambda)}\frac{\theta_p(tz/u_j)}{\theta_p(qz/u_j)}\prod_{j=1}^{\ell(\lambda)+1}\frac{\theta_p(qz/tu_j)}{\theta_p(z/u_j)}|\lambda\rangle_u,$$

where we set $u_i=q^{\lambda_i}t^{-i+1}u$ and $\theta_p(z)=-z^{-1/2}(z;p)_\infty(p/z;p)_\infty$.

Level $(1,N)$ Representation $\mathcal{F}_u^{(1,N)}$ of $U_{q,t,p}(\mathfrak{gl}_{1,tor})$

Theorem 3.3

The following gives a level $(1,N)$ representation on the Fock module $\mathcal{F}_u^{(1,N)}$ of α_m carrying a vacuum weight $u \in \mathbb{C}^*$.

$$x^+(z) = uz^{-N}\left(\frac{t}{q}\right)^{\frac{3N}{4}} \exp\left\{-\sum_{n>0}\frac{(t/q)^{n/4}}{1-(t/q)^n}\alpha_{-n}z^n\right\}\exp\left\{\sum_{n>0}\frac{(t/q)^{3n/4}}{1-(t/q)^n}\alpha_n z^{-n}\right\},$$

$$x^-(z) = u^{-1}z^{N}\left(\frac{t}{q}\right)^{-\frac{3N}{4}} \exp\left\{\sum_{n>0}\frac{(t/q)^{n/4}}{1-(t/q)^n}\alpha'_{-n}z^n\right\}\exp\left\{-\sum_{n>0}\frac{(t/q)^{3n/4}}{1-(t/q)^n}\alpha'_n z^{-n}\right\},$$

$$\psi^+(z) = \left(\frac{t}{q}\right)^{-\frac{N}{2}} \exp\left\{-\sum_{n>0}\frac{p^n(t/q)^{-n/4}}{1-p^n}\alpha_{-n}z^n\right\}\exp\left\{\sum_{n>0}\frac{(t/q)^{n/4}}{1-p^n}\alpha_n z^{-n}\right\},$$

$$\psi^-(z) = \left(\frac{t}{q}\right)^{\frac{N}{2}} \exp\left\{-\sum_{n>0}\frac{(t/q)^{n/4}}{1-p^n}\alpha_{-n}z^n\right\}\exp\left\{\sum_{n>0}\frac{p^n(t/q)^{-n/4}}{1-p^n}\alpha_n z^{-n}\right\},$$

where

$$\alpha'_m = \frac{1-p^{*m}}{1-p^m}\gamma^m\alpha_m \quad (m \in \mathbb{Z}_{\neq 0}), \quad \gamma^{1/2} = (t/q)^{1/4}, \text{ hence } \quad p^* = p\gamma^{-2} = pq/t.$$

Theorem 3.4 *(Level $(0,0)$ rep. in terms of the elliptic Ruijsenaars op.)*

For $f(x_1,\cdots,x_N)\in\mathbb{C}[[x_1^{\pm1},\cdots,x_N^{\pm1}]]$,
let $T_{q,x_i}f(\cdots,x_i,\cdots)=f(\cdots,qx_i,\cdots)$.

$$x^+(z)=a^+(p)\sum_{i=1}^N\prod_{j\neq i}\frac{\theta_p(tx_i/x_j)}{\theta_p(x_i/x_j)}\delta(x_i/z)T_{q,x_i},$$

$$x^-(z)=-a^-(p)\sum_{i=1}^N\prod_{j\neq i}\frac{\theta_p(t^{-1}x_i/x_j)}{\theta_p(x_i/x_j)}\delta(q^{-1}x_i/z)T_{q,x_i}^{-1},$$

$$\psi^\pm(z)=\prod_{j=1}^N\left.\frac{\theta_p(t^{-1}x_j/z)\theta_p(q^{-1}tx_j/z)}{\theta_p(x_j/z)\theta_p(q^{-1}x_j/z)}\right|_\pm,$$

or

$$\alpha_m=\frac{(1-t^{-m})(1-(q/t)^{-m})}{m}\sum_{j=1}^N x_j^m\qquad(m\in\mathbb{Z}\backslash\{0\}).$$

In particular, the zero-mode $x_0^+=\oint_{|z|=0}\frac{dz}{2\pi iz}x^+(z)$ acts as the elliptic Ruijsenaars difference operator.

Vertex operator $\Phi(u) : \mathcal{F}^{(1,N+1)}_{-uv} \to \mathcal{F}^{(0,1)}_u \widetilde{\otimes} \mathcal{F}^{(1,N)}_v$

$$\Delta^D(x)\Phi(u) = \Phi(u)x \qquad (\forall x \in \mathcal{U}_{q,t,p})$$

Theorem 4.1

$$\Phi(u) = \sum_\lambda |\lambda\rangle'_u \widetilde{\otimes} \Phi_\lambda(u),$$

$$\Phi_\lambda(u) = \frac{q^{n(\lambda')} t^*(\lambda,u,v,N) N_\lambda(p)}{c_\lambda} \Phi_\emptyset(u) \prod_{(i,j)\in\lambda} \widetilde{x}^-((t/q)^{1/4} t^{-i+1} q^{j-1} u),$$

$$\Phi_\emptyset(u) = \exp\left\{-\sum_{m>0} \frac{\alpha'_{-m}}{\kappa_m}((t/q)^{1/2}u)^m\right\} \exp\left\{\sum_{m>0} \frac{\alpha'_m}{\kappa_m}((t/q)^{1/2}u)^{-m}\right\}.$$

where $|\lambda\rangle'_u = \frac{c_\lambda(p)}{c'_\lambda(p)}|\lambda\rangle_u$, $\widetilde{x}^-(z) = uz^{-N}(t/q)^{3N/4} x^-(z)$, $\quad N_\lambda(0) = N'_\lambda(0) = 1$,

$$< P_\lambda, P_\lambda >_{q,t} = \frac{c'_\lambda}{c_\lambda} \quad \rightsquigarrow \quad \frac{c'_\lambda N_\lambda(p)}{c_\lambda N'_\lambda(p)} = \frac{\prod_{\square\in\lambda} \theta_p(q^{a(\square)+1} t^{\ell(\square)})}{\prod_{\square\in\lambda} \theta_p(q^{a(\square)} t^{\ell(\square)+1})} =: \frac{c'_\lambda(p)}{c_\lambda(p)},$$

$$t^*(\lambda,u,v,N) = (q^{-1}v)^{-|\lambda|}(-u)^{N|\lambda|}\left((-1)^{|\lambda|} q^{n(\lambda')+|\lambda|/2} t^{-n(\lambda)-|\lambda|/2}\right)^N$$

Cf. trig. case : Awata-Feigin-Shiraishi '12

The "shifted inverse"

$$\Phi^*(v) : \mathcal{F}_u^{(0,1)} \widetilde{\otimes} \mathcal{F}_v^{(1,N)} \to \mathcal{F}_{-uv}^{(1,N+1)}$$

$$\Phi^*_\lambda(v)\xi = \Phi^*(v)(|\lambda\rangle'_u \widetilde{\otimes} \xi) \qquad \xi \in \mathcal{F}_v^{(1,N)},$$

$$\Phi^*_\lambda(u) = \frac{q^{n(\lambda')} t(\lambda, v, p^{-1}u, N) N'_\lambda(p)}{c'_\lambda} : \widetilde{\Phi}_\lambda(p^{-1}u)^{-1} :,$$

where

$$\widetilde{\Phi}_\lambda(u) =: \Phi_\emptyset(u) \prod_{(i,j)\in\lambda} \widetilde{x}^-((t/q)^{1/4} t^{-i+1} q^{j-1} u) :$$

The Jordan quiver W-algebra $W_{p,p^*}(\Gamma(\widehat{A}_0))$

Theorem 5.1

The level $(1,N)$ rep. of $\mathcal{U}_{q,t,p}$ realizes $W_{p,p^*}(\Gamma(\widehat{A}_0))$ (Kimura-Pestun'15) with the $SU(4)$ Ω-deformation parameters p,p^*,q,t s.t. $p/p^*=t/q$.

- *screening currents :* $x^{\pm}(z)$
- *generating function :*

$$\begin{aligned} T(u) &= \sum_{\lambda}\Phi^*_{\lambda}(u)\Phi_{\lambda}(u)\\ &= \mathcal{C}\sum_{\lambda}\mathfrak{q}^{|\lambda|}Z^{\widehat{A}_0}_{\lambda}(t,q^{-1},p) :\prod_{\square\in A(\lambda)}Y(u/q^{\square})\prod_{\blacksquare\in R(\lambda)}Y((q/t)u/q^{\blacksquare})^{-1}: \end{aligned}$$

where $q^{\square}=t^{i-1}q^{-j+1}$ *for* $\square=(i,j)\in\lambda$, $\mathfrak{q}=p^{*-1}p^{N-1}(t/q)^{1/2}$,

$$Y(u)=:\exp\left\{\sum_{m\neq 0}\frac{1-p^m}{\kappa_m}\alpha'_m u^{-m}\right\}:,$$

$$Z^{\widehat{A}_0}_{\lambda}(t,q^{-1},p^*)=\prod_{\square\in\lambda}\frac{(1-pq^{a(\square)+1}t^{\ell(\square)})(1-pq^{-a(\square)}t^{-\ell(\square)-1})}{(1-q^{a(\square)+1}t^{\ell(\square)})(1-q^{-a(\square)}t^{-\ell(\square)-1})}.$$

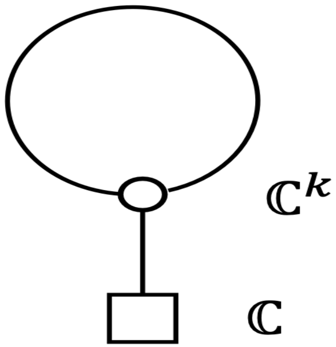
Cf. the same realization of $W_{p,p^*}(\mathfrak{g})$ by $U_{q,p}(\widehat{\mathfrak{g}})$, $\widehat{\mathfrak{g}}=A^{(1)}_N, D^{(1)}_N, B^{(1)}_N$ (K'14)

The 5d & 6d lifts of the $\mathcal{N}=2^*$ $U(1)$ Theory

$$T(u)=\mathcal{C}\sum_{\lambda}\mathfrak{q}^{|\lambda|}Z_{\lambda}^{\widehat{A}_0}(t,q^{-1},p):\prod_{\square\in A(\lambda)}Y(u/q^{\square})\prod_{\blacksquare\in R(\lambda)}Y((q/t)u/q^{\blacksquare})^{-1}:$$

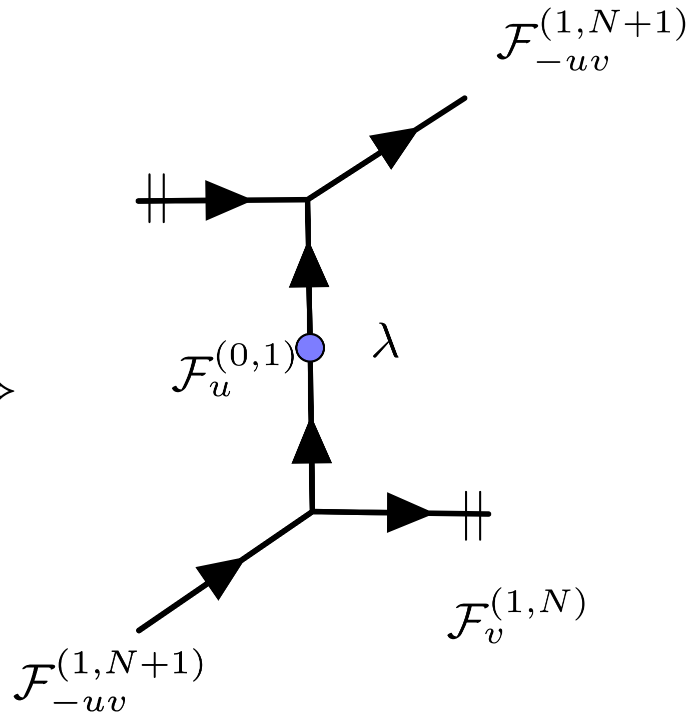
From this we immediately obtain the instanton PF of the 5d lift of the $\mathcal{N}=2^*$ $U(1)$ Theory

$$\langle 0|T(u)|0\rangle=\mathcal{C}\sum_{\lambda}\mathfrak{q}^{|\lambda|}Z_{\lambda}^{\widehat{A}_0}(t,q^{-1},p)=\mathcal{C}\sum_{k\geq 0}\mathfrak{q}^{k}\underbrace{\sum_{\substack{\lambda\\|\lambda|=k}}Z_{\lambda}^{\widehat{A}_0}(t,q^{-1},p)}_{\substack{\chi_y\text{-genus of }\mathrm{Hilb}_k(\mathbb{C}^2)\\(y=p)\quad\text{(Li-Liu-Zhou'04)}}}.$$



This result and $T(u) = \sum_\lambda \Phi^*_\lambda(u)\Phi_\lambda(u)$ indicate

$T(u) \quad \Longleftrightarrow$



(Hollowood-Iqbal-Vafa'08)

The 5d & 6d lifts of the $\mathcal{N}=2^*$ $U(1)$ Theory

The trace gives the instanton PF of the 6d lift of the $\mathcal{N}=2^*$ $U(1)$ theory

$$\mathrm{tr}_{\mathcal{F}^{(1,N+1)}_{-uv}} Q^{-d} T(u) = \mathcal{C}_Q \sum_{\lambda} \mathfrak{q}^{|\lambda|} Z^{\widehat{A}_0}_{\lambda}(t, q^{-1}, p; Q), \quad v \in \mathbb{C}^*$$

where

$$\mathcal{Z}^{\widehat{A}_0}_{\lambda}(t, q^{-1}, p; Q) = \prod_{\square \in \lambda} \frac{\theta_Q(p\, q^{a(\square)+1} t^{\ell(\square)})\theta_Q(p q^{-a(\square)} t^{-\ell(\square)-1})}{\theta_Q(q^{a(\square)+1} t^{\ell(\square)})\theta_Q(q^{-a(\square)} t^{-\ell(\square)-1})}.$$

Note :

$$\sum_{\lambda, |\lambda|=k} \mathcal{Z}^{\widehat{A}_0}_{\lambda}(t, q^{-1}, p; Q) \text{ is the equivariant elliptic genus of } \mathrm{Hilb}_k(\mathbb{C}^2).$$

(Haghighat-Iqbal-Kozcaz-Lockhart-Vafa'15)

The 5d & 6d Lifts of the $\mathcal{N}=2^* \ U(M)$ Theory

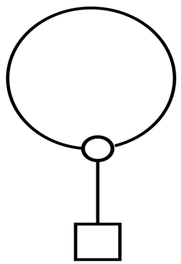
$$\langle 0|T(u_1)\cdots T(u_M)|0\rangle = \mathcal{C}_M \sum_{k=0}^{\infty} \mathfrak{q}_M^k \chi_p(\mathfrak{M}_{k,M}),$$

where $u_{j,i}=u_j/u_i$, $\mathfrak{q}_M=\mathfrak{q}p^{-(M-1)}=p^{*-1}p^{M+N}(t/q)^{1/2}$,

$$\chi_p(\mathfrak{M}_{k,M}) = \sum_{\substack{\lambda^{(1)},\cdots,\lambda^{(M)} \\ \sum|\lambda^{(j)}|=k}} \prod_{i,j}^{M} Z_{\lambda^{(i)},\lambda^{(j)}}(u_{j,i};t,q^{-1},p),$$

$$\mathcal{Z}_{\lambda,\mu}(u;t,q^{-1},p) = \prod_{\square\in\lambda} \frac{(1-p\,uq^{a_\mu(\square)+1}t^{\ell_\lambda(\square)})}{(1-uq^{a_\mu(\square)+1}t^{\ell_\lambda(\square)})} \prod_{\blacksquare\in\mu} \frac{(1-puq^{-a_\lambda(\blacksquare)}t^{-\ell_\mu(\blacksquare)-1})}{(1-uq^{a_\lambda(\blacksquare)}t^{-\ell_\mu(\blacksquare)-1})},$$

gives the instanton PF of the 5d lift of the $\mathcal{N}=2^* \ U(M)$ theory i.e. the gen. func. of the χ_y $(y=p)$ genus of the ADHM moduli space $\mathfrak{M}_{k,M}$.
(Haghighat-Iqbal-Kozcaz-Lockhart-Vafa '15)



The trace gives the instanton PF of the 6d lift of the $\mathcal{N}=2^* \ U(M)$ theory

$$\mathrm{tr}_{\mathcal{F}^{(1,N+1)}_{-u_1 v_1}} Q^{-d} T(u_1)\cdots T(u_M) = \mathcal{C}_{Q,M} \sum_{k=0}^{\infty} \mathfrak{q}_M^k \, \mathcal{E}_{p,Q}(\mathfrak{M}_{k,M}),$$

where

$$\mathcal{E}_{p,Q}(\mathfrak{M}_{k,M}) = \sum_{\substack{\lambda^{(1)},\cdots,\lambda^{(M)} \\ \sum_j |\lambda^{(j)}|=k}} \prod_{i,j=1}^{M} \mathcal{Z}_{\lambda^{(i)},\lambda^{(j)}}(u_{j,i}; t, q^{-1}, p; Q),$$

$$\mathcal{Z}_{\lambda^{(i)},\lambda^{(j)}}(u; t, q^{-1}, p; Q)$$
$$= \prod_{\square\in\lambda^{(i)}} \frac{\theta_Q(p\, u q^{a_{\lambda^{(j)}}(\square)+1} t^{\ell_{\lambda^{(i)}}(\square)})}{\theta_Q(u q^{a_{\lambda^{(j)}}(\square)+1} t^{\ell_{\lambda^{(i)}}(\square)})} \prod_{\blacksquare\in\lambda^{(j)}} \frac{\theta_Q(p u q^{-a_{\lambda^{(i)}}(\blacksquare)} t^{-\ell_{\lambda^{(j)}}(\blacksquare)-1})}{\theta_Q(u q^{a_{\lambda^{(i)}}(\blacksquare)} t^{-\ell_{\lambda^{(j)}}(\blacksquare)-1})}.$$

$\mathcal{E}_{p,Q}(\mathfrak{M}_{k,M})$: the equivariant elliptic genus of the ADHM moduli space $\mathfrak{M}_{k,M}$.

Hence we have shown a new AGT correspondence :

Instanton PF of the 5d & 6d lifts of the 4d $\mathcal{N}=2^*$ th.

$\Longleftrightarrow$ corr. fnc. of $W_{p,p^*}(\Gamma(\widehat{A}_0))$

via a realization of $W_{p,p^*}(\Gamma(\widehat{A}_0))$ by $U_{q,t,p}(\mathfrak{gl}_{1,tor})$