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Elliptic quantum toroidal algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ and affine quiver gauge theories

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Abstract
We introduce a new elliptic quantum toroidal algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$. Various representations in the quantum toroidal algebra $U_{q,t}(\mathfrak{gl}_{1,tor})$ are extended to the elliptic case including the level (0,0) representation realized by using the elliptic Ruijsenaars difference operator. Various intertwining operators of $U_{q,t,p}(\mathfrak{gl}_{1,tor})$-modules w.r.t. the Drinfeld comultiplication are also constructed. We show that $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ gives a realization of the affine quiver W-algebra $W_{q,t}(\Gamma(\widehat{A}_0))$ proposed by Kimura–Pestun. This realization turns out to be useful to derive the Nekrasov instanton partition functions, i.e., the χ_y- and elliptic genus, of the 5d and 6d lifts of the 4d $\mathcal{N} = 2^* \ U(M)$ theories and provide a new Alday–Gaiotto–Tachikawa correspondence.

Keywords Elliptic quantum group · Toroidal algebra · Affine quiver varieties

Mathematics Subject Classification 20C35

1 Introduction

The aim of this paper is to introduce a new elliptic quantum toroidal algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ associated with the toroidal algebra of type $\mathfrak{gl}_1$ and show its connection

Dedicated to Professor Michio Jimbo on the occasion of his 70th birthday.

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to the affine quiver W-algebra $W_{q,t}(\Gamma(\widehat{A}_0))$ and the affine quiver gauge theories in 5d and 6d.

We formulate $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ by generators and relations in the same scheme as the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{g}})$ associated with the affine Lie algebra $\widehat{\mathfrak{g}}$ [16, 40, 47, 51, 55]. The latter is the Drinfeld realization of the face-type elliptic quantum group [39] in terms of the Drinfeld generators, which are a deformation of the loop generators of the affine Lie algebras [13]. One of the important properties of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{g}})$ is that it gives a realization of the deformation of the W algebras of the Goddard–Kent–Olive (GKO) coset-type $\widehat{\mathfrak{g}}_{r-h^\vee-1} \oplus \widehat{\mathfrak{g}}_1 \supset \widehat{\mathfrak{g}}_{r-h^\vee}$ [30, 31]. Note that the GKO coset-type W algebras are isomorphic to those obtained by the quantum Hamiltonian reduction for simply laced $\widehat{\mathfrak{g}}$ but not for non-simply laced ones. See for example [12]. For simplicity, let us consider the deformed W algebra in the simply laced case and use the same notation $W_{q,t}(\mathfrak{g})$ as the one for the Hamiltonian reduction type [27]. Then, the level-1 representation of $U_{q,p}(\widehat{\mathfrak{g}})$ realizes $W_{q,t}(\mathfrak{g})$ [16, 40, 45, 47, 50] in the following way.[1]

a) the parameters (p, p^*) of $U_{q,p}(\widehat{\mathfrak{g}})$ are identified with (q, t) of $W_{q,t}(\mathfrak{g})$, where $p^* = pq^{-2}$. Here, q appearing in the both sides are different.
b) the elliptic Drinfeld currents $E_j(z)$, $F_j(z)$ of $U_{q,p}(\widehat{\mathfrak{g}})$ realize the screening currents $S_j^+(z)$, $S_j^-(z)$ of $W_{q,t}(\mathfrak{g})$.
c) the basic generating functions $\Lambda_i(z)$ of $W_{q,t}(\mathfrak{g})$ are realized by certain compositions of the intertwiners of the $U_{q,p}(\widehat{\mathfrak{g}})$-modules w.r.t. the Drinfeld comultiplication.

Furthermore, for $W_{q,t}(\mathfrak{sl}_N)$ a deformation of the primary fields constructed for example in [8] is realized as certain fusions of intertwiners of the $U_{q,p}(\widehat{\mathfrak{sl}}_N)$-modules.

The elliptic quantum toroidal algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ is an elliptic quantum group associated with the toroidal algebra $\mathfrak{gl}_1$. It is an elliptic analogue of the quantum toroidal algebra $U_{q,t}(\mathfrak{gl}_{1,tor})$ introduced by Miki as a deformation of the $W_{1+\infty}$ algebra [62]. Various representations of $U_{q,t}(\mathfrak{gl}_{1,tor})$ have been studied by many papers such as [5, 9, 17, 18, 20–23, 62]. It is also remarkable that $U_{q,t}(\mathfrak{gl}_{1,tor})$ is isomorphic to the elliptic Hall algebra introduced by Schiffmann and Vasserot [17, 18, 78–81]. See also [22, 23, 67] for the shuffle algebra formulation.

Representations of $U_{q,t}(\mathfrak{gl}_{1,tor})$ have many interesting applications to the 5d lifts of the 4d $\mathcal{N} = 2$ SUSY gauge theories, which are the gauge theories associated with the linear quivers, such as a calculation of the Nekrasov instanton partition functions and a study of the 5d analogue of the Alday–Gaiotto–Tachikawa (AGT) correspondence [3]. See for example [5, 6, 10, 11, 63, 86]. The essential properties of $U_{q,t}(\mathfrak{gl}_{1,tor})$ for applications to show AGT correspondence are summarized into the three points:

1) the deformed W-algebra $W_{q,t}(\mathfrak{sl}_N)$ is realized on the N-fold tensor product of $U_{q,t}(\mathfrak{gl}_{1,tor})$ [6, 9, 23, 62].
2) the intertwiners of $U_{q,t}(\mathfrak{gl}_{1,tor})$-modules w.r.t the Drinfeld comultiplication realize the refined topological vertex [4, 36] as the vertex operators on the bosonic Fock space [5].

[1] The same was confirmed for the non-simply laced case, say for $U_{q,p}(B_l^{(1)})$ [16, 50], which realizes the deformation of Fateev–Lukyanov's WB_l algebra [59].

3) a certain block of composition of the intertwiners of $U_{q,t}(\mathfrak{gl}_{1,tor})$-modules realizes the intertwiner (screened vertex operator) of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ [28, 29, 86].

Hence, by 2) taking a composition of the intertwiners of $U_{q,t}(\mathfrak{gl}_{1,tor})$-modules according to the web diagram [4, 5, 36], the expectation value of it gives an instanton partition function of the 5d lift of the 4d $\mathcal{N}=2$ theory and basically by 3) the same quantity is identified with an expectation value of the intertwiners of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$, i.e., a deformation of the conformal block. This is the 5d analogue of the AGT correspondence. It is also remarkable that thus obtained deformed conformal blocks coincides with the vertex functions introduced by Okounkov [1, 56, 74]. See for example (6.4) in [52] and take its trigonometric limit $q^\kappa \to 0$. There are some different approaches based on the elliptic Hall algebra [81] and the shuffle algebra [68]. For the affine Yangian case see [61]. See also [65].

Instead of taking an expectation value, one can take a trace of the composed intertwininig operators of $U_{q,t}(\mathfrak{gl}_{1,tor})$ [26, 77, 87]. Then, it gives an instanton partition function of the 6d lift of the theory [26, 37, 72] and the same arguments as 1)–3) yield that it is equivalent to the trace of the intertwiners of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$, i.e., the trace of the vertex operator of $W_{q,t}(\mathfrak{sl}_N)$ [29]. Note that the integral expressions for the latter are given as a part of the integral solution to the elliptic q-KZ equation [52, 55], which has exactly the same structure as discussed as quantum q-Langlands correspondence in [1]. This is the 6d analogue of the AGT correspondence. In this way, the instanton calculus and the AGT correspondence in 5d and 6d lifts of the 4d $\mathcal{N}=2$ SUSY gauge theories associated with the linear quivers can be treated by using the quantum toroidal algebra $U_{q,t}(\mathfrak{gl}_{1,tor})$.

In this paper, we show that the elliptic quantum toroidal algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ provides a relevant quantum group structure to treat the 5d and 6d lifts of the 4d $\mathcal{N}=2^*$ SUSY gauge theories, which are the $\mathcal{N}=2$ SUSY gauge theories coupled with the adjoint matter [69, 70]. These are gauge theories associated with the Jordan quiver. We also formulate the AGT correspondence between these 5d and 6d lifts and the affine quiver W algebra $W_{q,t}(\Gamma(\widehat{A}_0))$. For this purpose, we construct some useful representations of $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ and give a realization of $W_{q,t}(\Gamma(\widehat{A}_0))$ [41] in the same way as the above b) and c) for $U_{q,p}(\widehat{\mathfrak{g}})$. Namely the level $(1, N)$ elliptic currents $x^\pm(z)$ of $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ give the screening currents $S^\pm(z)$ of $W_{q,t}(\Gamma(\widehat{A}_0))$, and a certain composition of the "intertwiners" of $U_{q,t,p}(\mathfrak{gl}_{1,tor})$.[2] gives a realization of the generating function $T(u)$. There one of the key properties of $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ is that the level $(1, N)$ representation possesses the four parameters q, t, p, p^* satisfying $p/p^* = t/q$. They play the role of the $SU(4)$ Ω deformation parameters introduced by Nekrasov [70], i.e., two of three independent parameters are the 5d analogue the Ω deformation parameters ϵ_1, ϵ_2 [69] and the remaining one is a deformation of the adjoint mass parameter in the 4d $\mathcal{N}=2^*$ SUSY gauge theories.

Realizing the generating function $T(u)$ of $W_{q,t}(\Gamma(\widehat{A}_0))$, it becomes trivial that the vacuum expectation value of $T(u)$ gives the Nekrasov instanton partition function of the 5d lift of the 4d $\mathcal{N}=2^*$ $U(1)$ theory, i.e., the generating function of the χ_y genus

[2] A composition of the type I intertwiner $\Phi(u)$ of the $U_{q,t,p}(\mathfrak{gl}_{1,tor})$-module and its shifted inverse $\Phi^*(u)$. See Sect. 5

of the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ of n points on $\mathbb{C}^2$ [41, 42], where we take $y = p$. However, this fact and our realization of $T(u)$ as a composition of the "intertwiners" lead us to identify $T(u)$ with a basic refined topological vertex operator corresponding to the diagram Fig. 1 studied in [32, 34, 36]. Then, such realization makes calculation of compositions of $T(u)$ easy and allows us to apply them to various instanton calculus in the 5d and 6d lifts of the 4d $\mathcal{N} = 2^*$ theories. This includes $\mathcal{N} = 2^*$ $U(M)$ theories, whose instanton partition function is the generating function of the χ_y genus of the moduli space of the rank M instantons with charge n. Their 6d lifts give the generating functions of the elliptic genus of the same moduli space. Hence, we establish a new AGT correspondence between the 5d and 6d lifts of the 4d $\mathcal{N} = 2^*$ SUSY gauge theories and the affine quiver W algebra $W_{q,t}(\Gamma(\widehat{A}_0))$.

This paper is organized as follows. In Sect. 2, after reviewing basic facts in the quantum toroidal algebra $U_{q,t}(\mathfrak{gl}_{1,tor})$, we give a definition of the elliptic quantum toroidal algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$. The Z-algebra structure, an elliptic analogue of Miki's automorphism and Hopf algebroid structure are also given. In Sect. 3, we construct the level $(1, N)$, $(0, 1)$ and $(0, 0)$ representations. In particular, the level $(0, 0)$ representation is given by using the elliptic Ruijsenaars difference operator on $\mathbb{C}[[x_1, \dots, x_N]]$. In Sect. 4, we construct the type I and the type II dual vertex operator as intertwining operators of the $U_{q,t,p}(\mathfrak{gl}_{1,tor})$-modules constructed in Sect. 3. We also give a definition of their shifted inverse operators. In Sect. 5, we give a realization of the affine quiver W algebra $W_{q,t}(\Gamma(\widehat{A}_0))$ by using the level $(1, N)$ representation of $U_{q,t,p}(\mathfrak{gl}_{1,tor})$. In Sect. 6, several calculations of the instanton partition functions of the 5d and 6d lifts of the 4d $\mathcal{N} = 2^*$ SUSY gauge theories are given. The results indicate a new AGT correspondence between those gauge theories and $W_{q,t}(\Gamma(\widehat{A}_0))$. In Sect. 7, we summarize calculations of basic correlation functions of our intertwining operators, which should provide the $\mathcal{N} = 2^*$ analogues of the instanton partition functions of the 5d and 6d lifts of the 4d $\mathcal{N} = 2$ pure $SU(N)$ theory and $SU(N)$ theory coupled with $2N$ fundamental matters obtained, for example, in [4, 5]. Appendix A is a list of formulas, which is used to discuss the Z-algebra structure in Sect. 1. Appendix B is a direct check of the statement on the level $(0, 1)$ representation. In Appendices C and D, proofs of Theorem 4.2 and 4.7 are given. In Appendix F, some useful formulas of the Nekrasov function are collected.

Partial results in Sects. 2, 3, 4, 5 have been presented by H.K. at several workshops [54].

2 Elliptic quantum toroidal algebra $\mathcal{U}_{q,t,p}(\mathfrak{gl}_{1,tor})$

After reviewing the quantum toroidal algebra $U_{q,t}(\mathfrak{gl}_{1,tor})$, we introduce the elliptic quantum toroidal algebra $\mathcal{U}_{q,t,p}(\mathfrak{gl}_{1,tor})$. The Z-algebra structure, an elliptic analogue of Miki's automorphism and Hopf algebroid structure are also given.

2.1 The quantum toroidal algebra $U_{q,t}(\mathfrak{gl}_{1,tor})$

This section is a review of Miki's results [62] in our notation.

2.1.1 Definition

Definition 2.1 Let q, t be nonzero complex numbers such that $q, t, q/t$ are not roots of unity, and set

$$\kappa_m = (1-q^m)(1-t^{-m})(1-(t/q)^m),$$
$$G^{\pm}(z) = (1-q^{\pm 1}z)(1-t^{\mp 1}z)(1-(q/t)^{\mp 1}z).$$

The quantum toroidal algebra $\mathcal{U}_{q,t} = U_{q,t}(\mathfrak{gl}_{1,tor})$ is a $\mathbb{C}$-algebra generated by

$$a_m, \quad X_n^{\pm}, \quad \gamma^{\pm 1/2}, \quad (\psi_0^{\pm})^{\pm 1} \quad (m \in \mathbb{Z}\backslash\{0\},\ n \in \mathbb{Z}).$$

By using the generating functions

$$X^{\pm}(z) = \sum_{n\in\mathbb{Z}} X_n^{\pm} z^{-n},$$
$$\phi(z) = \psi_0^+ \exp\left\{\sum_{m>0} a_m \gamma^{m/2} z^{-m}\right\},$$
$$\psi(z) = \psi_0^- \exp\left\{-\sum_{m>0} a_{-m} \gamma^{m/2} z^{m}\right\},$$

the defining relations are

$$\gamma^{\pm 1/2} : \text{ central}, \tag{2.1}$$
$$[a_m, a_n] = -\delta_{m+n,0}\frac{\kappa_m}{m}(\gamma^m - \gamma^{-m})\gamma^{-|m|}, \tag{2.2}$$
$$[a_m, X^+(w)] = -\frac{\kappa_m}{m}\gamma^{-|m|-m/2} w^m X^+(w), \tag{2.3}$$
$$[a_m, X^-(w)] = \frac{\kappa_m}{m}\gamma^{-m/2} w^m X^-(w), \tag{2.4}$$
$$[X^+(z), X^-(w)] = \frac{(1-q)(1-1/t)}{(1-q/t)}\big(\delta(\gamma^{-1}z/w)\phi(w) - \delta(\gamma z/w)\psi(\gamma^{-1}w)\big), \tag{2.5}$$
$$z^3 G^{\pm}(w/z) X^{\pm}(z) X^{\pm}(w) = -w^3 G^{\pm}(z/w) X^{\pm}(w) X^{\pm}(z), \tag{2.6}$$
$$\mathrm{Sym}_{z_1,z_2,z_3} z_2 z_3^{-1}[X^{\pm}(z_1), [X^{\pm}(z_2), X^{\pm}(z_3)]] = 0, \tag{2.7}$$

where $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$.

Remark By rescaling the generators $X_m^{\pm}$ as

$$X'^{+}(z) = (1-t/q)X^+(z), \qquad X'^{-}(z) = (1-q/t)X^-(z),$$

the relation (2.5) can be rewritten as

$$[X'^{+}(z), X'^{-}(w)] = \kappa_1\big(\delta(\gamma^{-1}z/w)\phi(w) - \delta(\gamma z/w)\psi(\gamma^{-1}w)\big)$$

so that the whole relations of $\mathcal{U}_{q,t}$ are symmetric under any permutations among $q_1 = q, q_2 = t^{-1}, q_3 = t/q$. Note also that (2.2)–(2.4) are equivalent to

$$\phi(z)\phi(w) = \phi(w)\phi(z), \qquad \psi(z)\psi(w) = \psi(w)\psi(z), \tag{2.8}$$

$$\frac{G^+(\gamma^{-1}w/z)}{G^+(\gamma w/z)}\phi(z)\psi(w) = \frac{G^+(\gamma z/w)}{G^+(\gamma^{-1}z/w)}\psi(w)\phi(z), \tag{2.9}$$

$$z^3 G^+(\gamma^{-1}w/z)\phi(z)X^+(w) = -\gamma^{-3}w^3 G^+(\gamma z/w)^{-1}X^+(w)\phi(z), \tag{2.10}$$

$$z^3 G^+(w/z)\psi(z)X^+(w) = -w^3 G^+(z/w)X^+(w)\psi(z), \tag{2.11}$$

$$z^{-3}G^+(w/z)^{-1}\phi(z)X^-(w) = -w^{-3}G^+(z/w)^{-1}X^-(w)\phi(z), \tag{2.12}$$

$$z^{-3}G^+(\gamma^{-1}w/z)^{-1}\psi(z)X^-(w) = -\gamma^3 w^{-3}G^+(\gamma z/w)^{-1}X^-(w)\psi(z). \tag{2.13}$$

2.1.2 Automorphism of $\mathcal{U}_{q,t}$

Let us set $\omega = (1-q)(1-t^{-1})$ and define $Y_l^{\pm} \in \mathcal{U}_{q,t}$ $(l \in \mathbb{Z})$ by

$$Y_l^+ = \begin{cases} (\psi_0^+)^l (\text{ad}\, X_0^+)^{l-1} X_{-1}^+/(-\omega)^{l-1} & (l > 0) \\ -\dfrac{\gamma^{1/2}}{1-t/q}a_{-1} & (l = 0) \\ -(q/t)^{|l|}\gamma (\text{ad}\, X_0^-)^{|l|-1} X_{-1}^-/(-\omega)^{|l|-1} & (l < 0) \end{cases}$$

$$Y_l^- = \begin{cases} -(t/q)\gamma^{-1} (\text{ad}\, X_0^+)^{l-1} X_1^+/\omega^{l-1} & (l > 0) \\ \dfrac{\gamma^{1/2}}{1-q/t}a_1 & (l = 0) \\ (q/t)^{|l|-1}(\psi_0^+)^l (\text{ad}\, X_0^-)^{|l|-1} X_1^-/\omega^{|l|-1} & (l < 0) \end{cases},$$

where $(ad\, x)y = [x, y]$ for $x, y \in \mathcal{U}_{q,t}$. In addition, for $k \in \mathbb{Z}\backslash\{0\}$ we set

$$h_k = \begin{cases} [X_{-1}^+, \overbrace{X_0^+, \ldots, X_0^+}^{k-2}, X_1^+] & (k \geq 2) \\ X_0^+ & (k = 1) \\ -(q/t)X_0^- & (k = -1) \\ -(q/t)^{|k|}[X_1^-, \underbrace{X_0^-, \ldots, X_0^-}_{|k|-2}, X_{-1}^-] & (k \leq -2) \end{cases},$$

where for $x_1, x_2, \ldots, x_n \in \mathcal{U}_{q,t}$

$$[x_n, \ldots, x_1] = [x_n, [x_{n-1}, \ldots, [x_2, x_1]\cdots]].$$

In terms of h_k, we define $b_k \in \mathcal{U}_{q,t}$ $(k \in \mathbb{Z}\backslash\{0\})$ by the relations

$$\exp\left(\sum_{k>0} b_k (\psi_0^+)^{k/2} z^{-k}\right) = 1 + \kappa_1 \sum_{k\geq 1} \frac{h_k}{\omega^k} z^{-k},$$
$$\exp\left(-\sum_{k>0} b_{-k} (\psi_0^+)^{k/2} z^{k}\right) = 1 + \kappa_1 \sum_{k\geq 1} \frac{h_{-k}}{\omega^k} z^{k},$$

where we set $\omega = (1-q)(1-t^{-1})$. Then, we have

Theorem 2.2 (Miki) *The map*

$$\Psi : a_1 \mapsto (1-t/q)\gamma^{-1/2} X_0^+, \quad a_{-1} \mapsto -(1-q/t)\gamma^{-1/2} X_0^-,$$
$$X_0^+ \mapsto -\frac{\gamma^{1/2}}{1-t/q} a_{-1}, \ X_0^- \mapsto \frac{\gamma^{1/2}}{1-q/t} a_1,$$
$$\gamma \mapsto \psi_0^+, \quad \psi_0^+ \mapsto \gamma^{-1}$$

gives an automorphism of $\mathcal{U}_{q,t}$ satisfying $\Psi^4 = 1$. Moreover, Ψ maps

$$a_k \mapsto b_k, \quad X_l^\pm \mapsto Y_l^\pm, \quad b_k \mapsto a_{-k}, \quad Y_l^\pm \mapsto X_{-l}^\mp$$

For readers' convenience, we listed below the correspondence between our notations and those used in [62]. The symbols with M denote the ones in [62].

$$\gamma_M = q^{1/2}, \qquad q_M = (t/q)^{1/2}, \qquad C_M^{\pm 1} = \gamma^{\pm 1}, \qquad C_M^{\pm 1} = \psi_0^{\pm},$$
$$(q_M - q_M^{-1})(\gamma_M^k - \gamma_M^{-k}) a_k^M = \gamma^{|k|/2} a_k, \qquad (q_M - q_M^{-1})(\gamma_M^k - \gamma_M^{-k}) b_k^M = (\psi_0^+)^{|k|/2} b_k,$$
$$h_k^M = \left(\frac{t^{1/2}}{1-q}\right)^{|k|} h_k,$$
$$X_l^{M+} = \frac{t^{1/2}}{1-q} X_l^+, \quad X_l^{M-} = -\frac{t^{1/2} q/t}{1-q} X_l^-, \qquad Y_l^{M+} = \frac{t^{1/2}}{1-q} Y_l^+, \quad Y_l^{M-} = -\frac{t^{1/2} q/t}{1-q} Y_l^-.$$

2.2 The elliptic quantum toroidal algebra $U_{q,t,p}(\mathfrak{gl}_{1,tor})$

Let p be an indeterminate and set

$$(z;p)_n = (1-z)(1-zp)\cdots(1-zp^{n-1}), \quad (z;p)_0 = 1,$$

for $n \in \mathbb{N}$, $z \in \mathbb{C}$. We define $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ by generators and relations as follows.

Definition 2.3 The elliptic quantum toroidal algebra $\mathcal{U}_{q,t,p} = U_{q,t,p}(\mathfrak{gl}_{1,tor})$ is a $\mathbb{C}[[p]]$-algebra generated by $\alpha_m, x_n^\pm$, $(m \in \mathbb{Z}\backslash\{0\}, n \in \mathbb{Z})$ and invertible elements $\psi_0^\pm, \gamma^{1/2}$. The defining relations can be conveniently expressed in terms of the generating functions, which we call the elliptic currents,

$$x^{\pm}(z)=\sum_{n\in\mathbb{Z}}x^{\pm}_n z^{-n},$$

$$\psi^{+}(z)=\psi^{+}_0\exp\left(-\sum_{m>0}\frac{p^m}{1-p^m}\alpha_{-m}(\gamma^{-1/2}z)^{m}\right)\exp\left(\sum_{m>0}\frac{1}{1-p^m}\alpha_{m}(\gamma^{-1/2}z)^{-m}\right),$$

$$\psi^{-}(z)=\psi^{-}_0\exp\left(-\sum_{m>0}\frac{1}{1-p^m}\alpha_{-m}(\gamma^{1/2}z)^{m}\right)\exp\left(\sum_{m>0}\frac{p^m}{1-p^m}\alpha_{m}(\gamma^{1/2}z)^{-m}\right).$$

The defining relations are

$$\psi^{\pm}_0,\gamma^{1/2}\ :\ \text{central}, \tag{2.14}$$

$$[\alpha_m,\alpha_n]=-\frac{\kappa_m}{m}(\gamma^m-\gamma^{-m})\gamma^{-m}\frac{1-p^m}{1-p^{*m}}\delta_{m+n,0}, \tag{2.15}$$

$$[\alpha_m,x^{+}(z)]=-\frac{\kappa_m}{m}\frac{1-p^m}{1-p^{*m}}\gamma^{-3m/2}z^m x^{+}(z)\quad (m\neq 0). \tag{2.16}$$

$$[\alpha_m,x^{-}(z)]=\frac{\kappa_m}{m}\gamma^{-m/2}z^m x^{-}(z)\quad (m\neq 0), \tag{2.17}$$

$$[x^{+}(z),x^{-}(w)]=\frac{(1-q)(1-1/t)}{(1-q/t)}\big(\delta(\gamma^{-1}z/w)\psi^{+}(w)-\delta(\gamma z/w)\psi^{-}(\gamma^{-1}w)\big), \tag{2.18}$$

$$z^3G^{+}(w/z)g(w/z;\,p^*)x^{+}(z)x^{+}(w)=-w^3G^{+}(z/w)g(z/w;\,p^*)x^{+}(w)x^{+}(z), \tag{2.19}$$

$$z^3G^{-}(w/z)g(w/z;\,p)^{-1}x^{-}(z)x^{-}(w)=-w^3G^{-}(z/w)g(z/w;\,p)^{-1}x^{-}(w)x^{-}(z), \tag{2.20}$$

$$g(w/z;\,p^*)g(u/w;\,p^*)g(u/z;\,p^*)\left(\frac{w}{u}+\frac{w}{z}-\frac{z}{w}-\frac{u}{w}\right)x^{+}(z)x^{+}(w)x^{+}(u)$$
$$+\text{ permutations in } z,w,u=0, \tag{2.21}$$

$$g(w/z;\,p)^{-1}g(u/w;\,p)^{-1}g(u/z;\,p)^{-1}\left(\frac{w}{u}+\frac{w}{z}-\frac{z}{w}-\frac{u}{w}\right)x^{-}(z)x^{-}(w)x^{-}(u)$$
$$+\text{ permutations in } z,w,u=0, \tag{2.22}$$

where we set $p^*=p\gamma^{-2}$ and

$$g(z;s)=\exp\left(\sum_{m>0}\frac{\kappa_m}{m}\frac{s^m}{1-s^m}z^m\right)\in\mathbb{C}[[z]] \tag{2.23}$$

for $s=p,p^*$.

We treat these relations as formal Laurent series in z, w and u. All the coefficients in z,w,u are well defined in the p-adic topology [24, 25, 51].

It is sometimes convenient to set

$$\alpha'_m = \frac{1-p^{*m}}{1-p^m}\gamma^m\alpha_m \qquad (m\in\mathbb{Z}_{\neq 0})$$

which satisfy

$$[\alpha'_m,\alpha'_n] = -\frac{\kappa_m}{m}(\gamma^m-\gamma^{-m})\gamma^m\frac{1-p^{*m}}{1-p^m}\delta_{m+n,0}. \tag{2.24}$$

Remark In the same way as $\mathcal{U}_{q,t}$, by rescaling the generators $x^{\pm}_m$ as

$$x'^{+}(z) = (1-t/q)x^+(z), \qquad x'^{-}(z) = (1-q/t)x^-(z),$$

the relation (2.18) can be rewritten as

$$[x'^{+}(z), x'^{-}(w)] = \kappa_1\big(\delta(\gamma^{-1}z/w)\psi^+(w) - \delta(\gamma z/w)\psi^-(\gamma^{-1}w)\big)$$

so that the whole relations of $\mathcal{U}_{q,t,p}$ are symmetric under any permutations among $q_1=q, q_2=t^{-1}, q_3=t/q$.

Remark On $\mathcal{U}_{q,t,p}$-modules, where the central element $\gamma^{1/2}$ takes a complex value, we regard $p, p^*=p\gamma^{-2}$ as a generic complex number with $|p|<1, |p^*|<1$, and have

$$g(z;p) = \frac{(pqz;p)_\infty}{(pq^{-1}z;p)_\infty}\frac{(pt^{-1}z;p)_\infty}{(ptz;p)_\infty}\frac{(p(t/q)z;p)_\infty}{(p(t/q)^{-1}z;p)_\infty}.$$

Here, we set

$$(z;p)_\infty = \prod_{n=0}^{\infty}(1-zp^n) \qquad |z|<1.$$

From this, one can rewrite (2.15)–(2.17), (2.19) and (2.20) in the sense of analytic continuation as follows:

$$\begin{aligned}
\psi^{\pm}(z)\psi^{\pm}(w) &= \tfrac{g_\theta(w/z;p^*)}{g_\theta(w/z;p)}\psi^{\pm}(w)\psi^{\pm}(z),\\
\psi^{+}(z)\psi^{-}(w) &= \tfrac{g_\theta(\gamma^{-1}w/z;p^*)}{g_\theta(\gamma w/z;p)}\psi^{-}(w)\psi^{+}(z),\\
\psi^{+}(z)x^{+}(w) &= g_\theta(\gamma^{-1}w/z;p^*)x^{+}(w)\psi^{+}(z),\\
\psi^{-}(z)x^{+}(w) &= g_\theta(w/z;p^*)x^{+}(w)\psi^{-}(z),\\
\psi^{+}(z)x^{-}(w) &= g_\theta(z/w;p)x^{-}(w)\psi^{+}(z),\\
\psi^{-}(z)x^{-}(w) &= g_\theta(\gamma z/w;p)x^{-}(w)\psi^{-}(z),\\
x^{+}(z)x^{+}(w) &= g_\theta(w/z;p^*)x^{+}(w)x^{+}(z),\\
x^{-}(z)x^{-}(w) &= g_\theta(z/w;p)x^{-}(w)x^{-}(z),
\end{aligned}$$

where

$$g_\theta(z;p)=\frac{\theta_p(q^{-1}z)\theta_p((q/t)z)\theta_p(tz)}{\theta_p(qz)\theta_p((q/t)^{-1}z)\theta_p(t^{-1}z)}, \tag{2.25}$$

$$\theta_p(z)=(z;p)_\infty(p/z;p)_\infty, \tag{2.26}$$

$$g_\theta(z;p^*)=g_\theta(z;p)\big|_{p\mapsto p^*}. \tag{2.27}$$

Note that

$$g_\theta(z^{-1};p)=g_\theta(z;p)^{-1}, \qquad g_\theta(pz;p)=g_\theta(z;p).$$

2.3 The *Z* algebra structure

We next show that the Z algebra structure of $\mathcal{U}_{q,t,p}$ remains the same as the one of $\mathcal{U}_{q,t}$, i.e., it is not elliptically deformed. This is a common feature in the elliptic algebras $U_{q,p}(\widehat{\mathfrak{g}})$ [16].

In this subsection, we assume $\gamma\neq 1$. Set

$$E^{\pm}(\alpha,z)=\exp\left\{\pm\sum_{n>0}\frac{1}{\gamma^n-\gamma^{-n}}\alpha_{\pm n}(\gamma^{-1/2}z)^{\mp n}\right\}, \tag{2.28}$$

$$E^{\pm}(\alpha',z)=\exp\left\{\mp\sum_{n>0}\frac{1}{\gamma^n-\gamma^{-n}}\alpha'_{\pm n}(\gamma^{-1/2}z)^{\mp n}\right\}, \tag{2.29}$$

and define $\mathcal{Z}^{\pm}(z)$ by

$$\mathcal{Z}^{+}(z)=E^{-}(\alpha,z)x^{+}(z)E^{+}(\alpha,z), \tag{2.30}$$

$$\mathcal{Z}^{-}(z)=E^{-}(\alpha',z)x^{-}(z)E^{+}(\alpha',z). \tag{2.31}$$

Then, from (A.1)–(A.4) in Appendix A, we have

$$[\alpha_m,\mathcal{Z}^{\pm}(z)]=0,\quad (m\in\mathbb{Z}_{\neq 0}).$$

Furthermore, the relations in Definition 2.3 and Lemma A.1 lead to the following theorem.

Theorem 2.4

$$z^3G^{-}(w/z)g(w/z;\gamma^2)^{-1}\mathcal{Z}^{\pm}(z)\mathcal{Z}^{\pm}(w)=-w^3G^{-}(z/w)g(z/w;\gamma^2)^{-1}\mathcal{Z}^{\pm}(w)\mathcal{Z}^{\pm}(z), \tag{2.32}$$

$$g(\gamma w/z)g(\gamma w/z;\gamma^2)\mathcal{Z}^{+}(z)\mathcal{Z}^{-}(w)-g(\gamma z/w)g(\gamma z/w;\gamma^2)\mathcal{Z}^{-}(w)\mathcal{Z}^{+}(z)$$
$$=\frac{(1-q)(1-1/t)}{1-q/t}\left\{\delta(\gamma^{-1}z/w)\psi_0^{+}-\delta(\gamma z/w)\psi_0^{-}\right\}, \tag{2.33}$$

$$g(w/z)^{-1}g(u/w)^{-1}g(u/z)^{-1}g(w/z;\gamma^2)^{-1}g(u/w;\gamma^2)^{-1}g(u/z;\gamma^2)^{-1}$$

$$\times\left(\frac{w}{u}+\frac{w}{z}-\frac{z}{w}-\frac{u}{w}\right)\mathcal{Z}^+(z)\mathcal{Z}^+(w)\mathcal{Z}^+(u)+\textit{permutations in } z,w,u=0, \quad (2.34)$$

$$g(w/z;\gamma^2)^{-1}g(u/w;\gamma^2)^{-1}g(u/z;\gamma^2)^{-1}\left(\frac{w}{u}+\frac{w}{z}-\frac{z}{w}-\frac{u}{w}\right)\mathcal{Z}^-(z)\mathcal{Z}^-(w)\mathcal{Z}^-(u)$$
$$+\textit{permutations in } z,w,u=0. \quad (2.35)$$

Here, $g(z;\gamma^2)$ is given in (2.23) with $s=\gamma^2$ and we also set

$$g(z)=\exp\left(\sum_{m>0}\frac{\kappa_m}{m}z^m\right) \quad \in \mathbb{C}[[z]]. \quad (2.36)$$

We call the algebra generated by the coefficients of $\mathcal{Z}^\pm(z)=\sum_{\in n\in\mathbb{Z}}\mathcal{Z}_n^\pm z^{-n}$ in z satisfying the relations in Theorem 2.4 the Z algebra associated with $\mathcal{U}_{q,t,p}$.

Remark. Noting that the whole relations in this theorem are independent of the elliptic nome p, one finds that these relations coincides with those in Proposition 4.2 in [62] under the identification

$$\mathcal{K}_M=\gamma, \quad K_M^{-1}=\psi_0,$$
$$T^\pm(z)_M=\pm Z^\pm(z)/((1-q^{\pm1})(1-t^{\mp1}))$$

and the correspondence listed below Theorem 2.2.

Hence, the elliptic quantum toroidal algebra $\mathcal{U}_{q,t,p}$ possesses the same deformed Virasoro and W_3 algebra structures through the same Z algebra structure as $\mathcal{U}_{q,t}$ discussed in Theorem 4.1 and Remark 4.7 of [62].

2.4 Algebra homomorphism from $U_{q,t,p}(\mathfrak{gl}_{1,tor})$ to $U_{q,t}(\mathfrak{gl}_{1,tor})$

In [40], an homomorphism from the elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{g}})$ to the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ with $\widehat{\mathfrak{g}}$ being an untwisted affine Lie algebra was given. This allows us to extend any representations of $U_q(\widehat{\mathfrak{g}})$ with generic q to those of $U_{q,p}(\widehat{\mathfrak{g}})$ with generic p,q. Such homomorphism can be generalized to the cases with $\widehat{\mathfrak{g}}$ being any twisted affine Lie algebras, for example see [46], as well as any toroidal Lie algebras $\mathfrak{g}_{tor}$ [23, 54]. They include the cases with $\widehat{\mathfrak{g}}$ being (affine or toroidal) Lie super algebras. See for example [44].

Proposition 2.5 *Let us define $u^\pm(z,p)$ by*

$$u^+(\gamma^{-1/2}z,p)=\exp\left(-\sum_{m>0}\frac{p^{*m}}{1-p^{*m}}a_{-m}z^m\right),$$
$$u^-(\gamma^{1/2}z,p)=\exp\left(\sum_{m>0}\frac{p^m}{1-p^m}a_m z^{-m}\right),$$

where $p^* = p\gamma^{-2}$ *as above. Then, the following gives a homomorphism from* $\mathcal{U}_{q,t,p}$ *to* $\mathcal{U}_{q,t} \otimes \mathbb{C}[[p]]$.

$$x^+(z) \mapsto u^+(z,p)X^+(z), \tag{2.37}$$

$$x^-(z) \mapsto X^-(z)u^-(z,p), \tag{2.38}$$

$$\psi^+(z) \mapsto u^+(\gamma z,p)\phi(z)u^-(z,p), \tag{2.39}$$

$$\psi^-(z) \mapsto u^+(z,p)\psi(z)u^-(\gamma z,p). \tag{2.40}$$

In particular,(2.39) (*or* (2.40)) *is equivalent to*

$$\alpha_m \mapsto a_m, \qquad \alpha_{-m} \mapsto \frac{1-p^m}{1-p^{*m}}a_{-m} \qquad (m>0).$$

The statement is proved by using the following lemma.

Lemma 2.6

$$[a_n, u^+(z,p)] = \frac{\kappa_n}{n}\frac{\gamma^n-\gamma^{-n}}{1-p^{*n}}(p^*\gamma^{-1/2}z)^n u^+(z,p) \quad (n>0), \tag{2.41}$$

$$[a_{-n}, u^-(z,p)] = \frac{\kappa_n}{n}\frac{\gamma^n-\gamma^{-n}}{1-p^{n}}(p^{-1}\gamma^{1/2}z)^{-n} u^-(z,p) \quad (n>0), \tag{2.42}$$

$$u^+(z,p)X^+(w) = g(z/w;p^*)X^+(w)u^+(z,p), \tag{2.43}$$

$$u^+(z,p)X^-(w) = g(\gamma z/w;p^*)^{-1}X^-(w)u^+(z,p), \tag{2.44}$$

$$u^-(z,p)X^+(w) = g(\gamma^{-1}w/z;p)^{-1}X^+(w)u^-(z,p), \tag{2.45}$$

$$u^-(z,p)X^-(w) = g(w/z;p)X^-(w)u^-(z,p), \tag{2.46}$$

$$\phi(z)u^+(w,p) = g(\gamma w/z;p^*)g(\gamma^{-1}w/z;p^*)^{-1}u^+(w,p)\phi(z), \tag{2.47}$$

$$u^-(z,p)\psi(w) = g(\gamma w/z;p)g(\gamma^{-1}w/z;p)^{-1}\psi(w)u^-(z,p), \tag{2.48}$$

$$\phi(z)u^-(w,p) = u^-(w,p)\phi(z), \tag{2.49}$$

$$u^+(z,p)\psi(w) = \psi(w)u^+(z,p), \tag{2.50}$$

$$u^+(z,p)u^-(w,p) = g(p^*\gamma z/w;p^*)g(p\gamma^{-1}z/w;p)^{-1}u^-(w,p)u^+(z,p). \tag{2.51}$$

Here, $g(z;s),\ s=p,\ p^*$ *is given in* (2.23).

It is obvious that the homomorphism in Proposition 2.5 is invertible. Hence, one has the following isomorphism.

Theorem 2.7 *For generic* q,t,p,

$$\mathcal{U}_{q,t,p} \cong \mathcal{U}_{q,t} \otimes \mathbb{C}[[p]]$$

as a topological algebra.

2.5 Elliptic analogue of Miki's automorphism

By using a set of new generators $b_m, Y_l^{\pm}$ of $U_{q,t}(\mathfrak{gl}_{1,tor})$ defined in Sect. 2.1.2, let us consider their generating functions.

$$Y^{\pm}(z) = \sum_{l\in\mathbb{Z}} Y_l^{\pm} z^{-l},$$
$$\phi_b(z) = \gamma^{-1} \exp\left\{\sum_{m>0} b_m (\psi_0^+)^{m/2} z^{-m}\right\},$$
$$\psi_b(z) = \gamma \exp\left\{-\sum_{m>0} b_{-m} (\psi_0^+)^{m/2} z^{m}\right\}.$$

Applying the homomorphism in Sect. 2.4, one obtains a new elliptic quantum toroidal algebra $U^b_{q,t,p}(\mathfrak{gl}_{1,tor})$ generated by $\beta_m, y_l^{\pm}, \psi_0^+, \gamma^{-1}$, defined by

$$y^{\pm}(z) = \sum_{l\in\mathbb{Z}} y_l^{\pm} z^{-l},$$
$$\psi_b^+((\psi_0^+)^{1/2}z) = \gamma^{-1} \exp\left(-\sum_{m>0} \frac{p^m}{1-p^m}\beta_{-m} z^m\right) \exp\left(\sum_{m>0} \frac{1}{1-p^m}\beta_m z^{-m}\right),$$

where

$$y^+(z) := u_b^+(z,p) Y^+(z), \tag{2.52}$$
$$y^-(z) := Y^-(z) u_b^-(z,p), \tag{2.53}$$
$$\psi_b^+(z) := u_b^+(\psi_0^+ z, p)\phi_b(z) u_b^-(z,p), \tag{2.54}$$
$$\psi_b^-(z) := u_b^+(z,p)\psi_b(z) u_b^-(\psi_0^+ z, p) \tag{2.55}$$

with

$$u_b^+((\psi_0^+)^{-1/2}z, p) := \exp\left(-\sum_{m>0} \frac{p_b^{*m}}{1-p_b^{*m}} b_{-m} z^m\right),$$
$$u_b^-((\psi_0^+)^{1/2}z, p) := \exp\left(\sum_{m>0} \frac{p^m}{1-p^m} b_m z^{-m}\right),$$

and $p_b^* = p(\psi_0^+)^{-2}$.

Then, we obtain the following isomorphism.

Corollary 2.8

$$U_{q,t,p}(\mathfrak{gl}_{1,tor}) \cong U^b_{q,t,p}(\mathfrak{gl}_{1,tor}),$$

by

$$\alpha_m \mapsto \beta_m, \quad x^\pm_l \mapsto y^\pm_l, \quad \gamma \mapsto \psi^+_0, \quad \psi^+_0 \mapsto \gamma^{-1}.$$

Proof The statement follows from Theorems 2.2 and 2.7. □

2.6 Hopf algebroid structure

In this section, we introduce a Hopf algebroid structure into $\mathcal{U}_{q,t,p}$. This structure was introduced by Etingof and Varchenko [14, 15] and developed in [43, 49].

For $F(z,p) \in \mathbb{C}[[z,z^{-1}]][[p]]$, let $\widetilde{\otimes}$ denote the usual tensor product with the following extra condition

$$F(z,p^*)a\widetilde{\otimes}b = a\widetilde{\otimes}F(z,p)b \tag{2.56}$$

Here, $F(z,p^*)$ denotes the same F with replacing p by $p^* = p\gamma^{-2}$.

Define two moment maps $\mu_l, \mu_r : \mathbb{C}[[z,z^{-1}]][[p]] \to \mathcal{U}_{q,t,p}[[z,z^{-1}]]$ by

$$\mu_l(F(z,p)) = F(z,p), \qquad \mu_r(F(z,p)) = F(z,p^*).$$

Let $\gamma_{(1)} = \gamma\widetilde{\otimes}1$, $\gamma_{(2)} = 1\widetilde{\otimes}\gamma$ and $p^*_{(i)} = p\gamma^{-2}_{(i)}$ $(i=1,2)$. Let us define two algebra homomorphisms $\Delta : \mathcal{U}_{q,t,p} \to \mathcal{U}_{q,t,p}\widetilde{\otimes}\mathcal{U}_{q,t,p}$ and $\varepsilon : \mathcal{U}_{q,t,p} \to \mathbb{C}$ by

$$\Delta(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2}\widetilde{\otimes}\gamma^{\pm 1/2}, \tag{2.57}$$

$$\Delta(\psi^\pm(z)) = \psi^\pm(\gamma_{(2)}^{\mp 1/2}z)\widetilde{\otimes}\psi^\pm(\gamma_{(1)}^{\pm 1/2}z), \tag{2.58}$$

$$\Delta(x^+(z)) = 1\widetilde{\otimes}x^+(\gamma_{(1)}^{-1/2}z) + x^+(\gamma_{(2)}^{1/2}z)\widetilde{\otimes}\psi^-(\gamma_{(1)}^{-1/2}z), \tag{2.59}$$

$$\Delta(x^-(z)) = x^-(\gamma_{(2)}^{-1/2}z)\widetilde{\otimes}1 + \psi^+(\gamma_{(2)}^{-1/2}z)\widetilde{\otimes}x^-(\gamma_{(1)}^{1/2}z), \tag{2.60}$$

$$\Delta(\mu_l(F(z,p))) = \mu_l(F(z,p))\widetilde{\otimes}1, \qquad \Delta(\mu_r(F(z,p))) = 1\widetilde{\otimes}\mu_r(F(z,p)), \tag{2.61}$$

$$\varepsilon(\gamma^{1/2}) = \varepsilon(\psi^+_0) = 1, \qquad \varepsilon(\psi^\pm(z)) = 1, \qquad \varepsilon(x^\pm(z)) = 0, \tag{2.62}$$

$$\varepsilon(\mu_l(F(z,p))) = \varepsilon(\mu_r(F(z,p))) = F(z,p). \tag{2.63}$$

Note that Δ is the so-called Drinfeld comultiplication. Then, we have

Proposition 2.9 *The maps ε and Δ satisfy*

$$(\Delta\widetilde{\otimes}\mathrm{id})\circ\Delta = (\mathrm{id}\widetilde{\otimes}\Delta)\circ\Delta, \tag{2.64}$$

$$(\varepsilon\widetilde{\otimes}\mathrm{id})\circ\Delta = \mathrm{id} = (\mathrm{id}\widetilde{\otimes}\varepsilon)\circ\Delta. \tag{2.65}$$

We also define an algebra anti-homomorphism $S : \mathcal{U}_{q,t,p} \to \mathcal{U}_{q,t,p}$ by

$$S(\gamma^{1/2}) = \gamma^{-1/2},$$

$$S(\psi^\pm(z)) = \psi^\pm(z)^{-1},$$

$$S(x^+(z)) = -x^+(z)\psi^-(z)^{-1},$$

$$S(x^-(z)) = -\psi^+(z)x^-(z),$$
$$S(\mu_l(F(z,p))) = \mu_r(F(z,p)), \qquad S(\mu_r(F(z,p))) = \mu_l(F(z,p)).$$

Then, one can check the following.

Proposition 2.10

$$m \circ (\mathrm{id}\widetilde{\otimes}S) \circ \Delta(a) = \mu_l(\varepsilon(a)1) \qquad \forall a \in \mathcal{U}_{q,t,p},$$
$$m \circ (S\widetilde{\otimes}\mathrm{id}) \circ \Delta(a) = \mu_r(\varepsilon(a)1).$$

These Propositions indicate that $(\mathcal{U}_{q,t,p}, \Delta, \varepsilon, \mu_l, \mu_r, S)$ is a Hopf algebroid [14, 15, 43, 48, 49].

Remark For representations on which $\gamma^{1/2} = 1$ hence $p = p^*$, the following gives an opposite Drinfeld comultiplication.

$$\Delta^{op}(\psi^{\pm}(z)) = \psi^{\pm}(z)\widetilde{\otimes}\psi^{\pm}(z), \tag{2.66}$$
$$\Delta^{op}(x^+(z)) = x^+(z)\widetilde{\otimes}1 + \psi^-(z)\widetilde{\otimes}x^+(z), \tag{2.67}$$
$$\Delta^{op}(x^-(z)) = 1\widetilde{\otimes}x^-(z) + x^-(z)\widetilde{\otimes}\psi^+(z). \tag{2.68}$$

3 Representations of $U_{q,t,p}(\mathfrak{gl}_{1,tor})$

Let $\mathcal{V}$ be a $\mathcal{U}_{q,t,p}$ module. For $(k,l) \in \mathbb{C}^2$, we say that $\mathcal{V}$ has level (k,l), if the central elements $\gamma^{1/2}$ and ψ_0^+ act as

$$\gamma^{1/2}v = (t/q)^{k/4}v, \qquad \psi_0^+ v = (t/q)^{-l/2}v \qquad \forall v \in \mathcal{V}.$$

In the rest of this paper, we regard $p, p^* = p\gamma^{-2}$ as a generic complex number with $|p| < 1, |p^*| < 1$.

More generally, for $p_1, \dots p_r \in \mathbb{C}$ with $|p_i| < 1$ $(1 \le i \le r,\ r = 1, 2, \dots)$, we set

$$(z; p_1, \dots, p_r)_\infty = \prod_{n_1,\dots,n_r=0}^{\infty} (1 - zp_1^{n_1} \cdots p_r^{n_r}).$$

We use the following multiple elliptic Gamma functions defined by [66, 73]

$$\Gamma_r(z; p_1, \dots, p_r) = (z; p_1, \dots, p_r)_\infty^{(-1)^{r-1}} (p_1 \cdots p_r/z; p_1, \dots, p_r)_\infty.$$

Note

$$\Gamma_1(z; p_1) = \theta_{p_1}(z),$$
$$\Gamma_2(z; p_1, p_2) \equiv \Gamma(z; p_1, p_2) = \frac{(p_1p_2/z; p_1, p_2)_\infty}{(z; p_1, p_2)_\infty}$$

are Jacobi's odd theta function and the elliptic Gamma function [76], respectively. We have

Proposition 3.1

$$\Gamma_r(p_j z; p_1, \dots, p_r) = \Gamma_{r-1}(z; p_1, \dots, \check{p}_j, \dots, p_r)\Gamma_r(x; p_1, \dots, p_r)$$

for $r \geq 2$, where $\check{p}_j$ means the excluding of p_j.

3.1 Level (1, N) representation of $U_{q,t,p}(\mathfrak{gl}_{1,tor})$

For $u \in \mathbb{C}^*$, let $\mathcal{F}_u^{(1,N)} = \mathbb{C}[\alpha_{-m}\ (m > 0)]1_u^{(N)}$ be a Fock space on which the Heisenberg algebra $\{\alpha_m\ (m \in \mathbb{Z}_{\neq 0})\}$ and the central elements $\gamma^{1/2}$, ψ_0^+ act as

$$\begin{aligned}
&\gamma^{1/2} \cdot 1_u^{(N)} = (t/q)^{1/4} 1_u^{(N)}, \quad \psi_0^+ \cdot 1_u^{(N)} = (t/q)^{-N/2} 1_u^{(N)}, \quad \alpha_{-m} \cdot 1_u^{(N)} = 0,\\
&\alpha_{-m} \cdot \xi = \alpha_{-m}\xi,\\
&\alpha_m \cdot \xi = -\frac{\kappa_m}{m}(1-(q/t)^m)\frac{1-p^m}{1-p^{*m}}\frac{\partial}{\partial \alpha_{-m}}\xi
\end{aligned}$$

for $m > 0$, $\xi \in \mathcal{F}_u^{(1,N)}$. Note that $p^* = pq/t$ on $\mathcal{F}_u^{(1,N)}$.

Theorem 3.2 *The following assignment gives a level $(1, N)$ representation of $\mathcal{U}_{q,t,p}$ on $\mathcal{F}_u^{(1,N)}$.*

$$x^+(z) = uz^{-N}(t/q)^{3N/4}\exp\left\{-\sum_{n>0}\frac{(t/q)^{n/4}}{1-(t/q)^n}\alpha_{-n}z^n\right\}\exp\left\{\sum_{n>0}\frac{(t/q)^{3n/4}}{1-(t/q)^n}\alpha_n z^{-n}\right\}, \tag{3.1}$$

$$x^-(z) = u^{-1}z^{N}(t/q)^{-3N/4}\exp\left\{\sum_{n>0}\frac{(t/q)^{n/4}}{1-(t/q)^n}\alpha'_{-n}z^n\right\}\exp\left\{-\sum_{n>0}\frac{(t/q)^{3n/4}}{1-(t/q)^n}\alpha'_n z^{-n}\right\}, \tag{3.2}$$

$$\psi^+(z) = (t/q)^{-N/2}\exp\left\{-\sum_{n>0}\frac{p^n(t/q)^{-n/4}}{1-p^n}\alpha_{-n}z^n\right\}\exp\left\{\sum_{n>0}\frac{(t/q)^{n/4}}{1-p^n}\alpha_n z^{-n}\right\}, \tag{3.3}$$

$$\psi^-(z) = (t/q)^{N/2}\exp\left\{-\sum_{n>0}\frac{(t/q)^{n/4}}{1-p^n}\alpha_{-n}z^n\right\}\exp\left\{\sum_{n>0}\frac{p^n(t/q)^{-n/4}}{1-p^n}\alpha_n z^{-n}\right\}. \tag{3.4}$$

Proof Let us set $x^\pm(z) = u^\pm z^{\mp N}(t/q)^{\pm 3N/4}\widetilde{x}^\pm(z)$, $\psi^\pm(z) = (t/q)^{\mp N/2}\widetilde{\psi}^\pm(z)$. The statement follows from the operator product expansion (OPE) formulas listed below.

$$\begin{aligned}
&\widetilde{\psi}^+(z)x^+((t/q)^{1/2}w) = f(w/z;p^*) : \widetilde{\psi}^+(z)x^+((t/q)^{1/2}w) : \qquad |w/z| < 1,\\
&x^+((t/q)^{1/2}w)\widetilde{\psi}^+(z) = f(p^*z/w;p^*) : \widetilde{\psi}^+(z)x^+((t/q)^{1/2}w) : \qquad |z/w| < 1,\\
&\widetilde{\psi}^-(z)x^+((t/q)^{1/2}w) = f(p^*(t/q)^{1/2}w/z;p^*) : \widetilde{\psi}^-(z)x^+((t/q)^{1/2}w) : \qquad |w/z| < 1,\\
&x^+((t/q)^{1/2}w)\widetilde{\psi}^-(z) = f((t/q)^{-1/2}z/w;p^*) : \widetilde{\psi}^-(z)x^+((t/q)^{1/2}w) : \qquad |z/w| < 1,
\end{aligned}$$

$$
\begin{aligned}
\widetilde{\psi}^-(z)x^-(w) &= f(p(t/q)^{-1/2}w/z;p)^{-1} : \widetilde{\psi}^-(z)x^-(w) : \qquad |w/z| < 1,\\
x^-(w)\widetilde{\psi}^-(z) &= f((t/q)^{1/2}z/w;p)^{-1} : \widetilde{\psi}^-(z)x^-((t/q)^{1/2}w) : \qquad |z/w| < 1,\\
x^+(z)x^-(w) &= (w/z)^N \frac{(1-q\gamma w/z)(1-\gamma w/tz)}{(1-\gamma w/z)(1-q\gamma w/tz)} : \widetilde{x}^+(z)\widetilde{x}^-(w) : \qquad |w/z| < 1,\\
x^-(w)x^+(z) &= (w/z)^N \frac{(1-\gamma^{-1}z/qw)(1-t\gamma^{-1}z/w)}{(1-\gamma^{-1}z/w)(1-t\gamma^{-1}z/qw)} : \widetilde{x}^+(z)\widetilde{x}^-(w) : \qquad |z/w| < 1,
\end{aligned}
$$

and the formulas

$$
: \widetilde{x}^+(\gamma z)\widetilde{x}^-(w) := \widetilde{\psi}^+(z), \tag{3.5}
$$

$$
: \widetilde{x}^+(\gamma^{-1} z)\widetilde{x}^-(w) := \widetilde{\psi}^-(\gamma^{-1}z). \tag{3.6}
$$

Here, $f(z;s)$ is given by

$$
f(z;s) = \frac{(q^{-1}z;s)_\infty (tz;s)_\infty (qt^{-1}z;s)_\infty}{(qz;s)_\infty (t^{-1}z;s)_\infty (tq^{-1}z;s)_\infty}
$$

for $s = p, p^*$. □

A similar representation is given in Lemma A.16 in [23].

3.2 Vector representation and the q-Fock space representation

We next consider the elliptic analogue of a representation given in [17, 18, 22], i.e., the q-Fock space representation of $\mathcal{U}_{q,t,p}$.

For $u \in \mathbb{C}^*$, let $V(u)$ be a vector space spanned by $[u]_j$ $(j \in \mathbb{Z})$.

Proposition 3.3 *By the following action, $V(u)$ is a level-$(0,0)$ $\mathcal{U}_{q,t,p}$-module. We call $V(u)$ the vector representation.*

$$
\begin{aligned}
x^+(z)[u]_j &= a^+(p)\delta(q^j u/z)[u]_{j+1},\\
x^-(z)[u]_j &= a^-(p)\delta(q^{j-1} u/z)[u]_{j-1},\\
\psi^\pm(z)[u]_j &= \left.\frac{\theta_p(q^j t^{-1}u/z)\theta_p(q^{j-1}tu/z)}{\theta_p(q^j u/z)\theta_p(q^{j-1}u/z)}\right|_\pm [u]_j,\\
\psi_0^\pm[u]_j &= [u]_j,
\end{aligned}
$$

where we set

$$
a^+(p) = (1-t)\frac{(pt/q;p)_\infty (p/t;p)_\infty}{(p;p)_\infty (p/q;p)_\infty}, \tag{3.7}
$$

$$
a^-(p) = (1-t^{-1})\frac{(pq/t;p)_\infty (pt;p)_\infty}{(p;p)_\infty (pq;p)_\infty}. \tag{3.8}
$$

Proof Use a formula in Lemma 3.11. □

Let $\mathcal{F}_u$ be a vector space spanned by $|\lambda\rangle_u$ $(\lambda \in \mathcal{P}^+)$, where

$$\mathcal{P}^+ = \{\lambda = (\lambda_1, \lambda_2, \ldots) \mid \lambda_i \geq \lambda_{i+1},\ \lambda_i \in \mathbb{Z},\ \lambda_l = 0 \text{ for sufficiently large } l\ \}. \tag{3.9}$$

We denote by $\ell(\lambda)$ the length of $\lambda \in \mathcal{P}^+$, i.e., $\lambda_{\ell(\lambda)} > 0$ and $\lambda_{\ell(\lambda)+1} = 0$. We also set $|\lambda| = \sum_{i \geq 1} \lambda_i$ and denote by λ' the conjugate of λ.

Theorem 3.4 *The following action gives a level (0,1) representation of $\mathcal{U}_{q,t,p}$ on $\mathcal{F}_u$. We denote this by $\mathcal{F}_u^{(0,1)}$.*

$$\gamma^{1/2}|\lambda\rangle_u = |\lambda\rangle_u, \tag{3.10}$$

$$x^+(z)|\lambda\rangle_u = a^+(p) \sum_{i=1}^{\ell(\lambda)+1} A^+_{\lambda,i}(p)\delta(u_i/z)|\lambda + \mathbf{1}_i\rangle_u, \tag{3.11}$$

$$x^-(z)|\lambda\rangle_u = (q/t)^{1/2}a^-(p) \sum_{i=1}^{\ell(\lambda)} A^-_{\lambda,i}(p)\delta(q^{-1}u_i/z)|\lambda - \mathbf{1}_i\rangle_u, \tag{3.12}$$

$$\psi^+(z)|\lambda\rangle_u = (q/t)^{1/2} B^+_\lambda(u/z;\, p)|\lambda\rangle_u, \tag{3.13}$$

$$\psi^-(z)|\lambda\rangle_u = (q/t)^{-1/2} B^-_\lambda(z/u;\, p)|\lambda\rangle_u, \tag{3.14}$$

where we set

$$A^+_{\lambda,i}(p) = \prod_{j=1}^{i-1} \frac{\theta_p(tu_i/u_j)\theta_p(qt^{-1}u_i/u_j)}{\theta_p(qu_i/u_j)\theta_p(u_i/u_j)}, \tag{3.15}$$

$$A^-_{\lambda,i}(p) = \prod_{j=i+1}^{\ell(\lambda)} \frac{\theta_p(qt^{-1}u_j/u_i)}{\theta_p(u_j/u_i)} \prod_{j=i+1}^{\ell(\lambda)+1} \frac{\theta_p(tu_j/u_i)}{\theta_p(qu_j/u_i)}, \tag{3.16}$$

$$B^+_\lambda(u/z;\, p) = \prod_{i=1}^{\ell(\lambda)} \frac{\theta_p(u_i/tz)}{\theta_p(u_i/qz)} \prod_{i=1}^{\ell(\lambda)+1} \frac{\theta_p(tu_i/qz)}{\theta_p(u_i/z)}, \tag{3.17}$$

$$B^-_\lambda(z/u;\, p) = \prod_{i=1}^{\ell(\lambda)} \frac{\theta_p(tz/u_i)}{\theta_p(qz/u_i)} \prod_{i=1}^{\ell(\lambda)+1} \frac{\theta_p(qz/tu_i)}{\theta_p(z/u_i)}. \tag{3.18}$$

A direct proof showing these actions satisfy the defining relation of $\mathcal{U}_{q,t,p}$ is given in Appendix B. In Appendix C, we also give an inductive derivation of Theorem 3.4.

Remark 1 In the trigonometric case obtained by taking $p \to 0$, the level (0,1) representation in Theorem 3.4 is identified with the geometric representation of $\mathcal{U}_{q,t}$ on $\bigoplus_N \mathrm{K}_T(\mathrm{Hilb}_N(\mathbb{C}^2))$, the direct sum of the $T = \mathbb{C}^\times \times \mathbb{C}^\times$ equivariant K-theory of the Hilbert scheme of N points on $\mathbb{C}^2$ [17, 18, 22]. There the basis $\{|\lambda\rangle_u\}$ in $\mathcal{F}_u$ is identified with the fixed point basis $\{[\lambda]\}$ in $\mathrm{K}_T(\mathrm{Hilb}_N(\mathbb{C}^2))$. We conjecture [54] that the same is true in the elliptic case. Namely, if one could properly formulate a geometric action of $\mathcal{U}_{q,t,p}$ on the direct sum of the equivariant elliptic cohomology of the Hilbert scheme $\bigoplus_N \mathrm{E}_T(\mathrm{Hilb}_N(\mathbb{C}^2))$, it should be identified with the level (0,1) representation in Theorem 3.4 by identifying $|\lambda\rangle_u$ with the fixed point class $[\lambda]$ in $\bigoplus_N \mathrm{E}_T(\mathrm{Hilb}_N(\mathbb{C}^2))$, where the latter bases can be realized in terms of the elliptic stable envelopes [2] on $\mathrm{E}_T(\mathrm{Hilb}_N(\mathbb{C}^2))$ constructed in [84] by

$$[\lambda] = \sum_{\mu\in\mathcal{P}^+} \mathrm{Stab}_{\mathfrak{C}}^{-1}(\mu)|_\lambda \mathrm{Stab}_{\mathfrak{C}}(\mu)$$

in the similar way as the case of the equivariant elliptic cohomology of the partial flag variety [53]. See also [82, 83].

Remark 2 Again from the trigonometric limit of (C.4), one finds the eigenvalue of $\alpha_1(=a_1)$ on $|\lambda\rangle_u^{(N)}$ is given by

$$-(1-t/q)(1-t)t^{-N}\sum_{j=1}^{N} q^{\lambda_j}t^{N-j}.$$

Under the automorphism in Theorem 2.2, a_1 is identified with $(1-t/q)X_0^+$. Hence, we obtain the eigenvalue of X_0^+ as $-(1-t)t^{-N}\sum_{j=1}^N q^{\lambda_j}t^{N-j}$. Then, under the identification of $q^{N|\lambda|}|\lambda\rangle_u^{(N)}$ with the specialization of the Macdonald symmetric polynomial $P_\lambda(t^\rho q^\lambda)$, where the variables $x = (x_1,\ldots,x_N)$ are specialized by $t^\rho q^\lambda = (t^{N-1}q^{\lambda_1}, t^{N-2}q^{\lambda_2},\ldots,q^{\lambda_N})$. This is consistent with the identification that the trigonometric limit of (C.2) is the Pieri formula [60]:

$$e_1(x)P_\lambda(x) = \sum_{i=1}^{N} \psi_{\lambda+\mathbf{1}_i/\lambda} P_{\lambda+\mathbf{1}_i}(x)$$

where $e_1(x) = \sum_{j=1}^N x_j$ and

$$\psi_{\lambda+\mathbf{1}_i/\lambda} = \prod_{j=1}^{i-1} \frac{(1-tu_i/u_j)(1-qu_i/tu_j)}{(1-u_i/u_j)(1-qu_i/u_j)}.$$

See also [17, 18] for such identification and its extension to the case $N\to\infty$ in [22]. Hence, it is natural to expect that the action of the elliptic generator $x_0^+ = \oint_0 \frac{dz}{2\pi i z}x^+(z)$ on $|\lambda\rangle_u^{(N)}$ in (C.2) gives an elliptic analogue of the Pieri formula [54] and the eigenvalue of x_0^+ should be given in terms of an elliptic analogue of $e_1(t^\rho q^\lambda)$, which is unknown yet, times $t^{-N}a^+(p)$.

These two remarks suggest that x_0^+ behaves as the elliptic Ruijsenaars difference operator on $W^{(N)}(u)$ and the bases vector $|\lambda\rangle_u^{(N)}$ as an elliptic analogue of the Macdonald symmetric polynomials with the eigenvalue given by an elliptic analogue of $e_1(t^\rho q^\lambda)$, which should satisfy the elliptic analogue of the Pieri formula (C.2). See also [58, 64], where the fixed point classes in the equivariant homology groups of Hilbert schemes of points on $\mathbb{C}^2$ are identified with the Jack symmetric functions. In

Sect. 3.3, we give more explicit relation of x_0^+ to the elliptic Ruijsenaars difference operator.

3.2.1 Some elliptic formulas

Let us consider the combinatorial factors c_λ, c'_λ appearing in the inner product of the Macdonald symmetric functions as

$$\langle P_\lambda, P_\lambda \rangle_{q,t} = \frac{c'_\lambda}{c_\lambda}, \tag{3.19}$$

$$c_\lambda = \prod_{\square\in\lambda}(1-q^{a(\square)}t^{\ell(\square)+1}) = \prod_{1\le i\le j\le \ell(\lambda)} \frac{(q^{\lambda_i-\lambda_j}t^{j-i+1};q)_\infty}{(q^{\lambda_i-\lambda_{j+1}}t^{j-i+1};q)_\infty},$$

$$c'_\lambda = \prod_{\square\in\lambda}(1-q^{a(\square)+1}t^{\ell(\square)}) = \prod_{1\le i\le j\le \ell(\lambda)} \frac{(q^{\lambda_i-\lambda_j+1}t^{j-i};q)_\infty}{(q^{\lambda_i-\lambda_{j+1}+1}t^{j-i};q)_\infty}. \tag{3.20}$$

Here, $a(\square)\equiv a_\lambda(\square)=\lambda_i-j,\ \ell(\square)\equiv \ell_\lambda(\square)=\lambda'_j-i$ for $\square=(i,j)\in\lambda$. The second expressions for c_λ, c'_λ follow from the formula due to Macdonald [60]

$$(1-q)\sum_{(i,j)\in\lambda} q^{\lambda_i-j}t^{\lambda'_j-i+1} = t\sum_{1\le i\le \ell(\lambda)} q^{\lambda_i-\lambda_j}t^{j-i} - \sum_{1\le i<\ell(\lambda)+1} q^{\lambda_i-\lambda_j}t^{j-i}. \tag{3.21}$$

We introduce elliptic analogues of c_λ, c'_λ as follows.

$$c_\lambda(p) = \prod_{\square\in\lambda}\theta_p(q^{a(\square)}t^{\ell(\square)+1}) = \prod_{1\le i\le j\le \ell(\lambda)} \frac{\Gamma(q^{\lambda_i-\lambda_{j+1}}t^{j-i+1};q,p)}{\Gamma(q^{\lambda_i-\lambda_j}t^{j-i+1};q,p)}, \tag{3.22}$$

$$c'_\lambda(p) = \prod_{\square\in\lambda}\theta_p(q^{a(\square)+1}t^{\ell(\square)}) = \prod_{1\le i\le j\le \ell(\lambda)} \frac{\Gamma(q^{\lambda_i-\lambda_{j+1}+1}t^{j-i};q,p)}{\Gamma(q^{\lambda_i-\lambda_j+1}t^{j-i};q,p)}. \tag{3.23}$$

One can verify the following.

Proposition 3.5

$$\frac{c_{\lambda+\mathbf{1}_k}(p)}{c_\lambda(p)} = \prod_{i=1}^{k-1}\frac{\theta_p(tq^{-1}u_i/u_k)}{\theta_p(q^{-1}u_i/u_k)}\frac{\prod_{i=k+1}^{\ell(\lambda)+1}\theta_p(u_k/u_i)}{\prod_{i=k+1}^{\ell(\lambda)}\theta_p(tu_k/u_i)}, \tag{3.24}$$

$$\frac{c'_{\lambda+\mathbf{1}_k}(p)}{c'_\lambda(p)} = \prod_{i=1}^{k-1}\frac{\theta_p(u_i/u_k)}{\theta_p(t^{-1}u_i/u_k)}\frac{\prod_{i=k+1}^{\ell(\lambda)+1}\theta_p(qt^{-1}u_k/u_i)}{\prod_{i=k+1}^{\ell(\lambda)}\theta_p(qu_k/u_i)}. \tag{3.25}$$

The following formulas are useful in Sect. 4.

Proposition 3.6 *Let us consider*

$$A^{+'}_{\lambda,i}(p) = \prod_{j=i+1}^{\ell(\lambda)} \frac{\theta_p(tu_i/u_j)}{\theta_p(qu_i/u_j)} \prod_{j=i+1}^{\ell(\lambda)+1} \frac{\theta_p(qu_i/tu_j)}{\theta_p(u_i/u_j)}, \tag{3.26}$$

$$A^{-'}_{\lambda,i}(p) = \prod_{j=1}^{i-1} \frac{\theta_p(tu_j/u_i)\theta_p(qu_j/tu_i)}{\theta_p(qu_j/u_i)\theta_p(u_j/u_i)}. \tag{3.27}$$

We have

$$A^{+'}_{\lambda,i}(p) = \frac{c_\lambda(p)c'_{\lambda+\mathbf{1}_i}(p)}{c'_\lambda(p)c_{\lambda+\mathbf{1}_i}(p)} A^{+}_{\lambda,i}(p) \tag{3.28}$$

$$A^{-'}_{\lambda,i}(p) = \frac{c_\lambda(p)c'_{\lambda-\mathbf{1}_i}(p)}{c'_\lambda(p)c_{\lambda-\mathbf{1}_i}(p)} A^{-}_{\lambda,i}(p). \tag{3.29}$$

By a direct calculation, one can also verify the following formulas.

Proposition 3.7

$$A^{+'}_{\lambda,i}(p) = (q/t)A^{-}_{\lambda+\mathbf{1}_i,i}(p), \tag{3.30}$$

$$A^{-'}_{\lambda,i}(p) = (t/q)A^{+}_{\lambda-\mathbf{1}_i,i}(p). \tag{3.31}$$

Hence, we have

Proposition 3.8

$$\frac{c_\lambda(p)}{c_{\lambda+\mathbf{1}_i}(p)} \frac{c'_{\lambda+\mathbf{1}_i}(p)}{c'_\lambda(p)} A^{+}_{\lambda,i}(p) = (q/t)A^{-}_{\lambda+\mathbf{1}_i,i}(p), \tag{3.32}$$

$$\frac{c_\lambda(p)}{c_{\lambda-\mathbf{1}_i}(p)} \frac{c'_{\lambda-\mathbf{1}_i}(p)}{c'_\lambda(p)} A^{-}_{\lambda,i}(p) = (t/q)A^{+}_{\lambda-\mathbf{1}_i,i}(p). \tag{3.33}$$

We also need to introduce $N_\lambda(p)$ and $N'_\lambda(p)$ by

$$\frac{N_\lambda(p)}{N_{\lambda+\mathbf{1}_i}(p)} = \prod_{j=1}^{i-1} \frac{(pu_j/tu_i;p)_\infty(ptu_j/qu_i;p)_\infty}{(pu_j/qu_i;p)_\infty(pu_j/u_i;p)_\infty}$$
$$\times \prod_{j=i+1}^{\ell(\lambda)} \frac{(pqu_i/u_j;p)_\infty}{(ptu_i/u_j;p)_\infty} \prod_{j=i+1}^{\ell(\lambda)+1} \frac{(pu_i/u_j;p)_\infty}{(pqu_i/tu_j;p)_\infty}, \tag{3.34}$$

$$\frac{N'_\lambda(p)}{N'_{\lambda+\mathbf{1}_i}(p)} = \prod_{j=1}^{i-1} \frac{(pu_i/u_j;p)_\infty(pqu_i/u_j;p)_\infty}{(pqu_i/tu_j;p)_\infty(ptu_i/u_j;p)_\infty}$$

$$\times \prod_{j=i+1}^{\ell(\lambda)} \frac{(pu_j/tu_i;p)_\infty}{(pu_j/qu_i;p)_\infty} \prod_{j=i+1}^{\ell(\lambda)+1} \frac{(ptu_j/qu_i;p)_\infty}{(pu_j/u_i;p)_\infty}. \tag{3.35}$$

One can derive the following expressions.

$$\begin{aligned} N_\lambda(p) &= \prod_{1\le i\le j\le \ell(\lambda)} \frac{(pqu_i/u_j;q,p)_\infty}{(ptu_i/u_j;q,p)_\infty} \prod_{1\le i<j\le \ell(\lambda)+1} \frac{(pu_i/u_j;q,p)_\infty}{(pqu_i/tu_j;q,p)_\infty} \\ &= \prod_{\square\in\lambda} \frac{(pq^{a(\square)+1}t^{\ell(\square)};p)_\infty}{(pq^{a(\square)}t^{\ell(\square)+1};p)_\infty}, \end{aligned} \tag{3.36}$$

$$\begin{aligned} N'_\lambda(p) &= \prod_{1\le i\le j\le \ell(\lambda)} \frac{(pu_j/u_i;q,p)_\infty}{(pqu_j/tu_i;q,p)_\infty} \prod_{1\le i<j\le \ell(\lambda)+1} \frac{(pqu_j/u_i;q,p)_\infty}{(ptu_j/u_i;q,p)_\infty} \\ &= \prod_{\square\in\lambda} \frac{(pq^{-a(\square)}t^{-\ell(\square)-1};p)_\infty}{(pq^{-a(\square)-1}t^{-\ell(\square)};p)_\infty}. \end{aligned} \tag{3.37}$$

Then, one finds the following property.

Proposition 3.9

$$\frac{c'_\lambda}{c_\lambda}\frac{N_\lambda(p)}{N'_\lambda(p)} = \frac{c'_\lambda(p)}{c_\lambda(p)}. \tag{3.38}$$

3.3 Level (0,0) representation and elliptic Ruijsenaars operators

The two remarks after Proposition 3.9 suggest a connection of the level (0,1) or (0,0) representations of $\mathcal{U}_{q,t,p}$ to a possible elliptic analogue of the Macdonald symmetric functions, which are expected to be eigenfunctions of the elliptic Ruijsenaars difference operator. In this subsection, we show a direct relation between the level (0,0) representation and the elliptic Ruijsenaars difference operator

$$D = \sum_{i=1}^N \prod_{j\ne i} \frac{\theta_p(tx_i/x_j)}{\theta_p(x_i/x_j)} T_{q,x_i},$$

which acts on $\mathbb{C}[[x_1^{\pm1},\dots,x_N^{\pm1}]]$. Here, T_{q,x_i} denotes the shift operator

$$T_{q,x_i} f(x_1,\dots,x_i,\dots x_n) = f(x_1,\dots,qx_i,\dots x_n).$$

The following theorem is an elliptic analogue of Proposition 3.3 in [62].

Theorem 3.10 *The following assignment gives a level* $(0,0)$ *representation of* $\mathcal{U}_{q,t,p}$ *on* $\mathbb{C}[[x_1^{\pm 1},\dots,x_N^{\pm 1}]]$.

$$x^+(z) = a^+(p)\sum_{i=1}^{N}\prod_{j\neq i}\frac{\theta_p(tx_i/x_j)}{\theta_p(x_i/x_j)}\delta(x_i/z)T_{q,x_i}, \tag{3.39}$$

$$x^-(z) = -a^-(p)\sum_{i=1}^{N}\prod_{j\neq i}\frac{\theta_p(t^{-1}x_i/x_j)}{\theta_p(x_i/x_j)}\delta(q^{-1}x_i/z)T_{q,x_i}^{-1}, \tag{3.40}$$

$$\alpha_m = \frac{(1-t^{-m})(1-(q/t)^{-m})}{m}\sum_{j=1}^{N}x_j^m \qquad (m\in\mathbb{Z}\backslash\{0\}), \tag{3.41}$$

or

$$\psi^+(z) = \prod_{j=1}^{N}\frac{\theta_p(t^{-1}x_j/z)\theta_p(q^{-1}tx_j/z)}{\theta_p(x_j/z)\theta_p(q^{-1}x_j/z)}, \tag{3.42}$$

$$\psi^-(z) = \prod_{j=1}^{N}\frac{\theta_p(tz/x_j)\theta_p(qt^{-1}z/x_j)}{\theta_p(z/x_j)\theta_p(qz/x_j)}. \tag{3.43}$$

In particular, the zero-mode $x_0^+ = \oint_{|z|=0}\frac{dz}{2\pi iz}x^+(z)$ *acts as the elliptic Ruijsenaars difference operator*

$$x_0^+ = a^+(p)\sum_{i=1}^{N}\prod_{j\neq i}\frac{\theta_p(tx_i/x_j)}{\theta_p(x_i/x_j)}T_{q,x_i}$$

Proof Let us check the relation (2.18).

$$\begin{aligned}[x^+(z),x^-(w)] = -a^+(p)a^-(p)\delta(z/w)\sum_{i=1}^{N}\Bigg(&\prod_{j\neq i}\frac{\theta_p(tx_i/x_j)}{\theta_p(x_i/x_j)}\prod_{k\neq i}\frac{\theta_p(qx_i/tx_k)}{\theta_p(qx_i/x_k)}\delta(x_i/z)\\ &-\prod_{j\neq i}\frac{\theta_p(tx_i/qx_j)}{\theta_p(x_i/qx_j)}\prod_{k\neq i}\frac{\theta_p(x_i/tx_k)}{\theta_p(x_i/x_k)}\delta(q^{-1}x_i/z)\Bigg).\end{aligned}$$

Then, the relation (2.18) holds by the following lemma.

Lemma 3.11

$$
\begin{aligned}
&\prod_{j=1}^{N}\frac{\theta_p(qz/tx_j)\theta_p(tz/x_j)}{\theta_p(z/x_j)\theta_p(qz/x_j)}\Big|_{+}-\prod_{j=1}^{N}\frac{\theta_p(qz/tx_j)\theta_p(tz/x_j)}{\theta_p(z/x_j)\theta_p(qz/x_j)}\Big|_{-}\\
&=-\frac{\theta_p(q/t)\theta_p(t)}{(p;p)_\infty^2\theta_p(q)}\sum_{i=1}^{N}\left(\prod_{j\neq i}\frac{\theta_p(tx_i/x_j)}{\theta_p(qx_i/x_j)}\prod_{k\neq i}\frac{\theta_p(qx_i/tx_k)}{\theta_p(x_i/x_k)}\delta(x_i/z)\right.\\
&\qquad\left.-\prod_{j\neq i}\frac{\theta_p(tx_i/qx_j)}{\theta_p(x_i/qx_j)}\prod_{k\neq i}\frac{\theta_p(x_i/tx_k)}{\theta_p(x_i/qx_k)}\delta(q^{-1}x_i/z)\right).
\end{aligned}
$$

Proof Note the partial fraction expansion formula, see for example [75],

$$
\theta_p(s/b_{m+1})\prod_{j=1}^{m}\frac{\theta_p(s/b_j)}{\theta_p(s/a_j)}=-\sum_{k=1}^{m}\frac{1}{\theta_p(a_k/s)}\frac{\prod_{j=1}^{m+1}\theta_p(a_k/b_j)}{\prod_{\substack{j=1\\ \neq k}}^{m}\theta_p(a_k/a_j)} \tag{3.44}
$$

with the balancing condition $b_1\cdots b_{m+1}=a_1\cdots a_m s$. Let us consider the case $m=2N$ and take

$$
\begin{aligned}
&s=z,\quad a_j=x_j,\quad a_{N+j}=q^{-1}x_j,\quad b_j=(t/q)x_j,\quad b_{N+j}=\beta^{-1}x_j\\
&(j=1,\ldots,N),\quad b_{2N+1}=(\beta/t)^N z,
\end{aligned}
$$

for some constant β. Then, one has

$$
\begin{aligned}
&\prod_{j=1}^{N}\frac{\theta_p(qz/tx_j)}{\theta_p(z/x_j)}\prod_{j=1}^{N}\frac{\theta_p(\beta z/x_j)}{\theta_p(qz/x_j)}\\
&=-\sum_{i=1}^{N}\frac{\theta_p((t/\beta)^N x_i/z)\theta_p(q/t)}{\theta_p((t/\beta)^N)\theta_p(x_i/z)}\prod_{\substack{j=1\\ \neq i}}^{N}\frac{\theta_p(qx_i/tx_j)}{\theta_p(x_i/x_j)}\prod_{\substack{k=1\\ \neq i}}^{N}\frac{\theta_p(\beta x_i/x_k)}{\theta_p(qx_i/x_k)}\times\frac{\theta_p(\beta)}{\theta_p(q)}\\
&\quad-\sum_{i=1}^{N}\frac{\theta_p((t/\beta)^N x_i/qz)\theta_p(\beta/q)}{\theta_p((t/\beta)^N)\theta_p(x_i/qz)}\frac{\theta_p(t^{-1})}{\theta_p(q^{-1})}\prod_{\substack{j=1\\ \neq i}}^{N}\frac{\theta_p(x_i/tx_j)}{\theta_p(x_i/qx_j)}\prod_{\substack{k=1\\ \neq i}}^{N}\frac{\theta_p(\beta x_i/qx_k)}{\theta_p(x_i/x_k)}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\prod_{j=1}^{N}\frac{\theta_p(qz/tx_j)}{\theta_p(z/x_j)}\prod_{j=1}^{N}\frac{\theta_p(\beta z/x_j)}{\theta_p(qz/x_j)}\Big|_{+}-\prod_{j=1}^{N}\frac{\theta_p(qz/tx_j)}{\theta_p(z/x_j)}\prod_{j=1}^{N}\frac{\theta_p(\beta z/x_j)}{\theta_p(qz/x_j)}\Big|_{-}\\
&=-\frac{\theta_p(\beta)\theta_p(q/t)}{(p;p)_\infty^2\theta_p(q)}\sum_{i=1}^{N}\delta(x_i/z)\prod_{\substack{j=1\\ \neq i}}^{N}\frac{\theta_p(qx_i/tx_j)}{\theta_p(x_i/x_j)}\prod_{\substack{k=1\\ \neq i}}^{N}\frac{\theta_p(\beta x_i/x_k)}{\theta_p(qx_i/x_k)}
\end{aligned}
$$

$$
-\frac{\theta_p(t^{-1})\theta_p(\beta/q)}{(p;p)_\infty^2\theta_p(q^{-1})}\sum_{i=1}^{N}\delta(q^{-1}x_i/z)\prod_{\substack{j=1\\ \neq i}}^{N}\frac{\theta_p(x_i/tx_j)}{\theta_p(x_i/qx_j)}\prod_{\substack{k=1\\ \neq i}}^{N}\frac{\theta_p(\beta x_i/qx_k)}{\theta_p(x_i/x_k)}
$$

Then, taking the limit $\beta \to t$, one obtains the desired formula. □

4 The vertex operators

We construct the two vertex operators $\Phi(u)$ and $\Psi^*(u)$ of $\mathcal{U}_{q,t,p}$ called the type I and the type II dual vertex operators [38] as intertwining operators of $\mathcal{U}_{q,t,p}$-modules. These two vertex operators are the elliptic analogues of those constructed in [5], whose matrix elements reproduce the refined topological vertex in [4, 36, 85]. We also construct a shifted inverse of them denoted by $\Phi^*(u)$ and $\Psi(u)$, respectively. These vertex operators turn out to be useful to realize the affine quiver W algebra and instanton calculus in the affine quiver gauge theories. See Sects. 5 and 6.

4.1 The type I vertex operator

The type I vertex operator is the intertwining operator

$$
\Phi(u):\mathcal{F}^{(1,N+1)}_{-uv}\to\mathcal{F}^{(0,1)}_{u}\widetilde{\otimes}\mathcal{F}^{(1,N)}_{v}
$$

w.r.t. the comultiplication Δ satisfying

$$
\Delta(x)\Phi(u)=\Phi(u)x \qquad (\forall x\in\mathcal{U}_{q,t,p}). \tag{4.1}
$$

We define the components of $\Phi(u)$ by

$$
\Phi(u)|\xi\rangle=\sum_{\lambda\in\mathcal{P}^+}|\lambda\rangle'_u\widetilde{\otimes}\Phi_\lambda(u)|\xi\rangle \qquad \forall|\xi\rangle\in\mathcal{F}^{(1,N+1)}_{-uv}, \tag{4.2}
$$

where we set

$$
|\lambda\rangle'_u=\frac{c_\lambda(p)}{c'_\lambda(p)}|\lambda\rangle_u. \tag{4.3}
$$

Lemma 4.1 *The intertwining relation* (4.1) *reads*

$$
\Phi_\lambda(u)\psi^+((t/q)^{1/4}z)=(q/t)^{1/2}B^+_\lambda(u/z;p)\psi^+((t/q)^{1/4}z)\Phi_\lambda(u), \tag{4.4}
$$

$$
\Phi_\lambda(u)\psi^-((t/q)^{-1/4}z)=(q/t)^{-1/2}B^-_\lambda(z/u;p)\psi^-((t/q)^{-1/4}z)\Phi_\lambda(u), \tag{4.5}
$$

$$
\begin{aligned}
\Phi_\lambda(u)x^+((t/q)^{-1/4}z)&=x^+((t/q)^{-1/4}z)\Phi_\lambda(u)+qf(1;p)^{-1}\psi^-((t/q)^{-1/4}z)\\
&\quad\times\sum_{i=1}^{\ell(\lambda)+1}a^-(p)A^-_{\lambda,i}(p)\delta(q^{-1}u_i/z)\Phi_{\lambda-\mathbf{1}_i}(u),
\end{aligned} \tag{4.6}
$$

$$\Phi_\lambda(u)x^-((t/q)^{1/4}z) = (q/t)^{1/2}B^+_\lambda(u/z;p)x^-((t/q)^{1/4}z)\Phi_\lambda(u) + q^{-1}f(1;p)\\ \times(q/t)^{1/2}\sum_{i=1}^{\ell(\lambda)} a^+(p)A^+_{\lambda,i}(p)\delta(u_i/z)\Phi_{\lambda+\mathbf{1}_i}(u). \tag{4.7}$$

Proof For example, let us consider

$$\Delta(x^+(z))\Phi(u)|\xi\rangle = \Phi(u)x^+(z)|\xi\rangle \qquad \forall|\xi\rangle \in \mathcal{F}^{(1,N+1)}_{-uv}.$$

$$\begin{aligned}
\text{RHS} &= \sum_\lambda |\lambda\rangle'_u \widetilde{\otimes} \Phi_\lambda(u)x^+(z)|\xi\rangle,\\
\text{LHS} &= \sum_\lambda |\lambda\rangle'_u \widetilde{\otimes} x^+(z)\Phi_\lambda(u)|\xi\rangle + \sum_\lambda x^+((t/q)^{1/4}z)|\lambda\rangle'_u \widetilde{\otimes} \psi^-(z)\widetilde{\Phi}_\lambda(u)|\xi\rangle\\
&= \sum_\lambda |\lambda\rangle'_u \widetilde{\otimes} x^+(z)\Phi_\lambda(u)|\xi\rangle\\
&\quad + \sum_\lambda \sum_{i=1}^{\ell(\lambda)+1} a^+(p)A^{+'}_{\lambda,i}(p)\delta((t/q)^{-1/4}u_i/z)|\lambda+\mathbf{1}_i\rangle'_u \widetilde{\otimes} \psi^-(z)\Phi_\lambda(u)|\xi\rangle\\
&= \sum_\lambda |\lambda\rangle'_u \widetilde{\otimes} x^+(z)\Phi_\lambda(u)|\xi\rangle\\
&\quad + qf(1,p)^{-1}\sum_\lambda \sum_{i=1}^{\ell(\lambda)+1} a^-(p)A^-_{\lambda+\mathbf{1}_i,i}(p)\delta((t/q)^{-1/4}u_i/z)|\lambda+\mathbf{1}_i\rangle'_u \widetilde{\otimes} \psi^-(z)\Phi_\lambda(u)|\xi\rangle\\
&= \sum_\lambda |\lambda\rangle'_u \widetilde{\otimes} \Bigg\{ x^+(z)\Phi_\lambda(u)\\
&\quad + qf(1,p)^{-1}\sum_{i=1}^{\ell(\lambda)} a^-(p)A^-_{\lambda,i}(p)\delta((t/q)^{-1/4}q^{-1}u_i/z)\psi^-(z)\Phi_{\lambda-\mathbf{1}_i}(u)\Bigg\}|\xi\rangle
\end{aligned}$$

The third equality follows from (3.30). □

By using the representations in Theorems 3.2 and 3.4, one can solve these intertwining relations and obtain the following result.

Theorem 4.2

$$\begin{aligned}
\Phi_\lambda(u) &= \frac{q^{n(\lambda')}N_\lambda(p)t^*(\lambda,u,v,N)}{c_\lambda}\widetilde{\Phi}_\lambda(u),\\
\widetilde{\Phi}_\lambda(u) &=: \Phi_\emptyset(u)\prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i}\widetilde{x}^-((t/q)^{1/4}q^{j-1}t^{-i+1}u) :,\\
\Phi_\emptyset(u) &= \exp\left\{-\sum_{m>0}\frac{1}{\kappa_m}\alpha'_{-m}((t/q)^{1/2}u)^m\right\}\exp\left\{\sum_{m>0}\frac{1}{\kappa_m}\alpha'_m((t/q)^{1/2}u)^{-m}\right\}
\end{aligned}$$

where $x^-(z) = u^{-1}z^N(t/q)^{-3N/4}\widetilde{x}^-(z)$ *on* $\mathcal{F}_u^{(1,N)}$ *and*

$$n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i, \qquad n(\lambda') = \sum_{i\geq 1}(i-1)\lambda'_i = \sum_{i\geq 1}\frac{\lambda_i(\lambda_i-1)}{2}, \tag{4.8}$$

$$t^*(\lambda, u, v, N) = (q^{-1}v)^{-|\lambda|}(-u)^{N|\lambda|}f_\lambda(q,t)^N, \tag{4.9}$$

$$f_\lambda(q,t) = (-1)^{|\lambda|}q^{n(\lambda')+|\lambda|/2}t^{-n(\lambda)-|\lambda|/2}. \tag{4.10}$$

The factor $t^*(\lambda, u, v, N)$ was introduced in [5] and in particular $f_\lambda(q,t)$ is called the framing factor [4, 85]. Our vertex operator $\Phi(u)$ is the elliptic analogue of $\Phi^*(u)$ in [5]. One should note that our comultiplication is opposite from the one in [5]. A proof of the statement is given in Appendix D.

In later sections, the following formula is useful.

Proposition 4.3

$$\widetilde{\Phi}_\lambda(u) =: \exp\left(\sum_{m\neq 0}\frac{1-t^m}{\kappa_m}\mathcal{E}_{\lambda,m}\alpha'_m((t/q)^{1/2}u)^{-m}\right), \tag{4.11}$$

where we set

$$\mathcal{E}_{\lambda,m} = \frac{1}{1-t^m} + \sum_{j=1}^{\ell(\lambda)}(q^{-m\lambda_j}-1)t^{m(j-1)} \qquad (m\in\mathbb{Z}_{\neq 0}).$$

4.2 The type II dual vertex operator

The type II dual vertex operator is the intertwining operator

$$\Psi^*(v)\ \ \mathcal{F}_u^{(1,N)}\widetilde{\otimes}\mathcal{F}_v^{(0,1)} \to \mathcal{F}_{-vu}^{(1,N+1)} \tag{4.12}$$

satisfying

$$x\Psi^*(v) = \Psi^*(v)\Delta(x) \qquad \forall x\in\mathcal{U}_{q,t,p}. \tag{4.13}$$

We call $\Psi^*(v)$ the type II dual vertex operator. We define its components by

$$\Psi^*_\lambda(v)|\xi\rangle = \Psi^*(v)\left(|\xi\rangle\widetilde{\otimes}|\lambda\rangle_v\right) \qquad \forall|\xi\rangle\in\mathcal{F}_u^{(1,N)}. \tag{4.14}$$

Lemma 4.4 *The intertwining relation* (4.13) *is equivalent to*

$$\psi^+((q/t)^{1/4}z)\Psi^*_\lambda(v) = (q/t)^{1/2}B^+_\lambda(v/z;p^*)\Psi^*_\lambda(v)\psi^+((q/t)^{1/4})), \tag{4.15}$$

$$\psi^-((q/t)^{-1/4}z)\Psi^*_\lambda(v) = (q/t)^{-1/2}B^-_\lambda(z/v;p^*)\Psi^*_\lambda(v)\psi^-((q/t)^{-1/4})), \tag{4.16}$$

$$x^-((q/t)^{1/4}z)\Psi^*_\lambda(v) = \Psi^*_\lambda(v)x^-((q/t)^{1/4}z) + (q/t)^{1/2}a^-(p^*) \times \sum_{i=1}^{\ell(\lambda)} A^-_{\lambda,i}(p^*)\delta(q^{-1}v_i/z)\widetilde{\Psi}^*_{\lambda-\mathbf{1}_i}(v)\psi^+((q/t)^{1/4}z), \tag{4.17}$$

$$x^+((q/t)^{-1/4}z)\Psi^*_\lambda(v) = (q/t)^{-1/2}B^-_\lambda(z/v;p^*)\Psi^*_\lambda(v)x^+((q/t)^{-1/4}z) + a^+(p^*)\sum_{i=1}^{\ell(\lambda)+1} A^+_{\lambda,i}(p^*)\delta(v_i/z)\widetilde{\Psi}^*_{\lambda+\mathbf{1}_i}(v). \tag{4.18}$$

Here, $p^* = pq/t$ associated with $\mathcal{F}^{(1,N)}_u$.

One can prove this in the similar way to Lemma 4.1.

By using the representations in Theorem 3.2, 3.4 and (4.3), one can solve these intertwining relations and obtain the following result.

Theorem 4.5

$$\Psi^*_\lambda(v) = \frac{q^{n(\lambda')}t(\lambda,u,v,N)}{c_\lambda N'_\lambda(p^*)}\widetilde{\Psi}^*_\lambda(v),$$

$$\widetilde{\Psi}^*_\lambda(v) = \Psi^*_\emptyset(v)\prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i}\widetilde{x}^+((t/q)^{1/4}q^{j-1}t^{-i+1}v), \tag{4.19}$$

$$\Psi^*_\emptyset(v) = \exp\left\{\sum_{m>0}\frac{1}{\kappa_m}\alpha_{-m}((t/q)^{1/2}v)^m\right\}\exp\left\{-\sum_{m>0}\frac{1}{\kappa_m}\alpha_m((t/q)^{1/2}v)^{-m}\right\}. \tag{4.20}$$

where $x^+(z) = uz^{-N}(t/q)^{3N/4}\widetilde{x}^+(z)$ on $\mathcal{F}^{(1,N)}_u$, and

$$t(\lambda,u,v,N) = (-uv)^{|\lambda|}(-v)^{-(N+1)|\lambda|}f_\lambda(q,t)^{-N-1}$$

with $f_\lambda(q,t)$ given in (4.10).

Our vertex operator $\Psi^*(u)$ is the elliptic analogue of $\Phi(u)$ in [5]. A proof of the statement is similar to the one in Appendix D.

We have a similar formula to Proposition 4.3.

Proposition 4.6

$$\widetilde{\Psi}^*_\lambda(v) =: \exp\left(-\sum_{m\neq 0}\frac{1-t^m}{\kappa_m}\mathcal{E}_{\lambda,m}\alpha_m((t/q)^{1/2}v)^{-m}\right):. \tag{4.21}$$

4.3 The shifted inverse of $\Phi(u)$ and $\Psi^*(v)$

We next introduce the shifted inverse of $\Phi(u)$ and $\Psi^*(v)$ denoted by $\Phi^*(u)$ and $\Psi(v)$, respectively. In Sects. 5 and 6, we show that the vertex operators here and in the previous subsections are useful to derive physical and mathematical quantities associated with the Jordan quivers.

4.3.1 The vertex operator $\Phi^*(u)$

Let us consider the linear map

$$\Phi^*(u) : \mathcal{F}_u^{(0,1)} \widetilde{\otimes} \mathcal{F}_v^{(1,N)} \to \mathcal{F}_{-uv}^{(1,N+1)}, \tag{4.22}$$

whose components are defined by

$$\Phi^*(u)\left(|\lambda\rangle'_u \widetilde{\otimes} |\xi\rangle\right) = \Phi^*_\lambda(u)|\xi\rangle, \qquad \forall |\xi\rangle \in \mathcal{F}_v^{(1,N)}, \tag{4.23}$$

$$\Phi^*_\lambda(u) = \frac{q^{n(\lambda')} N'_\lambda(p) t(\lambda, v, up^{-1}, N)}{c'_\lambda} : \widetilde{\Phi}_\lambda(p^{-1}u)^{-1} : . \tag{4.24}$$

Note that from Proposition 4.3 we have

$$: \widetilde{\Phi}_\lambda(p^{-1}u)^{-1} :=: \exp\left(-\sum_{m\neq 0} \frac{1-t^m}{\kappa_m} p^m \mathcal{E}_{\lambda,m} \alpha'_m ((t/q)^{1/2}u)^{-m}\right) :$$

Proposition 4.7 *The vertex operator $\Phi^*_\lambda(u)$ satisfies the following relations.*

$$\psi^+((t/q)^{1/4}z)\Phi^*_\lambda(u) = (t/q)^{-1/2} B^+_\lambda(p^{-1}u/z;\, p)\Phi^*_\lambda(u)\psi^+((t/q)^{1/4}z), \tag{4.25}$$

$$\psi^-((t/q)^{-1/4}z)\Phi^*_\lambda(u) = (t/q)^{1/2} B^-_\lambda(pz/u;\, p)\Phi^*_\lambda(u)\psi^-((t/q)^{-1/4}z), \tag{4.26}$$

$$\begin{aligned} x^+((t/q)^{-1/4}z)\Phi^*_\lambda(u) &= \Phi^*_\lambda(u)x^+((t/q)^{-1/4}z) \\ &+ (t/q)^{-1/2}a^+(p)\sum_{i=1}^{\ell(\lambda)+1} A^{+'}_{\lambda,i}(p)\delta(p^{-1}tu_i/qz)\Phi^*_{\lambda+\mathbf{1}_i}(u)\psi^+((t/q)^{1/4}qz/t), \end{aligned} \tag{4.27}$$

$$\begin{aligned} x^-((t/q)^{1/4}z)\Phi^*_\lambda(u) &= (t/q)^{-1/2} B^+_\lambda(p^{-1}u/z;\, p)\Phi^*_\lambda(u)x^-((t/q)^{1/4}z) \\ &+ (t/q)a^-(p)\sum_{i=1}^{\ell(\lambda)} A^{-'}_{\lambda,i}(p)\delta(p^{-1}q^{-1}u_i/z)\Phi^*_{\lambda-\mathbf{1}_i}(u). \end{aligned} \tag{4.28}$$

A proof of this statement is given in Appendix E.

These relations allow us to expect that $\Phi^*(u)$ is the intertwiner satisfying

$$\Phi^*(u)\Delta(x) = x\Phi^*(u) \qquad \forall x \in \mathcal{U}_{q,t,p}. \tag{4.29}$$

However, it turns out this is not the case. In fact one can derive the following relations from (4.29), which are slightly different from those in Proposition 4.7.

Proposition 4.8 *The intertwining relation* (4.29) *reads*

$$\psi^+((t/q)^{1/4}z)\Phi^*_\lambda(u) = (t/q)^{-1/2}B^+_\lambda(u/z;\, p)\Phi^*_\lambda(u)\psi^+((t/q)^{1/4}z), \quad (4.30)$$
$$\psi^-((t/q)^{-1/4}z)\Phi^*_\lambda(u) = (t/q)^{1/2}B^-_\lambda(z/u;\, p)\Phi^*_\lambda(u)\psi^-((t/q)^{-1/4}z), \quad (4.31)$$
$$x^+((t/q)^{-1/4}z)\Phi^*_\lambda(u) = \Phi^*_\lambda(u)x^+((t/q)^{-1/4}z)$$
$$+a^+(p)\sum_{i=1}^{\ell(\lambda)+1} A^{+'}_{\lambda,i}(p)\delta(u_i/z)\Phi^*_{\lambda+\mathbf{1}_i}(u)\psi^-((t/q)^{-1/4}z), \quad (4.32)$$
$$x^-((t/q)^{1/4}z)\Phi^*_\lambda(u) = (t/q)^{-1/2}B^+_\lambda(u/z;\, p)\Phi^*_\lambda(u)x^-((t/q)^{1/4}z)$$
$$+(t/q)^{-1/2}a^-(p)\sum_{i=1}^{\ell(\lambda)} A^{-'}_{\lambda,i}(p)\delta(q^{-1}u_i/z)\Phi^*_{\lambda-\mathbf{1}_i}(u). \quad (4.33)$$

This discrepancy is probably due to a lack of understanding the dual representation to $\mathcal{F}^{(0,1)}_u$. It is hence an open problem to find a representation theoretical meaning of $\Phi^*(u)$.

4.3.2 The vertex operator $\Psi(v)$

Similar to $\Phi^*(u)$, we consider the linear map

$$\Psi(v) : \mathcal{F}^{(1,N+1)}_{-uv} \to \mathcal{F}^{(1,N)}_u \widetilde{\otimes} \mathcal{F}^{(0,1)}_v$$

and define its components by

$$\Psi(v)|\xi\rangle = \sum_{\lambda\in\mathcal{P}^+} \Psi_\lambda(v)|\xi\rangle \widetilde{\otimes} |\lambda\rangle_v \qquad \forall|\xi\rangle \in \mathcal{F}^{(1,N+1)}_{-uv}, \quad (4.34)$$
$$\Psi_\lambda(v) = \frac{q^{n(\lambda')}t^*(\lambda, p^*v, u, N)}{c'_\lambda N_\lambda(p^*)} : \widetilde{\Psi}^*_\lambda(p^*v)^{-1} : . \quad (4.35)$$

Note that from Proposition 4.6 we have

$$: \widetilde{\Psi}^*_\lambda(p^*v)^{-1} :=: \exp\left(-\sum_{m\neq 0} \frac{1-t^m}{\kappa_m} p^{*-m}\mathcal{E}_{\lambda,m}\alpha_m((t/q)^{1/2}v)^{-m}\right) : .$$

One finds the following relations satisfied.

Proposition 4.9

$$\Psi_\lambda(v)\psi^+((t/q)^{-1/4}z) = (t/q)^{-1/2}B^+_\lambda(p^*v/z;\, p^*)\psi^+((t/q)^{-1/4}z)\Psi_\lambda(v), \quad (4.36)$$

$$\Psi_\lambda(v)\psi^-((t/q)^{1/4}z) = (t/q)^{1/2}B^-_\lambda(p^{*-1}z/v;\,p^*)\psi^-((t/q)^{1/4}z)\Psi_\lambda(v), \tag{4.37}$$

$$\Psi_\lambda(v)x^-((t/q)^{-1/4}z) = x^-((t/q)^{-1/4}z)\Psi_\lambda(v) + (t/q)a^-(p^*)\sum_{i=1}^{\ell(\lambda)+1} A^{+'}_{\lambda,i}(p^*)\delta(p^*tv_i/qz)\psi^-((t/q)^{-1/4}z)\Psi_{\lambda+\mathbf{1}_i}(v), \tag{4.38}$$

$$\Psi_\lambda(v)x^+((t/q)^{1/4}z) = (t/q)^{1/2}B^-_\lambda(p^{*-1}z/v;\,p^*)x^+((t/q)^{1/4}z)\Psi_\lambda(u) + (t/q)^{-1/2}a^+(p^*)\sum_{i=1}^{\ell(\lambda)} A^{-'}_{\lambda,i}(p^*)\delta(p^*v_i/qz)\Psi_{\lambda-\mathbf{1}_i}(v). \tag{4.39}$$

A proof of this is similar to the one in Appendix E.

Again, these relations do not coincide with the intertwining relation

$$\Psi(v)x = \Delta(x)\Psi(v) \qquad \forall x \in \mathcal{U}_{q,t,p}, \tag{4.40}$$

which is equivalent to the following relations:

Proposition 4.10

$$\Psi_\lambda(v)\psi^+((t/q)^{-1/4}z) = (t/q)^{-1/2}B^+_\lambda(v/z;\,p^*)\psi^+((t/q)^{-1/4}z)\Psi_\lambda(v), \tag{4.41}$$

$$\Psi_\lambda(v)\psi^-((t/q)^{1/4}z) = (t/q)^{1/2}B^-_\lambda(z/v;\,p^*)\psi^-((t/q)^{1/4}z)\Psi_\lambda(v), \tag{4.42}$$

$$\Psi_\lambda(v)x^-((t/q)^{1/4}z) = x^-((t/q)^{1/4}z) + (t/q)^{1/2}a^-(p^*)\sum_{i=1}^{\ell(\lambda+\mathbf{1}_i)} A^{+'}_{\lambda,i}(p^*)\delta(q^{\lambda_i}t^{-i+1}v/z)\psi^+((t/q)^{-1/4}z)\Psi_{\lambda+\mathbf{1}_i}(v), \tag{4.43}$$

$$\Psi_\lambda(v)x^+((t/q)^{1/4}z) = (t/q)^{1/2}B^-_\lambda(z/v;\,p^*)x^+((t/q)^{1/4}z)\Psi_\lambda(v) + a^+(p^*)\sum_{i=1}^{\ell(\lambda-\mathbf{1}_i)+1} A^{-'}_{\lambda,i}(p^*)\delta(q^{-1}q^{\lambda_i}t^{-i+1}v/z)\Psi_{\lambda-\mathbf{1}_i}(v). \tag{4.44}$$

Hence, again it is an open problem to find a representation theoretical meaning of $\Psi(u)$.

5 Affine quiver W-algebra $W_{p,p^*}(\Gamma(\widehat{A}_0))$

One of the importance to consider the elliptic quantum group is that it gives a realization of the deformed W algebras such as $W_{p,p^*}(\mathfrak{g})$ [27] and provides an algebraic structure, i.e., a co-algebra structure, which enables us to define the intertwining operators (the vertex operators) as deformation of the primary fields in CFT. In this section, we realize the deformed W algebra $W_{p,p^*}(\Gamma(\widehat{A}_0))$ associated with the Jordan quiver $\widehat{A}_0$

[41] by using the level $(1, N)$ representation of $\mathcal{U}_{q,t,p}$ given in Sect. 3.1, where in particular $\gamma = (t/q)^{1/2}$, in the same scheme as it was done for $W_{p,p^*}(\mathfrak{g})$ in terms of the level 1 representation of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{g}})$ [16, 40, 45, 47, 50].

5.1 Screening currents

Let us set

$$s_m^+ = \frac{(t/q)^{m/2}}{1-(t/q)^m}\alpha_m, \qquad s_m^- = \frac{(t/q)^{m/2}}{1-(t/q)^m}\alpha'_m.$$

Then from (2.15) and (2.24) with $\gamma = (t/q)^{1/2}$, hence $p^* = pq/t$, one can show the following commutation relations

$$[s_m^+, s_n^+] = -\frac{1}{m}\frac{1-p^m}{1-p^{*m}}(1-q^m)(1-t^{-m})\delta_{m+n,0},$$
$$[s_m^-, s_n^-] = -\frac{1}{m}\frac{1-p^{*-m}}{1-p^{-m}}(1-q^m)(1-t^{-m})\delta_{m+n,0}.$$

Moreover, one can rewrite the elliptic currents $x^\pm(z)$ in Theorem 3.2 as

$$x^\pm((t/q)^{1/4}z) = ((t/q)^{N/2}u/z^N)^{\pm 1} : \exp\left\{\pm\sum_{m\neq 0} s_m^\pm\, z^{-m}\right\} : .$$

Hence, one of $x^\pm((t/q)^{1/4}z)$ coincides with the screening currents of $W_{p,p^*}(\Gamma(\widehat{A}_0))$ [41, 42] with the $SU(4)$ Ω-deformation parameters p, p^*, q, t [70] satisfying

$$p/p^* = t/q.$$

In our knowledge the elliptic quantum toroidal algebra $\mathcal{U}_{q,t,p}$ is the first quantum group structure which possesses the $SU(4)$ Ω-deformation parameters.

One should also note that in [41, 42] Kimura and Pestun constructed only one type of screening current. However, it is natural for the (deformed) W algebras that there are two types of screening currents [7, 12, 19, 27]. In this sense, our realization completes their construction.

5.2 Generating function

To obtain the generating function of $W_{p,p^*}(\Gamma(\widehat{A}_0))$, we apply the same scheme as used in [50]. Namely consider the composition of $\Phi(u)$ and $\Phi^*(u)$, which are given in Theorem 4.2 and Sect. 4.3.1, respectively.

$$T(u) = \Phi^*(u)\Phi(u) = \sum_{\lambda\in\mathcal{P}^+}\Phi^*_\lambda(u)\Phi_\lambda(u) \quad : \mathcal{F}^{(1,N+1)}_{-uv} \to \mathcal{F}^{(1,N+1)}_{-uv},$$

Note that one can choose $v \in \mathbb{C}^*$, $N \in \mathbb{Z}$ arbitrarily. Taking the normal ordering one obtains

$$\Phi^*_\lambda(u)\Phi_\lambda(u) = \mathcal{C}_\lambda(q,t,p) : \widetilde{\Phi}_\lambda(u)\widetilde{\Phi}^*_\lambda(u) : .$$

Then, one finds that the operator part is given by

$$: \widetilde{\Phi}^*_\lambda(u)\widetilde{\Phi}_\lambda(u) :=: \prod_{\square \in A(\lambda)} Y(u/q^{\square}) \prod_{\blacksquare \in R(\lambda)} Y((q/t)u/q^{\blacksquare})^{-1} : \tag{5.1}$$

with

$$Y(u) =: \exp\left\{\sum_{m\neq 0} y_m u^{-m}\right\} : . \tag{5.2}$$

Here, we set $q^{\square} \equiv t^{i-1}q^{-j+1}$ for $\square = (i,j) \in \lambda$, etc., and $y_m = \dfrac{1-p^m}{\kappa_m}(t/q)^{-m/2}\alpha'_m$. The symbols $R(\lambda)$ and $A(\lambda)$ denote the set of removable and addable boxes in the Young's diagram λ, respectively. The main structure of (5.1) is due to Proposition 4.3, which yields

$$: \widetilde{\Phi}^*_\lambda(u)\widetilde{\Phi}_\lambda(u) :=: \exp\left(\sum_{m\neq 0} \frac{(1-t^m)(1-p^m)}{\kappa_m}\mathcal{E}_{\lambda,m}\alpha'_m((t/q)^{1/2}u)^{-m}\right) :, \tag{5.3}$$

and the following combinatorial formula.

Proposition 5.1

$$\mathcal{E}_{\lambda,m} = \frac{1}{1-t^m}\left(\sum_{\square\in A(\lambda)} q^{m\square} - (t/q)^m \sum_{\blacksquare \in R(\lambda)} q^{m\blacksquare}\right) \tag{5.4}$$

Proof The statement follows from

$$\mathcal{E}_{\lambda,m} = \frac{1}{1-t^m}\left(1-(1-q^{-m})(1-t^m)\sum_{\square\in\lambda} q^{m\square}\right)$$

and

- $(i,\lambda_i) \in R(\lambda) \Longleftrightarrow \lambda_i > \lambda_{i+1}$
- if $(i,\lambda_i) \in R(\lambda) \Longleftrightarrow (i+1,\lambda_{i+1}+1) \in A(\lambda)$

for $\lambda \in \mathcal{P}^+$. □

Moreover, from (2.24) with $\gamma = (t/q)^{1/2}$, one finds the following commutation relation.

$$[y_m, y_n] = -\frac{1}{m}\frac{(1-p^{*m})(1-p^{-m})}{(1-q^m)(1-t^{-m})}\delta_{m+n,0}.$$

This agrees with the one in [41].

The coefficient part in $\Phi^*_\lambda(u)\Phi_\lambda(u)$ can be calculated by combining the normalization factors of the vertex operators and the OPE coefficient. The calculation of the latter coefficient is essentially due to the following formula [4] obtained by considering a q-analogue of (3.6) in [71].

Proposition 5.2

$$\begin{aligned}&-\frac{1-t^m}{1-q^m}\mathcal{E}_{\lambda,-m}\mathcal{E}_{\mu,m}\\&=\frac{t^m}{(1-q^m)(1-t^m)}+\sum_{\square\in\mu}q^{ma_\lambda(\square)}t^{m(\ell_\mu(\square)+1)}+\sum_{\blacksquare\in\lambda}q^{-m(a_\mu(\blacksquare)+1)}t^{-m\ell_\lambda(\blacksquare)}.\end{aligned} \tag{5.5}$$

Then, one finds

$$\mathcal{C}_\lambda(q,t,p)=\mathcal{C}\mathfrak{q}^{|\lambda|}\mathcal{Z}_\lambda^{\widehat{A}_0}(t,q^{-1},p),$$

where

$$\mathfrak{q}=p^{*-1}p^{N-1}(t/q)^{1/2}, \tag{5.6}$$

$$\mathcal{Z}_\lambda^{\widehat{A}_0}(t,q^{-1},p)=\prod_{\square\in\lambda}\frac{(1-p\,q^{a(\square)+1}t^{\ell(\square)})(1-pq^{-a(\square)}t^{-\ell(\square)-1})}{(1-q^{a(\square)+1}t^{\ell(\square)})(1-q^{-a(\square)}t^{-\ell(\square)-1})}, \tag{5.7}$$

$$\mathcal{C}=\frac{(p^{-1}t;q,t,p)_\infty}{(q;q,t,p)_\infty}. \tag{5.8}$$

Note that the sum $\sum_{\lambda,|\lambda|=n}\mathcal{Z}_\lambda^{\widehat{A}_0}(t,q^{-1},p)$ coincides with the equivariant χ_y-genus of the Hilbert scheme of n points on $\mathbb{C}^2$, $\mathrm{Hilb}_n(\mathbb{C}^2)$, [36, 57] with $y=p$. Note also that one can rewrite (5.7) as

$$\mathcal{Z}_\lambda^{\widehat{A}_0}(t,q^{-1},p)=\frac{N_{\lambda\lambda}(pq/t)}{N_{\lambda\lambda}(q/t)} \tag{5.9}$$

in terms of the 5d analogue of the Nekrasov function $N_{\lambda\mu}(x)$ given by

$$N_{\lambda\mu}(x)=\prod_{\square\in\lambda}(1-xq^{-a_\mu(\square)-1}t^{-\ell_\lambda(\square)})\prod_{\blacksquare\in\mu}(1-xq^{a_\lambda(\blacksquare)}t^{\ell_\mu(\blacksquare)+1}). \tag{5.10}$$

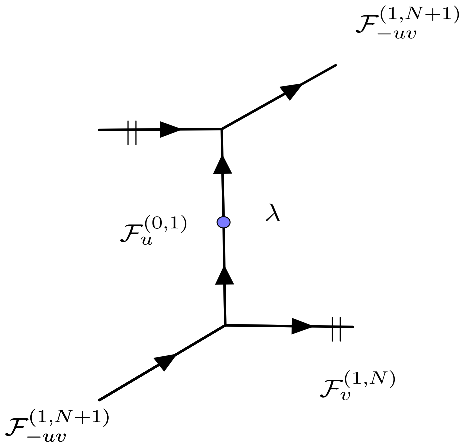

Fig. 1 Graphical expression of $\sum_\lambda \Phi^*_\lambda(u)\Phi_\lambda(u)$. The two horizontal lines with || are glued together

Hence, the whole operator

$$T(u) = \mathcal{C}\sum_\lambda \mathfrak{q}^{|\lambda|}\mathcal{Z}^{\widehat{A}_0}_\lambda(t, q^{-1}, p) : \prod_{\square\in A(\lambda)} Y(u/q^{\square}) \prod_{\blacksquare\in R(\lambda)} Y((q/t)u/q^{\blacksquare})^{-1} : \tag{5.11}$$

agrees with the generating function of $W_{p,p^*}(\Gamma(\widehat{A}_0))$ in [41, 42] up to an over all constant factor. Note that we have the symmetry

$$\begin{aligned}\mathcal{Z}^{\widehat{A}_0}_\lambda(t, q^{-1}, p) &= (pp^*)^{|\lambda|}\mathcal{Z}^{\widehat{A}_0}_\lambda(t, q^{-1}, p^{*-1})\\ &= p^{2|\lambda|}\mathcal{Z}^{\widehat{A}_0}_\lambda(t^{-1}, q, p^{-1}) = (t/q)^{|\lambda|}\mathcal{Z}^{\widehat{A}_0}_\lambda(t^{-1}, q, p^*).\end{aligned}$$

From (5.11), it is immediate to obtain the rank 1 instanton partition function of the 5d lift of the 4d $\mathcal{N} = 2^*$ theory [4, 32–34] by taking the vacuum expectation value:

$$\langle 0|T(u)|0\rangle = \mathcal{C}\sum_\lambda \mathfrak{q}^{|\lambda|}\mathcal{Z}^{\widehat{A}_0}_\lambda(t, q^{-1}, p). \tag{5.12}$$

It is then important to recognize that this result and our realization $T(u) = \sum_\lambda \Phi^*_\lambda(u)\Phi_\lambda(u)$ lead to the identification of $T(u)$ with the basic refined topological vertex depicted in Fig. 1, which was introduced in [32, 34, 36]. Once obtaining such basic operator, one can apply it to various calculations presented in the subsequent sections.

5.3 The higher rank extension

To extend $W_{p,p^*}(\Gamma(\widehat{A}_0))$ to the one associated with the higher rank instantons, one need to take a composition of $T(u)$'s. By using (5.3) and Propositions 5.1 and 5.2, one obtains the following expression.

$$T(u_1)\cdots T(u_M)=\mathcal{C}_M\sum_{k=0}^{\infty}\mathfrak{q}_M^k\sum_{\substack{\lambda^{(1)},\ldots,\lambda^{(M)}\\ \sum_j|\lambda^{(j)}|=k}}\prod_{i,j=1}^{M}\frac{N_{\lambda^{(i)}\lambda^{(j)}}(pqu_{i,j}/t)}{N_{\lambda^{(i)}\lambda^{(j)}}(qu_{i,j}/t)}$$
$$\times:\prod_{l=1}^{M}\prod_{\square\in A(\lambda^{(l)})}Y(u_l/q^{\square})\prod_{\blacksquare\in R(\lambda^{(l)})}Y((q/t)u_l/q^{\blacksquare})^{-1}:, \tag{5.13}$$

where we set $u_{j,i}=u_j/u_i$, and

$$\mathfrak{q}_M=\mathfrak{q}p^{-(M-1)}=p^{*-1}p^{M+N}(t/q)^{1/2}, \tag{5.14}$$
$$\mathcal{C}_M=\left(\frac{(p^{-1}t;q,t,p)_\infty}{(q;q,t,p)_\infty}\right)^M\prod_{1\le i<j\le M}\frac{(p^{-1}tu_{j,i};q,t)_\infty(pqu_{j,i};q,t)_\infty}{(tu_{j,i};q,t)_\infty(qu_{j,i};q,t)_\infty}. \tag{5.15}$$

In fact, the sum

$$\chi_p(\mathfrak{M}_{k,M})=\sum_{\substack{\lambda^{(1)},\ldots,\lambda^{(M)}\\ \sum_j|\lambda^{(j)}|=k}}\prod_{i,j=1}^{M}\frac{N_{\lambda^{(i)}\lambda^{(j)}}(pqu_{i,j}/t)}{N_{\lambda^{(i)}\lambda^{(j)}}(qu_{i,j}/t)} \tag{5.16}$$

coincides with the equivariant χ_y $(y=p)$ genus of the moduli space of rank M instantons with charge k [34].

Note that $T(u)$ gives a non-commutative 5d-analogue of Nekrasov's qq-character of the $\mathcal{N}=2^*$ $U(1)$ theory [41]. We expect that $T(u_1)\cdots T(u_M)$ gives a non-commutative 5d-analogue of Nekrasov's qq-character of the $\mathcal{N}=2^*$ $U(M)$ theory.

6 Instanton calculus in the Jordan quiver gauge theories

By using the generating function $T(u)$ of $W_{p,p^*}(\Gamma(\widehat{A}_0))$ constructed in the previous section, we show some demonstrations of deriving the instanton partition functions of the 5d and 6d lifts of the 4d $\mathcal{N}=2^*$ SUSY gauge theories. Hence, the elliptic quantum toroidal algebra $\mathcal{U}_{q,t,p}$ is a relevant quantum group structure of dealing with such theories.

6.1 Instanton partition functions for the 5d and 6d lifts of the $\mathcal{N} = 2^*$ $U(1)$ theory

As mentioned in Sect. 5.2, the vacuum expectation value of $T(u)$ gives the instanton partition functions of the 5d lift of the 4d $\mathcal{N} = 2^*$ $U(1)$ gauge theory.

$$\langle 0|T(u)|0\rangle = \mathcal{C} \sum_{\lambda} \mathfrak{q}^{|\lambda|} \mathcal{Z}_{\lambda}^{\widehat{A}_0}(t, q^{-1}, p). \tag{6.1}$$

We then identify $T(u) = \sum_{\lambda} \Phi_{\lambda}^*(u)\Phi_{\lambda}(u)$ with the basic topological vertex depicted in Fig. 1.

An immediate next application is to take a trace of $T(u)$. Let us introduce the degree counting operator

$$d = -\sum_{m>0} \frac{m^2}{\kappa_m(1-(t/q)^m)} \frac{1-p^m}{1-p^{*m}} \alpha'_{-m}\alpha'_m$$

such that

$$[d, \alpha'_m] = m\alpha'_m \qquad m \in \mathbb{Z}_{\neq 0}.$$

Then, the following trace yields the 6d version of the partition function of the rank 1 instantons [4, 32–34].

$$\mathrm{tr}_{\mathcal{F}^{(1,N+1)}_{-uv}}\, Q^d\, T(u) = \mathcal{C}_Q \sum_{\lambda} \mathfrak{q}^{|\lambda|} \mathcal{Z}_{\lambda}^{\widehat{A}_0}(t, q^{-1}, p; Q), \tag{6.2}$$

where

$$\mathcal{C}_Q = \frac{1}{(Q;Q)_\infty} \frac{(p^{-1}t; q, t, p)_\infty}{(q; q, t, p)_\infty} \frac{(p^{-1}tQ; q, t, Q)_\infty (pqQ; q, t, Q)_\infty}{(tQ; q, t, Q)_\infty (qQ; q, t, Q)_\infty}, \tag{6.3}$$

$$\mathcal{Z}_{\lambda}^{\widehat{A}_0}(t, q^{-1}, p; Q) = \frac{N^{\theta}_{\lambda\lambda}(pq/t; Q)}{N^{\theta}_{\lambda\lambda}(q/t; Q)}. \tag{6.4}$$

Here $N^{\theta}_{\lambda\mu}(x; Q)$ denotes the theta function analogue of the Nekrasov function given by

$$N^{\theta}_{\lambda\mu}(x; Q) = \prod_{\square\in\lambda} \theta_Q(xq^{-a_\mu(\square)-1}t^{-\ell_\lambda(\square)}) \prod_{\blacksquare\in\mu} \theta_Q(xq^{a_\lambda(\blacksquare)}t^{\ell_\mu(\blacksquare)+1}). \tag{6.5}$$

In fact the sum $\sum_{\lambda, |\lambda|=n} \mathcal{Z}_{\lambda}^{\widehat{A}_0}(t, q^{-1}, p; Q)$ gives the equivariant elliptic genus of $\mathrm{Hilb}_n(\mathbb{C}^2)$ [57].

6.2 The 5d and 6d lifts of the $\mathcal{N}=2^*$ $U(M)$ theory

The higher rank instanton partition functions can be obtained from the composition $T(u_1)\cdots T(u_M)$ in (5.13). The vacuum expectation value gives the instanton partition function of the 5d lift of the 4d $\mathcal{N}=2^*$ $U(M)$ theory [34].

$$\langle 0|T(u_1)\cdots T(u_M)|0\rangle = \mathcal{C}_M \sum_{k=0}^{\infty} \mathfrak{q}_M^k\, \chi_p(\mathfrak{M}_{k,M}), \tag{6.6}$$

where $\chi_p(\mathfrak{M}_{k,M})$ is given by (5.16).

Furthermore, taking the trace of (5.13), one obtains

$$\mathrm{tr}_{\mathcal{F}^{(1,N+1)}_{-u_1v_1}} Q^d\, T(u_1)\cdots T(u_M) = \mathcal{C}_{Q,M} \sum_{k=0}^{\infty} \mathfrak{q}_M^k\, \mathcal{E}_{p,Q}(\mathfrak{M}_{k,M}), \tag{6.7}$$

where $u_1v_1 = u_2v_2 = \cdots = u_Mv_M$ with arbitrary $v_1,\cdots,v_M \in \mathbb{C}^*$. We here also set

$$\mathcal{E}_{p,Q}(\mathfrak{M}_{k,M}) = \sum_{\substack{\lambda^{(1)},\ldots,\lambda^{(M)}\\ \sum_j |\lambda^{(j)}|=k}} \prod_{i,j=1}^{M} \frac{N^{\theta}_{\lambda^{(i)}\lambda^{(j)}}(pqu_{i,j}/t;Q)}{N^{\theta}_{\lambda^{(i)}\lambda^{(j)}}(qu_{i,j}/t;Q)}, \tag{6.8}$$

$$\begin{aligned}\mathcal{C}_{Q,M} &= \frac{1}{(Q;Q)_\infty}\left(\frac{(t;q,t)_\infty \Gamma_3(p^{-1}t;q,t,Q)}{(p^{-1}t;q,t)_\infty \Gamma_3(t;q,t,Q)}\right)^M\\ &\quad\times \prod_{1\le i<j\le M} \frac{\Gamma_3(p^{-1}tu_{j,i};q,t,Q)\Gamma_3(pqu_{j,i};q,t,Q)}{\Gamma_3(tu_{j,i};q,t,Q)\Gamma_3(qu_{j,i};q,t,Q)},\end{aligned} \tag{6.9}$$

The sum $\mathcal{E}_{p,Q}(\mathfrak{M}_{k,M})$ gives the equivariant elliptic genus of the moduli space of rank M instantons with charge k [33, 35]. Hence, (6.7) gives the instanton partition function of the 6d lift of the $\mathcal{N}=2^*$ $U(M)$ theory.

7 Correlation functions of $\Phi(u)$ and $\Psi^*(v)$

We here give some $\mathcal{U}_{q,t,p}$-analogues of the formulas of those obtained in Propositions 5.1 and 5.2 in [5]. We expect that they give the $\mathcal{N}=2^*$ theory (Jordan quiver gauge theory) analogue of the partition function of the 5d lift of the pure $SU(N)$ theory and the 5d and 6d lifts of the $SU(N)$ theory with $2N$ fundamental matters, respectively. The latter theories are the A_1 linear quiver gauge theories.

7.1 OPE formulas

From Propositions 4.3, 4.6 and 5.2, we have the following OPE formulas for the operator parts of the type I $\Phi(u)$ and the type II dual $\Psi^*(v)$ intertwining operators.

Proposition 7.1

$$\widetilde{\Phi}_\lambda(u)\widetilde{\Phi}_\mu(v) = \frac{\mathcal{G}(v/u,p)}{N_{\mu\lambda}(v/u;p)} :\widetilde{\Phi}_\lambda(u)\widetilde{\Phi}_\mu(v):,$$
$$\widetilde{\Psi}^*_\lambda(u)\widetilde{\Psi}^*_\mu(v) = \frac{\mathcal{G}^*(qv/tu,p)}{N^*_{\mu\lambda}(qv/tu;p)} :\widetilde{\Psi}^*_\lambda(u)\widetilde{\Psi}^*_\mu(v):,$$
$$\widetilde{\Phi}_\lambda(u)\widetilde{\Psi}^*_\mu(v) = \frac{N_{\mu\lambda}((t/q)^{-1/2}v/u)}{\mathcal{G}(v/u)} :\widetilde{\Phi}_\lambda(u)\widetilde{\Psi}^*_\mu(v):,$$
$$\widetilde{\Psi}^*_\lambda(u)\widetilde{\Phi}_\mu(v) = \frac{N_{\mu\lambda}((t/q)^{-1/2}v/u)}{\mathcal{G}(v/u)} :\widetilde{\Psi}^*_\lambda(u)\widetilde{\Phi}_\mu(v): .$$

Here $N_{\lambda\mu}(u;p)$ and $N^*_{\lambda\mu}(u;p)$ denote p-deformations of $N_{\lambda\mu}(u)$ in (5.10) given by

$$N_{\lambda\mu}(u;p) = \prod_{\square\in\lambda} \frac{(uq^{-a_\mu(\square)-1}t^{-\ell_\lambda(\square)};p)_\infty}{(p^*uq^{-a_\mu(\square)-1}t^{-\ell_\lambda(\square)};p)_\infty} \prod_{\blacksquare\in\mu} \frac{(uq^{a_\lambda(\blacksquare)}t^{\ell_\mu(\blacksquare)+1};p)_\infty}{(p^*uq^{a_\lambda(\blacksquare)}t^{\ell_\mu(\blacksquare)+1};p)_\infty}, \tag{7.1}$$

$$N^*_{\lambda\mu}(u;p) = N_{\lambda\mu}(u;p)\Big|_{p\leftrightarrow p^*}. \tag{7.2}$$

We also set

$$\mathcal{G}(u) = \exp\left\{\sum_{n>0}\frac{1}{n}\frac{t^n}{(1-q^n)(1-t^n)}u^n\right\} = \frac{1}{(tu;q,t)_\infty},$$
$$\mathcal{G}(u,p) = \exp\left\{\sum_{n>0}\frac{1}{n}\frac{(1-p^{*n})t^n}{(1-q^n)(1-t^n)(1-p^n)}u^n\right\} = \frac{(p^*tu;q,t,p)_\infty}{(tu;q,t,p)_\infty},$$
$$\mathcal{G}^*(u,p) = \mathcal{G}(u,p)\Big|_{p\leftrightarrow p^*}.$$

7.2 The four points operator

Remember

$$\Phi(-x)\ \ \mathcal{F}_u^{(1,N+1)} \to \mathcal{F}_{-x}^{(0,1)}\widetilde{\otimes}\mathcal{F}_{u/x}^{(1,N)},$$
$$\Psi^*(-x)\ \ \mathcal{F}_u^{(1,N)}\widetilde{\otimes}\mathcal{F}_{-x}^{(0,1)} \to \mathcal{F}_{ux}^{(1,N+1)}.$$

Let us consider the following composition.

$$\phi(w) := (\Psi^*(-w)\widetilde{\otimes}\mathrm{id})\circ(\mathrm{id}\widetilde{\otimes}\Phi(-w))\ \ \mathcal{F}_v^{(1,M)}\widetilde{\otimes}\mathcal{F}_u^{(1,L)} \to \mathcal{F}_{vw}^{(1,M+1)}\widetilde{\otimes}\mathcal{F}_{u/w}^{(1,L-1)}.$$

This is an intertwining operator satisfying

$$\Delta(x)\phi(w) = \phi(w)\Delta(x) \qquad \forall x \in \mathcal{U}_{q,t,p}.$$

We call $\phi(w)$ the basic four points operator.

Proposition 7.2 *The action of $\phi(w)$ on $|\xi_v^M\rangle\widetilde{\otimes}|\eta_u^L\rangle \in \mathcal{F}_v^{(1,M)}\widetilde{\otimes}\mathcal{F}_u^{(1,L)}$ is given by*

$$\phi(w)\ |\xi_v^M\rangle\widetilde{\otimes}|\eta_u^L\rangle = \sum_\lambda \frac{c_\lambda(p^*)}{c'_\lambda(p^*)}\ \Psi_\lambda^*(-w)|\xi_v^M\rangle\widetilde{\otimes}\Phi_\lambda(-w)|\eta_u^L\rangle.$$

Proof Use (4.2), (4.3), (4.14) and (2.56). □

Proposition 7.3 *The composition of $\phi(w)$'s gives the following generalized four points intertwining operator.* (Fig. 2)

$$\phi(w_N)\circ\cdots\circ\phi(w_1)\ \ \mathcal{F}_v^{(1,M)}\widetilde{\otimes}\mathcal{F}_u^{(1,L)} \ \rightarrow\ \mathcal{F}_{vw_1\cdots w_N}^{(1,M+N)}\widetilde{\otimes}\mathcal{F}_{u/w_1\cdots w_N}^{(1,L-N)}.$$

The action on $|\xi_v^M\rangle\widetilde{\otimes}|\eta_u^L\rangle \in \mathcal{F}_v^{(1,M)}\widetilde{\otimes}\mathcal{F}_u^{(1,L)}$ is given by

$$\begin{aligned}&\phi(w_N)\circ\cdots\circ\phi(w_1)\ |\xi_v^M\rangle\widetilde{\otimes}|\eta_u^L\rangle\\ &= \sum_{\lambda^{(1)},\ldots,\lambda^{(N)}}\prod_{a=1}^N \frac{c_{\lambda^{(a)}}(p^*)}{c'_{\lambda^{(a)}}(p^*)}\ \Psi^*_{\lambda^{(N)}}(-w_N)\cdots\Psi^*_{\lambda^{(1)}}(-w_1)|\xi_v^M\rangle\widetilde{\otimes}\Phi_{\lambda^{(N)}}(-w_N)\cdots\Phi_{\lambda^{(1)}}(-w_1)|\eta_u^L\rangle.\end{aligned}$$

Corollary 7.4 *For a vector $|\omega^{M+N}_{vw_1\cdots w_N}\rangle\widetilde{\otimes}|\zeta^{L-N}_{u/w_1\cdots w_N}\rangle \in \mathcal{F}_{vw_1\cdots w_N}^{(1,M+N)}\widetilde{\otimes}\mathcal{F}_{u/w_1\cdots w_N}^{(1,L-N)}$, one has the following expectation value.*

$$\begin{aligned}&\langle\omega^{M+N}_{vw_1\cdots w_N}|\widetilde{\otimes}\langle\zeta^{L-N}_{u/w_1\cdots w_N}|\ \phi(w_N)\circ\cdots\circ\phi(w_1)\ |\xi_v^M\rangle\widetilde{\otimes}|\eta_u^L\rangle\\ &= \sum_{\lambda^{(1)},\ldots,\lambda^{(N)}}\prod_{a=1}^N \frac{c_{\lambda^{(a)}}(p^*)}{c'_{\lambda^{(a)}}(p^*)}\ \langle\omega^{M+N}_{vw_1\cdots w_N}|\Psi^*_{\lambda^{(N)}}(-w_N)\cdots\Psi^*_{\lambda^{(1)}}(-w_1)|\xi_v^M\rangle\\ &\quad\widetilde{\otimes}\langle\zeta^{L-N}_{u/w_1\cdots w_N}|\Phi_{\lambda^{(N)}}(-w_N)\cdots\Phi_{\lambda^{(1)}}(-w_1)|\eta_u^L\rangle.\end{aligned}$$

Note that this gives the $\mathcal{N}=2^*$ theory analogue of the partition function of the 5d pure $SU(N)$ theory obtained, for example, in [4, 5].

7.3 The six points and higher operators

Next let us consider the operator

$$\phi\left(w;\frac{y}{x}\right) = (\Phi(-y)\widetilde{\otimes}\mathrm{id})\circ(\Psi^*(-w)\widetilde{\otimes}\Psi^*(-x))\circ(\mathrm{id}\widetilde{\otimes}\Phi(-w)\widetilde{\otimes}\mathrm{id})$$

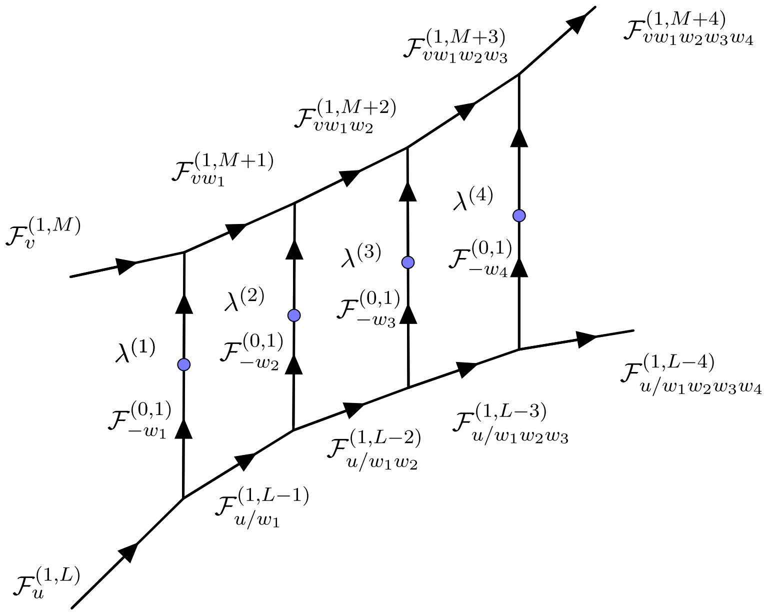

Fig. 2 Graphical expression of $\phi(w_4)\circ\cdots\circ\phi(w_1)$

$$: \ \mathcal{F}^{(1,0)}_v\widetilde{\otimes}\mathcal{F}^{(1,0)}_u\widetilde{\otimes}\mathcal{F}^{(0,1)}_{-x} \ \to \ \mathcal{F}^{(0,1)}_{-y}\widetilde{\otimes}\mathcal{F}^{(1,0)}_{vw/y}\widetilde{\otimes}\mathcal{F}^{(1,0)}_{ux/w}.$$

This is an intertwining operator satisfying

$$\phi\left(w;\begin{matrix}y\\x\end{matrix}\right)(\Delta\widetilde{\otimes}\mathrm{id})\Delta(a)=(\Delta\widetilde{\otimes}\mathrm{id})\Delta(a)\,\phi\left(w;\begin{matrix}y\\x\end{matrix}\right)\qquad \forall a\in\mathcal{U}_{q,t,p}.$$

We call $\phi\left(w;\begin{matrix}y\\x\end{matrix}\right)$ the six points operator.

Proposition 7.5 *For* $\mu\in\mathcal{P}^+$*, the action of* $\phi\left(w;\begin{matrix}y\\x\end{matrix}\right)$ *on* $|\xi^0_v\rangle\widetilde{\otimes}|\eta^0_u\rangle\widetilde{\otimes}|\mu\rangle_{-x}\in\mathcal{F}^{(1,0)}_v\widetilde{\otimes}\mathcal{F}^{(1,0)}_u\widetilde{\otimes}\mathcal{F}^{(0,1)}_{-x}$ *is given by*

$$\begin{aligned}&\phi\left(w;\begin{matrix}y\\x\end{matrix}\right)\ |\xi^0_v\rangle\widetilde{\otimes}|\eta^0_u\rangle\widetilde{\otimes}|\mu\rangle_{-x}\\&=\sum_{\lambda,\sigma\in\mathcal{P}^+}\frac{c_\lambda(p^*)}{c'_\lambda(p^*)}\ |\sigma\rangle'_{-y}\widetilde{\otimes}\Phi_\sigma(-y)\Psi^*_\lambda(-w)|\xi^0_v\rangle\widetilde{\otimes}\Psi^*_\mu(-x)\Phi_\lambda(-w)|\eta^0_u\rangle.\end{aligned}$$

Let us set

$$\phi^{(j)}\left(w;\begin{matrix}y\\x\end{matrix}\right):=\underbrace{\mathrm{id}\widetilde{\otimes}\cdots\widetilde{\otimes}\mathrm{id}}_{j-1}\widetilde{\otimes}\phi\left(w;\begin{matrix}y\\x\end{matrix}\right)\widetilde{\otimes}\underbrace{\mathrm{id}\widetilde{\otimes}\cdots\widetilde{\otimes}\mathrm{id}}_{N-j}.$$

Proposition 7.6 *By composing* $\phi^{(j)}\left(w_j; \begin{matrix} y_j \\ x_j \end{matrix}\right)$ $(j = 1, \dots, N)$, *we obtain the following* $2(N+2)$ *points intertwining operator* (Fig. 3).

$$\phi^{(N)}\left(w_N; \begin{matrix} y_N \\ x_N \end{matrix}\right) \circ \cdots \circ \phi^{(1)}\left(w_1; \begin{matrix} y_1 \\ x_1 \end{matrix}\right)$$
$$: \mathcal{F}_v^{(1,0)} \widetilde{\otimes} \mathcal{F}_u^{(1,0)} \widetilde{\otimes} \mathcal{F}_{-x_1}^{(0,1)} \widetilde{\otimes} \cdots \widetilde{\otimes} \mathcal{F}_{-x_N}^{(0,1)} \to \mathcal{F}_{-y_1}^{(0,1)} \widetilde{\otimes} \cdots \widetilde{\otimes} \mathcal{F}_{-y_N}^{(0,1)} \widetilde{\otimes} \mathcal{F}^{(1,0)}_{v \frac{w_1 \cdots w_N}{y_1 \cdots y_N}} \widetilde{\otimes} \mathcal{F}^{(1,0)}_{u \frac{x_1 \cdots x_N}{w_1 \cdots w_N}}.$$

For $\mu = (\mu^{(1)}, \dots, \mu^{(N)}) \in (\mathcal{P}^+)^N$, *the action on* $|\xi_v^0\rangle \widetilde{\otimes} |\eta_u^0\rangle \widetilde{\otimes} |\mu^{(1)}\rangle_{-x_1} \widetilde{\otimes} \cdots \widetilde{\otimes} |\mu^{(N)}\rangle_{-x_N} \in \mathcal{F}_v^{(1,0)} \widetilde{\otimes} \mathcal{F}_u^{(1,0)} \widetilde{\otimes} \mathcal{F}_{-x_1}^{(0,1)} \widetilde{\otimes} \cdots \widetilde{\otimes} \mathcal{F}_{-x_N}^{(0,1)}$ *is given by*

$$\phi^{(N)}\left(w_N; \begin{matrix} y_N \\ x_N \end{matrix}\right) \circ \cdots \circ \phi^{(1)}\left(w_1; \begin{matrix} y_1 \\ x_1 \end{matrix}\right) |\xi_v^0\rangle \widetilde{\otimes} |\eta_u^0\rangle \widetilde{\otimes} |\mu^{(1)}\rangle_{-x_1} \widetilde{\otimes} \cdots \widetilde{\otimes} |\mu^{(N)}\rangle_{-x_N}$$
$$= \sum_{\substack{\lambda^{(1)}, \dots, \lambda^{(N)} \\ \sigma^{(1)}, \dots, \sigma^{(N)}}} \prod_{a=1}^{N} \frac{c_{\lambda^{(a)}}(p^*)}{c'_{\lambda^{(a)}}(p^*)} |\sigma^{(1)}\rangle'_{-y_1} \widetilde{\otimes} \cdots \widetilde{\otimes} |\sigma^{(N)}\rangle'_{-y_N}$$
$$\widetilde{\otimes}\, \Phi_{\sigma^{(N)}}(-y_N) \Psi^*_{\lambda^{(N)}}(-w_N) \cdots \Phi_{\sigma^{(1)}}(-y_1) \Psi^*_{\lambda^{(1)}}(-w_1) |\xi_v^0\rangle$$
$$\widetilde{\otimes}\, \Psi^*_{\mu^{(N)}}(-x_N) \Phi_{\lambda^{(N)}}(-w_N) \cdots \Psi^*_{\mu^{(1)}}(-x_1) \Phi_{\lambda^{(1)}}(-w_1) |\eta_u^0\rangle,$$

where $\lambda = (\lambda^{(1)}, \dots, \lambda^{(N)}), \sigma = (\sigma^{(1)}, \dots, \sigma^{(N)}) \in (\mathcal{P}^+)^N$.

Corollary 7.7 *For a vector*

$$|\nu^{(1)}\rangle'_{-y_1} \widetilde{\otimes} \cdots \widetilde{\otimes} |\nu^{(N)}\rangle'_{-y_N} \widetilde{\otimes} |\omega^0_{v \frac{w_1 \cdots w_N}{y_1 \cdots y_N}}\rangle \widetilde{\otimes} |\zeta^0_{u \frac{x_1 \cdots x_N}{w_1 \cdots w_N}}\rangle \in \mathcal{F}_{-y_1}^{(0,1)} \widetilde{\otimes} \cdots \widetilde{\otimes} \mathcal{F}_{-y_N}^{(0,1)}$$
$$\widetilde{\otimes}\, \mathcal{F}^{(1,0)}_{v \frac{w_1 \cdots w_N}{y_1 \cdots y_N}} \widetilde{\otimes} \mathcal{F}^{(1,0)}_{u \frac{x_1 \cdots x_N}{w_1 \cdots w_N}},$$

we have the expectation value

$${}_{-y_1}\text{‘}\langle \nu^{(1)}| \widetilde{\otimes} \cdots \widetilde{\otimes}\, {}_{-y_N}\text{‘}\langle \nu^{(N)}| \widetilde{\otimes} \langle \omega^0_{v \frac{w_1 \cdots w_N}{y_1 \cdots y_N}}| \widetilde{\otimes} \langle \zeta^0_{u \frac{x_1 \cdots x_N}{w_1 \cdots w_N}}| \phi^{(N)}\left(w_N; \begin{matrix} y_N \\ x_N \end{matrix}\right) \circ \cdots \circ \phi^{(1)}\left(w_1; \begin{matrix} y_1 \\ x_1 \end{matrix}\right)$$
$$\times |\xi_v^0\rangle \widetilde{\otimes} |\eta_u^0\rangle \widetilde{\otimes} |\mu^{(1)}\rangle_{-x_1} \widetilde{\otimes} \cdots \widetilde{\otimes} |\mu^{(N)}\rangle_{-x_N}$$
$$= \sum_{\lambda^{(1)}, \dots, \lambda^{(N)}} \prod_{a=1}^{N} \frac{c_{\lambda^{(a)}}(p^*)}{c'_{\lambda^{(a)}}(p^*)} \langle \omega^0_{v \frac{w_1 \cdots w_N}{y_1 \cdots y_N}} | \Phi_{\nu^{(N)}}(-y_N) \Psi^*_{\lambda^{(N)}}(-w_N) \cdots \Phi_{\nu^{(1)}}(-y_1) \Psi^*_{\lambda^{(1)}}(-w_1) |\xi_v^0\rangle$$
$$\widetilde{\otimes} \langle \zeta^0_{u \frac{x_1 \cdots x_N}{w_1 \cdots w_N}} | \Psi^*_{\mu^{(N)}}(-x_N) \Phi_{\lambda^{(N)}}(-w_N) \cdots \Psi^*_{\mu^{(1)}}(-x_1) \Phi_{\lambda^{(1)}}(-w_1) |\eta_u^0\rangle.$$

Here ${}_y\text{‘}\langle \mu|$ $(\mu \in \mathcal{P}^+, y \in \mathbb{C}^*)$ *is defined by*

$${}_y\text{‘}\langle \mu || \nu \rangle'_y = \delta_{\mu,\nu} \qquad \nu \in \mathcal{P}^+$$

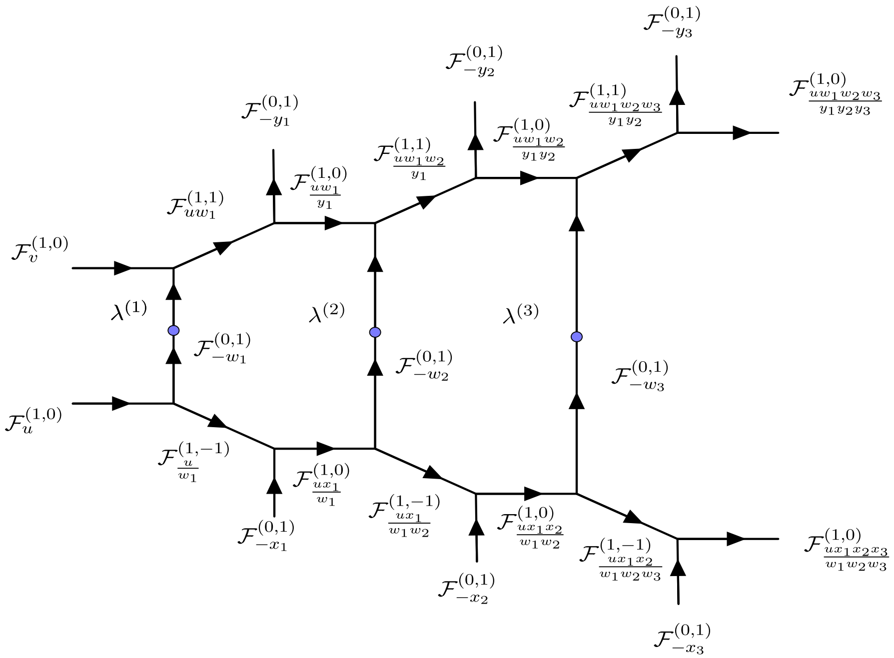

Fig. 3 Graphical expression of $\phi^{(3)}\left(w_3; \frac{y_3}{x_3}\right) \circ \phi^{(2)}\left(w_2; \frac{y_2}{x_2}\right) \circ \phi^{(1)}\left(w_1; \frac{y_1}{x_1}\right)$

Corollary 7.8 *When $x_1 \cdots x_N = y_1 \cdots y_N = w_1 \cdots w_N$, one can take the following trace*

$$\begin{aligned}
&\mathrm{tr}_{\mathcal{F}_v^{(1,0)}} Q^d \widetilde{\otimes}\, \mathrm{tr}_{\mathcal{F}_u^{(1,0)}} Q^d \left({}_{-y_1}\text{‘}\langle \nu^{(1)}| \widetilde{\otimes} \cdots \widetilde{\otimes}_{-y_N}\text{‘}\langle \nu^{(N)}| \phi^{(N)}\left(w_N; \frac{y_N}{x_N}\right) \circ \cdots \right.\\
&\left. \circ \phi^{(1)}\left(w_1; \frac{y_1}{x_1}\right) |\mu^{(1)}\rangle_{-x_1} \widetilde{\otimes} \cdots \widetilde{\otimes} |\mu^{(N)}\rangle_{-x_N} \right)\\
&= \sum_{\lambda^{(1)},\ldots,\lambda^{(N)}} \prod_{a=1}^{N} \frac{c_{\lambda^{(a)}}(p^*)}{c'_{\lambda^{(a)}}(p^*)}\, \mathrm{tr}_{\mathcal{F}_v^{(1,0)}} \left(Q^d \Phi_{\nu^{(N)}}(-y_N) \Psi^*_{\lambda^{(N)}}(-w_N) \cdots \Phi_{\nu^{(1)}}(-y_1) \Psi^*_{\lambda^{(1)}}(-w_1) \right)\\
&\quad \widetilde{\otimes}\, \mathrm{tr}_{\mathcal{F}_u^{(1,0)}} \left(Q^d \Psi^*_{\mu^{(N)}}(-x_N) \Phi_{\lambda^{(N)}}(-w_N) \cdots \Psi^*_{\mu^{(1)}}(-x_1) \Phi_{\lambda^{(1)}}(-w_1) \right).
\end{aligned}$$

Note that Corollary 7.7 gives the $\mathcal{N} = 2^*$ theory analogue of the partition function of the 5d $SU(N)$ theory with $2N$ fundamental matters obtained, for example, in [4, 5] for the case $\mu = \nu = (\emptyset, \ldots, \emptyset)$. The trace in Corollary 7.8 gives its 6d lift. It can be evaluated as follows:

Proposition 7.9

$$\mathrm{tr}_{\mathcal{F}_v^{(1,0)}} Q^d \widetilde{\otimes}\, \mathrm{tr}_{\mathcal{F}_u^{(1,0)}} Q^d \left({}_{-y_1}\langle \nu^{(1)}| \widetilde{\otimes} \cdots \widetilde{\otimes}_{-y_N} \langle \nu^{(N)}| \phi^{(N)}\left(w_N; \frac{y_N}{x_N}\right) \circ \cdots \circ \right.$$

$$
\begin{aligned}
&\phi^{(1)}\left(w_1;\frac{y_1}{x_1}\right)|\mu^{(1)}\rangle'_{-x_1}\widetilde{\otimes}\cdots\widetilde{\otimes}|\mu^{(N)}\rangle'_{-x_N}\Big)\\
&=\mathcal{C}_N(q,t,p,Q)\mathcal{N}^N_\mu(u,x,w;q,t,p,Q)\mathcal{N}^N_\nu(v,y,w;q,t,p,Q)\\
&\times\prod_{1\le a<b\le N}\left(\frac{\Gamma_4(qw_a/w_b;q,t,p^*,Q)}{\Gamma_4(p^*tw_a/w_b;q,t,p^*,Q)}\frac{\Gamma_4(tw_a/w_b;q,t,p^*,Q)}{\Gamma_4(p^*qw_a/w_b;q,t,p^*,Q)}\right)\\
&\times\sum_{\lambda^{(1)},\dots,\lambda^{(N)}}\prod_{a=1}^N\left(-\frac{v}{u}\prod_{k=1}^{a-1}\frac{w_k^2}{x_ky_k}\right)^{|\lambda_a|}q^{-n(\lambda^{(a)'})}t^{n(\lambda^{(a)})}\\
&\times\prod_{1\le a<b\le N}N^\theta_{\lambda^{(a)}\lambda^{(b)}}(w_a/w_b;p^*)N^\theta_{\lambda^{(a)}\lambda^{(b)}}(qw_a/tw_b;p^*)\\
&\times\prod_{a,b=1}^N\Big(N^\theta_{\lambda^{(b)}\nu^{(a)}}(\sqrt{q/t}w_b/y_a;Q)N^\theta_{\mu^{(b)}\lambda^{(a)}}(\sqrt{q/t}x_b/w_a;Q)N^\Gamma_{\lambda^{(a)}\lambda^{(b)}}\\
&(w_a/w_b;p^*,Q)N^\Gamma_{\lambda^{(a)}\lambda^{(b)}}(qw_a/tw_b;p^*,Q)\Big).
\end{aligned}
$$

Here, we set $p^{**}=p^*(q/t)=p(q/t)^2$ *and*

$$
\begin{aligned}
\mathcal{C}_N(q,t,p,Q)=\frac{1}{(Q;Q)^2_\infty}&\left(\frac{(p^*tQ;q,t,p,Q)_\infty}{(tQ;q,t,p,Q)_\infty}\frac{(pqQ;q,t,p^*,Q)_\infty}{(qQ;q,t,p^*,Q)_\infty}\right.\\
&\left.\frac{(p^{**}tQ;q,t,p^*,Q)_\infty}{(tQ;q,t,p^*,Q)_\infty}\frac{(p^{**}tQ;q,t,p^{**},Q)_\infty}{(qQ;q,t,p^{**},Q)_\infty}\right)^N,
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{N}^N_\mu(u,x,w;q,t,p,Q)\\
&=\prod_{a=1}^N\left(-t^{-1}u\prod_{k=1}^a\frac{x_k}{w_k}\right)^{|\mu^{(a)}|}t^{-n(\mu^{(a)})}c'_{\mu^{(a)}}(p^{**})\\
&\times\prod_{1\le a<b\le N}\Gamma(\sqrt{qt}x_a/w_b;q,t)_\infty\frac{\Gamma_4(qx_a/x_b;q,t,p^{**},Q)}{\Gamma_4(p^{**}x_a/x_b;q,t,p^{**},Q)}N^\theta_{\mu^{(a)}\mu^{(b)}}(x_a/x_b;p^{**})\\
&\times\prod_{a,b=1}^N N^\Gamma_{\mu^{(a)}\mu^{(b)}}(qx_a/tx_b;p^{**},Q)\Gamma_3(\sqrt{qt}x_a/w_b;q,t,Q),\\
&\mathcal{N}^N_\nu(v,y,w;q,t,p,Q)\\
&=\prod_{a=1}^N\left(-v^{-1}\prod_{k=1}^a\frac{y_k}{w_k}\right)^{|\nu^{(a)}|}t^{-n(\nu^{(a)})}c'_{\nu^{(a)}}(p)\\
&\times\prod_{1\le a<b\le N}\Gamma(\sqrt{qt}w_b/y_a;q,t)\frac{\Gamma_4(ty_a/y_b;q,t,p,Q)}{\Gamma_4(pqy_a/y_b;q,t,p,Q)}N^\theta_{\nu^{(a)}\nu^{(b)}}(qy_a/ty_b;p)\\
&\times\prod_{a,b=1}^N N^\Gamma_{\nu^{(a)}\nu^{(b)}}(y_a/y_b;p,Q)\Gamma_3(\sqrt{qt}w_a/y_b;q,t,Q).
\end{aligned}
$$

Here, $N^{\Gamma}_{\lambda\mu}(x;p,Q)$ *denotes the elliptic Gamma-analogues of the Nekrasov function defined by*

$$N^{\Gamma}_{\lambda\mu}(x;p,Q)=\prod_{\square\in\lambda}\Gamma(xq^{-a_\mu(\square)-1}t^{-\ell_\lambda(\square)};p,Q)\prod_{\blacksquare\in\mu}\Gamma(xq^{a_\lambda(\blacksquare)}t^{\ell_\mu(\blacksquare)+1};p,Q). \tag{7.3}$$

Proof The statement follows from Proposition 7.1 and the trace formula given, for example, in Appendix E of [55]. □

As for the functions $N^{\theta}_{\lambda\mu}(x;p)$ and $N^{\Gamma}_{\lambda\mu}(x;p,Q)$, the following properties are useful.

Proposition 7.10

$$\begin{aligned}
N^{\theta}_{\lambda\mu}(x;p)&=\left(x\sqrt{\frac{t}{q}}\right)^{|\lambda|+|\mu|}\frac{f_\lambda(q,t)}{f_\mu(q,t)}N^{\theta}_{\mu\lambda}(q/tx;p),\\
N^{\Gamma}_{\lambda\mu}(x;p,Q)&=\frac{1}{N^{\Gamma}_{\mu\lambda}(pQq/tx;p,Q)}\\
&=\left(x\sqrt{\frac{t}{q}}\right)^{|\lambda|+|\mu|}\frac{f_\lambda(q,t)}{f_\mu(q,t)}\frac{1}{N^{\theta}_{\lambda\mu}(x;p)N^{\theta}_{\lambda\mu}(x;Q)N^{\Gamma}_{\mu\lambda}(q/tx;p,Q)}.
\end{aligned}$$

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

A Formulas for $E^{\pm}(\alpha,z)$ and $E^{\pm}(\alpha',z)$

Lemma A.1 *The operators* $E^{\pm}(\alpha,z)$ *and* $E^{\pm}(\alpha',z)$ *defined in* (2.28) *and* (2.29) *satisfy the following relations.*

$$[\alpha_{-n},E^+(\alpha,z)]=\frac{\kappa_n}{n}\frac{1-p^n}{1-p^{*n}}\gamma^{-n/2}z^{-n}E^+(\alpha,z)\quad(n>0), \tag{A.1}$$

$$[\alpha_{n},E^-(\alpha,z)]=\frac{\kappa_n}{n}\frac{1-p^n}{1-p^{*n}}\gamma^{-3n/2}z^{n}E^-(\alpha,z)\quad(n>0), \tag{A.2}$$

$$[\alpha_{-n},E^+(\alpha',z)]=-\frac{\kappa_n}{n}\gamma^{n/2}z^{-n}E^+(\alpha',z)\quad(n>0), \tag{A.3}$$

$$[\alpha_{n},E^-(\alpha',z)]=-\frac{\kappa_n}{n}\gamma^{-n/2}z^{n}E^-(\alpha',z)\quad(n>0), \tag{A.4}$$

$$E^+(\alpha, z)E^-(\alpha, w) = g(w/z)^{-1}g(w/z;\gamma^2)^{-1}g(w/z;p^*)^{-1}E^-(\alpha, w)E^+(\alpha, z), \tag{A.5}$$

$$E^+(\alpha', z)E^-(\alpha', w) = g(w/z;\gamma^2)^{-1}g(w/z;p)E^-(\alpha', w)E^+(\alpha', z), \tag{A.6}$$

$$E^+(\alpha, z)E^-(\alpha', w) = g(\gamma w/z)g(\gamma w/z;\gamma^2)E^-(\alpha', w)E^+(\alpha, z), \tag{A.7}$$

$$E^+(\alpha', z)E^-(\alpha, w) = g(\gamma w/z)g(\gamma w/z;\gamma^2)E^-(\alpha, w)E^+(\alpha', z), \tag{A.8}$$

$$E^+(\alpha, z)x^+(w) = g(w/z)g(w/z;\gamma^2)g(w/z;p^*)x^+(w)E^+(\alpha, z), \tag{A.9}$$

$$E^+(\alpha, z)x^-(w) = g(\gamma w/z)^{-1}g(\gamma w/z;\gamma^2)^{-1}x^-(w)E^+(\alpha, z), \tag{A.10}$$

$$E^-(\alpha, z)x^+(w) = g(z/w)^{-1}g(z/w;\gamma^2)^{-1}g(z/w;p^*)^{-1}x^+(w)E^-(\alpha, z), \tag{A.11}$$

$$E^-(\alpha, z)x^-(w) = g(\gamma z/w)g(\gamma z/w;\gamma^2)x^-(w)E^-(\alpha, z), \tag{A.12}$$

$$E^+(\alpha', z)x^+(w) = g(\gamma w/z)^{-1}g(\gamma w/z;\gamma^2)^{-1}x^+(w)E^+(\alpha', z), \tag{A.13}$$

$$E^+(\alpha', z)x^-(w) = g(w/z;\gamma^2)g(w/z;p)^{-1}f(pw/z;p)x^-(w)E^+(\alpha', z), \tag{A.14}$$

$$E^-(\alpha', z)x^+(w) = g(\gamma z/w)g(\gamma z/w;\gamma^2)x^+(w)E^-(\alpha', z), \tag{A.15}$$

$$E^-(\alpha', z)x^-(w) = g(z/w;\gamma^2)^{-1}g(z/w;p)x^-(w)E^-(\alpha', z). \tag{A.16}$$

Here, $g(z;s),\ s = p, p^*, \gamma^2$ *and* $g(z)$ *are given in* (2.23) *and* (2.36)*, respectively.*

Proof The statement follows from direct calculations using (2.15)–(2.17) and (2.24). □

B Direct Check of the Level (0,1) Representation

Let us check the relation (2.18).

$$\begin{aligned} & x^+(z)x^-(w)|\lambda\rangle_u \\ &= a^+(p)a^-(p)(t/q)^{-1/2}\sum_{i=1}^{\ell}\sum_{\substack{j=1\\ j\neq i}}^{\ell+1}\delta(q^{-1}u_i/w)\delta(u_j/z)A^-_{\lambda,i}(p)A^+_{\lambda-\mathbf{1}_i,j}(p)|\lambda-\mathbf{1}_i+\mathbf{1}_j\rangle_u \\ &\quad + a^+(p)a^-(p)(t/q)^{-1/2}\sum_{i=1}^{\ell}\delta(q^{-1}u_i/w)\delta(z/w)A^-_{\lambda,i}(p)A^+_{\lambda-\mathbf{1}_i,i}(p)|\lambda\rangle_u, \end{aligned} \tag{B.1}$$

$$\begin{aligned} & x^-(w)x^+(z)|\lambda\rangle_u \\ &= a^+(p)a^-(p)(t/q)^{-1/2}\sum_{i=1}^{\ell}\sum_{\substack{j=1\\ j\neq i}}^{\ell+1}\delta(q^{-1}u_i/w)\delta(u_j/z)A^+_{\lambda,j}(p)A^-_{\lambda+\mathbf{1}_j,i}(p)|\lambda+\mathbf{1}_j-\mathbf{1}_i\rangle_u \\ &\quad + a^+(p)a^-(p)(t/q)^{-1/2}\sum_{i=1}^{\ell+1}\delta(u_i/w)\delta(z/w)A^+_{\lambda,i}(p)A^-_{\lambda+\mathbf{1}_i,i}(p)|\lambda\rangle_u. \end{aligned} \tag{B.2}$$

One finds that the first terms in (B.1) and (B.2) coincide. Hence,

$$
\begin{aligned}
&[x^+(z), x^-(w)]|\lambda\rangle_u \\
&= a^+(p)a^-(p)(t/q)^{-1/2}\delta(z/w) \\
&\times \left(\sum_{i=1}^{\ell} \delta(q^{-1}u_i/w)\delta(z/w)A^-_{\lambda,i}(p)A^+_{\lambda-\mathbf{1}_i,i}(p) - \sum_{i=1}^{\ell+1} \delta(u_i/w)A^+_{\lambda,i}(p)A^-_{\lambda+\mathbf{1}_i,i}(p) \right) |\lambda\rangle_u.
\end{aligned}
$$

One also shows

$$
\begin{aligned}
a^+(p)a^-(p) &= \frac{(1-q)(1-t^{-1})}{1-q/t}\frac{\theta_p(q/t)\theta_p(t)}{(p;p)^2_\infty\theta_p(q)}, \\
A^-_{\lambda,i}(p)A^+_{\lambda-\mathbf{1}_i,i}(p) &= \prod_{\substack{j=1\\ \neq i}}^{\ell}\frac{\theta_p(qu_j/tu_i)}{\theta_p(u_j/u_i)}\prod_{\substack{j=1\\ \neq i}}^{\ell+1}\frac{\theta_p(tu_j/u_i)}{\theta_p(qu_j/u_i)} \\
&= (t/q)\prod_{\substack{j=1\\ \neq i}}^{\ell}\frac{\theta_p(tu_i/qu_j)}{\theta_p(u_i/u_j)}\prod_{\substack{j=1\\ \neq i}}^{\ell+1}\frac{\theta_p(u_i/tu_j)}{\theta_p(u_i/qu_j)} \\
&= -(t/q)\frac{(p;p)^2_\infty\theta_p(q)}{\theta_p(q/t)\theta_p(t)}\mathrm{Res}_{z=q^{-1}u_i}B^-(z/u;p)\frac{dz}{z}, \\
A^+_{\lambda,i}(p)A^-_{\lambda+\mathbf{1}_i,i}(p) &= \prod_{\substack{j=1\\ \neq i}}^{\ell}\frac{\theta_p(u_j/tu_i)}{\theta_p(u_j/qu_i)}\prod_{\substack{j=1\\ \neq i}}^{\ell+1}\frac{\theta_p(tu_j/qu_i)}{\theta_p(u_j/u_i)} \\
&= (t/q)\prod_{\substack{j=1\\ \neq i}}^{\ell}\frac{\theta_p(tu_i/u_j)}{\theta_p(qu_i/u_j)}\prod_{\substack{j=1\\ \neq i}}^{\ell+1}\frac{\theta_p(qu_i/tu_j)}{\theta_p(u_i/u_j)} \\
&= (t/q)\frac{(p;p)^2_\infty\theta_p(q)}{\theta_p(q/t)\theta_p(t)}\mathrm{Res}_{z=u_i}B^-(z/u;p)\frac{dz}{z}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&[x^+(z), x^-(w)]|\lambda\rangle_u \\
&= \frac{(1-q)(1-t^{-1})}{1-q/t}\delta(z/w)\left(-\sum_{i=1}^{\ell}\delta(q^{-1}u_i/w)\mathrm{Res}_{z=q^{-1}u_i}(t/q)^{1/2}B^-(z/u;p)\frac{dz}{z} \right. \\
&\left. -\sum_{i=1}^{\ell+1}\delta(u_i/w)\mathrm{Res}_{z=u_i}(t/q)^{1/2}B^-(z/u;p)\frac{dz}{z} \right)|\lambda\rangle_u.
\end{aligned}
$$

Using the partial fraction expansion formula (3.44), one can show

$$
\begin{aligned}
&B^-_\lambda(z/u;\,p)|_{|u/z|<1} - B^-_\lambda(z/u;\,p)|_{|u/z|>1}\\
&\quad = -\sum_{k=1}^{\ell}\delta(q^{-1}u_k/z)\mathrm{Res}_{z=q^{-1}u_k}B^-_\lambda(z/u;\,p)\frac{dz}{z} - \sum_{k=1}^{\ell+1}\delta(u_k/z)\mathrm{Res}_{z=q^{-1}u_k}B^-_\lambda(z/u;\,p)\frac{dz}{z}
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&[x^+(z), x^-(w)]|\lambda\rangle_u\\
&\quad = \frac{(1-q)(1-t^{-1})}{1-q/t}\delta(z/w)\\
&\qquad \times\left((t/q)^{1/2}B^-_\lambda(z/u;\,p)|_{|u/z|<1} - (t/q)^{1/2}B^-_\lambda(z/u;\,p)|_{|u/z|>1}\right)|\lambda\rangle_u\\
&\quad = \frac{(1-q)(1-t^{-1})}{1-q/t}\delta(z/w)\left((t/q)^{1/2}(t/q)^{-1}B^+_\lambda(u/z;\,p)|_{|u/z|<1}\right.\\
&\qquad \left.-(t/q)^{1/2}B^-_\lambda(z/u;\,p)|_{|u/z|>1}\right)|\lambda\rangle_u\\
&\quad = \frac{(1-q)(1-t^{-1})}{1-q/t}\delta(z/w)\left(\psi^+(z)-\psi^-(z)\right)|\lambda\rangle_u.
\end{aligned}
$$

□

C Inductive derivation of Theorem 3.4

We consider a tensor product of the vector representations in Proposition 3.3. Define

$$
V^{(N)}(u) = V(u)\widetilde{\otimes}V(u(t/q)^{-1})\widetilde{\otimes}V(u(t/q)^{-2})\widetilde{\otimes}\cdots\widetilde{\otimes}V(u(t/q)^{-N+1}).
$$

Set

$$
\begin{aligned}
&\mathcal{P}^{(N)} = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathbb{Z}^N \mid \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N\},\\
&\quad |\lambda\rangle^{(N)}_u = [u]_{\lambda_1}\widetilde{\otimes}[u(t/q)^{-1}]_{\lambda_2-1}\widetilde{\otimes}[u(t/q)^{-2}]_{\lambda_3-2}\widetilde{\otimes}\cdots\widetilde{\otimes}[u(t/q)^{-N+1}]_{\lambda_N-N+1}
\end{aligned}
$$

and define $W^{(N)}(u)$ to be a subspace of $V^{(N)}(u)$ spanned by $\{|\lambda\rangle^{(N)}_u \mid \lambda \in \mathcal{P}^{(N)}\}$. An action of $\mathcal{U}_{q,t,p}$ on the tensor product space can be constructed by using the comultiplication Δ^{op} in (2.66)–(2.68) repeatedly.

One can verify the following two propositions.

Proposition C.1 *By applying Δ^{op} repeatedly, the following gives a level-(0,0) $\mathcal{U}_{q,t,p}$-module structure on $W^{(N)}(u)$.*

$$
\gamma^{1/2}|\lambda\rangle^{(N)}_u = |\lambda\rangle^{(N)}_u, \tag{C.1}
$$

$$
x^+(z)|\lambda\rangle^{(N)}_u = a^+(p)\sum_{i=1}^{N}A^{(N)+}_{\lambda,i}(p)\delta(u_i/z)|\lambda+\mathbf{1}_i\rangle^{(N)}_u, \tag{C.2}
$$

$$x^-(z)|\lambda\rangle_u^{(N)} = a^-(p)\sum_{i=1}^N A_{\lambda,i}^{(N)-}(p)\delta(q^{-1}u_i/z)|\lambda-\mathbf{1}_i\rangle_u^{(N)}, \tag{C.3}$$

$$\psi^+(z)|\lambda\rangle_u^{(N)} = B_\lambda^{(N)+}(u/z;p)|\lambda\rangle_u^{(N)}, \tag{C.4}$$

$$\psi^-(z)|\lambda\rangle_u^{(N)} = B_\lambda^{(N)-}(z/u;p)|\lambda\rangle_u^{(N)}, \tag{C.5}$$

where $u_i = q^{\lambda_i}t^{-i+1}u$, $\lambda \pm \mathbf{1}_i = (\lambda_1, \ldots, \lambda_i \pm 1, \ldots, \lambda_N)$ *and we set*

$$A_{\lambda,i}^{(N)+}(p) = \prod_{j=1}^{i-1} \frac{\theta_p(tu_i/u_j)\theta_p(qt^{-1}u_i/u_j)}{\theta_p(qu_i/u_j)\theta_p(u_i/u_j)}$$

$$A_{\lambda,i}^{(N)-}(p) = \prod_{j=i+1}^{N} \frac{\theta_p(tu_j/u_i)\theta_p(qt^{-1}u_j/u_i)}{\theta_p(qu_j/u_i)\theta_p(u_j/u_i)}$$

$$B_\lambda^{(N)+}(u/z;p) = \prod_{j=1}^{N} \frac{\theta_p(t^{-1}u_j/z)\theta_p(q^{-1}tu_j/z)}{\theta_p(u_j/z)\theta_p(q^{-1}u_j/z)}$$

$$B_\lambda^{(N)-}(z/u;p) = \prod_{j=1}^{N} \frac{\theta_p(tz/u_j)\theta_p(qt^{-1}z/u_j)}{\theta_p(z/u_j)\theta_p(qz/u_j)}.$$

In particular, we have

$$\alpha_m|\lambda\rangle_u^{(N)} = \frac{(1-t^{-m})(1-(q/t)^{-m})}{m}\sum_{j=1}^N u_j^m|\lambda\rangle_u^{(N)} \qquad (m\in\mathbb{Z}\backslash\{0\}). \tag{C.6}$$

An inductive limit $N\to\infty$ can be taken in the same way as in the trigonometric case [17, 18]. Let

$$\mathcal{P}^{(N),+} = \{\lambda\in\mathcal{P}^{(N)} \mid \lambda_N \geq 0\,\}.$$

and define $W^{(N),+}(u)$ to be the subspace of $W^{(N)}(u)$ spanned by $\{|\lambda\rangle_u^{(N)},\ \lambda\in\mathcal{P}^{(N),+}\}$. Let us define $\tau_N : \mathcal{P}^{(N),+}\to\mathcal{P}^{(N+1),+}$ by

$$\tau_N(\lambda) = (\lambda_1,\lambda_2,\ldots,\lambda_N,0).$$

This induces the embedding $W^{(N),+}\hookrightarrow W^{(N+1),+}$. We then define a semi-infinite tensor product space $\mathcal{F}_u$ by the inductive limit

$$\mathcal{F}_u = \lim_{N\to\infty} W^{(N),+}(u).$$

The action of $\mathcal{U}_{q,t,p}$ on $\mathcal{F}_u$ via Δ^{op} is defined inductively as follows. Define

$$x^{+[N]}(z) = x^+(z), \qquad x^{-[N]}(z) = (q/t)^{1/2}\frac{\theta_p(q^{-1}t^{-N+1}u/z)}{\theta_p(t^{-N}u/z)}x^-(z),$$

$$\psi^{+[N]}(z) = (q/t)^{1/2}\frac{\theta_p(q^{-1}t^{-N+1}u/z)}{\theta_p(t^N u/z)}\psi^+(z),$$
$$\psi^{-[N]}(z) = (q/t)^{-1/2}\frac{\theta_p(qt^{N-1}z/u)}{\theta_p(t^{-N}z/u)}\psi^+(z).$$

Then, we have

Lemma C.2 *For $x = x^\pm, \psi^\pm$, we have*

$$\tau_N(x^{[N]}(z)|\lambda\rangle_u) = x^{[N+1]}(z)\tau_N(|\lambda\rangle_u).$$

Thanks to this lemma, one can define the action of $\mathcal{U}_{q,t,p}$ on $\mathcal{F}_u$ by

$$x(z)|\lambda\rangle_u = \lim_{N\to\infty} x^{[N]}(z)|(\lambda_1,\lambda_2,\ldots,\lambda_N)\rangle.$$

This gives the level (0, 1) representation in Theorem 3.4.

D Proof of Theorem 4.2

Proof of (4.4): From Theorem 3.2 one has

$$\Phi_\emptyset(u)\psi^+((t/q)^{1/4}z)\Big|_{\mathcal{F}^{(1,N+1)}} = (t/q)^{-(N+1)/2}\frac{(pqz/tu;p)_\infty}{(pz/u;p)_\infty} :\Phi_\emptyset(u)\widetilde{\psi}^+((t/q)^{1/4}z):,$$
$$\psi^+((t/q)^{1/4}z)\Big|_{\mathcal{F}^{(1,N)}}\Phi_\emptyset(u) = (t/q)^{-N/2}\frac{(u/z;p)_\infty}{(tu/qz;p)_\infty} :\widetilde{\psi}^+((t/q)^{1/4}z)\Phi_\emptyset(u):.$$

Hence, we have

$$\Phi_\emptyset(u)\psi^+((t/q)^{1/4}z) = (t/q)^{-1/2}\frac{\theta_p(tu/qz)}{\theta_p(u/z)}\psi^+((t/q)^{1/4}z)\Phi_\emptyset(u).$$

Then, (4.4) follows from

$$\begin{aligned}
&:\widetilde{\psi}^+((t/q)^{1/4}z)^{-1}\widetilde{\Phi}_\lambda(u)\widetilde{\psi}^+((t/q)^{1/4}z): \\
&\quad =: \widetilde{\psi}^+((t/q)^{1/4}z)^{-1}\Phi_\emptyset(u)\widetilde{\psi}^+((t/q)^{1/4}z) \\
&\qquad \times \prod_{(i,j)\in\lambda}\widetilde{\psi}^+((t/q)^{1/4}z)^{-1}\widetilde{x}^-((t/q)^{1/4}u_{i,j})\widetilde{\psi}^+((t/q)^{1/4}z):,
\end{aligned}$$

where $u_{i,j} = q^{j-1}t^{-i+1}u$, and the following proposition.

Proposition D.1

$$\prod_{i=1}^{\ell(\lambda)}\prod_{j=1}^{\lambda_i} g(q^{j-1}t^{-i+1}z;\,p)=\frac{\theta_p(z)}{\theta_p((t/q)z)}B^{+}_{\lambda}(z;\,p).$$

□

Proof of (4.6): From Theorem 3.2 one has

$$\Phi_{\emptyset}(u)\widetilde{x}^{+}((t/q)^{-1/4}z)=(1-qz/tu):\Phi_{\emptyset}(u)\widetilde{x}^{+}((t/q)^{-1/4}z):,$$
$$\widetilde{x}^{+}((t/q)^{-1/4}z)\Phi_{\emptyset}(u)=(1-tu/qz):\Phi_{\emptyset}(u)\widetilde{x}^{+}((t/q)^{-1/4}z):,$$
$$\widetilde{x}^{+}((t/q)^{-1/4}z)\widetilde{x}^{-}((t/q)^{1/4}w)=\frac{(1-tw/z)(1-w/qz)}{(1-tw/qz)(1-w/z)}:\widetilde{x}^{+}((t/q)^{-1/4}z)\widetilde{x}^{-}((t/q)^{1/4}w):$$

Hence, we have

$$\widetilde{\Phi}_{\lambda}(u)\widetilde{x}^{+}((t/q)^{-1/4}z)=(1-t^{\ell(\lambda)}qz/tu)\prod_{j=1}^{\ell(\lambda)}\frac{1-qz/tu_j}{1-qz/u_j}:\widetilde{\Phi}_{\lambda}(u)\widetilde{x}^{+}((t/q)^{-1/4}z): \quad \text{for } |z/u_j|<1,$$
$$\widetilde{x}^{+}((t/q)^{-1/4}z)\widetilde{\Phi}_{\lambda}(u)=(1-t^{-\ell(\lambda)}tu/qz)\prod_{j=1}^{\ell(\lambda)}\frac{1-tu_j/qz}{1-u_j/qz}:\widetilde{\Phi}_{\lambda}(u)\widetilde{x}^{+}((t/q)^{-1/4}z): \quad \text{for } |u_j/z|<1$$

and

$$\begin{aligned}
&:\widetilde{\Phi}_{\lambda}(u)\widetilde{x}^{+}((t/q)^{-1/4}q^{-1}u_k):\\
&\quad =:\Phi_{\emptyset}(u)\prod_{(i,j)\in\lambda}\widetilde{x}^{-}((t/q)^{1/4}u_{i,j})\widetilde{x}^{+}((t/q)^{-1/4}q^{-1}u_k):\\
&\quad =:\Phi_{\emptyset}(u)\prod_{(i,j)\in\lambda-\mathbf{1}_k}\widetilde{x}^{-}((t/q)^{1/4}u_{i,j})\cdot\widetilde{x}^{-}((t/q)^{1/4}u_{k,\lambda_k})\widetilde{x}^{+}((t/q)^{-1/4}q^{-1}u_k):\\
&\quad =:\widetilde{\Phi}_{\lambda-\mathbf{1}_k}(u)\widetilde{\psi}^{-}((t/q)^{-1/4}q^{-1}u_k):.
\end{aligned}\tag{D.1}$$

The last equality follows from

$$:\widetilde{x}^{-}((t/q)^{1/4}z)\widetilde{x}^{+}((t/q)^{-1/4}z):=\widetilde{\psi}^{-}((t/q)^{-1/4}z).$$

Therefore, noting

$$x^{+}(z)\Big|_{\mathcal{F}^{(1,N)}_{v}}=vz^{-N}(t/q)^{3N/4}\widetilde{x}^{+}(z),$$
$$x^{+}(z)\Big|_{\mathcal{F}^{(1,N+1)}_{-uv}}=-uvz^{-(N+1)}(t/q)^{3(N+1)/4}\widetilde{x}^{+}(z),$$

one obtains

$$\begin{aligned}
&\left(-uvz^{-(N+1)}(t/q)^{N+1}\right)^{-1}\left(\widetilde{\Phi}_\lambda(u)x^+((t/q)^{-1/4}z)\Big|_{\mathcal{F}^{(1,N+1)}_{-uv}}-x^+((t/q)^{-1/4}z)\Big|_{\mathcal{F}^{(1,N)}_{v}}\widetilde{\Phi}_\lambda(u)\right)\\
&=\left((1-t^{\ell(\lambda)}qz/tu)\prod_{j=1}^{\ell(\lambda)}\frac{1-qz/tu_j}{1-qz/u_j}\Big|_{|z/u_j|<1}\right.\\
&\quad\left.+(t/q)^{-1}z/u(1-t^{-\ell(\lambda)}tu/qz)\prod_{j=1}^{\ell(\lambda)}\frac{1-tu_j/qz}{1-u_j/qz}\Big|_{|u_j/z|<1}\right)\\
&\quad\times:\widetilde{\Phi}_\lambda(u)\widetilde{x}^+((t/q)^{1/4}(t/q)^{-1/2}z):\\
&=\left((1-t^{\ell(\lambda)}qz/tu)\prod_{j=1}^{\ell(\lambda)}\frac{1-qz/tu_j}{1-qz/u_j}\Big|_{|z/u_j|<1}\right.\\
&\quad\left.-(1-t^{\ell(\lambda)}qz/tu)\prod_{j=1}^{\ell(\lambda)}\frac{1-qz/tu_j}{1-qz/u_j}\Big|_{|u_j/z|<1}\right)\\
&\quad\times:\widetilde{\Phi}_\lambda(u)\widetilde{x}^+((t/q)^{1/4}(t/q)^{-1/2}z):\\
&=\sum_{k=1}^{\ell(\lambda)}\delta(q^{-1}u_k/z)\prod_{j=1}^{k-1}\frac{1-u_k/tu_j}{1-u_k/u_j}\\
&\quad\times\prod_{j=k+1}^{\ell(\lambda)}\frac{1-u_k/tu_j}{1-u_k/u_j}(1-u_k/tu_{\ell(\lambda)+1}):\widetilde{\Phi}_\lambda(u)\widetilde{x}^+((t/q)^{1/4}(t/q)^{-1/2}z):\\
&=\sum_{k=1}^{\ell(\lambda)}\delta(q^{-1}u_k/z)(1-t^{-1})(q/t)t^{-k+1}\frac{c_\lambda}{c_{\lambda-\mathbf{1}_k}}A^-_{\lambda,k}:\widetilde{\Phi}_{\lambda-\mathbf{1}_k}(u)\widetilde{\psi}^-((t/q)^{-1/4}q^{-1}u_k):.
\end{aligned}\tag{D.2}$$

In the last equality we used (D.1). The third equality follows from the formula

$$\begin{aligned}
&(1-s/b_{n+1})\prod_{j=1}^{n}\frac{1-s/b_j}{1-s/a_j}\Big|_{|s/a_j|<1}-(1-s/b_{n+1})\prod_{j=1}^{n}\frac{1-s/b_j}{1-s/a_j}\Big|_{|a_j/s|<1}\\
&\quad=\sum_{k=1}^{n}\delta(s/a_k)(1-a_k/b_{n+1})\frac{\prod_{j=1}^{n}(1-a_k/b_j)}{\prod_{\substack{j=1\\ j\neq k}}^{n}(1-a_k/a_j)}.
\end{aligned}\tag{D.3}$$

This is obtained from the partial fraction formula for $\prod_{j=1}^{n}\frac{1-s/b_j}{1-s/a_j}$ and

$$\frac{1}{1-z}\Big|_{|z|<1}-\frac{1}{1-z}\Big|_{|z|>1}=\delta(z).$$

On the other hand, we have

$$\widetilde{\psi}^-((t/q)^{-1/4}z)\Phi_\emptyset(u) = \frac{(pu/z;\,p)_\infty}{(ptu/qz;\,p)_\infty} : \widetilde{\psi}^-((t/q)^{-1/4}z)\Phi_\emptyset(u) :,$$
$$\widetilde{\psi}^-((t/q)^{-1/4}z)\widetilde{x}^-((t/q)^{1/4}w) = h(w/z) : \widetilde{\psi}^-((t/q)^{-1/4}z)\widetilde{x}^-((t/q)^{1/4}w) :,$$

where

$$h(w) = \frac{(pqw;\,p)_\infty(ptw/q;\,p)_\infty(pw/t;\,p)_\infty}{(pw/q;\,p)_\infty(pqw/t;\,p)_\infty(ptw;\,p)_\infty}.$$

We have

$$\prod_{(i,j)\in\lambda-\mathbf{1}_k} h(u_{i,j}/z)\Big|_{z=q^{-1}u_k} = -t\frac{a^-(p)}{a^+(p)}\frac{(ptu/u_k;\,p)_\infty}{(pqu/u_k;\,p)_\infty}\prod_{j=1}^{\ell(\lambda)}\frac{(pu_j/u_k;\,p)_\infty}{(pqu_j/tu_k;\,p)_\infty}$$
$$\times\prod_{j=1}^{\ell(\lambda)+1}\frac{(pqu_j/u_k;\,p)_\infty}{(ptu_j/u_k;\,p)_\infty}.$$

Hence,

$$\widetilde{\psi}^-((t/q)^{-1/4}z)\widetilde{\Phi}_{\lambda-\mathbf{1}_k}(u)\Big|_{z=q^{-1}u_k} =^-\, t\frac{a^-(p)}{a^+(p)}\prod_{j=1}^{\ell(\lambda)}\frac{(pu_j/u_k;\,p)_\infty}{(pqu_j/tu_k;\,p)_\infty}$$
$$\times\prod_{j=1}^{\ell(\lambda)+1}\frac{(pqu_j/u_k;\,p)_\infty}{(ptu_j/u_k;\,p)_\infty} : \widetilde{\psi}^-((t/q)^{-1/4}z)\widetilde{\Phi}_{\lambda-\mathbf{1}_k}(u) : \Big|_{z=q^{-1}u_k}.$$

Substituting this into (D.2), one obtains

$$\widetilde{\Phi}_\lambda(u)x^+((t/q)^{-1/4}z)\Big|_{\mathcal{F}^{(1,N+1)}_{-uv}} - x^+((t/q)^{-1/4}\Big|_{\mathcal{F}^{(1,N)}_{v}}\widetilde{\Phi}_\lambda(u)$$
$$= uvz^{-(N+1)}(t/q)^{N+1}\sum_{k=1}^{\ell(\lambda)}\delta(q^{-1}u_k/z)t^{-k}a^+(p)\frac{c_\lambda}{c_{\lambda-\mathbf{1}_k}}A^-_{\lambda,k}(p)\frac{N_{\lambda-\mathbf{1}_k}(p)}{N_\lambda(p)}(t/q)^{-N/2}$$
$$\times\psi^-((t/q)^{-1/4}z)\widetilde{\Phi}_{\lambda-\mathbf{1}_k}(u)$$
$$= \frac{c_\lambda}{q^{n(\lambda')}N_\lambda(p)t^*(\lambda,u,v,N)}q\sum_{k=1}^{\ell(\lambda)}\delta(q^{-1}u_k/z)t^{-1}a^+(p)A^-_{\lambda,k}(p)\psi^-((t/q)^{-1/4}z)$$
$$\times\frac{q^{n((\lambda-\mathbf{1}_k)')}N_{\lambda-\mathbf{1}_k}(p)t^*(\lambda-\mathbf{1}_k,u,v,N)}{c_{\lambda-1_k}}\widetilde{\Phi}_{\lambda-\mathbf{1}_k}(u).$$

Here, we used

$$\frac{t^*(\lambda-\mathbf{1}_k,u,v,N)}{t^*(\lambda,u,v,N)}=q^{-1}v(q^{-1}u_k)^{-N}(t/q)^{N/2}.$$

□

Proof of (4.7): From Theorem 3.2 we have

$$\widetilde{\Phi}_\lambda(u)\widetilde{x}^-((t/q)^{1/4}z)=\prod_{i=1}^{\ell(\lambda)}\frac{(tz/u_i;p)_\infty}{(pqz/u_i;p)_\infty}$$
$$\times\prod_{i=1}^{\ell(\lambda)+1}\frac{(pqz/tu_i;p)_\infty}{(z/u_i;p)_\infty}:\widetilde{x}^-((t/q)^{1/4}z)\widetilde{\Phi}_\lambda(u):\quad\text{for } |z/u|<1,$$
$$\widetilde{x}^-((t/q)^{1/4}z)\widetilde{\Phi}_\lambda(u)=\prod_{i=1}^{\ell(\lambda)}\frac{(q^{-1}u_i/z;p)_\infty}{(pt^{-1}u_i/z;p)_\infty}$$
$$\times\prod_{i=1}^{\ell(\lambda)+1}\frac{(pu_i/z;p)_\infty}{(tu_i/qz;p)_\infty}:\widetilde{x}^-((t/q)^{1/4}z)\widetilde{\Phi}_\lambda(u):\quad\text{for } |u/z|<1$$

and

$$B^+_\lambda(u/z,p)\widetilde{x}^-((t/q)^{1/4}z)\widetilde{\Phi}_\lambda(u)=-z/u\prod_{i=1}^{\ell(\lambda)}\frac{(tz/u_i;p)_\infty}{(pqz/u_i;p)_\infty}$$
$$\times\prod_{i=1}^{\ell(\lambda)+1}\frac{(pqz/tu_i;p)_\infty}{(z/u_i;p)_\infty}:\widetilde{x}^-((t/q)^{1/4}z)\widetilde{\Phi}_\lambda(u):.$$

Hence, noting

$$x^-(z)\Big|_{\mathcal{F}^{(1,N+1)}_{-uv}}=(-uv)^{-1}z^{N+1}(t/q)^{-3(N+1)/4}\widetilde{x}^-(z),$$
$$x^-(z)\Big|_{\mathcal{F}^{(1,N)}_{v}}=v^{-1}z^{N}(t/q)^{-3N/4}\widetilde{x}^-(z),$$

one gets

$$\widetilde{\Phi}_\lambda(u)x^-((t/q)^{1/4}z)-(t/q)^{-1/2}B^+_\lambda(z,p)x^-((t/q)^{1/4}z)\widetilde{\Phi}_\lambda(u)$$
$$=-(uv)^{-1}z^{N+1}(t/q)^{-(N+1)/2}$$
$$\times\left(\prod_{i=1}^{\ell(\lambda)}\frac{(tz/u_i;p)_\infty}{(pqz/u_i;p)_\infty}\prod_{i=1}^{\ell(\lambda)+1}\frac{(pqz/tu_i;p)_\infty}{(z/u_i;p)_\infty}\Big|_{|z/u|<1}-\prod_{i=1}^{\ell(\lambda)}\frac{(tz/u_i;p)_\infty}{(pqz/u_i;p)_\infty}\right.$$
$$\left.\times\prod_{i=1}^{\ell(\lambda)+1}\frac{(pqz/tu_i;p)_\infty}{(z/u_i;p)_\infty}\Big|_{|u/z|<1}\right):\widetilde{\Phi}_\lambda(u)\widetilde{x}^-((t/q)^{1/4}z):$$

$$= -(uv)^{-1}z^{N+1}(t/q)^{-(N+1)/2}\left(\frac{\prod_{i=1}^{\ell(\lambda)}(1-tz/u_i)}{\prod_{i=1}^{\ell(\lambda)+1}(1-z/u_i)}\Big|_{|z/u_i|<1} - \frac{\prod_{i=1}^{\ell(\lambda)}(1-tz/u_i)}{\prod_{i=1}^{\ell(\lambda)+1}(1-z/u_i)}\Big|_{|u_i/z|<1}\right)\prod_{i=1}^{\ell(\lambda)}\frac{(ptz/u_i;p)_\infty}{(pqz/u_i;p)_\infty}\prod_{i=1}^{\ell(\lambda)+1}\frac{(pqz/tu_i;p)_\infty}{(pz/u_i;p)_\infty} \times :\widetilde{\Phi}_\lambda(u)\widetilde{x}^-((t/q)^{1/4}z):. \tag{D.4}$$

Applying the formula

$$\frac{\prod_{j=1}^{n-1}(1-s/b_j)}{\prod_{j=1}^{n}(1-s/a_j)}\Big|_{|s/a|<1} - \frac{\prod_{j=1}^{n-1}(1-s/b_j)}{\prod_{j=1}^{n}(1-s/a_j)}\Big|_{|a/s|<1} = \sum_{k=1}^{n}\delta(a_k/s)\frac{\prod_{j=1}^{n-1}(1-a_k/b_j)}{\prod_{\substack{j=1\\ \neq k}}^{n}(1-a_k/a_j)} \tag{D.5}$$

and using

$$:\widetilde{\Phi}_\lambda(u)\widetilde{x}^-((t/q)^{1/4}z):\Big|_{z=u_k} = \widetilde{\Phi}_{\lambda+\mathbf{1}_k}(u),$$

the RHS of (D.4) is

$$(uv)^{-1}z^{N+1}(t/q)^{-(N+1)/2}\sum_{k=1}^{\ell(\lambda)+1}\delta(u_k/z)$$
$$\prod_{i=1}^{\ell(\lambda)}\frac{(tu_k/u_i;p)_\infty}{(pqu_k/u_i;p)_\infty}\frac{\prod_{i=1}^{\ell(\lambda)+1}(pqu_k/tu_i;p)_\infty}{\prod_{\substack{i=1\\ \neq k}}^{\ell(\lambda)+1}(u_k/u_i;p)_\infty}\frac{1}{(p;p)_\infty}\widetilde{\Phi}_{\lambda+\mathbf{1}_k}(u)$$
$$= (uv)^{-1}z^{N+1}(t/q)^{-(N+1)/2}\sum_{k=1}^{\ell(\lambda)+1}\delta(u_k/z)t^k a^-(p)\frac{c_\lambda}{c_{\lambda+\mathbf{1}_k}}A^+_{\lambda,k}(p)\frac{N_{\lambda+\mathbf{1}_k}(p)}{N_\lambda(p)}\widetilde{\Phi}_{\lambda+\mathbf{1}_k}(u)$$
$$= \frac{c_\lambda}{q^{n(\lambda')}N_\lambda(p)t^*(\lambda,u,v,N)}q^{-1}f(1;p)a^+(p)(t/q)^{-1/2} \times \sum_{k=1}^{\ell(\lambda)+1}\delta(u_k/z)A^+_{\lambda,k}(p)\frac{q^{n((\lambda+\mathbf{1}_k)')}N_{\lambda+\mathbf{1}_k}(p)t^*(\lambda+\mathbf{1}_k,u,v,N)}{c_{\lambda+\mathbf{1}_k}}\widetilde{\Phi}_{\lambda+\mathbf{1}_k}(u).$$

□

E Proof of Proposition 4.7

Proof of (4.30): From (4.4), we have

$$\psi^+((t/q)^{1/4}z)\widetilde{\Phi}_\lambda(p^{-1}u)^{-1} = (q/t)^{1/2}B^+_\lambda(p^{-1}u/z;p)\widetilde{\Phi}_\lambda(p^{-1}u)^{-1}\psi^+((t/q)^{1/4}z).$$

□

Proof of (4.32): Similar to the calculations in Appendix D, we have

$$\begin{aligned}
&\widetilde{\Phi}_\lambda(p^{-1}u)^{-1}\widetilde{x}^+((t/q)^{-1/4}z) = \frac{1}{1-pt^l qz/tu}\\
&\qquad\times\prod_{j=1}^{l}\frac{1-pqz/u_j}{1-pqz/tu_j} : \widetilde{\Phi}_\lambda(p^{-1}u)^{-1}\widetilde{x}^+((t/q)^{-1/4}z) :\\
&\qquad \text{for } |z/u_j|<1,\\
&\widetilde{x}^+((t/q)^{-1/4}z)\widetilde{\Phi}_\lambda(p^{-1}u)^{-1} = \frac{1}{1-p^{-1}t^{-l}tu/qz}\\
&\qquad\times\prod_{j=1}^{l}\frac{1-p^{-1}u_j/qz}{1-p^{-1}tu_j/qz} : \widetilde{\Phi}_\lambda(p^{-1}u)^{-1}\widetilde{x}^+((t/q)^{-1/4}z) :\\
&\qquad \text{for } |u_j/z|<1
\end{aligned}$$

and

$$\begin{aligned}
&: \widetilde{\Phi}_\lambda(p^{-1}u)^{-1}\widetilde{x}^+((t/q)^{-1/4}p^{-1}tu_k/q) :\\
&\ \ =: \Phi_\emptyset(p^{-1}u)^{-1}\prod_{(i,j)\in\lambda}\widetilde{x}^-((t/q)^{1/4}p^{-1}u_{i,j})^{-1}\widetilde{x}^+((t/q)^{-1/4}tu_k/q) :\\
&\qquad =: \Phi_\emptyset(p^{-1}u)^{-1}\prod_{(i,j)\in\lambda+\mathbf{1}_k}\widetilde{x}^-((t/q)^{1/4}p^{-1}u_{i,j})^{-1}\\
&\qquad\quad\times\ \widetilde{x}^-((t/q)^{1/4}u_{k,\lambda_k+1})\widetilde{x}^+((t/q)^{-1/4}p^{-1}tu_k/q) :\\
&\qquad =: \widetilde{\Phi}_{\lambda+\mathbf{1}_k}(p^{-1}u)\widetilde{\psi}^+((t/q)^{1/4}p^{-1}u_k) : .
\end{aligned}\tag{E.1}$$

Therefore, noting

$$\begin{aligned}
x^+(z)\Big|_{\mathcal{F}_v^{(1,N)}} &= vz^{-N}(t/q)^{3N/4}\widetilde{x}^+(z),\\
x^+(z)\Big|_{\mathcal{F}_{-p^{-1}uv}^{(1,N+1)}} &= -p^{-1}uvz^{-(N+1)}(t/q)^{3(N+1)/4}\widetilde{x}^+(z),
\end{aligned}$$

one obtains

$$\begin{aligned}
&x^+((t/q)^{-1/4}z)\Big|_{\mathcal{F}_{-p^{-1}uv}^{(1,N+1)}}\widetilde{\Phi}_\lambda(p^{-1}u)^{-1} - \widetilde{\Phi}_\lambda(p^{-1}u)^{-1}x^+((t/q)^{-1/4}z)\Big|_{\mathcal{F}_v^{(1,N)}}\\
&= -p^{-1}uvz^{-(N+1)}(t/q)^{N+1}\\
&\quad\times\left(\frac{1}{1-p^{-1}t^{-l}tu/qz}\prod_{j=1}^{\ell}\frac{1-p^{-1}u_j/qz}{1-p^{-1}tu_j/qz}\Big|_{|u_j/z|<1}\right.\\
&\qquad\left.+p(t/q)^{-1}z/u\frac{1}{1-pt^\ell qz/tu}\prod_{j=1}^{\ell}\frac{1-pqz/u_j}{1-pqz/tu_j}\Big|_{|z/u_j|<1}\right)\\
&\quad\times : \widetilde{\Phi}_\lambda(p^{-1}u)^{-1}\widetilde{x}^+((t/q)^{-1/4}z) :
\end{aligned}$$

$$
\begin{aligned}
&= -p^{-1}uvz^{-(N+1)}(t/q)^{N+1}pqz/tu \\
&\times\left(-\frac{1}{1-t^{\ell}qz/tu}\prod_{j=1}^{\ell}\frac{1-pqz/u_j}{1-pqz/tu_j}\Big|_{|u_j/z|<1}+\frac{1}{1-pt^{\ell}qz/tu}\prod_{j=1}^{\ell}\frac{1-pqz/u_j}{1-pqz/tu_j}\Big|_{|z/u_j|<1}\right) \\
&\times :\widetilde{\Phi}_{\lambda}(p^{-1}u)^{-1}\widetilde{x}^{+}((t/q)^{-1/4}z): \\
&= -p^{-1}uvz^{-(N+1)}(t/q)^{N+1}pqz/tu \\
&\times\sum_{k=1}^{\ell+1}\delta(p^{-1}tu_k/qz)\frac{\prod_{j=1}^{\ell}(1-tu_k/u_j)}{\prod_{\substack{j=1\\ \neq k}}^{\ell+1}(1-u_k/u_j)} :\widetilde{\Phi}_{\lambda+\mathbf{1}_k}(p^{-1}u)^{-1}\widetilde{\psi}^{-}((t/q)^{-1/4}u_k):.
\end{aligned}
\tag{E.2}
$$

The last equality follows from (D.5).

On the other hand, we have

$$
\begin{aligned}
&\widetilde{\Phi}_{\emptyset}(u)^{-1}\widetilde{\psi}^{-}((t/q)^{-1/4}z) = \frac{(z/u;p)_\infty}{(qz/tu;p)_\infty} :\widetilde{\psi}^{-}((t/q)^{-1/4}z)\widetilde{\Phi}_{\emptyset}(u)^{-1}:, \\
&\widetilde{x}^{-}((t/q)^{1/4}u)^{-1}\widetilde{\psi}^{-}((t/q)^{-1/4}z) = h(p^{-1}z/u)^{-1} :\widetilde{\psi}^{-}((t/q)^{-1/4}z)\widetilde{x}^{-}((t/q)^{1/4}u)^{-1}:,
\end{aligned}
$$

where

$$
h(w) = \frac{(pqw;p)_\infty(ptw/q;p)_\infty(pw/t;p)_\infty}{(pw/q;p)_\infty(pqw/t;p)_\infty(ptw;p)_\infty}.
$$

We have

$$
\begin{aligned}
&\widetilde{\Phi}_{\lambda+\mathbf{1}_k}(p^{-1}u)^{-1}\widetilde{\psi}^{-}((t/q)^{-1/4}u_k) \\
&\quad = \frac{(pu_k/u;p)_\infty}{(pqu_k/tu;p)_\infty}\prod_{(i,j)\in\lambda+\mathbf{1}_k} h(u_k/u_{i,j})^{-1} :\widetilde{\Phi}_{\lambda+\mathbf{1}_i}(p^{-1}u)^{-1}\widetilde{\psi}^{-}((t/q)^{-1/4}u_k):
\end{aligned}
$$

We have $u_{k,(\lambda+\mathbf{1}_k)_k} = u_k$,

$$
\begin{aligned}
\prod_{(i,j)\in\lambda+\mathbf{1}_k} h(u_k/u_{i,j})^{-1} &= \prod_{(i,j)\in\lambda} h(u_k/u_{i,j})^{-1}\times h(u_k/u_{k,(\lambda+\mathbf{1}_k)_k})^{-1} \\
&= -t\frac{a^{-}(p)}{a^{+}(p)}\frac{(pqu_k/tu;p)_\infty}{(pu_k/u;p)_\infty}\prod_{j=1}^{l}\frac{(pqu_k/u_j;p)_\infty}{(ptu_k/u_j;p)_\infty}\prod_{j=1}^{l+1}\frac{(pu_k/u_j;p)_\infty}{(pqu_k/tu_j;p)_\infty}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\widetilde{\Phi}_{\lambda+\mathbf{1}_k}(p^{-1}u)^{-1}\widetilde{\psi}^{-}((t/q)^{-1/4}u_k) \\
&\quad = -t\frac{a^{-}(p)}{a^{+}(p)}\prod_{j=1}^{l}\frac{(pqu_k/u_j;p)_\infty}{(ptu_k/u_j;p)_\infty}\prod_{j=1}^{l+1}\frac{(pu_k/u_j;p)_\infty}{(pqu_k/tu_j;p)_\infty}.: \\
&\qquad \times\widetilde{\Phi}_{\lambda+\mathbf{1}_k}(p^{-1}u)^{-1}\widetilde{\psi}^{-}((t/q)^{-1/4}u_k):.
\end{aligned}
$$

Therefore,

$$\frac{\prod_{j=1}^{\ell}(1-tu_k/u_j)}{\prod_{\substack{j=1\\ \neq k}}^{\ell+1}(1-u_k/u_j)} :\widetilde{\Phi}_{\lambda+\mathbf{1}_k}(p^{-1}u)^{-1}\widetilde{\psi}^-((t/q)^{-1/4}u_k):$$
$$= t^{k-1}a^+(p)\frac{c'_\lambda N'_{\lambda+\mathbf{1}_k}(p)}{c'_{\lambda+\mathbf{1}_k}N'_\lambda(p)}A^{+'}_{\lambda,k}(p)\widetilde{\Phi}_{\lambda+\mathbf{1}_k}(p^{-1}u)^{-1}\widetilde{\psi}^-((t/q)^{-1/4}u_k).$$

Substituting this into (D.2), one obtains

$$x^+((t/q)^{-1/4}z)\Big|_{\mathcal{F}^{(1,N+1)}_{-p^{-1}uv}}\widetilde{\Phi}_\lambda(p^{-1}u)^{-1}-\widetilde{\Phi}_\lambda(p^{-1}u)^{-1}x^+((t/q)^{-1/4}z)\Big|_{\mathcal{F}^{(1,N)}_v}$$
$$=\frac{c'_\lambda}{q^{n(\lambda')}t(\lambda,v,p^{-1}u,N)N'_\lambda(p)}(t/q)^{-1/2}a^+(p)$$
$$\times\sum_{k=1}^{\ell+1}\delta(p^{-1}tu_k/qz)A^{+'}_{\lambda,k}(p)\frac{q^{n((\lambda+\mathbf{1}_k)')}t(\lambda+\mathbf{1}_k,v,p^{-1}u,N)N'_{\lambda+\mathbf{1}_k}(p)}{c'_{\lambda+\mathbf{1}_k}}$$
$$\times\widetilde{\Phi}_{\lambda+\mathbf{1}_k}(p^{-1}u)^{-1}\psi^+((t/q)^{1/4}qz/t).$$

Here, we used

$$\frac{t(\lambda+\mathbf{1}_k,v,p^{-1}u,N)}{t(\lambda,v,p^{-1}u,N)}=-p^{-1}uvz^{-(N+1)}(t/q)^{3(N+1)/2},$$
$$pqz/tu\cdot t^{k-1}=q^{\lambda_k}=\frac{q^{n((\lambda+\mathbf{1}_k)')}}{q^{n(\lambda')}}.$$

□

Proof of (4.28): We have

$$\widetilde{\Phi}_\lambda(u)^{-1}\widetilde{x}^-((t/q)^{1/4}z)=\prod_{i=1}^{l(\lambda)}\frac{(pqz/u_i;p)_\infty}{(tz/u_i;p)_\infty}$$
$$\times\prod_{i=1}^{l(\lambda)+1}\frac{(z/u_i;p)_\infty}{(pqz/tu_i;p)_\infty}:\widetilde{x}^-((t/q)^{1/4}z)\widetilde{\Phi}_\lambda(u)^{-1}:\quad \text{for } |z/u|<1,$$
$$\widetilde{x}^-((t/q)^{1/4}z)\widetilde{\Phi}_\lambda(u)^{-1}=\prod_{i=1}^{l(\lambda)}\frac{(pu_i/tz;p)_\infty}{(u_i/qz;p)_\infty}$$
$$\times\prod_{i=1}^{l(\lambda)+1}\frac{(tu_i/qz;p)_\infty}{(pu_i/z;p)_\infty}:\widetilde{x}^-((t/q)^{1/4}z)\widetilde{\Phi}_\lambda(u)^{-1}:\quad \text{for } |u/z|<1$$

and

$$
\begin{aligned}
&B^+_\lambda(p^{-1}u/z,p)\widetilde{\Phi}_\lambda(p^{-1}u)^{-1}\widetilde{x}^-((t/q)^{1/4}z)\\
&= -pz/u\prod_{i=1}^{l(\lambda)}\frac{(u_i/tz;p)_\infty}{(p^{-1}u_i/qz;p)_\infty}\prod_{i=1}^{l(\lambda)+1}\frac{(p^{-1}tu_i/qz;p)_\infty}{(u_i/z;p)_\infty}:\widetilde{\Phi}_\lambda(p^{-1}u)^{-1}\widetilde{x}^-((t/q)^{1/4}z):.
\end{aligned}
$$

Noting

$$
\begin{aligned}
x^-((t/q)^{1/4}z)\Big|_{\mathcal{F}^{(1,N+1)}_{-p^{-1}uv}} &= (-p^{-1}uv)^{-1}z^{N+1}(t/q)^{-(N+1)/2}\widetilde{x}^-((t/q)^{1/4}z),\\
x^-((t/q)^{1/4}z)\Big|_{\mathcal{F}^{(1,N)}_{v}} &= v^{-1}z^{N}(t/q)^{-N/2}\widetilde{x}^-((t/q)^{1/4}z),
\end{aligned}
$$

one gets

$$
\begin{aligned}
&x^-((t/q)^{1/4}z)\Big|_{\mathcal{F}^{(1,N+1)}_{-p^{-1}uv}}\widetilde{\Phi}_\lambda(p^{-1}u)^{-1}-(t/q)^{-1/2}B^+_\lambda(p^{-1}u/z,p)\widetilde{\Phi}_\lambda(p^{-1}u)^{-1}x^-((t/q)^{1/4}z)\Big|_{\mathcal{F}^{(1,N)}_{v}}\\
&= -(uv)^{-1}pz^{N+1}(t/q)^{-(N+1)/2}\\
&\quad\times\left(\prod_{i=1}^{l(\lambda)}\frac{(u_i/tz;p)_\infty}{(p^{-1}u_i/qz;p)_\infty}\prod_{i=1}^{l(\lambda)+1}\frac{(p^{-1}tu_i/qz;p)_\infty}{(u_i/z;p)_\infty}\Big|_{|u/z|<1}\right.\\
&\quad\left.-\prod_{i=1}^{l(\lambda)}\frac{(u_i/tz;p)_\infty}{(p^{-1}u_i/qz;p)_\infty}\prod_{i=1}^{l(\lambda)+1}\frac{(p^{-1}tu_i/qz;p)_\infty}{(u_i/z;p)_\infty}\Big|_{|z/u|<1}\right)\\
&\quad\times:\widetilde{\Phi}_\lambda(p^{-1}u)^{-1}\widetilde{x}^-((t/q)^{1/4}z):\\
&= -(uv)^{-1}pz^{N+1}(t/q)^{-(N+1)/2}\\
&\quad\times\left((1-p^{-1}tu_{\ell+1}/qz)\prod_{i=1}^{l(\lambda)}\frac{(1-p^{-1}tu_i/qz)}{(1-p^{-1}u_i/qz)}\Big|_{|u_i/z|<1}\right.\\
&\quad\left.-(1-p^{-1}tu_{\ell+1}/qz)\prod_{i=1}^{l(\lambda)}\frac{(1-p^{-1}tu_i/qz)}{(1-p^{-1}u_i/qz)}\Big|_{|z/u_i|<1}\right)\\
&\quad\times\prod_{i=1}^{l(\lambda)}\frac{(u_i/tz;p)_\infty}{(u_i/qz;p)_\infty}\prod_{i=1}^{l(\lambda)+1}\frac{(tu_i/qz;p)_\infty}{(u_i/z;p)_\infty}:\widetilde{\Phi}_\lambda(p^{-1}u)^{-1}\widetilde{x}^-((t/q)^{1/4}z):.
\end{aligned}
\tag{E.3}
$$

Applying the formula (D.3) and using

$$
:\widetilde{\Phi}_\lambda(p^{-1}u)^{-1}\widetilde{x}^-((t/q)^{1/4}z):\Big|_{z=p^{-1}u_{k,\lambda_k}}=\widetilde{\Phi}_{\lambda-\mathbf{1}_k}(p^{-1}u)^{-1},
$$

the RHS of (E.3) is

$$
\begin{aligned}
&-(uv)^{-1}pz^{N+1}(t/q)^{-(N+1)/2}ta^-(p)\\
&\quad\times\sum_{k=1}^{\ell(\lambda)}\delta(pqz/u_k)\prod_{\substack{i=1\\ \neq k}}^{\ell(\lambda)}\frac{(pqu_i/tu_k;p)_\infty}{(u_i/u_k;p)_\infty}\prod_{\substack{i=1\\ \neq k}}^{l(\lambda)+1}\frac{(tu_i/u_k;p)_\infty}{(pqu_i/u_k;p)_\infty}\times\widetilde{\Phi}_{\lambda-\mathbf{1}_k}(p^{-1}u)^{-1}
\end{aligned}
$$

$$= -(uv)^{-1} p z^{N+1} (t/q)^{-(N+1)/2}$$
$$\times \sum_{k=1}^{\ell(\lambda)} \delta(pqz/u_k) q^{-\lambda_k} a^-(p) \frac{c'_\lambda N'_{\lambda-\mathbf{1}_k}(p)}{c'_{\lambda-\mathbf{1}_k} N'_\lambda(p)} A^{-\,'}_{\lambda,k}(p) \widetilde{\Phi}_{\lambda-\mathbf{1}_k}(p^{-1}u)^{-1}$$
$$= \frac{c'_\lambda}{q^{n(\lambda')} t(\lambda, v, p^{-1}u, N) N'_\lambda(p)} (t/q) a^-(p)$$
$$\times \sum_{k=1}^{\ell(\lambda)} \delta(pqz/u_k) A^{-\,'}_{\lambda,k}(p) \frac{q^{n((\lambda-\mathbf{1}_k)')} t(\lambda+\mathbf{1}_k, v, p^{-1}u, N) N'_{\lambda-\mathbf{1}_k}(p)}{c'_{\lambda-\mathbf{1}_k}} \widetilde{\Phi}_{\lambda-\mathbf{1}_k}(p^{-1}u)^{-1}.$$

□

F Useful formulas for the Nekrasov function

For readers' convenience, we list some useful formulas for the Nekrasov function $N_{\lambda\mu}(x)$. Let $\lambda, \mu \in \mathcal{P}^+$. Then for any integer $\ell \geq \ell(\lambda), \ell(\mu)$, one has the following formulas [4, 5]:

$$N_{\lambda\mu}(x) = \prod_{\square\in\lambda} (1 - xq^{-a_\mu(\square)-1} t^{-\ell_\lambda(\square)}) \prod_{\blacksquare\in\mu} (1 - xq^{a_\lambda(\blacksquare)} t^{\ell_\mu(\blacksquare)+1})$$
$$= \prod_{1\leq i\leq j\leq\ell} \frac{(xq^{-\mu_i+\lambda_{j+1}} t^{i-j}; q)_\infty}{(xq^{-\mu_i+\lambda_j} t^{i-j}; q)_\infty} \prod_{1\leq r\leq s\leq\ell} \frac{(xq^{\lambda_r-\mu_s} t^{-r+s+1}; q)_\infty}{(xq^{\lambda_r-\mu_{s+1}} t^{-r+s+1}; q)_\infty}$$
$$= \prod_{1\leq i\leq j\leq\ell} (xq^{-\mu_i+\lambda_{j+1}} t^{i-j}; q)_{\lambda_j-\lambda_{j+1}} \prod_{1\leq r\leq s\leq\ell} (xq^{\lambda_r-\mu_s} t^{-r+s+1}; q)_{\mu_s-\mu_{s+1}}$$
$$= \prod_{i,j=1}^{\ell} \frac{(xq^{\lambda_i-\mu_j} t^{-i+j+1}; q)_\infty}{(xq^{\lambda_i-\mu_j} t^{-i+j}; q)_\infty} \prod_{r=1}^{\ell} \frac{(xq^{-\mu_r} t^{-\ell+r}; q)_\infty}{(xq^{\lambda_r} t^{-r+\ell+1}; q)_\infty},$$

where

$$(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}).$$

The second equality follows from the formula

$$(1-q) \sum_{(i,j)\in\lambda} q^{\mu_i-j} t^{\lambda'_j-i+1} = t \sum_{1\leq i\leq\ell(\lambda)} q^{\mu_i-\lambda_i} t^{j-i} - \sum_{1\leq i<\ell(\lambda)+1} q^{\mu_i-\lambda_j} t^{j-i}, \tag{F.1}$$

which is derived by using [60]

$$(1-q) \sum_{j=1}^{\lambda_i} q^{-j} t^{\lambda'_j} = \sum_{j=i}^{\ell(\lambda)} q^{-\lambda_j} t^j (1 - q^{\lambda_j-\lambda_{j+1}}). \tag{F.2}$$

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