A short introduction to the Form Factor Program for $\mathrm{T\bar{T}}$- deformed Massive Integrable Quantum Field Theories

10th Workshop on Conformal Field Theory and Integrable Models Bologna-05/09/23

Based on **arXiv:2306.11064** and **arXiv:2306.01640** with
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Introduction

- Irrelevant perturbations
- $\mathrm{T}\bar{\mathrm{T}}$ - deformations
- Some useful results
- The Form Factor Program
- Correlation Functions

FFs in $\mathrm{T}\bar{\mathrm{T}}$ - deformed IQFTs

- The Minimal Form Factor
- The General Solution
- The Cumulant Expansion

The Ising Model

- Some General Considerations
- c-theorem and Δ-sum rule

Conclusions

- Summing up
- Outlook

$$\mathcal{A} = \left[\mathcal{A}_{CFT} + \mu \int \Phi_\Delta(z) d^2 z\right] + \boxed{\sum_i \alpha_i \int \mathcal{O}_{\delta_i}(z) d^2 z} + \text{marginal terms}$$

Complete UV theory

- $d = 2\Delta < 2$

Shatters UV completeness

- $d_i = 2\delta_i > 2$

- **The theory is effective**
- **Perturbative expansion in α causes the growth of divergencies, the theory is not renormalizable**

$$\frac{d}{d\alpha}\mathcal{A}_\alpha = \int (\mathrm{T\bar{T}})_{(\alpha)}\, d^2 z$$

- The theory is solvable
- Preservation of symmetries
- Universality
- Its generalizations span the entire space of IQFTs

The studies on these deformations cover a vast area of research: from integrable field theories (Smirnov & Zamolodchikov'16;Cavagliá, Negro, Szécsényi & Tateo'16;...), spin chains (Bargheer, Beisert, Loebbert'09; Poszgay, Jiang, Takács'19;...), generalized hydrodynamics (Cardy & Doyon'20; Medenjak, Policastro & Yoshimura'20;...) to string theory (Giveon, Itzhaki, Kutasov'17;...), Ads/CFT correspondence (McGough, Mezei, Verlinde'16;...), Quantum Gravity (JT gravity) (Dubovsky, Gorbenko, Hernandez-Chifflet'18;...), confining/effective string (Caselle, Fioravanti, Gliozzi & Tateo'13; Chen, Dubovsky, Hernandez-Chifflet'18;...) and more...

Focusing on massive integrable quantum field theories our work is mostly based on the following already established results:

1. Deforming an integrable field theory results in a modification of its S-matrix by a **CDD-factor**
(Smirnov & Zamolodchikov'16);

$$S_{\boldsymbol{\alpha}}(\theta) = \Phi_{\boldsymbol{\alpha}}(\theta) S_{\mathbf{0}}(\theta) \qquad \Phi_{\boldsymbol{\alpha}}(\theta) = \exp\left[-i \sum_{s \in \mathcal{S}} \alpha_s m^{2s} \sinh(s\theta)\right]$$

$$\{\boldsymbol{\alpha} = (\alpha_s)_{s \in \mathcal{S} \subset \mathbb{N}}\}$$

2. Once the S-matrix is known, the Bootstrap Program can be carried out.
In particular, a TBA/NLIE analysis (Cavagliá, Negro, Szécsényi & Tateo'16; Camilo, Fleury, Léncses, Negro & Zamolodchikov'21) shows intriguing non-Wilsonian behaviours (Hagedorn spectrum, non-locality) depending on the sign of the deformation;

3. From a GHD perspective (Cardy & Doyon'20) the deformation can be interpreted as a situation in which the fundamental particles in the system acquire a finite length for one sign and a negative length (i.e. free space is added) for the other.

Introduction: The Form Factor Program

Form factors are tensor valued functions, representing matrix elements of some local or semi-local operator located at the origin between a multi-particle *in*-state and the vacuum.

$$F_n^{\mathcal{O}}(\theta_1,\dots,\theta_n;\boldsymbol{\alpha}) = \langle 0|\mathcal{O}(0)|\theta_1,\dots,\theta_n|0\rangle$$
$$= \langle 0|\mathcal{O}(0)|Z^\dagger(\theta_1),\dots,Z^\dagger(\theta_n)|0\rangle$$

ZF-ALGEBRA

$$Z^\dagger(\theta_1)Z^\dagger(\theta_2) = S(\theta_{12})Z^\dagger(\theta_2)Z^\dagger(\theta_1)$$

(Karowski & Weisz'78, Smirnov'90;...)

WE ARE CONSIDERING ONLY DIAGONAL THEORIES WITHOUT BOUND STATES!

BRAIDING:

$$F_n^{\mathcal{O}}(\theta_1,\dots,\theta_i,\theta_{i+1},\dots,\theta_n;\boldsymbol{\alpha}) = S_{\boldsymbol{\alpha}}(\theta_i-\theta_{i+1})F_n^{\mathcal{O}}(\theta_1,\dots,\theta_{i+1},\theta_i,\dots,\theta_n;\boldsymbol{\alpha})$$

MONODROMY:

$$F_n^{\mathcal{O}}(\theta_1+2\pi i,\theta_2,\dots,\theta_n;\boldsymbol{\alpha}) = \gamma_{\mathcal{O}}F_n^{\mathcal{O}}(\theta_2,\dots,\theta_n,\theta_1;\boldsymbol{\alpha})$$

$\gamma_{\mathcal{O}}$ is the factor of **local commutativity**
(Yurov & Zamolodchikov'91)

KINEMATICAL RESIDUE EQUATION:

$$\lim_{\bar\theta\to\theta}(\bar\theta-\theta)F_{n+2}^{\mathcal{O}}(\bar\theta+i\pi,\theta,\theta_1,\dots,\theta_n;\boldsymbol{\alpha}) = i\Big(1-\gamma_{\mathcal{O}}\prod_{j=1}^n S_{\boldsymbol{\alpha}}(\theta-\theta_j)\Big)F_n^{\mathcal{O}}(\theta_1,\dots,\theta_n;\boldsymbol{\alpha})$$

ASYMPTOTIC BOUND:

$$\lim_{|\theta_i|\to\infty} F_n^{\mathcal{O}}(\theta_1,\dots,\theta_i,\dots,\theta_n;0) \sim e^{y_{\mathcal{O}}|\theta_i|}\ , \text{ with } y_{\mathcal{O}}\le\Delta_{\mathcal{O}}$$

(Delfino & Mussardo'95)

Form factors are building blocks for correlation functions. Let us consider a two-point correlation function

$$\langle 0|\mathcal{O}_1(0)\mathcal{O}_2(r)|0\rangle$$

inserting an orthogonal projector between the two operators and using relativistic invariance

$$I := \sum_{n=o}^{\infty} \int_{-\infty}^{\infty} \frac{d\theta_1 \ldots d\theta_n}{n!(2\pi)^n} |\theta_n, \ldots, \theta_1|0\rangle\langle 0|\theta_1 \ldots, \theta_n| \qquad \langle 0|\mathcal{O}(\mathrm{x})|\theta_1, \ldots, \theta_n|0\rangle = \left(\prod_{j=1}^{n} e^{ip^{\nu}(\theta_j)k_{\nu}}\right) F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n; \boldsymbol{\alpha})$$

we arrive to an expression **in terms of *n*-particles form factors**

$$\langle 0|\mathcal{O}_1(0)\mathcal{O}_2(r)|0\rangle = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{d\theta_1 \ldots d\theta_n}{n!(2\pi)^n} F_n^{\mathcal{O}_1}(\theta_1, \ldots, \theta_n; \boldsymbol{\alpha}) \times (F_n^{\mathcal{O}_2^\dagger}(\theta_1, \ldots, \theta_n; \boldsymbol{\alpha}))^* e^{-r\sum_{j=1}^{n} m_j \cosh\theta_j}$$

$$\langle 0|\theta_n, \ldots, \theta_1|\mathcal{O}(0)|0\rangle = \left(F_n^{\mathcal{O}^\dagger}(\theta_1, \ldots, \theta_n; \boldsymbol{\alpha})\right)^*$$

This is a rapidly convergent expansion at large distance (mr≫1) but it can also be used to probe the UV scale. In addition, features of the underlying CFT can be tested by employing FFs of fields in the massive QFTs.

For n=2 Watson's equations we end up with a set of decoupled equations with the S-matrix eigenvalues (Karowski & Weisz'78):

$$F_2^{\mathcal{O}}(\theta_{12};\boldsymbol{\alpha}) = S_{\boldsymbol{\alpha}}(\theta_{12})F_2^{\mathcal{O}}(-\theta_{12};\boldsymbol{\alpha}) \qquad F_2^{\mathcal{O}}(i\pi+\theta_{12};\boldsymbol{\alpha}) = F_2^{\mathcal{O}}(i\pi-\theta_{12};\boldsymbol{\alpha})$$

Theorem: If $F_2^{\mathcal{O}}(\theta)$ is meromorphic in the physical strip $0 \leqslant \mathrm{Im}\theta \leqslant \pi$ with possible poles (or zeros) only on the immaginary axis and $F_2^{\mathcal{O}} = O(\exp\exp|\theta|)$ for $|\mathrm{Re}\theta| \to +\infty$, the solutions of the equations above are uniquely determined by the poles at $\theta = i\alpha_k$ (and zeros) up to a normalization constant. They can be written

$$F_2^{\mathcal{O}}(\theta_{12};\boldsymbol{\alpha}) = K_2^{\mathcal{O}}(\theta_{12};\boldsymbol{\alpha})F_{\mathrm{min}}^{\mathcal{O}}(\theta_{12};\boldsymbol{\alpha})$$

Minimal solution with no poles (or zeros) in the physical strip

$$K(\theta) = K(-\theta) = K(2\pi i+\theta)$$

The first step in finding a solution for the Watson's equations is the identification of the *minimal Form Factor*.

$$F_{\mathrm{min}}(\theta;\boldsymbol{\alpha}) = S_{\boldsymbol{\alpha}}(\theta)F_{\mathrm{min}}(-\theta;\boldsymbol{\alpha}) = F_{\mathrm{min}}(2\pi i-\theta;\boldsymbol{\alpha})$$

$$F_{\mathrm{min}}(\theta;0) = S_0(\theta)F_{\mathrm{min}}(-\theta;0) = F_{\mathrm{min}}(2\pi i-\theta;0) \longrightarrow \varphi(\theta;\boldsymbol{\alpha}) = \Phi_{\boldsymbol{\alpha}}(\theta)\varphi(-\theta;\boldsymbol{\alpha}) = \varphi(2\pi i-\theta;\boldsymbol{\alpha})$$

We found out that the most general minimal form factor for a generalized $\mathrm{T\bar{T}}$-deformation is (us;23):

$$F_{\min}(\theta;\boldsymbol{\alpha}) := \varphi(\theta;\boldsymbol{\alpha})C(\theta;\boldsymbol{\beta})F_{\min}(\theta;\mathbf{0})$$

Minimal solution of the undeformed theory

$$\varphi(\theta;\boldsymbol{\alpha}) := \exp\left[\frac{\theta - i\pi}{2\pi}\sum_{s\in\mathcal{S}}\alpha_s\sinh(s\theta)\right]$$

"Minimal" solution of the equation for the minimal CCD-Form Factor

from now on

$m = 1$

$$C(\theta;\boldsymbol{\beta}) := \exp\left(\sum_{s\in\mathcal{S}'}\beta_s\cosh(s\theta)\right)$$

"CDD-factor" of the minimal solution, i.e. a function that can always be added to any solution

Turns out that a general solution can be directly worked out imposing **factorization**. Following the structure already seen in the two-particle case

$$F_n^{\mathcal{O}}(\{\theta_i\}_n;\boldsymbol{\alpha}) = F_n^{\mathcal{O}}(\{\theta_i\}_n;\mathbf{0})\Upsilon_n^{\mathcal{O}}(\{\theta_i\}_n;\boldsymbol{\alpha})$$

$$\Upsilon_n^{\mathcal{O}}(\{\theta_i\}_n;\boldsymbol{\alpha}) = \Theta_n^{\mathcal{O}}(\{\theta_i\}_n;\boldsymbol{\alpha})\prod_{i<j}^{n}\varphi(\theta_{ij};\boldsymbol{\alpha})$$

WITH
$$\Upsilon_n^{\mathcal{O}}(\{\theta_i\}_n;\mathbf{0}) = 1$$

Our solution for the Ising field theory: while the order/disorder operators (semi-local fields) follow exactly the general solution, for the trace of the stress-energy tensor (a local field) a multiplicative piece should be considered to ensure the right *undeformed*-limit.

$$F_{2n}^{\mu}(\theta_1,\ldots,\theta_{2n};\boldsymbol{\alpha}) = i^n\langle\mu\rangle_{\boldsymbol{\alpha}}\sqrt{\prod_{i=1}^{2n}\cos\left(\sum_{s\in\mathcal{S}}\frac{\alpha_s}{2}\sum_{j=1}^{2n}\sinh(s\theta_{ij})\right)}\prod_{i<j}\tanh\frac{\theta_{ij}}{2}\varphi(\theta_{ij};\boldsymbol{\alpha})$$

$$F_{2n}^{\Theta}(\theta_1,\ldots,\theta_{2n};\boldsymbol{\alpha}) = 2\pi i\, i^n\frac{\sqrt{\prod\limits_{i=1}^{2n}\sin\left(\sum\limits_{s\in\mathcal{S}}\frac{\alpha_s}{2}\sum\limits_{j=1}^{2n}\sinh(s\theta_{ij})\right)}}{\left|\sum\limits_{s\in\mathcal{S}}\frac{\alpha_s}{4}\sqrt{\left(\frac{\sigma_{1,s}^{(2n)}\sigma_{2n-1,s}^{(2n)}}{\sigma_{2n,s}^{(2n)}}-4\right)\frac{\sigma_{2n}^{(2n)}\sigma_{1,s}^{(2n)}\sigma_{2n-1,s}^{(2n)}}{\sigma_{2n,s}^{(2n)}\sigma_{1}^{(2n)}\sigma_{2n-1}^{(2n)}}}\right|}\prod_{i<j}\tanh\frac{\theta_{ij}}{2}\varphi(\theta_{ij};\boldsymbol{\alpha})$$

A powerful tool to study the asymptotics of the correlation functions is their cumulant expansion (Smirnov'90). For fields with non-vanishing VEV we can write

$$\log \frac{\langle \mathcal{O}(0)\mathcal{O}(r)\rangle_{\boldsymbol{\alpha}}}{\langle \mathcal{O}\rangle_{\boldsymbol{\alpha}}^2} = \sum_{\ell=1}^{\infty} c_\ell^{\mathcal{O}}(r;\boldsymbol{\alpha})$$

where the cumulants are

$$c_\ell^{\mathcal{O}}(r;\boldsymbol{\alpha}) := \frac{1}{\ell!}\int_{-\infty}^{\infty}\frac{d\theta_1}{2\pi}\cdots\int_{-\infty}^{\infty}\frac{d\theta_\ell}{2\pi}\, e^{-r\sum_{i=1}^{\ell}\cosh\theta_i}\, h_\ell^{\mathcal{O}}(\theta_1,\ldots,\theta_\ell;\boldsymbol{\alpha})$$
$$= \frac{1}{\pi\ell!}\int_{-\infty}^{\infty}\frac{d\theta_1}{2\pi}\cdots\int_{-\infty}^{\infty}\frac{d\theta_{\ell-1}}{2\pi}\, K_0(r d_\ell)\, h_\ell^{\mathcal{O}}(\theta_1,\ldots,\theta_{\ell-1},0;\boldsymbol{\alpha})$$

Modified Bessel Function (K_0)

$$d_\ell = \sqrt{\left(\sum_{j=1}^{\ell-1}\cosh\theta_i + 1\right)^2 - \left(\sum_{j=1}^{\ell-1}\sinh\theta_i\right)^2}$$

For instance

$$\langle\mu\rangle_{\boldsymbol{\alpha}}^2 h_2^{\mu}(\theta_1,\theta_2;\boldsymbol{\alpha}) := |F_2^{\mu}(\theta_1,\theta_2;\boldsymbol{\alpha})|^2,$$

$$\langle\mu\rangle_{\boldsymbol{\alpha}}^4 h_4^{\mu}(\theta_1,\theta_2,\theta_3,\theta_4;\boldsymbol{\alpha}) := \langle\mu\rangle_{\boldsymbol{\alpha}}^2 |F_4^{\mu}(\theta_1,\theta_2,\theta_3,\theta_4;\boldsymbol{\alpha})|^2 - |F_2^{\mu}(\theta_1,\theta_2;\boldsymbol{\alpha})|^2 |F_2^{\mu}(\theta_3,\theta_4;\boldsymbol{\alpha})|^2, \ldots$$

Let us consider the *two*-particle contribution of the expansion for the **disorder field** and for the **trace of the stress energy tensor**. From a first look is already clear that the general behaviour is dictated by the exponential factor; in particular by the sign of the deformation parameter **α**.

$$c_2^{\mu}(r;\boldsymbol{\alpha}) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dx K_0(2r\cosh\tfrac{x}{2})\cos^2\left(\sum_{s\in\mathcal{S}}\tfrac{\alpha_s}{2}\sinh(sx)\right)\prod_{s\in\mathcal{S}} e^{\frac{\alpha_s x}{\pi}\sinh(sx)}\tanh^2\tfrac{x}{2}$$

$$c_2^{\Theta}(r;\boldsymbol{\alpha}) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dx\, K_0(2r\cosh\tfrac{x}{2})\left[\frac{\sin\left(\sum_{s\in\mathcal{S}}\frac{\alpha_s}{2}\sinh(sx)\right)}{\sum_{s\in\mathcal{S}}\frac{\alpha_s}{2}\sinh(sx)}\right]^2\prod_{s\in\mathcal{S}} e^{\frac{\alpha_s x}{\pi}\sinh(sx)}\sinh^2\tfrac{x}{2}$$

For simplicity we focus on a pure $\mathrm{T\bar{T}}$- deformation (only one deformation parameter) to study the asymptotic behaviour

$$c_2^{\mu}(r \gg 1;\alpha>0) \approx \frac{1}{4\pi^2}\sqrt{\frac{\pi}{2r}}\int_{-\infty}^{\infty}\frac{dx}{\sqrt{\cosh\frac{x}{2}}}\cos^2\left(\tfrac{\alpha}{2}\sinh x\right)e^{-2r\cosh\frac{x}{2}+\frac{\alpha x}{\pi}\sinh x}\tanh^2\tfrac{x}{2}$$

For α>0 the cumulant expansion is strongly divergent. In the IR limit a characteristic cut-off at large momenta can be estimated

$$\Lambda = 2W_0(\pi r/\alpha)$$

Lambert W-function

For $\alpha<0$ the cumulant expansion is rapidly convergent. Both UV and IR regimes can be probed without any issue

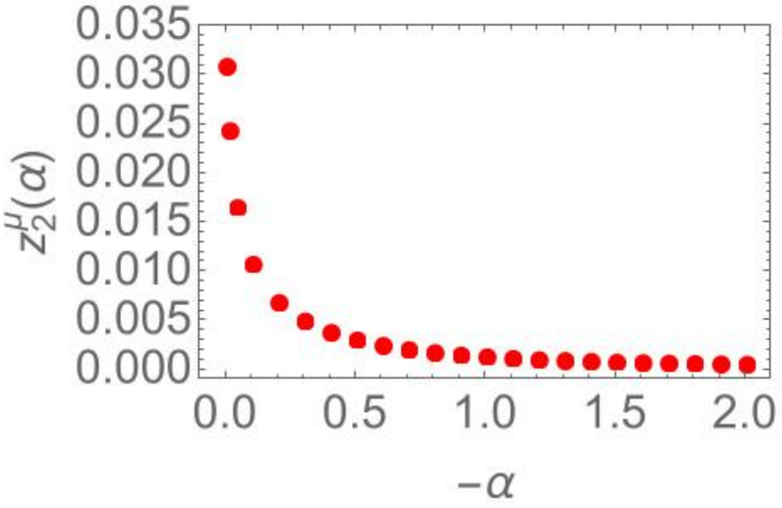
$$c_2^\mu(r \ll 1; \alpha < 0) \approx -\frac{\log r}{4\pi^2} \int_{-\infty}^{\infty} dx \cos^2\left(\frac{\alpha}{2}\sinh x\right) e^{\frac{\alpha x}{\pi}\sinh x} \tanh^2 \frac{x}{2} := -4 z_2^\mu(\alpha) \log r$$

We can expand the Bessel function for small distances to investigate the scaling properties of the correlation functions. **We found that the correlators seems to be still following a power-law but that now depends on the deformation parameter**

$$\langle \mathcal{O}(0)\mathcal{O}(r)\rangle \sim r^{-4z^{\mathcal{O}}(\alpha)}$$

$$z^{\mathcal{O}}(\alpha) < \Delta^{\mathcal{O}}$$

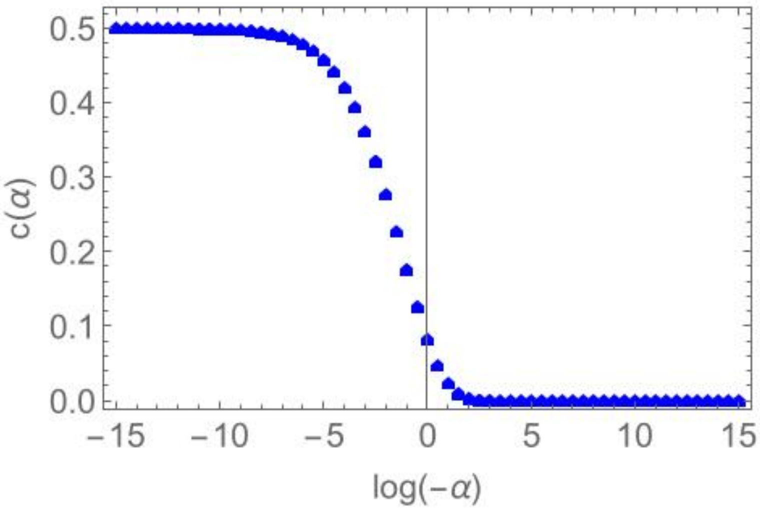
UV dimension of the field in the unperturbed theory



(A.B. Zamolodchikov'86)

$$c^{UV} - c^{IR} = \frac{3}{2}\int_0^\infty dr\, r^3 \langle \Theta(0)\Theta(r)\rangle_c$$

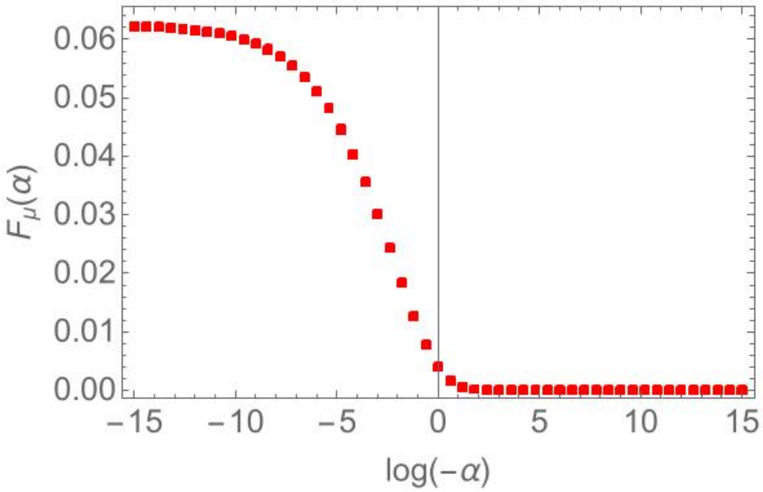
$$c(\alpha) := c^{UV} - c^{IR} \approx \frac{3}{8}\int_{-\infty}^{+\infty} dx \left[\frac{\sin(\frac{\alpha}{2}\sinh x)}{\frac{\alpha}{2}\sinh x}\right]^2 \frac{\tanh^2 \frac{x}{2}}{\cosh^2 \frac{x}{2}} e^{\frac{\alpha}{\pi} x \sinh x}$$



(Delfino, Simonetti & Cardy'96)

$$\Delta_{\mathcal{O}}^{UV} - \Delta_{\mathcal{O}}^{IR} = -\frac{1}{4\pi\langle\mathcal{O}\rangle}\int_0^\infty dr\, r \langle \Theta(0)\mathcal{O}(r)\rangle_c$$

$$F_\mu(\alpha) := \Delta_\mu^{UV} - \Delta_\mu^{IR} = \frac{1}{16\pi}\int_{-\infty}^{+\infty} dx \left|\frac{\sin(\alpha\sinh x)}{\alpha\sinh x}\right| \frac{\tanh^2 \frac{x}{2}}{\cosh \frac{x}{2}} e^{\frac{\alpha}{\pi} x \sinh x}$$



Summing up:

- We developed a systematic programme for the computation of Form Factors of $\mathrm{T}\bar{\mathrm{T}}$-deformed IQFTs employing standard techniques;

- The analysis on the correlations functions seems to be in agreement with already established results, especially with the picture proposed in [Cardy & Doyon'20] :

    - For **α**>0 we have a "UV regime" characterized by particles of finite length; therefore the short distance cannot be probed, the two-point function diverges;
    - For **α**<0 the particles have "negative length". We have a rapidly convergent spectral expansion that is in a way similar to a critical system but clearly doesn't flow to a fixed point.

Outlook:

- The study of these correlation functions and their asymptotic is just at its start; we want to deepen our understanding, especially about the UV regime.
- We already started the study of the entanglement properties of these theories (see arXiv:2306.11064 by us and also ArXiv:2306.07784 by Hou, He & Jiang) and as in the previous point there is still a lot more to understand.
- Obvious extension to more involved models
- IQFTs whose S-matrices are pure CDD-factors: there should be a relation between our results and the standard ones (see the next talk by Olalla!)

Thank You

We can now proceed imposing a general Ansatz for the kinematical residue equations

$$F_n^{\mathcal{O}}(\{\theta_i\}_n;\boldsymbol{\alpha}) = H_n^{\boldsymbol{\alpha}} Q_n^{\mathcal{O}}(\{\theta_i\}_n;\boldsymbol{\alpha}) \prod_{i<j}^{n} \frac{F_{\text{min}}(\theta_{ij};\boldsymbol{\alpha})}{e^{\theta_i}+e^{\theta_j}}$$

Normalization constants

Symmetric and 2πi-periodic polynomials in all rapidities

$$x_j := e^{\theta_j}; \quad \begin{cases} \sigma_j^{(n)}(x_1,\ldots,x_n) = \sum_{1\leq i_1<i_2<\cdots<i_j\leq n} x_{i_1}x_{i_2}\cdots x_{i_j} \\ \sigma_0^{(n)}(x_1,\ldots,x_n) = 1 \end{cases}$$

Encodes the pole structure

Form factors posses kinematic poles when the rapidities of conjugate particles differ by iπ. By using the kinematical residue equation, starting from the two-particle one, we can recursively generate higher particle ones.

FFs in $\mathrm{T}\bar{\mathrm{T}}$-deformed IQFTs: The minimal Form Factor

Turns out that a general solution can be directly worked out imposing **factorization**. Following the structure already seen in the two-particle case

$$F_n^{\mathcal{O}}(\{\theta_i\}_n;\boldsymbol{\alpha}) = F_n^{\mathcal{O}}(\{\theta_i\}_n;\mathbf{0})\Upsilon_n^{\mathcal{O}}(\{\theta_i\}_n;\boldsymbol{\alpha})$$

$$\Upsilon_n^{\mathcal{O}}(\{\theta_i\}_n;\boldsymbol{\alpha}) = \Theta_n^{\mathcal{O}}(\{\theta_i\}_n;\boldsymbol{\alpha})\prod_{i<j}^{n}\varphi(\theta_{ij};\boldsymbol{\alpha})$$

WITH

$$\Upsilon_n^{\mathcal{O}}(\{\theta_i\}_n;\mathbf{0}) = 1$$

The most general solution reads

$$\Theta_n^{\mathcal{O}}(\{\theta_i\}_n;\boldsymbol{\alpha}) = \prod_{i=1}^{n}\sqrt{\frac{\prod\limits_{j=1}^{n} S_{\boldsymbol{\alpha}}(\theta_{ij})^{1/2} - \gamma_{\mathcal{O}}\prod\limits_{j=1}^{n} S_{\boldsymbol{\alpha}}(\theta_{ij})^{-1/2}}{\prod\limits_{j=1}^{n} S_{\mathbf{0}}(\theta_{ij})^{1/2} - \gamma_{\mathcal{O}}\prod\limits_{j=1}^{n} S_{\mathbf{0}}(\theta_{ij})^{-1/2}}} = \prod_{i=1}^{n}\sqrt{\frac{\sin\left(\frac{1}{2}\sum\limits_{j=1}^{n}[\delta(\theta_{ij}) - i\log\Phi_{\boldsymbol{\alpha}}(\theta_{ij})] - \frac{\omega_{\mathcal{O}}}{2}\right)}{\sin\left(\frac{1}{2}\sum\limits_{j=1}^{n}\delta(\theta_{ij}) - \frac{\omega_{\mathcal{O}}}{2}\right)}}$$

$S_{\mathbf{0}}(\theta) = e^{i\delta(\theta)}$

Extra care should be given in the case of local fields, where additional non trivial multiplicative factors can appear. The correct form should always reproduce the original FF in the undeformed limit !

$\gamma_{\mathcal{O}} = e^{i\omega_{\mathcal{O}}}$