GGEs, modular transforms and defects

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Motivation: desire to understand characters of W-algebras.

Simplest case: W_3 algebra.

Generators L_m, W_m

Commuting modes $[L_0, W_0] = 0$

"Standard" character $\chi_i = \mathrm{Tr}_i\left(q^{L_0 - c/24}\right)$

$q = e^{2\pi i \tau}$: Under $\tau \to -\frac{1}{\tau}$ $\quad \chi_i\left(-\frac{1}{\tau}\right) = \sum S_{ij}\, \chi_j(\tau)$

Consider instead: $\mathrm{Tr}_i\left(q^{L_0 - c/24}\, z^{W_0}\right) \qquad q = e^{2\pi i \tau}, \quad z = e^{\alpha}$

Q: does this have nice properties?

$$\mathrm{Tr}\left(q^{L_0 - c/24}\, W_0^{\,m}\right) \quad \text{all have "nice" properties}$$

[Gaberdiel, Iles + GW]

But: unable to see any pattern.

L_0, W_0 are first in infinite set of conserved charges of quantum Boussinesq hierarchy

- Is there a simpler model to look at?

Consider instead KdV hierarchy (conserved charges of $\varphi_{1,3}$ perturbation)

Infinite set of charges $I_1, I_3, \ldots$

Simplest case: a single free fermion.

Lowest charge:

$$I_1 = \underbrace{\int_0^1 T(w)\, dw}_{\text{Torus}} = \underbrace{L_0 - c/24}_{\text{after map to plane}} = \sum_{k>0} k\, \psi_{-k} \psi_k + \underbrace{\begin{cases} -1/48 & NS. \\ +\frac{1}{24} & R. \end{cases}}_{c_1^{(NS/R)}}$$

$$\mathrm{Tr}\left(q^{I_1}\right) = q^{h - c/24} \prod_k \underbrace{\left(1 + q^k\right)}_{1 \text{ if mode } \psi_{-k} \text{ absent; } q^k \text{ if present.}}$$

Well known nice modular properties

4 sets of functions:

$$\chi_{NS} = \mathrm{Tr}_{NS}\left(q^{L_0 - c/24} \right) = q^{-1/48} \prod_{k=1/2,\,3/2,\,\dots} (1+q^k)$$

$$\chi_{\widetilde{NS}} = \mathrm{Tr}_{NS}\left((-1)^F q^{L_0 - c/24} \right) = q^{-1/48} \prod_{k=1/2,\,3/2,\,\dots} (1-q^k)$$

$$\chi_{R} = \mathrm{Tr}_{R}\left(q^{L_0 - c/24} \right) = q^{1/24} \prod_{k=1,2,\dots} (1+q^k)$$

$$\chi_{\widetilde{R}} = \mathrm{Tr}_{R}\left((-1)^F q^{L_0 - c/24} \right) = 0$$

Modular group

$$\tau \to \left(\frac{a\tau+b}{c\tau+d}\right)$$

generated by

$\tau \to \tau+1$ $\qquad$ $\tau \to -1/\tau$

$NS \leftrightarrow \widetilde{NS}$ $\qquad$ $NS \leftrightarrow NS$

$R \leftrightarrow R$ $\qquad$ $\widetilde{NS} \leftrightarrow R$

subgroup $\Gamma(2)$

b, c even

generated by

$\tau \to \tau+2$ $\qquad$ $\tau \to \frac{1}{2\tau+1}$

Sectors are invariant

Next charge: $\int_0^1 (T\,T)\,dw$

Simpler with:

$$I_3 = \frac{6}{7}\int_0^1 (T\,T)\,dw \qquad \text{(torus)}$$

$$= \frac{6}{7}\left(\Lambda_0 - \frac{49}{120}L_0 + \frac{49}{11520}\right) \qquad \text{(plane)}$$

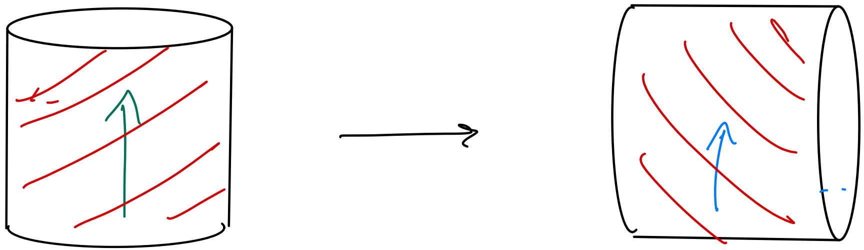
$$= \sum_{k>0} k^3\,\psi_{-k}\psi_k + c_3^{(NS/R)} \qquad c_3 = \begin{cases} \frac{7}{1920} & NS \\ -\frac{1}{240} & R \end{cases}$$

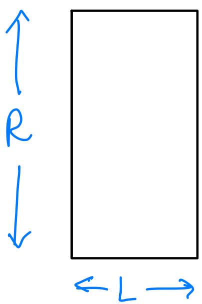
$-\frac{1}{2}\zeta(-3)$

$$\mathrm{Tr}\left(q^{I_1} z^{I_3}\right) = q^{c_1} z^{c_3} \prod_{k>0}\left(1 + q^k z^{k^3}\right)$$

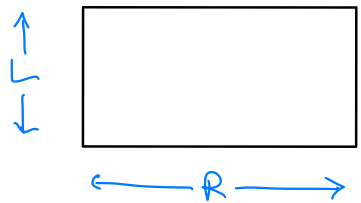
Qn: does this have nice modular properties?

$$H = \frac{2\pi}{L}\left(L_0 - \frac{c}{24}\right) - \frac{\alpha}{L} I_3$$





$$\tau = \frac{iR}{L}$$



$$\hat{\tau} = \frac{iL}{R} = -\frac{1}{\tau}$$

$$Z = \mathrm{Tr}\left(e^{-RH}\right)$$

?

Higher charges are $I_{2n-1} = \sum_{k>0} k^{2n-1} \psi_{-k}\psi_k + c_{2n-1}^{(NS/R)}$

$$c_{2n-1}^{(NS/R)} = \begin{cases} \frac{1}{2}(1-2^{1-2n})\zeta(1-2n) & NS \\ -\frac{1}{2}\zeta(1-2n) & R \end{cases}$$

Q: $\mathrm{Tr}\left(\hat{q}^{L_0 - c/24} e^{\alpha I_3}\right) \overset{?}{=} \mathrm{Tr}\left(q^{L_0 - c/24} e^{Q}\right)$

$$Q = \sum \alpha_{2p+1} I_{2p+1}$$

$$= e^{\sum \alpha_{2p+1} c_{2p+1}} \cdot q^{h - c/24} \cdot \prod_{k>0} \left(1 + q^k e^{\sum \alpha_{2p+1} k^{2p+1}}\right)$$

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$$Q = \sum \alpha_{2p+1} I_{2p+1}$$

$$= e^{-\sum \alpha_{2p+1} c_{2p+1}} \cdot q^{h - c/24} \cdot \prod_{k>0}\left(1 + q^k e^{\sum \alpha_{2p+1} k^{2p+1}}\right)$$

A: No, can show $\alpha_{2p+1} = \dfrac{\alpha^p}{p!(2\pi i)^{p-1}}\left[\dfrac{(3p)!}{(2p+1)!} c^{p-1}(c\tau+d)^{3p+1}\right]$

Sum $\sum \alpha_{2p+1} k^{2p+1}$ only converges for a finite number of k.

In general, it is an asymptotic expansion in α

We can find a function with the same asymptotic expansion,

$$\sum \alpha_{2p+1} k^{2p+1} \sim F\left(\alpha c (c\tau+d)^2 k^2\right)$$

But this function has a branch point, so the value is not determined for large (αk^2)

The situation for the ground state energy is better.

$$E_0 = -\sum \alpha_{2p+1} C_{2p+1}^{NS/R}$$

The ground state energy has the same asymptotic expansion as

$$-\sum \alpha_{2p+1} C^{R}_{2p+1} = \frac{c\tau+d}{ic}\int_0^\infty \frac{dt}{2\pi}\frac{t}{e^t-1}\, g(t) \qquad (\text{R sector})$$

$$g(t) = {}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{3}{2};\frac{27}{8\pi i}\frac{\alpha c(c\tau+d)^3 t^2}{(2\pi i)^2}\right) - 1$$

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This has TBA form:

$$\int_0^\infty \frac{dt}{2\pi} \frac{t}{e^t-1} g(t) = \int_0^\infty \frac{dk}{2\pi} \log\left(1 - e^{-k + \frac{\alpha k^3}{8\pi^3}}\right)$$

Ground state energy for fermion with $p(k) = k$, $E(k) = k - \frac{\alpha k^3}{8\pi^3}$.
and trivial scattering.

If ground state energy is

$$\int_0^\infty \frac{dt}{2\pi} \frac{t}{e^t - 1} \, g(t)$$

Expect excited state energies to be given by extra terms corresponding to the poles in the integrand (deformation of contour argument). [Dorey, Tateo]

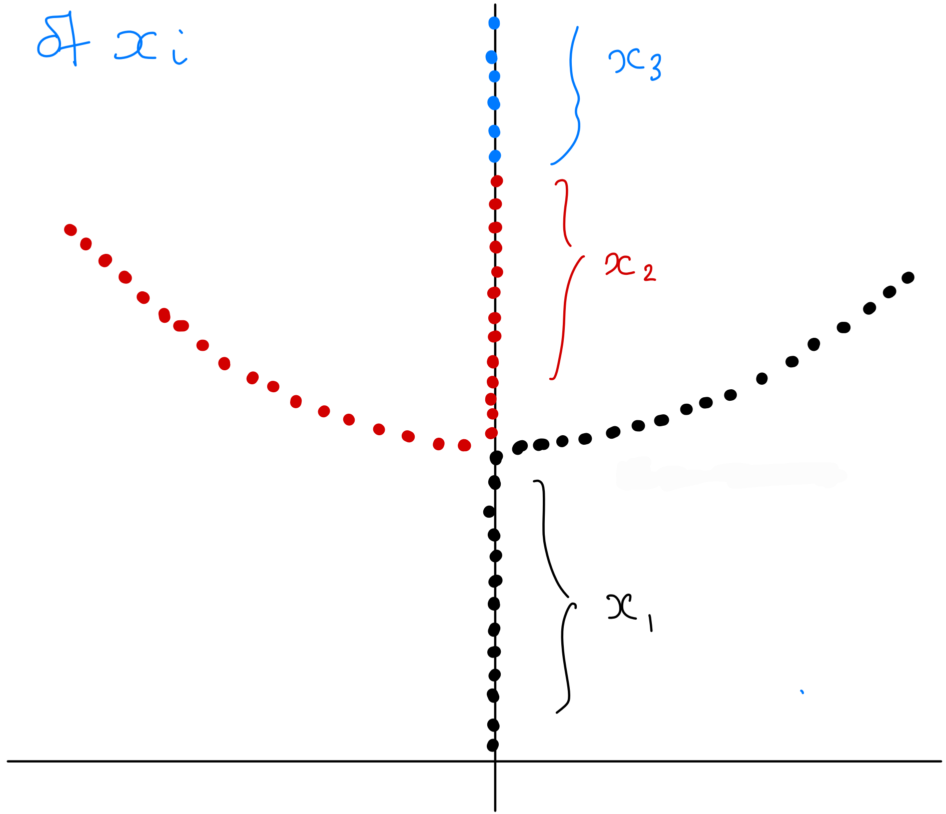
Final expression:

$$\mathrm{Tr}\left(q^{L_0 - \frac{c}{24}} e^{\alpha I_3}\right) = q^{-\mathcal{E}_0} \prod \left(1 + e^{\tau x_1}\right) \underbrace{\left(1 + e^{\tau x_2}\right)\left(1 + e^{-\tau x_3}\right)}_{\text{extra terms.}}$$

x_i solve $x - \frac{\alpha}{8\pi^3} x^3 = 2\pi n i$; take soln with +ve real part.

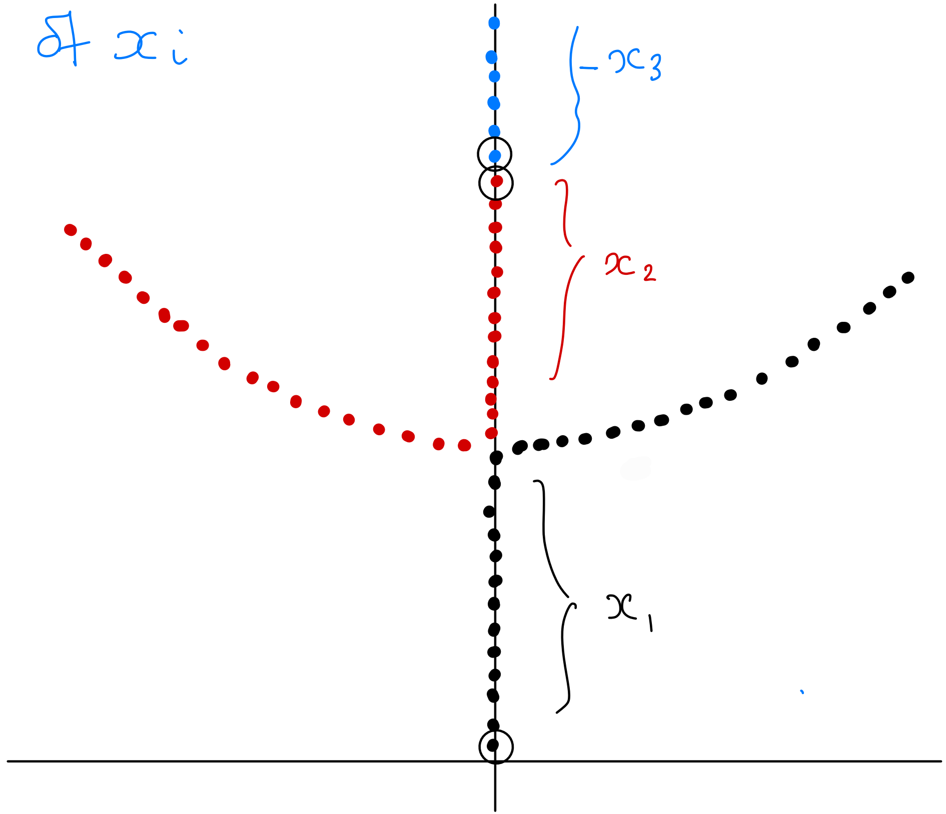
$\left(t = k - \frac{\alpha k^3}{8\pi^3} \text{ solved by } k = t\left[{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}; \frac{t^2}{4\pi}\right) - 1 \right] \right)$

positions of x_i



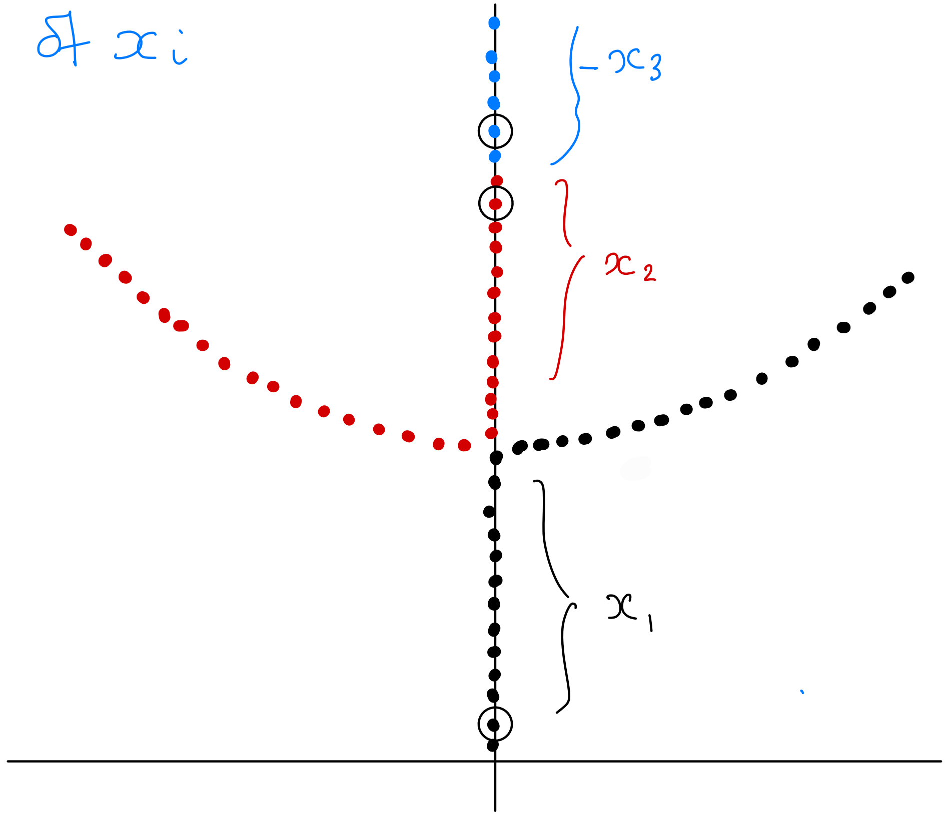
positions of x_i

$k=1$



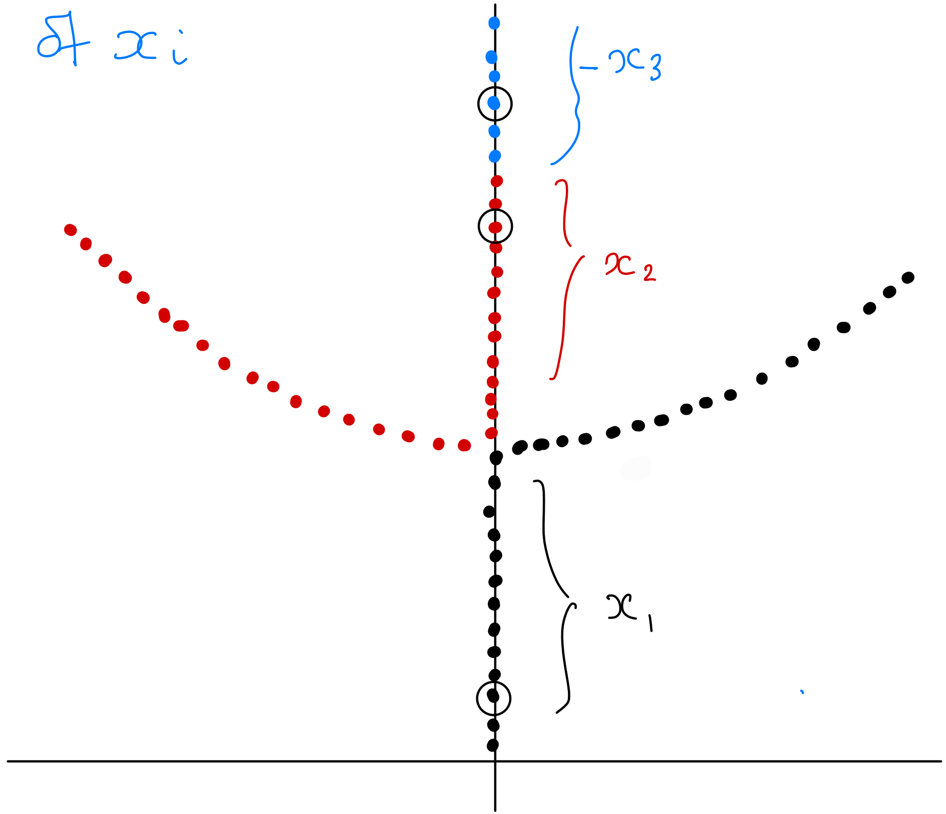
positions of x_i

$k=2$



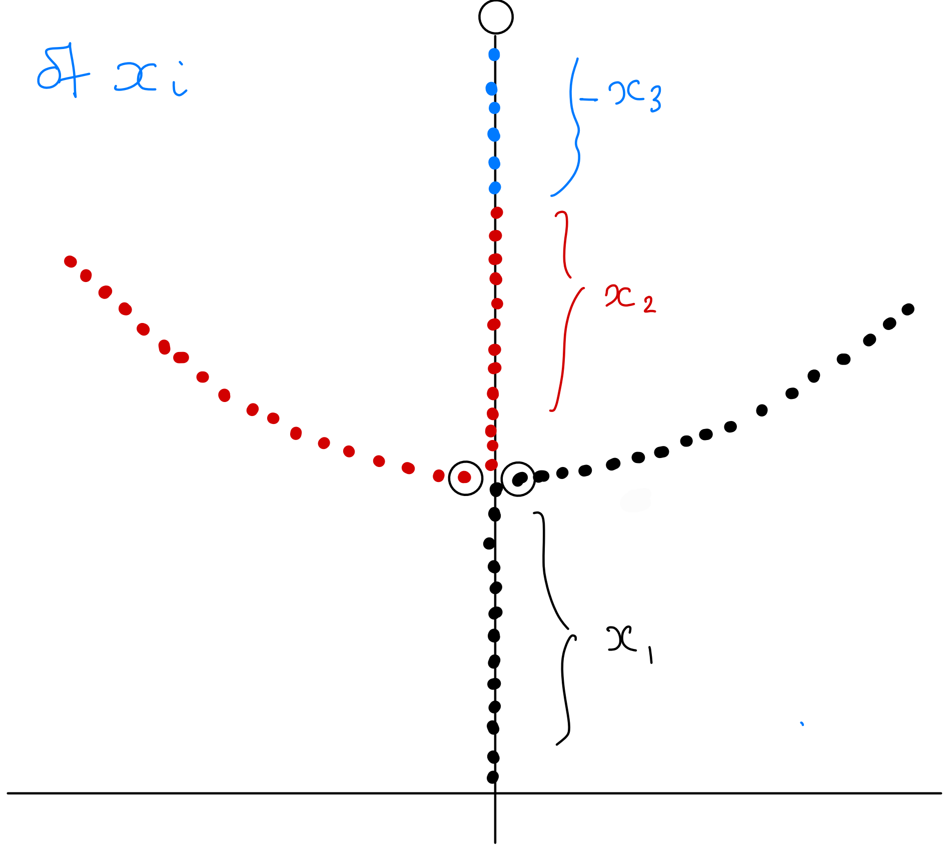
positions of x_i

$k = 3$



positions of x_i

$k = 14$



1. The argument extends to polynomial GGEs

$$\mathrm{Tr}\left(\hat{q}^{L_0 - c/24}\, e^{\underbrace{\sum \hat{\alpha}_{2p+1} I_{2p+1}}_{\text{Highest term is } I_N}} \right)$$

$$= q^{\mathcal{E}_0} \prod \underbrace{(1 + e^{\tau x_1}) \cdots\cdots (1 + e^{\pm \tau x_n})}_{N \text{ terms.}}$$

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2. This was a conjecture — can now prove this.

c.f. Earlier result for "power partitions" $\prod (1-q^{k^n})$ [Zagier 2021]

which is a specialisation of this result.

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which is a specialisation of this result.

3. Have physical interpretation in terms of defects

4. Have a formal expression for the defect operator

Power partitions

$$\frac{1}{\prod (1-q^k)}$$ counts partitions into integers

$$\frac{1}{\prod (1-q^{k^n})}$$ counts partitions into n-th powers

Power partitions (Zagier 21) (Hardy Ramanujan 1918 for $s=2$)

$$\eta(\tau) = q^{-1/24} \prod (1-q^k)$$

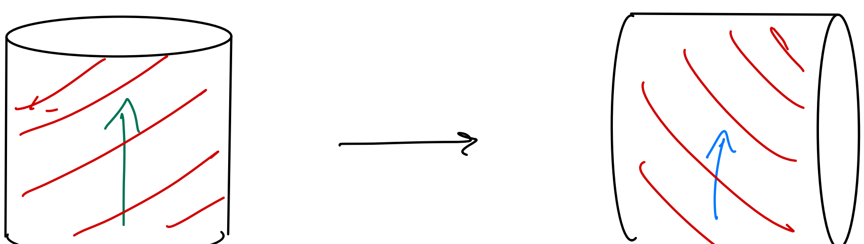
$$\eta_s(\tau) = q^{-\frac{1}{2}\zeta(-s)} \prod (1-q^{k^s}) \qquad \left(-\frac{1}{2}\zeta(-3) = -\frac{1}{240}\right)$$

$$\eta_s\left(-\frac{1}{\tau}\right) = (2\pi)^{\frac{s-1}{2}} \sqrt{\frac{\tau}{i}} \underbrace{\prod_{\substack{\operatorname{Im}(z)>0 \\ z^s = \pm\tau}} \eta_{\frac{1}{s}}(z)}_{\text{product of } s \text{ terms } \eta_{\frac{1}{s}}(z_1)\cdots\eta_{\frac{1}{2}}(z_s)}$$

Physical Interpretation: re-examine the ground state energy

$$E_0 = -\int_0^\infty \frac{dk}{2\pi} \log\left(1 + e^{-\varepsilon(k)}\right) \qquad (+ \text{ sign for NS})$$

$$\varepsilon = \underbrace{-k + \frac{\alpha k^3}{8\pi^3}}_{\varepsilon_0(k)} + \int dk\, \cancel{\varphi(k-k')} \ldots$$



GGE

$$\varepsilon_0(k) = k - \frac{\alpha k^3}{8\pi^3}$$

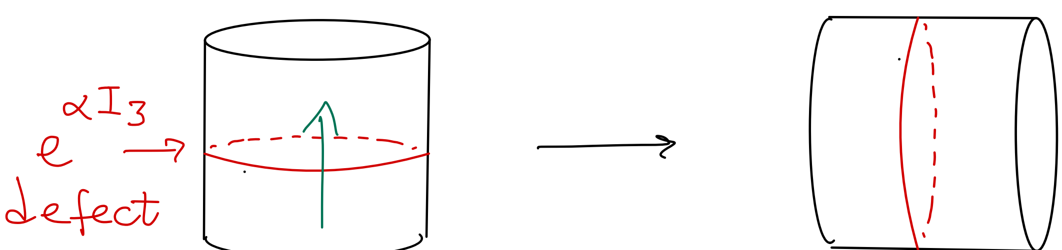
ground state energy E_0

Physical Interpretation: re-examine the ground state energy

$$E = \int_0^\infty \frac{dk}{2\pi} \log\left(1 + e^{-k + \frac{\alpha k^3}{8\pi^3}}\right) \qquad (+ \text{ sign for NS})$$

$$= \int_0^\infty \frac{dk}{2\pi} \log\left(1 + T(-ik)\, e^{-k}\right)$$

$T(k)$ = phase for transmission through a defect [Bajnok + Simon]



Regular system

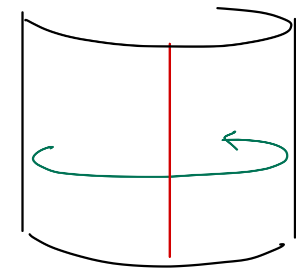
$\mathcal{E}_0(k) = k$

ground state energy for system with defect

[Hernandez-Chifflet et al]

Defect insertion with $T(k) = e^{-i\frac{\alpha k^3}{2\pi^3}}$

leads to a new quantisation condition on momentum of fermion modes.



$$e^{ik - \frac{i\alpha k^3}{8\pi^3}} = 1$$

or

$$\boxed{k - \frac{\alpha k^3}{8\pi^3} = 2n\pi i}$$

The triple product is just the product over allowed fermion modes in the presence of the defect.

Conclusions and outlook

- The GGE behaves exactly like (is) a defect under modular transformations

- The modular properties are not like those of characters

- The form of the modular transform agrees with the physics of the defect.

- Could consider extension to massive GGEs
  (Partion fn and traces of conserved quantities have good modular properties
  [Saleur + Itzykson, Kostov, Bergman Gaberdiel Green, Berg Bringman Gannon, Downing, Murthy GW]

- Would like to consider Lee-Yang, W_3 models, etc...

Thank you!

Defect operator construction

Want to write $H = H_0 + D(0)$

such that the modular transform is a

product over the spectrum of H.

Not a simple perturbation of a

free fermion because of "extra modes".

Idea:

Adapt ideas of G. Zs. Toth for irrelevant boundary perturbations.

Perturbation term (here $D(\sigma)$) changes quantisation condition on fermion momentum

Gives solution exactly in terms of required k

But: Very formal

- Constructed knowing which transmission factor is required

- Does not appear to be simply related to original GGE

(Details to appear)

Power Partions

Simplest if $s=2$

$$\eta_2 = \prod_{1,2\ldots}(1+q^{k^2})$$

Take log

$$\log \tilde{\eta}_2 = \sum_{k=1,2} \log(1+q^{k^2})$$

$$= \underbrace{\frac{1}{2}\sum_{k\in\mathbb{Z}} \log(1+q^{k^2})}_{} - \frac{1}{2}\log 2$$

lattice sum $\Rightarrow$ Poisson (re)summation

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{p\in\mathbb{Z}} \tilde{f}(p)$$

Fourier transform

$$\tilde{f}(p) = \int_{-\infty}^{\infty} f(k)\, e^{-2\pi i k p}\, dk$$

$$\sum_{k\in\mathbb{Z}} \log(1+q^{k^2}) = \sum_{p\in\mathbb{Z}} \int_{-\infty}^{\infty} \log(1+q^{k^2})\, e^{-2\pi i k p}\, dk$$

$$= \underbrace{\sum_{p<0} + \sum_{p>0} \int_{-\infty}^{\infty} \log(1+q^{k^2})\, e^{-2\pi i k p}\, dk}_{\text{move contours, picking up residues as you go}} + (\text{term } p=0).$$