Q-operators for Open Quantum Spin Chains

Robert Weston
Maxwell Institute for Mathematical Sciences
and Heriot-Watt University, Edinburgh

10th Bologna Workshop, Sep 2023

Plan

- Q-operators for closed chains
- Q-operators for open chains
  - the challenge
  - the resolution: universal-K

Intro

Pause

- Sklyanin-K/universal-K connection
- Light sketch of the rest
- Discussion

A few details

Joint work with Alec Cooper and Bart Vlaar:
arXiv:2001.10760, 2301.03997

Introduction

Q-operators for closed spin chains

- $Q(z)$ introduced in 72 by Baxter for 8-V model

(i) Diagonalizable & polynomial

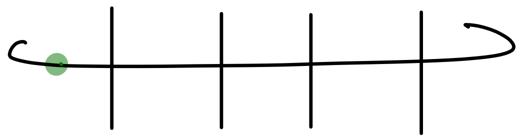
(ii) $[T(z), Q(z)] = [Q(z), Q(z')] = 0$

(iii) $T(z)\,Q(z) = a(z)\,Q(qz) + b(z)\,Q(q^{-1}z)$

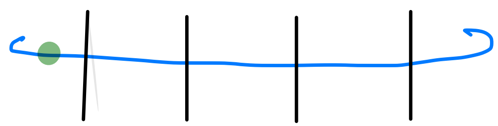
$\Rightarrow$ Bethe Eqns

- QISM / Quantum group picture [Sklyanin, BLZ 96]

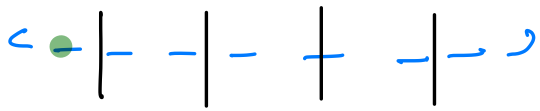
2 dim $U_q(\hat{sl}_2)$ repn

$T(z) =$ 

∞ dim $U_q(b_+)$ repns.

$Q(z) =$ 

Univ $\mathcal{R} \in U_q(b_+) \otimes U_q(b_-)$

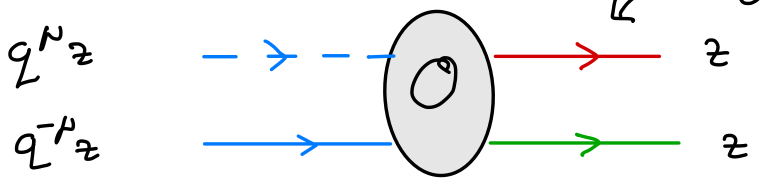
$\bar{Q}(z) =$ 

[also need twist •]

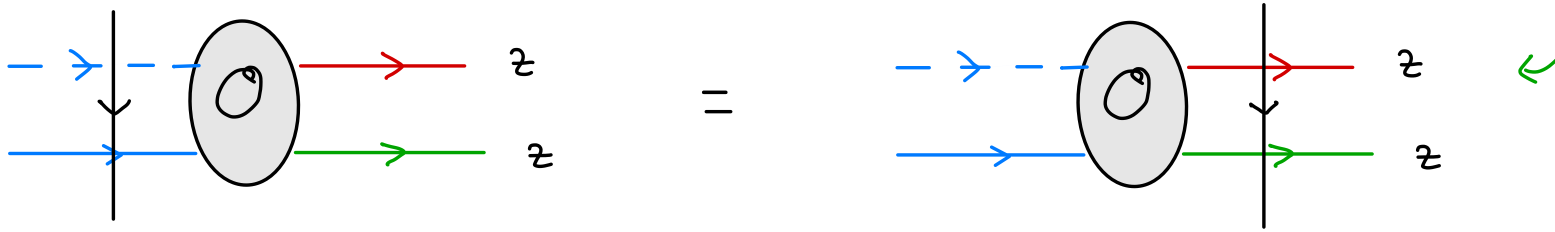
Factorization

$\exists$ $U_q(b_+)$ isomorphism $\mathcal{O}$

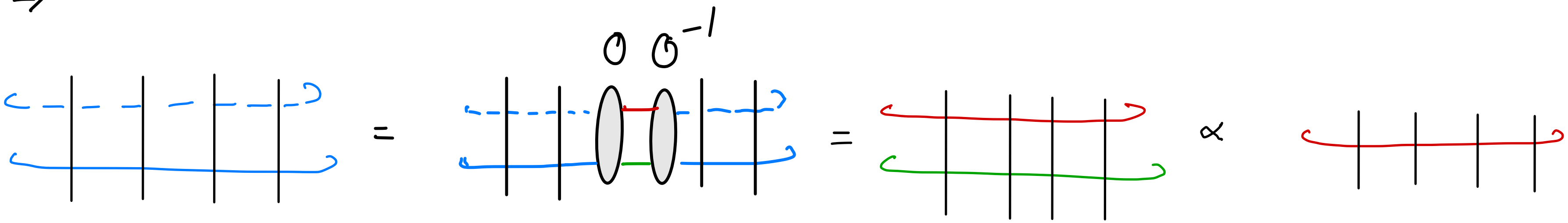
$U_q(\hat{sl}_2)$ Verma module [Khoroshkin, Tsuboi 14]



$U_q(b_+)$ filtered module [Bazhanov, Lukowski, Meneghelli, Staudacher, 10]



$\Rightarrow$



Factorization

or $\quad Q(q^{-\mu}z)\,\overline{Q}(q^{\mu}z) = \overline{T}_{\mu}(z)$

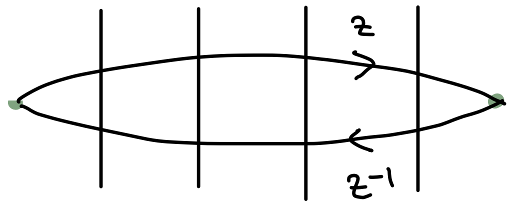
& $\quad \overline{T}^{(m)}(z) = \overline{T}_{\mu}(z) - \overline{T}_{-\mu}(z) \quad ; \quad \mu = -\frac{(m+1)}{2}$

$\leftrightarrow$ spin $m/2$ transfer matrix ; Generalised [HJ 11, FH 13]

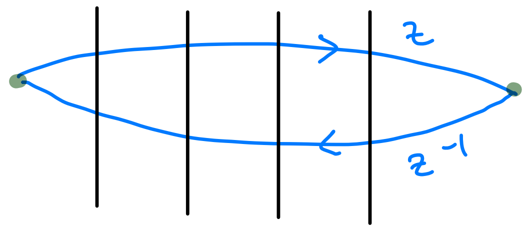
• Q-operators for open spin chains

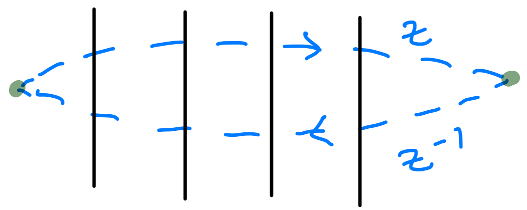
Looks easy!

[Frassek/Szécsényi 15, Baseilhac/Tsuboi 17,
Vlaar/Weston 20, Tsuboi 20]

$T(z) =$ 

[Sklyanin 88]

$Q(z) =$ 

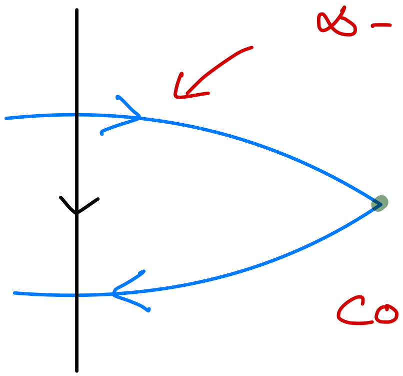
$\bar{Q}(z) =$ 

but until recently, full alg. picture of origin of
TQ relns missing.

The challenge: too many algebras!

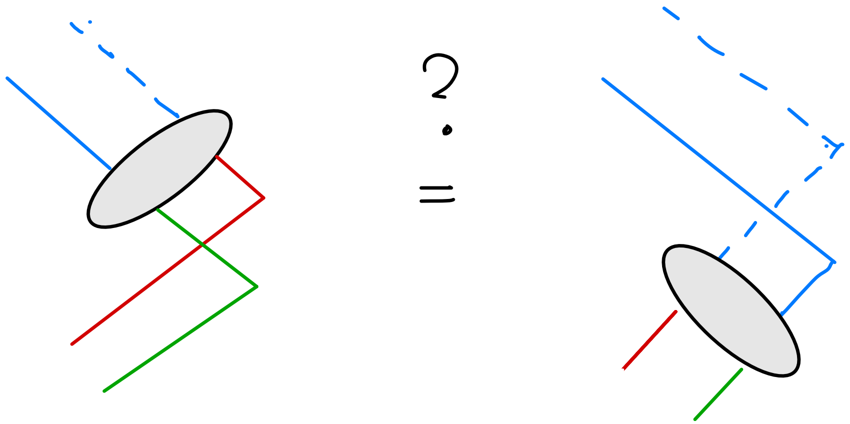
$U_q(\hat{sl}_2)$ repn

∞-dim $U_q(b_+)$ repn.

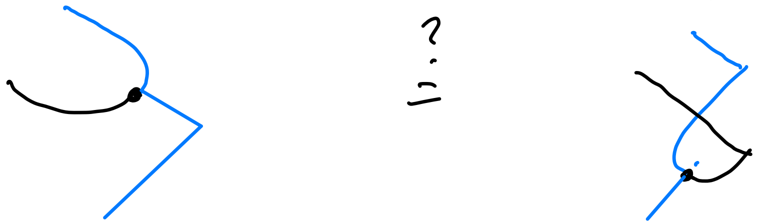


co-ideal subalgebra intertwiner

– doesn't act on our $U_q(b_+)$ repns!



Boundary factorization

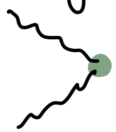


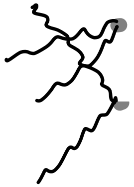
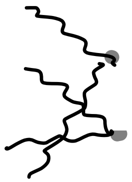
Boundary fusion

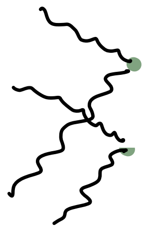
The resolution [Cooper/Vlaar/W 23]

- Exploit recent universal $\mathbb{K}$ matrix
  [Bao/Wang 18, Balagović/Kolb 19, Appel/Vlaar 20,22] CRM Lectures

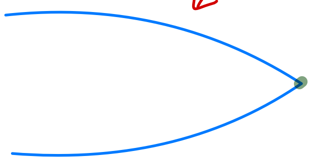
- A = coideal subalgebra

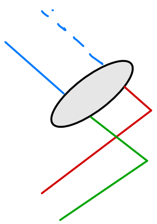
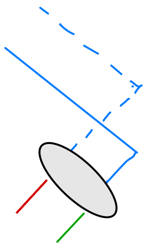
  $\mathbb{K} \in U_q(b_+) =$  , $[\mathbb{K}, A] = 0$

   =  RE

- $\Delta(\mathbb{K}) \in U_q(b_+) \otimes U_q(b_+) =$ 

∞-dim $U_q(b_+)$ repn

So  ∞-dim repn of $\mathbb{K} \in U_q(b_+)$

 =  holds as equality of $U_q(b_+)$ intertwiners via Schur's Lemma.

Pause

More Details

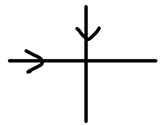
Sklyanin-K/universal-K connection

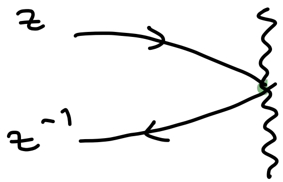
(i) Sklyanin

- U = underlying q-group ; $U_q(\mathfrak{g})$ ($U_q(\widehat{sl}_2)$ here)

  A = 'boundary q-group'

- Sklyanin [88] introduced A via FRT construction

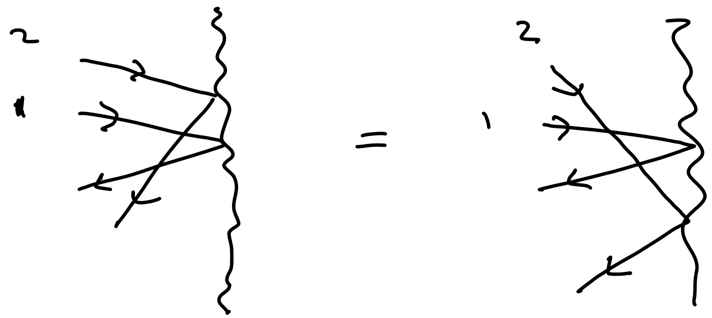
  $R(z) \in \mathrm{End}(V \otimes V)$; 

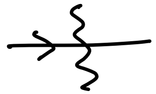
  $K^{(2)}(z) \in \mathrm{End}(V) \otimes A$ = 

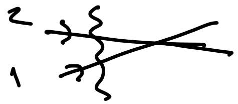
Defines A

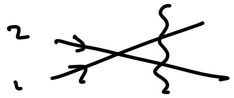
satifying RE

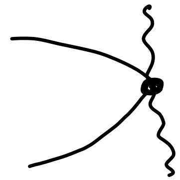
$$R(z_1/z_2)\, K_1^{(2)}(z_1)\, R(z_1 z_2)\, K_2^{(2)}(z_2) = K_2^{(2)}(z_2)\, R(z_1 z_2)\, K_1^{(2)}(z_1)\, R(z_1/z_2)$$



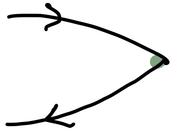
- c.f. $L(z) =$  $\in \mathrm{End}(V) \otimes \mathcal{U}$

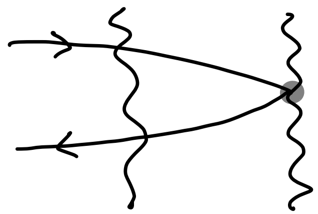
  $R(z_1|z_2)\, L_1(z_1)\, L_2(z_2)$ 

  $= L_2(z_2)\, L_1(z_1)\, R(z_1|z_2)$ 

- $K(z) = (\mathbb{1} \otimes \varepsilon)\, K^{(2)}(z) = (\mathbb{1} \otimes \varepsilon)$ 

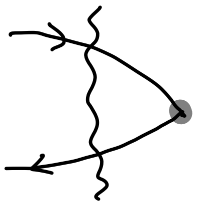
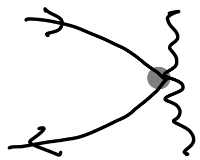
  counit.

  $=$  $\in \mathrm{End}(V)$

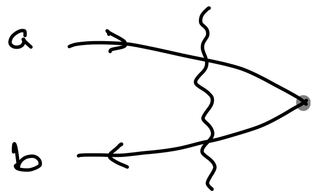
- Coproduct $(\mathbb{1} \otimes \Delta)\, K^{(2)} =$  $\in \mathrm{End}(V) \otimes \mathcal{U} \otimes A \quad (*)$

$\Delta : A \to \mathcal{U} \otimes A$, so coideal subalgebra.

- $(\mathbb{1} \otimes \varepsilon)\, \Delta = \mathbb{1}$; so $(\mathbb{1} \otimes \mathbb{1} \otimes \varepsilon)\ (*)$

  $\Rightarrow$  $=$  $= K^{(2)}$

- So A gen by matrix elements

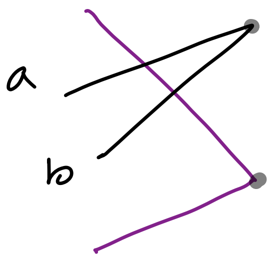
$$x_{ab} = \langle a | K^{(2)} | b \rangle =$$ 

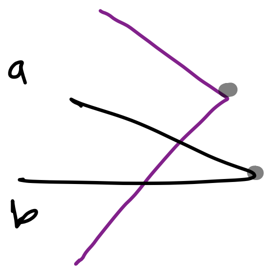
$\in A$

<u>Prop</u> Given repn π_W of A on W, we have

$$\boxed{K_W \, \pi_W(x_{ab}) = \pi_W(x_{ab}) \, K_W}$$

[Delius / Mackay 03]

<u>Proof</u> <u>RE</u> 

$=$ 

—

Hence, K_W is A intertwiner (& $K^{(2)} \in \mathrm{End}(V) \otimes A$)

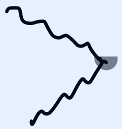
(ii) The universal K-matrix picture

[Bao/Wang 18, Balagović/Kolb 19, Appel/Vlaar 20/22]

- The objects

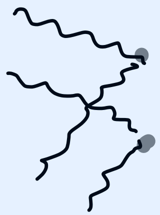
U = underlying q-group ;
A = coideal subalgebra ;
B = upper Borel subalg.

- universal $\mathbb{K}$ is constructed for wide class of coideal sub. A associated with quantum affine algebra :

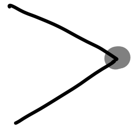
* $\mathbb{K} \in B$  ; with

* $\mathbb{K} x = x \mathbb{K}$ with $x \in A$

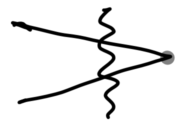
* $\mathbb{K}$ satisfies universal (twisted) RE .

* $\Delta(\mathbb{K}) =$  $\in B \otimes B$

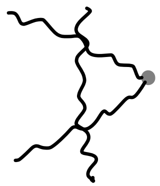
- If we have $\mathcal{B}$ repn π,

then $K = \pi(\mathbb{K}) =$ 

$$\boxed{\text{If } \pi \text{ not an } A \text{ repn, no } K\pi(x) = \pi(x)K, \text{ for } x \in A}$$

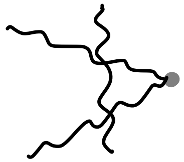
- Connection to Sklyanin's $K^{(2)} =$  is

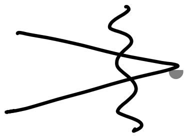
just

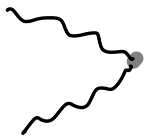
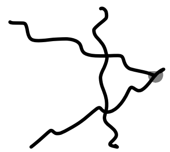
$\mathbb{K}^{(2)} =$  $\in \mathcal{B} \otimes A$

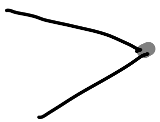
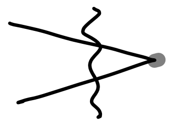
$$K^2 = (\pi_{\mathcal{B}} \otimes \mathbb{1})\, \mathbb{K}^2 \in \mathrm{End}(V) \otimes A .$$

Summary

VA $\mathbb{K}^{(2)} =$  $\in \mathcal{B} \otimes \mathcal{A}$

Skl $K^{(2)} =$  $\in \mathrm{End}(V) \otimes \mathcal{A}$

VA $\mathbb{K} =$  $= (\underline{\Pi} \otimes \varepsilon)$  $\in \mathcal{B}$

Skl $K =$  $= (\underline{\Pi} \otimes \varepsilon)$  $\in \mathrm{End}(V)$

Light sketch of the rest

- We consider $U_q(\widehat{sl}_2)$ case and

  $\mathcal{A}$ = augmented q-Onsager algebra

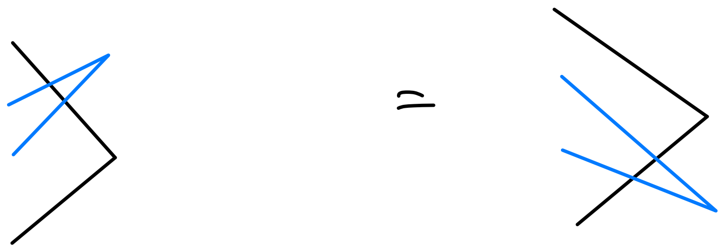
  with $K_{\frac{1}{2}}(z) = \begin{pmatrix} \xi z^2 - 1 & \\ & \xi - z^2 \end{pmatrix}$

  [Sklyanin 88, Baseilhac/Belliard 13]

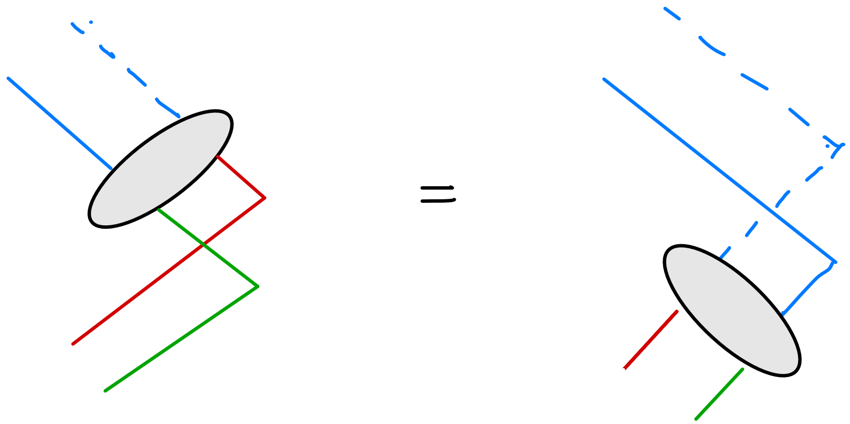
- $K_B = \pi_B \, \mathbb{K}$ well defined for all $U_q(b+)$ level-0 repns (including the 4 ∞-dim ones required for factorization).

- In practice find K_B by solving RE involving

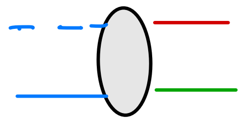
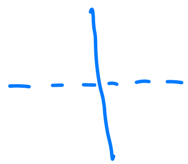
e.g.

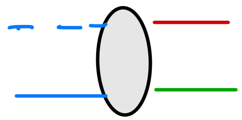


1-dim soln. space

-  follows from Schur's Lemma

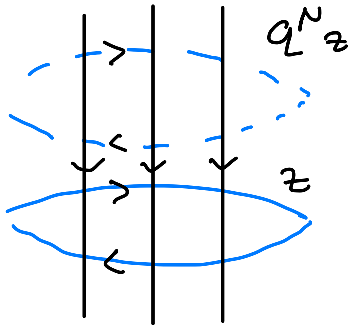
R-matrices also well defined. as repn

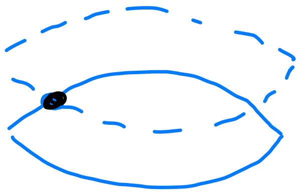
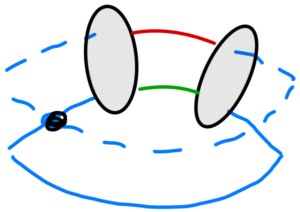
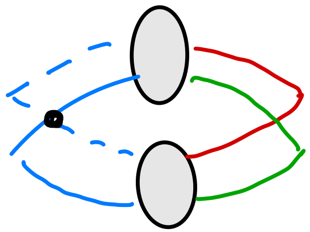
- All repns ,  ,  , etc have relatively simple q-osc. expressions.

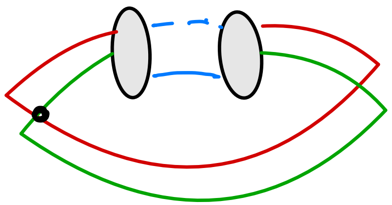
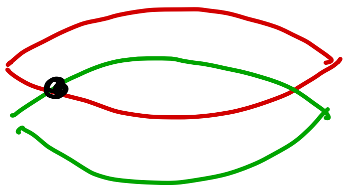
e.g.  $= e_{q^2}(q^2 \bar{a}_1^\dagger a_2)\, q^{\nu(D_2-D_1)/2}$

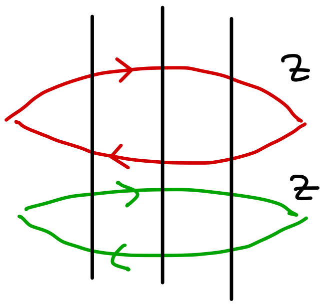
[Khoroshkin / Tsuboi 14]

- Final step

$$Q(q^{-N}z)\,\bar{Q}(q^{N}z) =$$ 

$=$  $=$  $=$ 

$=$  $=$ 

$=$ 

$\propto \quad T_N(z)$

reproduces known Bethe Eqns.

Discussion

- Question: does this generalise to

(i) Non-diagonal K matrices for sl_2 case

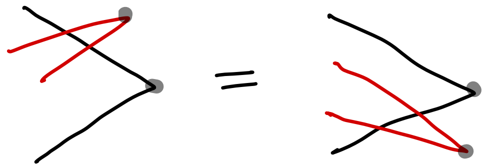
(ii) Other $U_q(\mathfrak{g})$?

(iii) Rational / Yangian limit ?

- Partial answers

(i) We think so, although RE alg. becomes modified by twist.

However, technically more difficult to complexity of solving



(ii) We hope so; ——— generalises to prefundamental repns [Jimbo / Hernandez, Frenkel / Hernandez] and 'TQ relns' in closed case conj. by [Frenkel / Reshetikhin], proved by [Frenkel / Hernandez].

(iii) Yes – in $\mathfrak{sl}_2$ case studied. Reproduces Q, $\overline{Q}$ of [Frassek / Szécsényi, 15]. [Alec Cooper's thesis].