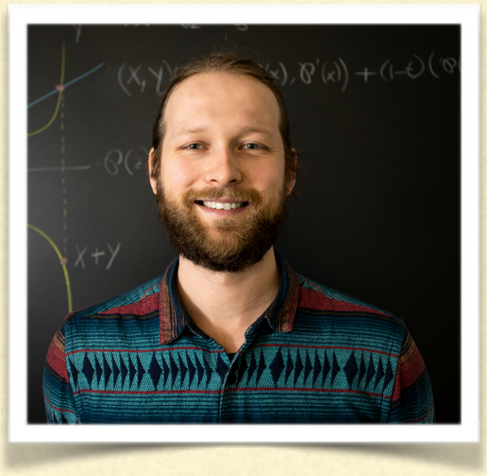
Regge spectroscopy of higher twist states in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

Rob Klabbers, Michelangelo Preti, and István M. Szécsényi



[arXiv:2307.15107]

10th Bologna Workshop on Conformal Field Theory and Integrable Models

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HUMBOLDT-UNIVERSITÄT ZU BERLIN

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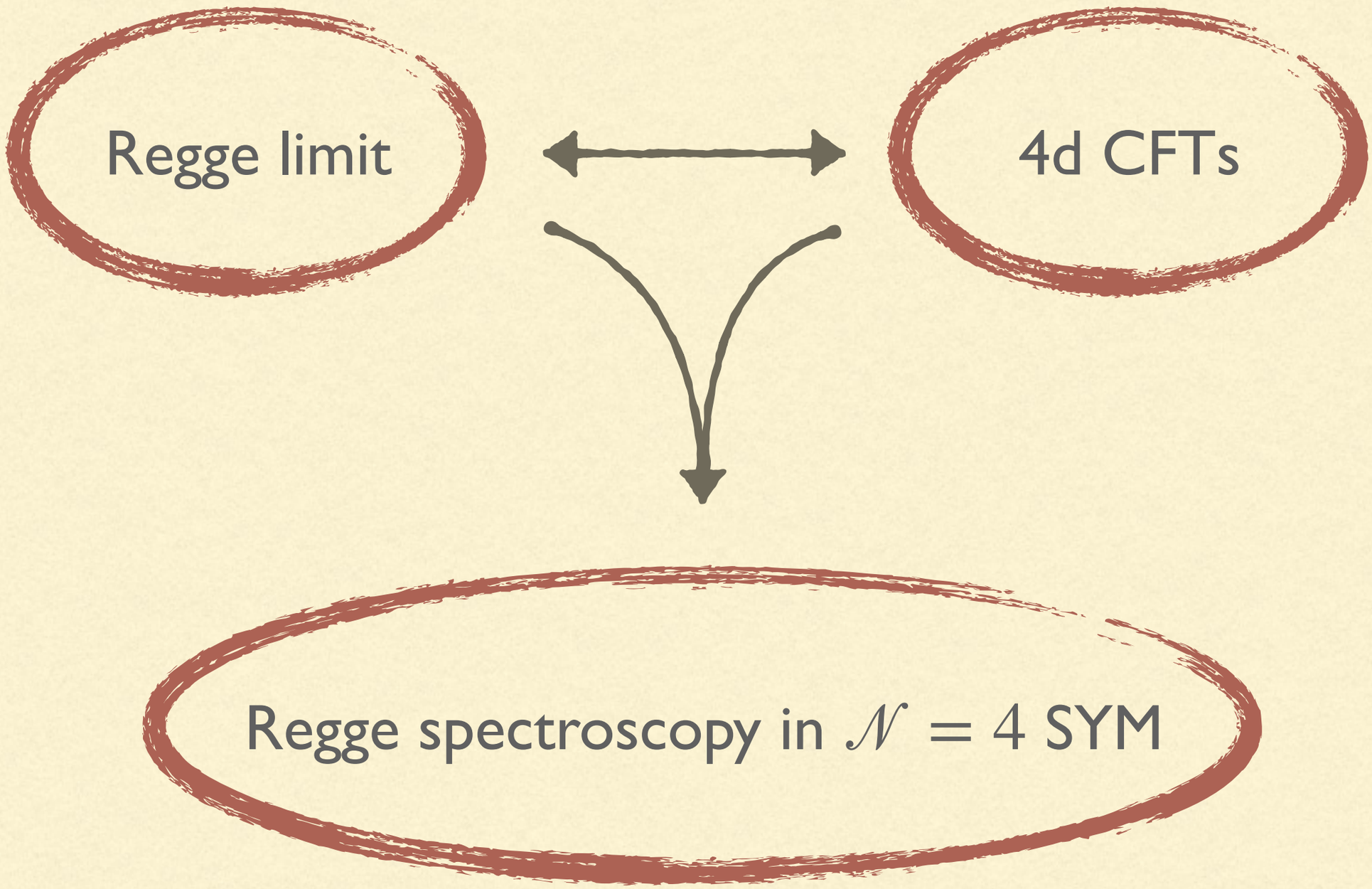
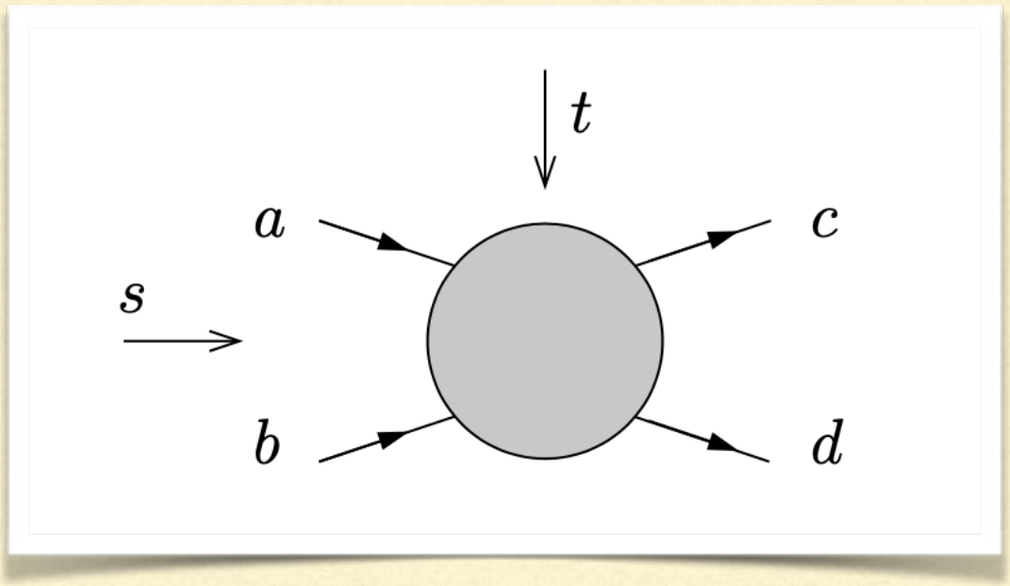

Regge limit
4d CFTs
Regge spectroscopy in $\mathcal{N} = 4$ SYM

Regge limit



Regge limit: $s \gg |t|$

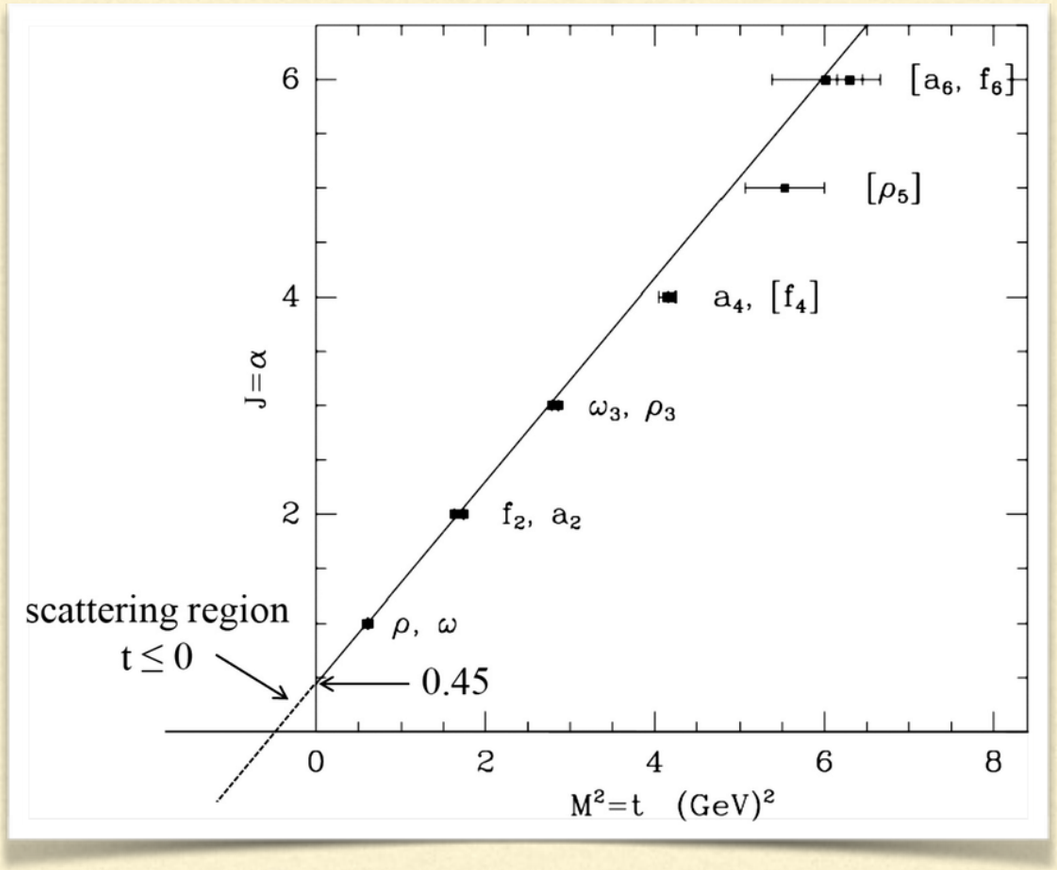
[Regge, '59]

$$\mathcal{A}^{ab}_{ab} \sim s^{\alpha(t)}$$

$$\sigma_{\mathrm{TOT}} \sim s^{\alpha(0)-1}$$

$\alpha(0)$ is called intercept

Regge trajectories

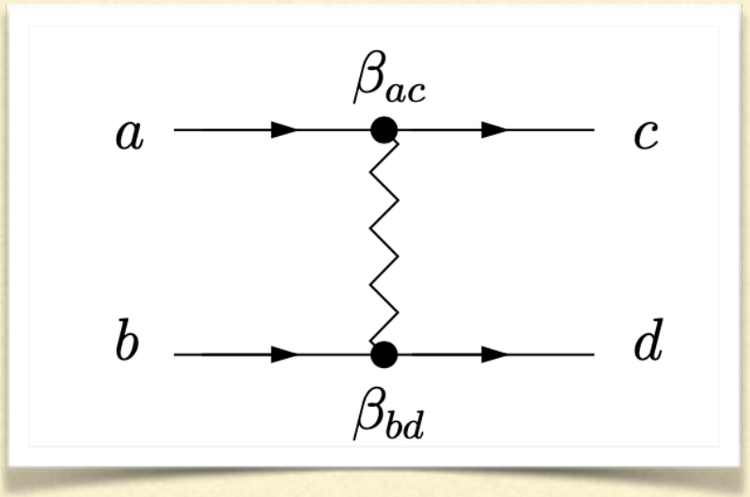


Chew-Frautschi plot

[Chew, Frautschi, '61]

$$\alpha(t) = 0.45 + 0.9t$$

Reggeon



QCD and the Pomeron

Pomeranchuk theorem: [Pomeranchuk '61]

$$\frac{\sigma_{\mathrm{TOT}}^{pp}}{\sigma_{\mathrm{TOT}}^{p\bar{p}}} \to 1 \quad \text{for } s \to \infty$$

Pomeron trajectory: $\alpha_{\mathbb{P}}(t) = 1.09 + 0.25t$

Odderon trajectory: $\sigma_{\mathrm{TOT}}^{pp} - \sigma_{\mathrm{TOT}}^{p\bar{p}} \sim s^{\alpha_{\mathbb{O}}(0)}$

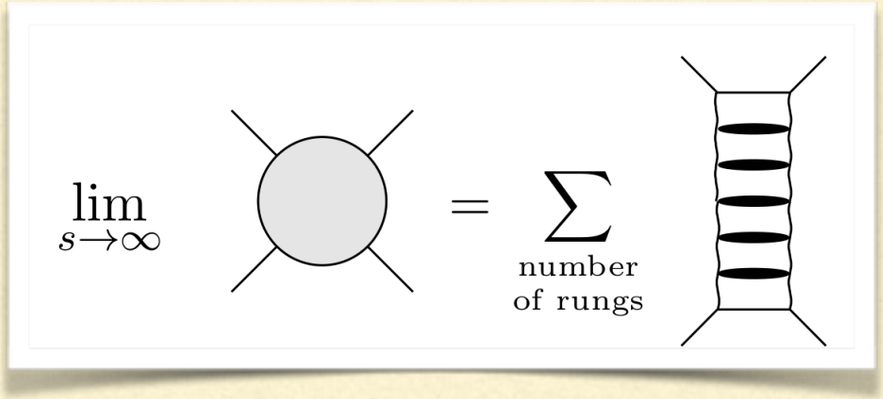
Experimental observation

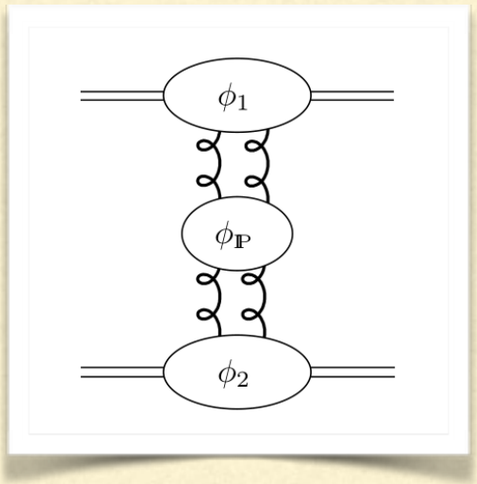
[Abazov et al. (TOTEM, D0) '21]

Integrability in high-energy QCD

On high-energy, the Pomeron can be described by the Balitsky–Fadin– Kuraev–Lipatov (BFKL) equation

[Kuraev, Lipatov, Fadin '77], [Balitsky, Lipatov '78]





The solution is connected to the integrable XXX Heisenberg spin chain noncompact $SL(2,\mathbb{C})$ and spin 0

[Faddeev, Korchemsky '95], [Korchemsky '95]

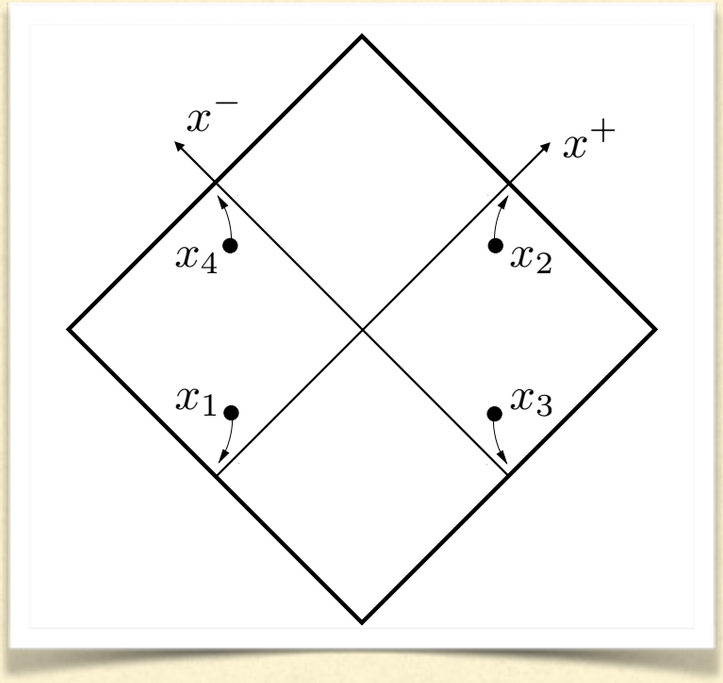
Conformal Regge theory

No amplitudes, rather 4 point functions: $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle$

Δ instead of m^2

Regge limit

[Costa, Goncalves, Penedones '12]



Intercept: $\Delta(S_0) - \frac{d}{2} = 0$

Conformal Regge theory

$$\frac{\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle}{\langle \mathcal{O}_1\mathcal{O}_2\rangle\langle \mathcal{O}_3\mathcal{O}_3\rangle} \sim 1 + f_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}(S_0) f_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}}(S_0) e^{t(S_0-1)} + \dots$$

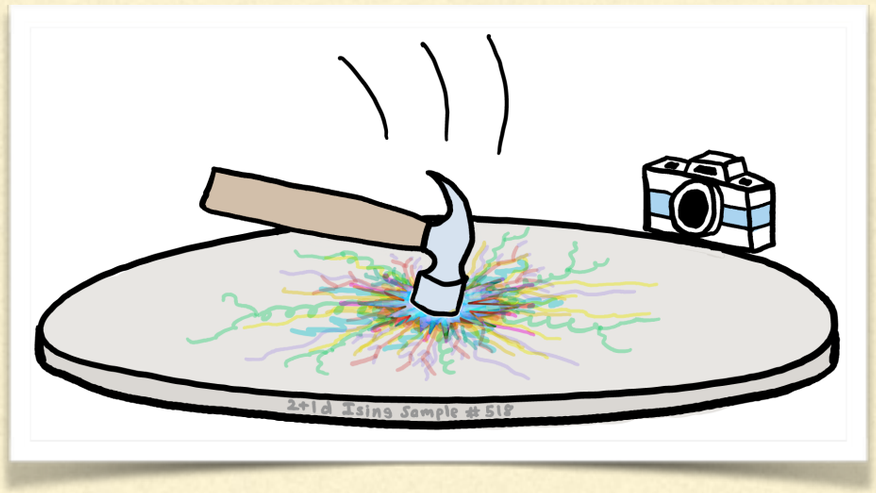
Analytically continued OPE

Light-ray operators

[Kravchuk, Simmons-Duffin '18]

Asymptotic detector operators

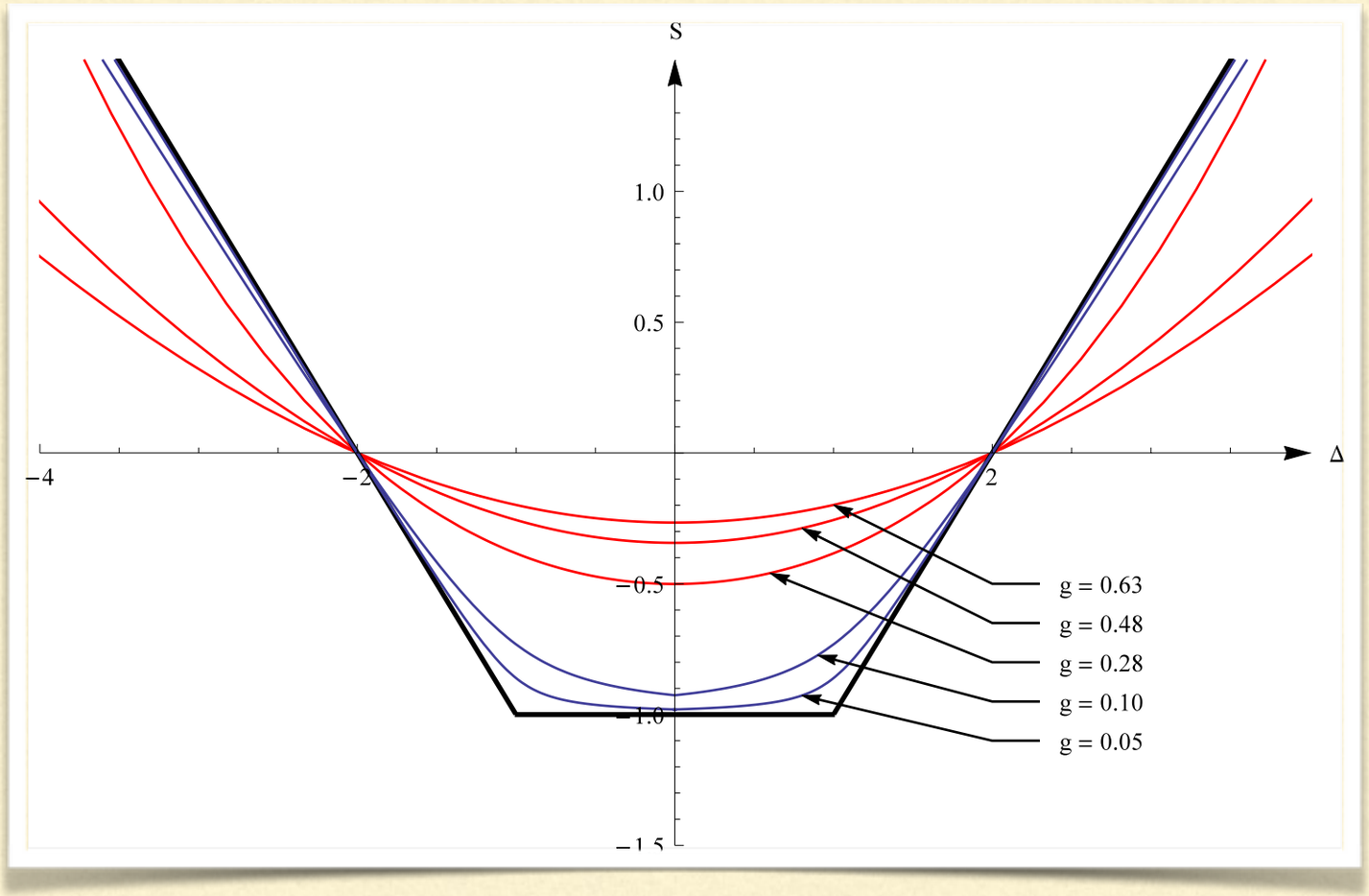
[Caron-Huot, Kologlu, Kravchuk,
Meltzer Simmons-Duffin '22]



$\mathcal{N} = 4$ SYM

- 4D superconformal theory
- $PSU(2,2|4)$ symmetry: $(J_1, J_2, J_3 | \Delta, S, S_2)$
- Integrable in the planar limit
- $\Delta = \tau + S + \gamma(S)$ $\qquad \gamma(S) = \sum_{i=1}^{\infty} \gamma_n(S) g^{2n}$
- Maximal transcendentality [Kotikov, Lipatov, '01]

Pomeron in $\mathcal{N} = 4$ SYM



[Gromov, Levkovich-Maslyuk, Sizov, Valatka '14]
[Alfimov, Gromov, Kazakov '15]

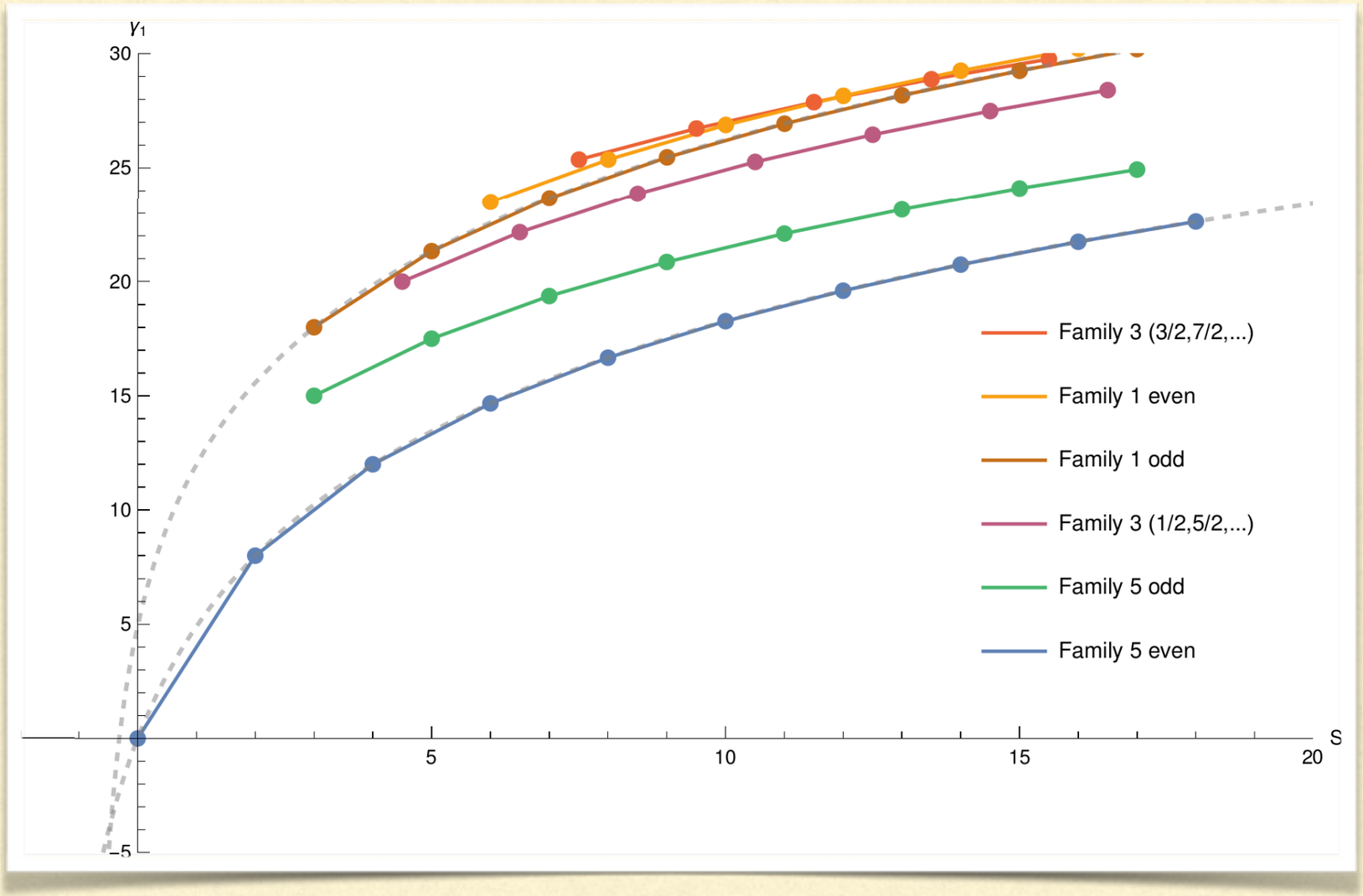
$$\mathrm{Tr}(ZD^S Z) + \text{perm}\,.$$

$$(2,0,0|2+S+\gamma,S,0)$$

$$\alpha(\Delta) = -1 + 4\left(2\psi(1) - \psi\left(\frac{1-\Delta}{2}\right) - \psi\left(\frac{1+\Delta}{2}\right)\right) g^2 + \mathcal{O}(g^4)$$

$$\alpha_{\mathbb{P}}(0) = \alpha(0) + 2$$

Twist-3 states $\mathcal{N} = 4$ SYM



$$\mathcal{O}_S = \mathrm{Tr}(D^S ZZZ) + \mathrm{perm}\,.$$

$$(3,0,0\,|\,3+S+\gamma, S, 0)$$

parity singlet

Quantum Spectral Curve

$\mathbf{P}_a(u), \mathbf{Q}_i(u) \quad a, i = 1,\dots 4$

[Gromov, Kazakov, Leurent, Volin '13]
[Alfimov, Gromov, Sizov '18]
[Marboe, Volin '18], …

QQ-relations

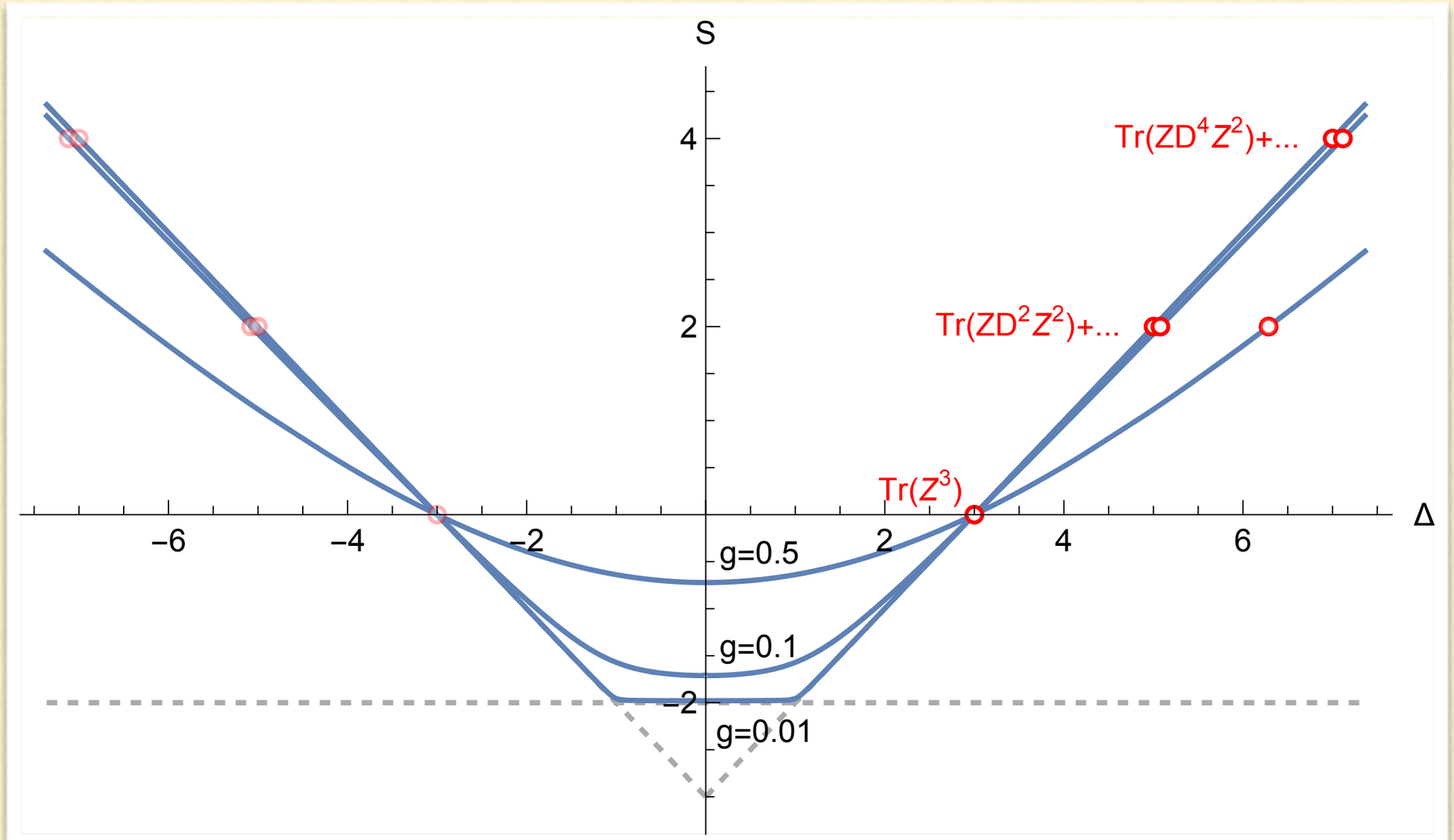
Asymptotic for large u: $\mathbf{P}_a(u) \sim A_a u^{-\tilde{M}_a}, \ \mathbf{Q}_i(u) \sim B_i u^{\hat{M}_i - 1}$

Analytic properties:

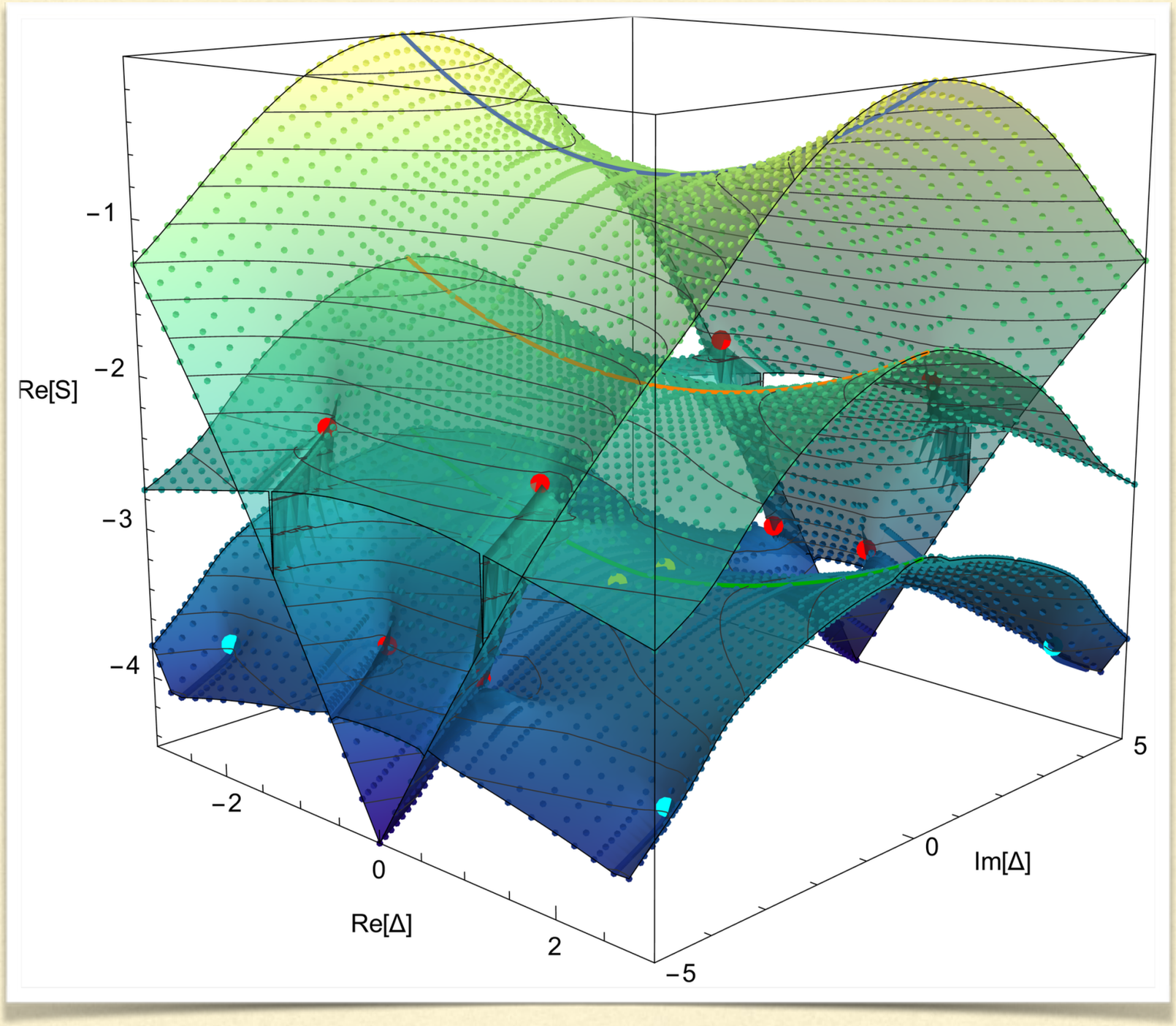
$$\tilde{\mathbf{Q}}^i(u) = M^{ij}(u)\mathbf{Q}_j(-u)\,,$$
$$\tilde{\mathbf{Q}}_i(u) = -\left(M^{-1}\right)_{ji}(u)\mathbf{Q}^j(-u)\,.$$

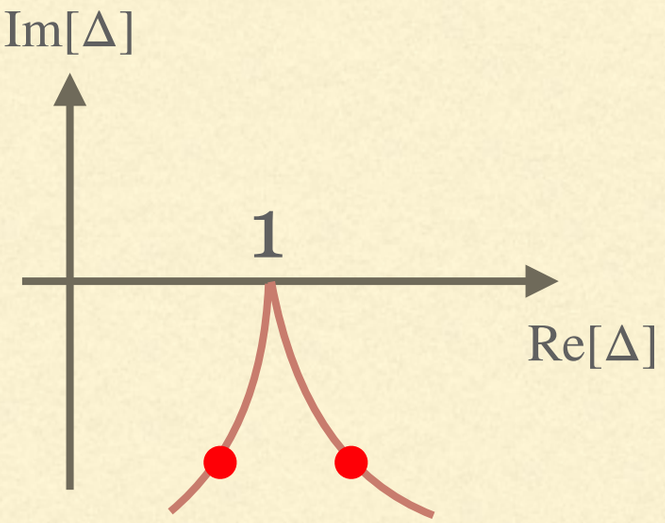
$$M = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 & 0 \\ \ell_2 & 0 & 0 & 0 \\ \ell_3 & 0 & \ell_4 & \ell_5 \\ 0 & 0 & \ell_5 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \ell_6 & 0 \\ 0 & 0 & 0 & 0 \\ \ell_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{2\pi u} + \begin{pmatrix} 0 & 0 & \ell_7 & 0 \\ 0 & 0 & 0 & 0 \\ \ell_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{-2\pi u}\,.$$

Leading trajectory

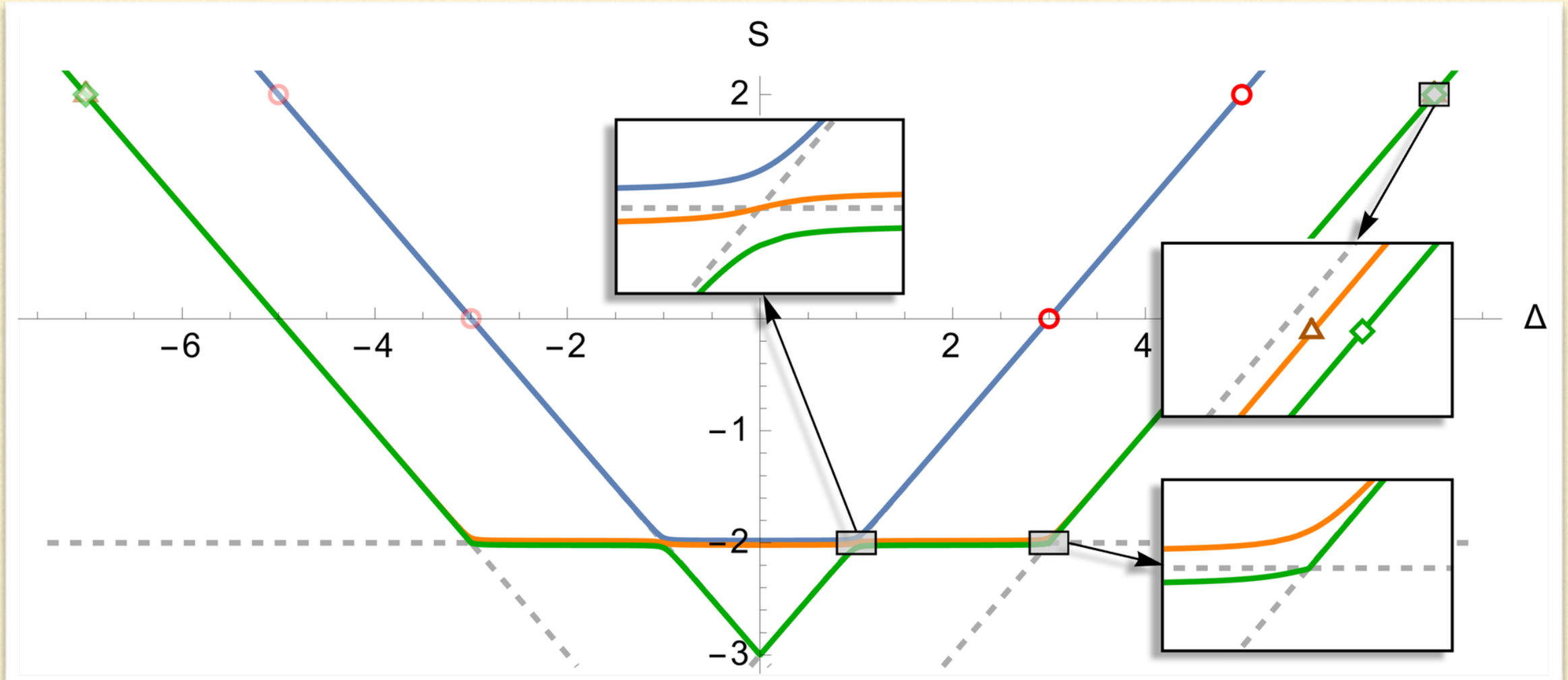


Riemann surface





Trajectories at weak coupling



Resolution of the degeneracy

Perturbative solution to QSC [Alfimov, Gromov, Kazakov '15]

$$S = -2 + \sum_{i=1} I_i(\Delta) g^i$$ [Marboe, Volin '18]

$\mathbf{P}_a$ at $\mathcal{O}(g)$ → Diff. eq. for $\mathbf{Q}_i$ → factorisation

$$\frac{2(u+i)+igI_1}{(u+i)^{3/2}}\mathbf{Q}_i(u+i) + \frac{2(u-i)-igI_1}{(u-i)^{3/2}}\mathbf{Q}_i(u-i) - \frac{1+8u^2-\Delta^2+2gI_1}{2u^{5/2}}\mathbf{Q}_i(u) = 0$$

Resolution of the degeneracy

→ Solution for $\mathbf{Q}_1$ and $\mathbf{Q}_3$ $\qquad$ I_1 is still free

New regularity condition: $\mathbf{Q}_i\sqrt{x}^{-1} + \tilde{\mathbf{Q}}_i\sqrt{x}$

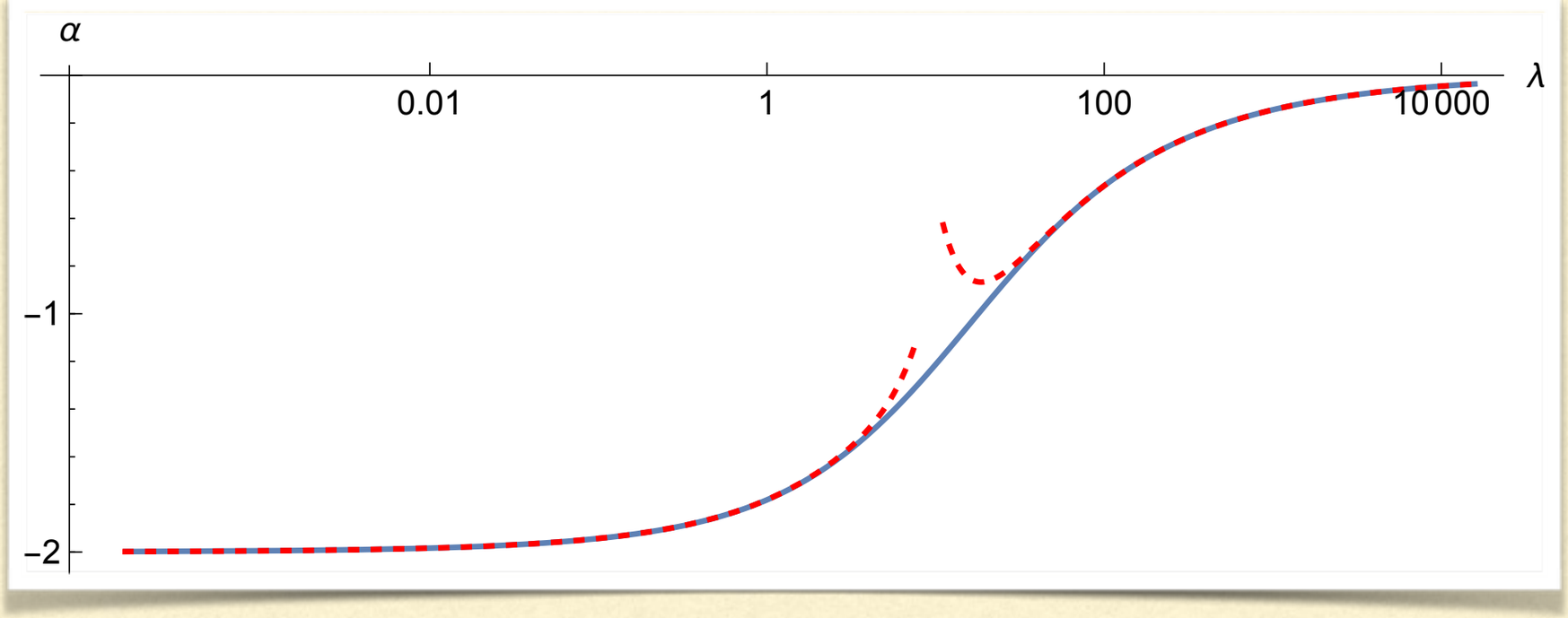
Gluing: $\tilde{\mathbf{Q}}_1(u) = -\mathbf{Q}_3(-u)/\ell_2, \quad \tilde{\mathbf{Q}}_3(u) = \mathbf{Q}_1(-u)\,\ell_2$

→ $I_1^2 = 4$ → $I_1 = \pm 2$

Linear g dependence is present (!)

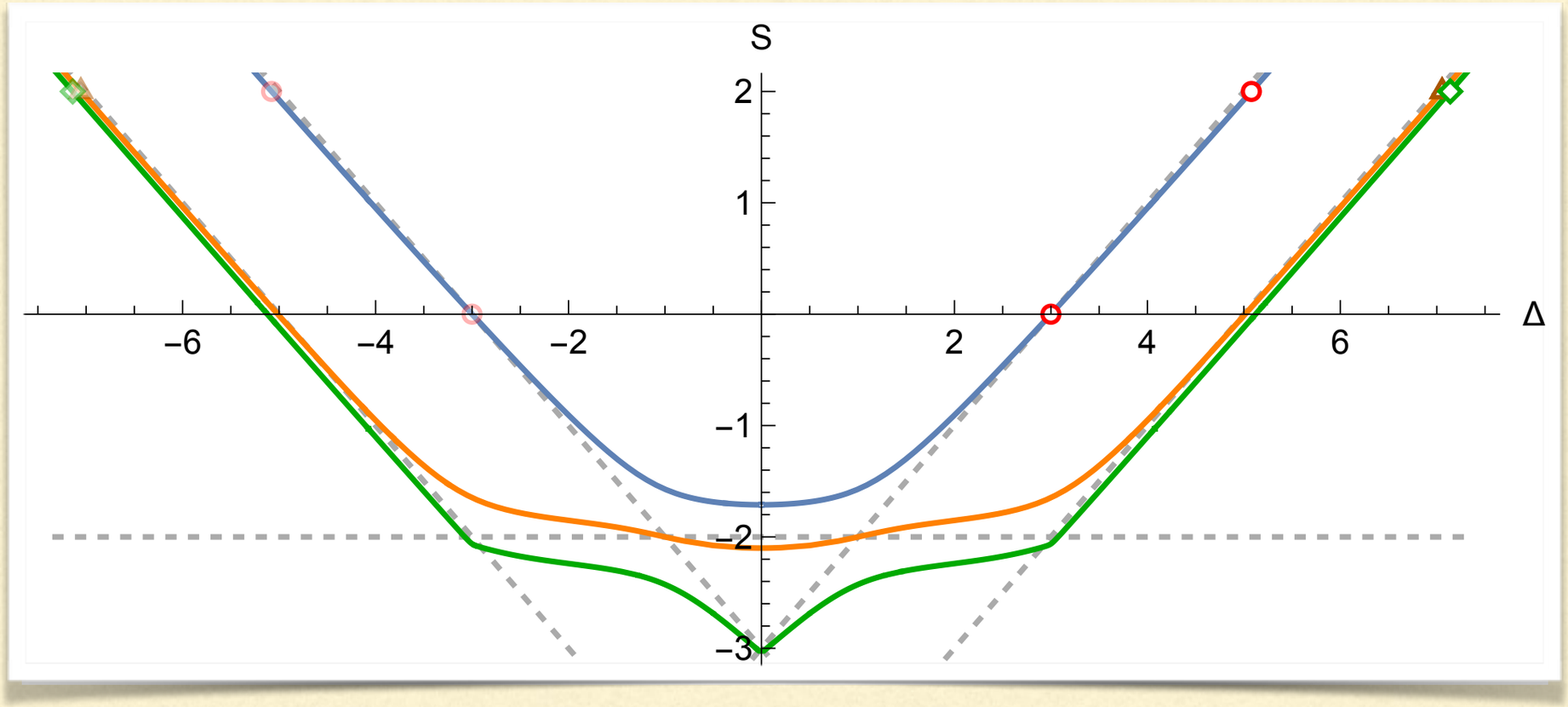
Intercept

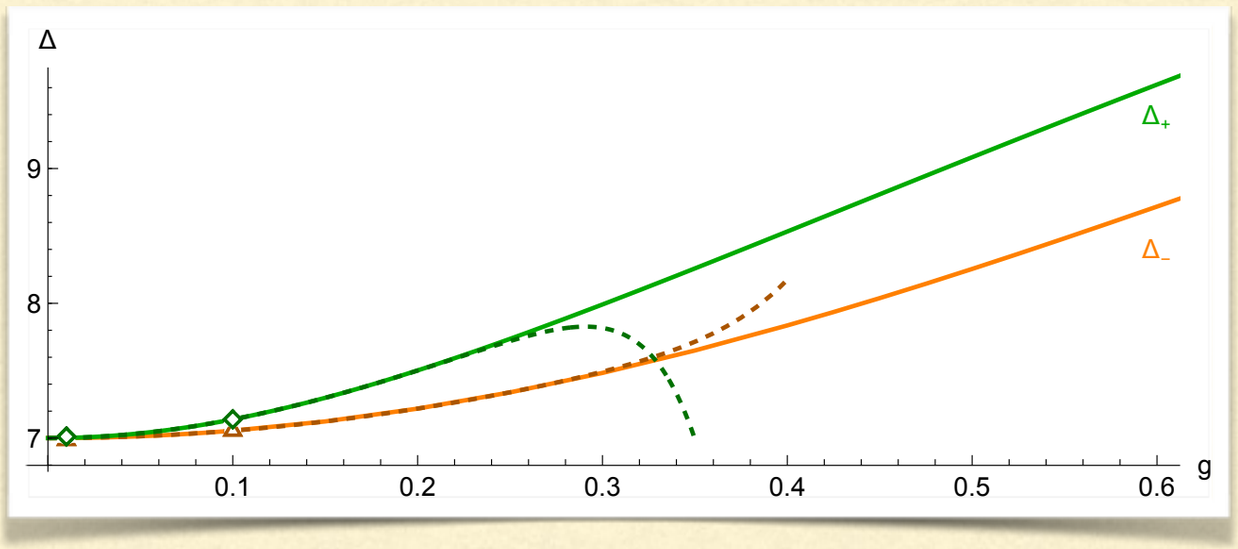
$$\alpha(0) = -2 + 2g + 16\log 2\; g^2 - \frac{2\pi^2}{3} g^3$$
$$-204.773771582926 61 g^4 + 136.293336388 13 g^5$$
$$+4733.390789 74 g^6 - 6116.795 85 g^7 + \ldots,$$



[Brower, Costa, Djuric, Raben, Tan '15] [Gromov, Levkovich-Maslyuk, Sizov, Valatka '14]

Identifying the extra trajectories





$$(3,0,0\,|\,5+S+\gamma,S,0)$$

$$\mathcal{O}\_=1.45\mathrm{Tr}(\mathcal{D}\Psi ZZZ\bar{\Psi})+0.553\mathrm{Tr}(\mathcal{D}\Psi ZZ\bar{\Psi}Z)+0.553\mathrm{Tr}(\mathcal{D}\Psi Z\bar{\Psi}ZZ)+1.45\mathrm{Tr}(\mathcal{D}\Psi\bar{\Psi}ZZZ)-0.106\mathrm{Tr}(\mathcal{D}Z\Psi ZZ\bar{\Psi})-0.106\mathrm{Tr}(\mathcal{D}Z\Psi\bar{\Psi}ZZ)+0.106\mathrm{Tr}(\mathcal{D}Z\Psi ZZ\bar{\Psi})+0.106\mathrm{Tr}(\mathcal{D}Z\Psi\bar{\Psi}ZZ)-$$

$$1.00\mathrm{Tr}(\mathcal{D}ZZ\Psi Z\bar{\Psi})-1.00\mathrm{Tr}(\mathcal{D}ZZ\Psi\bar{\Psi}Z)+1.00\mathrm{Tr}(\mathcal{D}ZZ\Psi Z\bar{\Psi})+1.00\mathrm{Tr}(\mathcal{D}ZZ\Psi\bar{\Psi}Z)-1.79\mathrm{Tr}(\mathcal{D}ZZZ\Psi\bar{\Psi})+1.79\mathrm{Tr}(\mathcal{D}ZZZ\Psi\bar{\Psi})-0.894\mathrm{Tr}(\mathcal{D}ZZZX\mathcal{D}\bar{X})+0.894\mathrm{Tr}(\mathcal{D}ZZZY\mathcal{D}\bar{Y})+0.894\mathrm{Tr}(\mathcal{D}ZZZ\bar{Y}\mathcal{D}Y)-$$

$$0.894\mathrm{Tr}(\mathcal{D}ZZZ\bar{X}\mathcal{D}X)-3.58\mathrm{Tr}(\mathcal{D}ZZZ\mathcal{D}Z\bar{Z})-0.0528\mathrm{Tr}(\mathcal{D}ZZZ\mathcal{D}X\bar{X})+0.0528\mathrm{Tr}(\mathcal{D}ZZZ\mathcal{D}Y\bar{Y})+0.0528\mathrm{Tr}(\mathcal{D}ZZZ\mathcal{D}\bar{Y}Y)-0.0528\mathrm{Tr}(\mathcal{D}ZZZ\mathcal{D}\bar{X}X)-0.106\mathrm{Tr}(\mathcal{D}ZZZ\bar{\Psi}\Psi)+0.106\mathrm{Tr}(\mathcal{D}ZZZ\bar{\Psi}\Psi)-$$

$$0.500\mathrm{Tr}(\mathcal{D}ZZXZ\mathcal{D}\bar{X})+1.84\mathrm{Tr}(\mathcal{D}ZZX\mathcal{D}Z\bar{X})-0.500\mathrm{Tr}(\mathcal{D}ZZX\mathcal{D}\bar{X}Z)+0.500\mathrm{Tr}(\mathcal{D}ZZYZ\mathcal{D}\bar{Y})-1.84\mathrm{Tr}(\mathcal{D}ZZY\mathcal{D}Z\bar{Y})+0.500\mathrm{Tr}(\mathcal{D}ZZY\mathcal{D}\bar{Y}Z)+0.500\mathrm{Tr}(\mathcal{D}ZZ\bar{Y}Z\mathcal{D}Y)-1.84\mathrm{Tr}(\mathcal{D}ZZ\bar{Y}\mathcal{D}ZY)+$$

$$0.500\mathrm{Tr}(\mathcal{D}ZZ\bar{Y}\mathcal{D}YZ)-0.500\mathrm{Tr}(\mathcal{D}ZZ\bar{X}Z\mathcal{D}X)+1.84\mathrm{Tr}(\mathcal{D}ZZ\bar{X}\mathcal{D}ZX)-0.500\mathrm{Tr}(\mathcal{D}ZZ\bar{X}\mathcal{D}XZ)-2.00\mathrm{Tr}(\mathcal{D}ZZ\mathcal{D}ZZ\bar{Z})+0.658\mathrm{Tr}(\mathcal{D}ZZ\mathcal{D}ZX\bar{X})-0.658\mathrm{Tr}(\mathcal{D}ZZ\mathcal{D}ZY\bar{Y})-0.658\mathrm{Tr}(\mathcal{D}ZZ\mathcal{D}Z\bar{Y}Y)+$$

$$0.658\mathrm{Tr}(\mathcal{D}ZZ\mathcal{D}Z\bar{X}X)-2.00\mathrm{Tr}(\mathcal{D}ZZ\mathcal{D}Z\bar{Z}Z)-0.500\mathrm{Tr}(\mathcal{D}ZZ\mathcal{D}X\bar{X}Z)+0.500\mathrm{Tr}(\mathcal{D}ZZ\mathcal{D}Y\bar{Y}Z)+0.500\mathrm{Tr}(\mathcal{D}ZZ\mathcal{D}\bar{Y}YZ)-0.500\mathrm{Tr}(\mathcal{D}ZZ\mathcal{D}\bar{X}XZ)-1.00\mathrm{Tr}(\mathcal{D}ZZ\bar{\Psi}\Psi Z)+1.00\mathrm{Tr}(\mathcal{D}ZZ\bar{\Psi}\Psi Z)-$$

$$0.0528\mathrm{Tr}(\mathcal{D}ZXZZ\mathcal{D}\bar{X})+1.84\mathrm{Tr}(\mathcal{D}ZXZ\mathcal{D}Z\bar{X})+1.84\mathrm{Tr}(\mathcal{D}ZX\mathcal{D}Z\bar{X}Z)-0.0528\mathrm{Tr}(\mathcal{D}ZX\mathcal{D}\bar{X}ZZ)+0.0528\mathrm{Tr}(\mathcal{D}ZYZZ\mathcal{D}\bar{Y})-1.84\mathrm{Tr}(\mathcal{D}ZYZ\mathcal{D}Z\bar{Y})-1.84\mathrm{Tr}(\mathcal{D}ZY\mathcal{D}Z\bar{Y}Z)+0.0528\mathrm{Tr}(\mathcal{D}ZY\mathcal{D}\bar{Y}ZZ)+$$

$$0.0528\mathrm{Tr}(\mathcal{D}Z\bar{Y}ZZ\mathcal{D}Y)+0.0528\mathrm{Tr}(\mathcal{D}Z\bar{Y}\mathcal{D}YZZ)-0.0528\mathrm{Tr}(\mathcal{D}Z\bar{X}ZZ\mathcal{D}X)-0.0528\mathrm{Tr}(\mathcal{D}Z\bar{X}\mathcal{D}XZZ)-0.211\mathrm{Tr}(\mathcal{D}Z\mathcal{D}ZZZ\bar{Z})+0.500\mathrm{Tr}(\mathcal{D}Z\mathcal{D}ZZX\bar{X})-0.500\mathrm{Tr}(\mathcal{D}Z\mathcal{D}ZZY\bar{Y})-$$

$$0.500\mathrm{Tr}(\mathcal{D}Z\mathcal{D}ZZ\bar{Y}Y)+0.500\mathrm{Tr}(\mathcal{D}Z\mathcal{D}ZZ\bar{X}X)+0.658\mathrm{Tr}(\mathcal{D}Z\mathcal{D}ZXZ\bar{X})+0.500\mathrm{Tr}(\mathcal{D}Z\mathcal{D}ZX\bar{X}Z)-0.658\mathrm{Tr}(\mathcal{D}Z\mathcal{D}ZYZ\bar{Y})-0.500\mathrm{Tr}(\mathcal{D}Z\mathcal{D}ZY\bar{Y}Z)-0.658\mathrm{Tr}(\mathcal{D}Z\mathcal{D}Z\bar{Y}ZY)-0.500\mathrm{Tr}(\mathcal{D}Z\mathcal{D}Z\bar{Y}YZ)+$$

$$0.658\mathrm{Tr}(\mathcal{D}Z\mathcal{D}Z\bar{X}ZX)+0.500\mathrm{Tr}(\mathcal{D}Z\mathcal{D}Z\bar{X}XZ)-0.211\mathrm{Tr}(\mathcal{D}Z\mathcal{D}Z\bar{Z}ZZ)-0.0528\mathrm{Tr}(\mathcal{D}Z\mathcal{D}XZZ\bar{X})-0.500\mathrm{Tr}(\mathcal{D}Z\mathcal{D}XZ\bar{X}Z)-0.894\mathrm{Tr}(\mathcal{D}Z\mathcal{D}X\bar{X}ZZ)+0.0528\mathrm{Tr}(\mathcal{D}Z\mathcal{D}YZZ\bar{Y})+0.500\mathrm{Tr}(\mathcal{D}Z\mathcal{D}YZ\bar{Y}Z)+$$

$$0.894\mathrm{Tr}(\mathcal{D}Z\mathcal{D}Y\bar{Y}ZZ)+0.0528\mathrm{Tr}(\mathcal{D}Z\mathcal{D}\bar{Y}ZZY)+0.500\mathrm{Tr}(\mathcal{D}Z\mathcal{D}\bar{Y}ZYZ)+0.894\mathrm{Tr}(\mathcal{D}Z\mathcal{D}\bar{Y}YZZ)-0.0528\mathrm{Tr}(\mathcal{D}Z\mathcal{D}\bar{X}ZZX)-0.500\mathrm{Tr}(\mathcal{D}Z\mathcal{D}\bar{X}ZXZ)-0.894\mathrm{Tr}(\mathcal{D}Z\mathcal{D}\bar{X}XZZ)-1.79\mathrm{Tr}(\mathcal{D}Z\bar{\Psi}\Psi ZZ)-$$

$$1.00\mathrm{Tr}(\mathcal{D}Z\bar{\Psi}Z\Psi Z)-0.106\mathrm{Tr}(\mathcal{D}Z\bar{\Psi}ZZ\Psi)+1.79\mathrm{Tr}(\mathcal{D}Z\bar{\Psi}\Psi ZZ)+1.00\mathrm{Tr}(\mathcal{D}Z\bar{\Psi}Z\Psi Z)+0.106\mathrm{Tr}(\mathcal{D}Z\bar{\Psi}ZZ\Psi)+1.45\mathrm{Tr}(\mathcal{D}XZZZ\mathcal{D}\bar{X})+0.553\mathrm{Tr}(\mathcal{D}XZZ\mathcal{D}\bar{X}Z)+0.553\mathrm{Tr}(\mathcal{D}XZ\mathcal{D}\bar{X}ZZ)+$$

$$1.45\mathrm{Tr}(\mathcal{D}X\mathcal{D}\bar{X}ZZZ)-1.45\mathrm{Tr}(\mathcal{D}YZZZ\mathcal{D}\bar{Y})-0.553\mathrm{Tr}(\mathcal{D}YZZ\mathcal{D}\bar{Y}Z)-0.553\mathrm{Tr}(\mathcal{D}YZ\mathcal{D}\bar{Y}ZZ)-1.45\mathrm{Tr}(\mathcal{D}Y\mathcal{D}\bar{Y}ZZZ)-0.106\mathrm{Tr}(\Psi\mathcal{F}ZZ\bar{\Psi})-1.00\mathrm{Tr}(\Psi\mathcal{F}Z\bar{\Psi}Z)-1.79\mathrm{Tr}(\Psi\mathcal{F}\bar{\Psi}ZZ)-0.500\mathrm{Tr}(\bar{\Psi}\Psi\Psi Z\bar{\Psi})-$$

$$0.658\mathrm{Tr}(\bar{\Psi}\Psi\Psi\bar{\Psi}Z)-0.658\mathrm{Tr}(\bar{\Psi}\Psi Z\Psi\bar{\Psi})+0.0528\mathrm{Tr}(\bar{\Psi}\Psi ZZ\mathcal{D}\bar{X})+1.00\mathrm{Tr}(\bar{\Psi}\Psi Z\bar{X}\mathcal{D}Z)+1.00\mathrm{Tr}(\bar{\Psi}\Psi Z\mathcal{D}Z\bar{X})+1.84\mathrm{Tr}(\bar{\Psi}\Psi Z\bar{\Psi}\Psi)+0.842\mathrm{Tr}(\bar{\Psi}\Psi\bar{X}Z\mathcal{D}Z)+1.00\mathrm{Tr}(\bar{\Psi}\Psi\bar{X}\mathcal{D}ZZ)+0.842\mathrm{Tr}(\bar{\Psi}\Psi\mathcal{D}ZZ\bar{X})+$$

$$1.00\mathrm{Tr}(\bar{\Psi}\Psi\mathcal{D}Z\bar{X}Z)+0.0528\mathrm{Tr}(\bar{\Psi}\Psi\mathcal{D}\bar{X}ZZ)+1.84\mathrm{Tr}(\bar{\Psi}\Psi\bar{\Psi}\Psi Z)+1.84\mathrm{Tr}(\bar{\Psi}\Psi\bar{\Psi}Z\Psi)+0.500\mathrm{Tr}(\bar{\Psi}\Psi\Psi Z\bar{\Psi})+0.658\mathrm{Tr}(\bar{\Psi}\Psi\Psi\bar{\Psi}Z)+0.658\mathrm{Tr}(\bar{\Psi}\Psi Z\Psi\bar{\Psi})-0.0528\mathrm{Tr}(\bar{\Psi}\Psi ZZ\mathcal{D}Y)-1.00\mathrm{Tr}(\bar{\Psi}\Psi ZY\mathcal{D}Z)-$$

$$1.00\mathrm{Tr}(\bar{\Psi}\Psi Z\mathcal{D}ZY)-1.84\mathrm{Tr}(\bar{\Psi}\Psi Z\bar{\Psi}\Psi)-0.842\mathrm{Tr}(\bar{\Psi}\Psi YZ\mathcal{D}Z)-1.00\mathrm{Tr}(\bar{\Psi}\Psi Y\mathcal{D}ZZ)-0.842\mathrm{Tr}(\bar{\Psi}\Psi\mathcal{D}ZZY)-1.00\mathrm{Tr}(\bar{\Psi}\Psi\mathcal{D}ZYZ)-0.0528\mathrm{Tr}(\bar{\Psi}\Psi\mathcal{D}YZZ)-1.84\mathrm{Tr}(\bar{\Psi}\Psi\bar{\Psi}\Psi Z)-1.84\mathrm{Tr}(\bar{\Psi}\Psi\bar{\Psi}Z\Psi)+$$

$$0.106\mathrm{Tr}(\bar{\Psi}\Psi ZZ\mathcal{D}Z)+1.00\mathrm{Tr}(\bar{\Psi}\Psi Z\mathcal{D}ZZ)+1.79\mathrm{Tr}(\bar{\Psi}\Psi\mathcal{D}ZZZ)-1.00\mathrm{Tr}(\bar{\Psi}Z\mathcal{F}\bar{\Psi}Z)-0.500\mathrm{Tr}(\bar{\Psi}Z\Psi\Psi\bar{\Psi})+0.500\mathrm{Tr}(\bar{\Psi}Z\Psi Z\mathcal{D}\bar{X})+0.842\mathrm{Tr}(\bar{\Psi}Z\Psi\bar{X}\mathcal{D}Z)+0.842\mathrm{Tr}(\bar{\Psi}Z\Psi\mathcal{D}Z\bar{X})+0.500\mathrm{Tr}(\bar{\Psi}Z\Psi\mathcal{D}\bar{X}Z)+$$

$$1.84\mathrm{Tr}(\bar{\Psi}Z\Psi\bar{\Psi}\Psi)+0.500\mathrm{Tr}(\bar{\Psi}Z\Psi\Psi\bar{\Psi})-0.500\mathrm{Tr}(\bar{\Psi}Z\Psi Z\mathcal{D}Y)-0.842\mathrm{Tr}(\bar{\Psi}Z\Psi Y\mathcal{D}Z)-0.842\mathrm{Tr}(\bar{\Psi}Z\Psi\mathcal{D}ZY)-0.500\mathrm{Tr}(\bar{\Psi}Z\Psi\mathcal{D}YZ)-1.84\mathrm{Tr}(\bar{\Psi}Z\Psi\bar{\Psi}\Psi)+1.00\mathrm{Tr}(\bar{\Psi}Z\Psi\mathcal{D}ZZ)-0.106\mathrm{Tr}(\bar{\Psi}ZZ\mathcal{F}\bar{\Psi})+$$

$$0.894\mathrm{Tr}(\bar{\Psi}ZZ\Psi\mathcal{D}\bar{X})-0.894\mathrm{Tr}(\bar{\Psi}ZZ\Psi\mathcal{D}Y)+0.106\mathrm{Tr}(\bar{\Psi}ZZ\Psi\mathcal{D}Z)+0.894\mathrm{Tr}(\bar{\Psi}ZZY\mathcal{D}\Psi)-0.894\mathrm{Tr}(\bar{\Psi}ZZ\bar{X}\mathcal{D}\Psi)-0.0528\mathrm{Tr}(\bar{\Psi}ZZ\mathcal{D}\Psi\bar{X})+0.0528\mathrm{Tr}(\bar{\Psi}ZZ\mathcal{D}\Psi Y)+1.79\mathrm{Tr}(\bar{\Psi}ZZ\mathcal{D}Z\Psi)-$$

$$0.0528\mathrm{Tr}(\bar{\Psi}ZZ\mathcal{D}Y\Psi)+0.0528\mathrm{Tr}(\bar{\Psi}ZZ\mathcal{D}\bar{X}\Psi)-1.79\mathrm{Tr}(\bar{\Psi}ZZ\bar{\Psi}\mathcal{F})+0.342\mathrm{Tr}(\bar{\Psi}ZY\Psi\mathcal{D}Z)+0.500\mathrm{Tr}(\bar{\Psi}ZYZ\mathcal{D}\Psi)+0.500\mathrm{Tr}(\bar{\Psi}ZY\mathcal{D}\Psi Z)-1.00\mathrm{Tr}(\bar{\Psi}ZY\mathcal{D}Z\Psi)-0.342\mathrm{Tr}(\bar{\Psi}Z\bar{X}\Psi\mathcal{D}Z)-0.500\mathrm{Tr}(\bar{\Psi}Z\bar{X}Z\mathcal{D}\Psi)-$$

$$0.500\mathrm{Tr}(\bar{\Psi}Z\bar{X}\mathcal{D}\Psi Z)+1.00\mathrm{Tr}(\bar{\Psi}Z\bar{X}\mathcal{D}Z\Psi)-0.500\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}\Psi\bar{X}Z)+0.500\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}\Psi YZ)-0.342\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}Z\Psi\bar{X})+0.342\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}Z\Psi Y)+1.00\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}Z\Psi Z)+1.00\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}ZZ\Psi)-1.00\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}ZY\Psi)+$$

$$1.00\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}Z\bar{X}\Psi)-0.500\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}Y\Psi Z)+0.500\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}\bar{X}\Psi Z)-1.00\mathrm{Tr}(\bar{\Psi}Z\bar{\Psi}\mathcal{F}Z)-0.658\mathrm{Tr}(\bar{\Psi}Z\bar{\Psi}\Psi\Psi)+0.658\mathrm{Tr}(\bar{\Psi}Z\bar{\Psi}\Psi\Psi)-1.00\mathrm{Tr}(\bar{\Psi}Z\bar{\Psi}Z\mathcal{F})+0.342\mathrm{Tr}(\bar{\Psi}Y\Psi Z\mathcal{D}Z)+0.342\mathrm{Tr}(\bar{\Psi}Y\Psi\mathcal{D}ZZ)+$$

$$0.342\mathrm{Tr}(\bar{\Psi}YZ\Psi\mathcal{D}Z)+0.0528\mathrm{Tr}(\bar{\Psi}YZZ\mathcal{D}\Psi)-0.842\mathrm{Tr}(\bar{\Psi}YZ\mathcal{D}Z\Psi)+0.0528\mathrm{Tr}(\bar{\Psi}Y\mathcal{D}\Psi ZZ)-0.842\mathrm{Tr}(\bar{\Psi}Y\mathcal{D}Z\Psi Z)-1.00\mathrm{Tr}(\bar{\Psi}Y\mathcal{D}ZZ\Psi)-0.342\mathrm{Tr}(\bar{\Psi}\bar{X}\Psi Z\mathcal{D}Z)-0.342\mathrm{Tr}(\bar{\Psi}\bar{X}\Psi\mathcal{D}ZZ)-$$

$$0.342\mathrm{Tr}(\bar{\Psi}\bar{X}Z\Psi\mathcal{D}Z)-0.0528\mathrm{Tr}(\bar{\Psi}\bar{X}ZZ\mathcal{D}\Psi)+0.842\mathrm{Tr}(\bar{\Psi}\bar{X}Z\mathcal{D}Z\Psi)-0.0528\mathrm{Tr}(\bar{\Psi}\bar{X}\mathcal{D}\Psi ZZ)+0.842\mathrm{Tr}(\bar{\Psi}\bar{X}\mathcal{D}Z\Psi Z)+1.00\mathrm{Tr}(\bar{\Psi}\bar{X}\mathcal{D}ZZ\Psi)-0.0528\mathrm{Tr}(\bar{\Psi}\mathcal{D}\Psi ZZ\bar{X})-0.500\mathrm{Tr}(\bar{\Psi}\mathcal{D}\Psi Z\bar{X}Z)-$$

$$0.894\mathrm{Tr}(\bar{\Psi}\mathcal{D}\Psi\bar{X}ZZ)+0.0528\mathrm{Tr}(\bar{\Psi}\mathcal{D}\Psi ZZY)+0.500\mathrm{Tr}(\bar{\Psi}\mathcal{D}\Psi ZYZ)+0.894\mathrm{Tr}(\bar{\Psi}\mathcal{D}\Psi YZZ)-0.342\mathrm{Tr}(\bar{\Psi}\mathcal{D}Z\Psi Z\bar{X})-0.342\mathrm{Tr}(\bar{\Psi}\mathcal{D}Z\Psi\bar{X}Z)+0.342\mathrm{Tr}(\bar{\Psi}\mathcal{D}Z\Psi ZY)+0.342\mathrm{Tr}(\bar{\Psi}\mathcal{D}Z\Psi YZ)+$$

$$0.106\mathrm{Tr}(\bar{\Psi}\mathcal{D}Z\Psi ZZ)-0.342\mathrm{Tr}(\bar{\Psi}\mathcal{D}ZZ\Psi\bar{X})+0.342\mathrm{Tr}(\bar{\Psi}\mathcal{D}ZZ\Psi Y)+0.106\mathrm{Tr}(\bar{\Psi}\mathcal{D}ZZZ\Psi)-0.842\mathrm{Tr}(\bar{\Psi}\mathcal{D}ZZY\Psi)+0.842\mathrm{Tr}(\bar{\Psi}\mathcal{D}ZZ\bar{X}\Psi)-0.842\mathrm{Tr}(\bar{\Psi}\mathcal{D}ZY\Psi Z)-1.00\mathrm{Tr}(\bar{\Psi}\mathcal{D}ZYZ\Psi)+$$

$$0.842\mathrm{Tr}(\bar{\Psi}\mathcal{D}Z\bar{X}\Psi Z)+1.00\mathrm{Tr}(\bar{\Psi}\mathcal{D}Z\bar{X}Z\Psi)-0.894\mathrm{Tr}(\bar{\Psi}\mathcal{D}Y\Psi ZZ)-0.500\mathrm{Tr}(\bar{\Psi}\mathcal{D}YZ\Psi Z)-0.0528\mathrm{Tr}(\bar{\Psi}\mathcal{D}YZZ\Psi)+0.894\mathrm{Tr}(\bar{\Psi}\mathcal{D}\bar{X}\Psi ZZ)+0.500\mathrm{Tr}(\bar{\Psi}\mathcal{D}\bar{X}Z\Psi Z)+0.0528\mathrm{Tr}(\bar{\Psi}\mathcal{D}\bar{X}ZZ\Psi)-$$

$$0.106\mathrm{Tr}(\bar{\Psi}\bar{\Psi}\mathcal{F}ZZ)-0.500\mathrm{Tr}(\bar{\Psi}\bar{\Psi}\Psi\Psi Z)-0.658\mathrm{Tr}(\bar{\Psi}\bar{\Psi}\Psi Z\Psi)+0.500\mathrm{Tr}(\bar{\Psi}\bar{\Psi}\Psi\Psi Z)+0.658\mathrm{Tr}(\bar{\Psi}\bar{\Psi}\Psi Z\Psi)-0.500\mathrm{Tr}(\bar{\Psi}\bar{\Psi}Z\Psi\Psi)+0.500\mathrm{Tr}(\bar{\Psi}\bar{\Psi}Z\Psi\Psi)-0.106\mathrm{Tr}(\bar{\Psi}\bar{\Psi}ZZ\mathcal{F})-0.0528\mathrm{Tr}(\bar{\Psi}\Psi ZZ\mathcal{D}\bar{Y})-$$

$$1.00\mathrm{Tr}(\bar{\Psi}\Psi Z\bar{Y}\mathcal{D}Z)-1.00\mathrm{Tr}(\bar{\Psi}\Psi Z\mathcal{D}Z\bar{Y})-0.842\mathrm{Tr}(\bar{\Psi}\Psi\bar{Y}Z\mathcal{D}Z)-1.00\mathrm{Tr}(\bar{\Psi}\Psi\bar{Y}\mathcal{D}ZZ)-0.842\mathrm{Tr}(\bar{\Psi}\Psi\mathcal{D}ZZ\bar{Y})-1.00\mathrm{Tr}(\bar{\Psi}\Psi\mathcal{D}Z\bar{Y}Z)-0.0528\mathrm{Tr}(\bar{\Psi}\Psi\mathcal{D}\bar{Y}ZZ)+0.0528\mathrm{Tr}(\bar{\Psi}\Psi ZZ\mathcal{D}X)+1.00\mathrm{Tr}(\bar{\Psi}\Psi ZX\mathcal{D}Z)+$$

$$1.00\mathrm{Tr}(\bar{\Psi}\Psi Z\mathcal{D}ZX)+0.842\mathrm{Tr}(\bar{\Psi}\Psi XZ\mathcal{D}Z)+1.00\mathrm{Tr}(\bar{\Psi}\Psi X\mathcal{D}ZZ)+0.842\mathrm{Tr}(\bar{\Psi}\Psi\mathcal{D}ZZX)+1.00\mathrm{Tr}(\bar{\Psi}\Psi\mathcal{D}ZXZ)+0.0528\mathrm{Tr}(\bar{\Psi}\Psi\mathcal{D}XZZ)-0.106\mathrm{Tr}(\bar{\Psi}\Psi ZZ\mathcal{D}Z)-1.00\mathrm{Tr}(\bar{\Psi}\Psi Z\mathcal{D}ZZ)-1.79\mathrm{Tr}(\bar{\Psi}\Psi\mathcal{D}ZZZ)-$$

$$0.500\mathrm{Tr}(\bar{\Psi}Z\Psi Z\mathcal{D}\bar{Y})-0.842\mathrm{Tr}(\bar{\Psi}Z\Psi\bar{Y}\mathcal{D}Z)-0.842\mathrm{Tr}(\bar{\Psi}Z\Psi\mathcal{D}Z\bar{Y})-0.500\mathrm{Tr}(\bar{\Psi}Z\Psi\mathcal{D}\bar{Y}Z)+0.500\mathrm{Tr}(\bar{\Psi}Z\Psi Z\mathcal{D}X)+0.842\mathrm{Tr}(\bar{\Psi}Z\Psi X\mathcal{D}Z)+0.842\mathrm{Tr}(\bar{\Psi}Z\Psi\mathcal{D}ZX)+0.500\mathrm{Tr}(\bar{\Psi}Z\Psi\mathcal{D}XZ)-$$

$$1.00\mathrm{Tr}(\bar{\Psi}Z\Psi\mathcal{D}ZZ)-0.894\mathrm{Tr}(\bar{\Psi}ZZ\Psi\mathcal{D}\bar{Y})+0.894\mathrm{Tr}(\bar{\Psi}ZZ\Psi\mathcal{D}X)-0.106\mathrm{Tr}(\bar{\Psi}ZZ\Psi\mathcal{D}Z)-0.894\mathrm{Tr}(\bar{\Psi}ZZX\mathcal{D}\Psi)+0.894\mathrm{Tr}(\bar{\Psi}ZZ\bar{Y}\mathcal{D}\Psi)+0.0528\mathrm{Tr}(\bar{\Psi}ZZ\mathcal{D}\Psi\bar{Y})-0.0528\mathrm{Tr}(\bar{\Psi}ZZ\mathcal{D}\Psi X)-$$

$$1.79\mathrm{Tr}(\bar{\Psi}ZZ\mathcal{D}Z\Psi)+0.0528\mathrm{Tr}(\bar{\Psi}ZZ\mathcal{D}X\Psi)-0.0528\mathrm{Tr}(\bar{\Psi}ZZ\mathcal{D}\bar{Y}\Psi)-0.342\mathrm{Tr}(\bar{\Psi}ZX\Psi\mathcal{D}Z)-0.500\mathrm{Tr}(\bar{\Psi}ZXZ\mathcal{D}\Psi)-0.500\mathrm{Tr}(\bar{\Psi}ZX\mathcal{D}\Psi Z)+1.00\mathrm{Tr}(\bar{\Psi}ZX\mathcal{D}Z\Psi)+0.342\mathrm{Tr}(\bar{\Psi}Z\bar{Y}\Psi\mathcal{D}Z)+$$

$$0.500\mathrm{Tr}(\bar{\Psi}Z\bar{Y}Z\mathcal{D}\Psi)+0.500\mathrm{Tr}(\bar{\Psi}Z\bar{Y}\mathcal{D}\Psi Z)-1.00\mathrm{Tr}(\bar{\Psi}Z\bar{Y}\mathcal{D}Z\Psi)+0.500\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}\Psi\bar{Y}Z)-0.500\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}\Psi XZ)+0.342\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}Z\Psi\bar{Y})-0.342\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}Z\Psi X)-1.00\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}Z\Psi Z)-1.00\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}ZZ\Psi)+$$

$$1.00\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}ZX\Psi)-1.00\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}Z\bar{Y}\Psi)+0.500\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}X\Psi Z)-0.500\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}\bar{Y}\Psi Z)-0.342\mathrm{Tr}(\bar{\Psi}X\Psi Z\mathcal{D}Z)-0.342\mathrm{Tr}(\bar{\Psi}X\Psi\mathcal{D}ZZ)-0.342\mathrm{Tr}(\bar{\Psi}XZ\Psi\mathcal{D}Z)-0.0528\mathrm{Tr}(\bar{\Psi}XZZ\mathcal{D}\Psi)+$$

$$0.842\mathrm{Tr}(\bar{\Psi}XZ\mathcal{D}Z\Psi)-0.0528\mathrm{Tr}(\bar{\Psi}X\mathcal{D}\Psi ZZ)+0.842\mathrm{Tr}(\bar{\Psi}X\mathcal{D}Z\Psi Z)+1.00\mathrm{Tr}(\bar{\Psi}X\mathcal{D}ZZ\Psi)+0.342\mathrm{Tr}(\bar{\Psi}\bar{Y}\Psi Z\mathcal{D}Z)+0.342\mathrm{Tr}(\bar{\Psi}\bar{Y}\Psi\mathcal{D}ZZ)+0.342\mathrm{Tr}(\bar{\Psi}\bar{Y}Z\Psi\mathcal{D}Z)+0.0528\mathrm{Tr}(\bar{\Psi}\bar{Y}ZZ\mathcal{D}\Psi)-$$

$$0.842\mathrm{Tr}(\bar{\Psi}\bar{Y}Z\mathcal{D}Z\Psi)+0.0528\mathrm{Tr}(\bar{\Psi}\bar{Y}\mathcal{D}\Psi ZZ)-0.842\mathrm{Tr}(\bar{\Psi}\bar{Y}\mathcal{D}Z\Psi Z)-1.00\mathrm{Tr}(\bar{\Psi}\bar{Y}\mathcal{D}ZZ\Psi)+0.0528\mathrm{Tr}(\bar{\Psi}\mathcal{D}\Psi ZZ\bar{Y})+0.500\mathrm{Tr}(\bar{\Psi}\mathcal{D}\Psi Z\bar{Y}Z)+0.894\mathrm{Tr}(\bar{\Psi}\mathcal{D}\Psi\bar{Y}ZZ)-0.0528\mathrm{Tr}(\bar{\Psi}\mathcal{D}\Psi ZZX)-$$

$$0.500\mathrm{Tr}(\bar{\Psi}\mathcal{D}\Psi ZXZ)-0.894\mathrm{Tr}(\bar{\Psi}\mathcal{D}\Psi XZZ)+0.342\mathrm{Tr}(\bar{\Psi}\mathcal{D}Z\Psi Z\bar{Y})+0.342\mathrm{Tr}(\bar{\Psi}\mathcal{D}Z\Psi\bar{Y}Z)-0.342\mathrm{Tr}(\bar{\Psi}\mathcal{D}Z\Psi ZX)-0.342\mathrm{Tr}(\bar{\Psi}\mathcal{D}Z\Psi XZ)-0.106\mathrm{Tr}(\bar{\Psi}\mathcal{D}Z\Psi ZZ)+0.342\mathrm{Tr}(\bar{\Psi}\mathcal{D}ZZ\Psi\bar{Y})-$$

$$0.342\mathrm{Tr}(\bar{\Psi}\mathcal{D}ZZ\Psi X)-0.106\mathrm{Tr}(\bar{\Psi}\mathcal{D}ZZZ\Psi)+0.842\mathrm{Tr}(\bar{\Psi}\mathcal{D}ZZX\Psi)-0.842\mathrm{Tr}(\bar{\Psi}\mathcal{D}ZZ\bar{Y}\Psi)+0.842\mathrm{Tr}(\bar{\Psi}\mathcal{D}ZX\Psi Z)+1.00\mathrm{Tr}(\bar{\Psi}\mathcal{D}ZXZ\Psi)-0.842\mathrm{Tr}(\bar{\Psi}\mathcal{D}Z\bar{Y}\Psi Z)-1.00\mathrm{Tr}(\bar{\Psi}\mathcal{D}Z\bar{Y}Z\Psi)+$$

$$0.894\mathrm{Tr}(\bar{\Psi}\mathcal{D}X\Psi ZZ)+0.500\mathrm{Tr}(\bar{\Psi}\mathcal{D}XZ\Psi Z)+0.0528\mathrm{Tr}(\bar{\Psi}\mathcal{D}XZZ\Psi)-0.894\mathrm{Tr}(\bar{\Psi}\mathcal{D}\bar{Y}\Psi ZZ)-0.500\mathrm{Tr}(\bar{\Psi}\mathcal{D}\bar{Y}Z\Psi Z)-0.0528\mathrm{Tr}(\bar{\Psi}\mathcal{D}\bar{Y}ZZ\Psi)+1.45\mathrm{Tr}(\bar{\Psi}ZZZ\mathcal{D}\Psi)+0.553\mathrm{Tr}(\bar{\Psi}ZZ\mathcal{D}\Psi Z)+$$

$$0.553\mathrm{Tr}(\bar{\Psi}Z\mathcal{D}\Psi ZZ)+1.45\mathrm{Tr}(\bar{\Psi}\mathcal{D}\Psi ZZZ)+1.45\mathrm{Tr}\left(\mathcal{D}^2ZZZZ\bar{Z}\right)-0.0528\mathrm{Tr}\left(\mathcal{D}^2ZZZX\bar{X}\right)+0.0528\mathrm{Tr}\left(\mathcal{D}^2ZZZY\bar{Y}\right)+0.0528\mathrm{Tr}\left(\mathcal{D}^2ZZZ\bar{Y}Y\right)-0.0528\mathrm{Tr}\left(\mathcal{D}^2ZZZ\bar{X}X\right)+0.553\mathrm{Tr}\left(\mathcal{D}^2ZZZ\bar{Z}Z\right)-$$

$$0.500\mathrm{Tr}\left(\mathcal{D}^2ZZXZ\bar{X}\right)+0.500\mathrm{Tr}\left(\mathcal{D}^2ZZYZ\bar{Y}\right)+0.500\mathrm{Tr}\left(\mathcal{D}^2ZZ\bar{Y}ZY\right)-0.500\mathrm{Tr}\left(\mathcal{D}^2ZZ\bar{X}ZX\right)+0.553\mathrm{Tr}\left(\mathcal{D}^2ZZ\bar{Z}ZZ\right)-0.894\mathrm{Tr}\left(\mathcal{D}^2ZXZZ\bar{X}\right)-0.500\mathrm{Tr}\left(\mathcal{D}^2ZXZ\bar{X}Z\right)-0.0528\mathrm{Tr}\left(\mathcal{D}^2ZX\bar{X}ZZ\right)+$$

$$0.894\mathrm{Tr}\left(\mathcal{D}^2ZYZZ\bar{Y}\right)+0.500\mathrm{Tr}\left(\mathcal{D}^2ZYZ\bar{Y}Z\right)+0.0528\mathrm{Tr}\left(\mathcal{D}^2ZY\bar{Y}ZZ\right)+0.894\mathrm{Tr}\left(\mathcal{D}^2Z\bar{Y}ZZY\right)+0.500\mathrm{Tr}\left(\mathcal{D}^2Z\bar{Y}ZYZ\right)+0.0528\mathrm{Tr}\left(\mathcal{D}^2Z\bar{Y}YZZ\right)-0.894\mathrm{Tr}\left(\mathcal{D}^2Z\bar{X}ZZX\right)-0.500\mathrm{Tr}\left(\mathcal{D}^2Z\bar{X}ZXZ\right)-$$

$$0.0528\mathrm{Tr}\left(\mathcal{D}^2Z\bar{X}XZZ\right)+1.45\mathrm{Tr}\left(\mathcal{D}^2Z\bar{Z}ZZZ\right)-0.894\mathrm{Tr}(\mathcal{D}\bar{\Psi}\Psi ZZ\bar{X})-0.500\mathrm{Tr}(\mathcal{D}\bar{\Psi}\Psi Z\bar{X}Z)-0.0528\mathrm{Tr}(\mathcal{D}\bar{\Psi}\Psi\bar{X}ZZ)+0.894\mathrm{Tr}(\mathcal{D}\bar{\Psi}\Psi ZZY)+0.500\mathrm{Tr}(\mathcal{D}\bar{\Psi}\Psi ZYZ)+0.0528\mathrm{Tr}(\mathcal{D}\bar{\Psi}\Psi YZZ)-$$

$$1.45\mathrm{Tr}(\mathcal{D}\bar{\Psi}\Psi ZZZ)-0.500\mathrm{Tr}(\mathcal{D}\bar{\Psi}Z\Psi Z\bar{X})+0.500\mathrm{Tr}(\mathcal{D}\bar{\Psi}Z\Psi ZY)-0.553\mathrm{Tr}(\mathcal{D}\bar{\Psi}Z\Psi ZZ)-0.0528\mathrm{Tr}(\mathcal{D}\bar{\Psi}ZZ\Psi\bar{X})+0.0528\mathrm{Tr}(\mathcal{D}\bar{\Psi}ZZ\Psi Y)-0.553\mathrm{Tr}(\mathcal{D}\bar{\Psi}ZZ\Psi Z)-1.45\mathrm{Tr}(\mathcal{D}\bar{\Psi}ZZZ\Psi)+$$

$$0.0528\mathrm{Tr}(\mathcal{D}\bar{\Psi}ZZY\Psi)-0.0528\mathrm{Tr}(\mathcal{D}\bar{\Psi}ZZ\bar{X}\Psi)+0.500\mathrm{Tr}(\mathcal{D}\bar{\Psi}ZYZ\Psi)-0.500\mathrm{Tr}(\mathcal{D}\bar{\Psi}Z\bar{X}Z\Psi)+0.0528\mathrm{Tr}(\mathcal{D}\bar{\Psi}Y\Psi ZZ)+0.500\mathrm{Tr}(\mathcal{D}\bar{\Psi}YZ\Psi Z)+0.894\mathrm{Tr}(\mathcal{D}\bar{\Psi}YZZ\Psi)-0.0528\mathrm{Tr}(\mathcal{D}\bar{\Psi}\bar{X}\Psi ZZ)-$$

$$0.500\mathrm{Tr}(\mathcal{D}\bar{\Psi}\bar{X}Z\Psi Z)-0.894\mathrm{Tr}(\mathcal{D}\bar{\Psi}\bar{X}ZZ\Psi)+0.894\mathrm{Tr}(\mathcal{D}\bar{\Psi}\Psi ZZ\bar{Y})+0.500\mathrm{Tr}(\mathcal{D}\bar{\Psi}\Psi Z\bar{Y}Z)+0.0528\mathrm{Tr}(\mathcal{D}\bar{\Psi}\Psi\bar{Y}ZZ)-0.894\mathrm{Tr}(\mathcal{D}\bar{\Psi}\Psi ZZX)-0.500\mathrm{Tr}(\mathcal{D}\bar{\Psi}\Psi ZXZ)-0.0528\mathrm{Tr}(\mathcal{D}\bar{\Psi}\Psi XZZ)+$$

$$1.45\mathrm{Tr}(\mathcal{D}\bar{\Psi}\Psi ZZZ)+0.500\mathrm{Tr}(\mathcal{D}\bar{\Psi}Z\Psi Z\bar{Y})-0.500\mathrm{Tr}(\mathcal{D}\bar{\Psi}Z\Psi ZX)+0.553\mathrm{Tr}(\mathcal{D}\bar{\Psi}Z\Psi ZZ)+0.0528\mathrm{Tr}(\mathcal{D}\bar{\Psi}ZZ\Psi\bar{Y})-0.0528\mathrm{Tr}(\mathcal{D}\bar{\Psi}ZZ\Psi X)+0.553\mathrm{Tr}(\mathcal{D}\bar{\Psi}ZZ\Psi Z)+1.45\mathrm{Tr}(\mathcal{D}\bar{\Psi}ZZZ\Psi)-$$

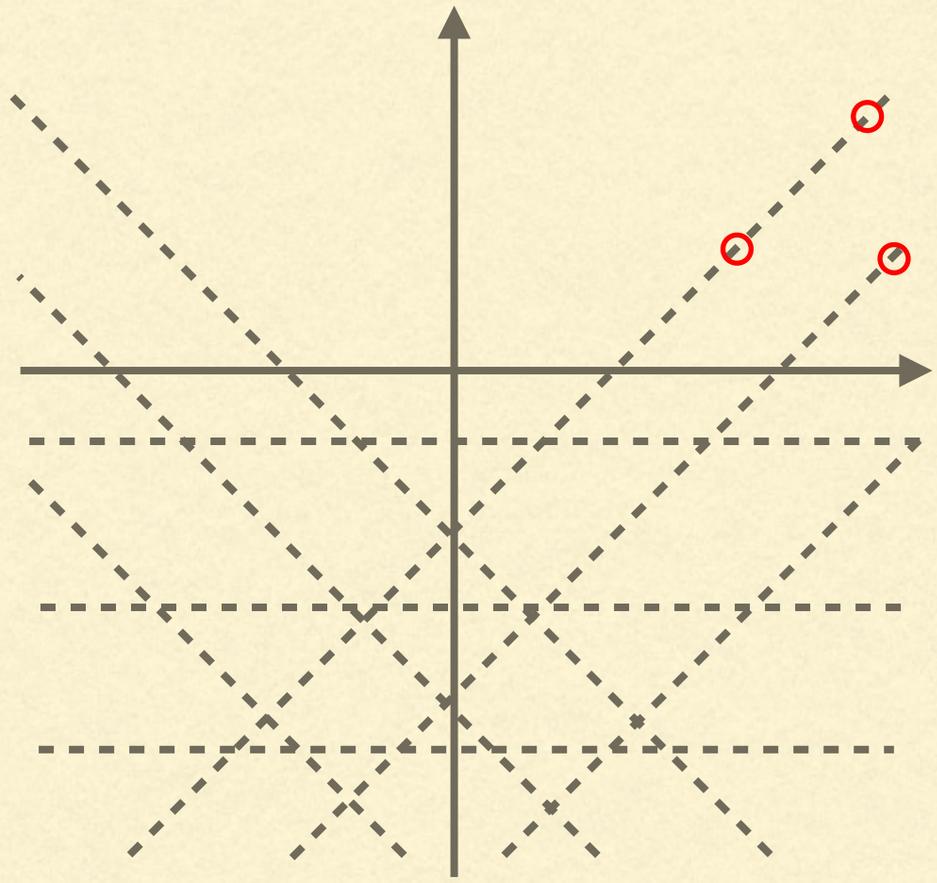
$$0.0528\mathrm{Tr}(\mathcal{D}\bar{\Psi}ZZX\Psi)+0.0528\mathrm{Tr}(\mathcal{D}\bar{\Psi}ZZ\bar{Y}\Psi)-0.500\mathrm{Tr}(\mathcal{D}\bar{\Psi}ZXZ\Psi)+0.500\mathrm{Tr}(\mathcal{D}\bar{\Psi}Z\bar{Y}Z\Psi)-0.0528\mathrm{Tr}(\mathcal{D}\bar{\Psi}X\Psi ZZ)-0.500\mathrm{Tr}(\mathcal{D}\bar{\Psi}XZ\Psi Z)-0.894\mathrm{Tr}(\mathcal{D}\bar{\Psi}XZZ\Psi)+0.0528\mathrm{Tr}(\mathcal{D}\bar{\Psi}\bar{Y}\Psi ZZ)+$$

$$0.500\mathrm{Tr}(\mathcal{D}\bar{\Psi}\bar{Y}Z\Psi Z)+0.894\mathrm{Tr}(\mathcal{D}\bar{\Psi}\bar{Y}ZZ\Psi)+1.45\mathrm{Tr}(\mathcal{F}\mathcal{F}ZZZ)+0.106\mathrm{Tr}(\mathcal{F}\Psi\Psi ZZ)+1.00\mathrm{Tr}(\mathcal{F}\Psi Z\Psi Z)+1.79\mathrm{Tr}(\mathcal{F}\Psi ZZ\Psi)-0.106\mathrm{Tr}(\mathcal{F}\Psi\Psi ZZ)-1.00\mathrm{Tr}(\mathcal{F}\Psi Z\Psi Z)-1.79\mathrm{Tr}(\mathcal{F}\Psi ZZ\Psi)+$$

$$0.553\mathrm{Tr}(\mathcal{F}Z\mathcal{F}ZZ)+1.00\mathrm{Tr}(\mathcal{F}Z\Psi Z\Psi)-1.00\mathrm{Tr}(\mathcal{F}Z\Psi Z\Psi)+0.553\mathrm{Tr}(\mathcal{F}ZZ\mathcal{F}Z)+0.106\mathrm{Tr}(\mathcal{F}ZZ\Psi\Psi)-0.106\mathrm{Tr}(\mathcal{F}ZZ\Psi\Psi)+1.45\mathrm{Tr}(\mathcal{F}ZZZ\mathcal{F})$$

General properties

States with fixed $(J_1, J_2, J_3 \,|\, S_2)$
+ discrete symmetries
form the surface



2nd order branch points,
but extra degeneracies
at weak coupling

Linear g dependence → Perturbative inversion of $\Delta(S_0) = 0$ breaks down

→ $g \mapsto -g$ connection between trajectories

Summary

- Analytic continuation for the twist-3 $\mathcal{O}_S$ trajectory
- Extended the QSC numerics
- Explored the Riemann surface and its connectivity
- Explicitly observed degeneracy of horizontal trajectories
- Resolved the degeneracy analytically
  → linear g dependence

Outlook

- Analytic continuation for the other twist-3 families

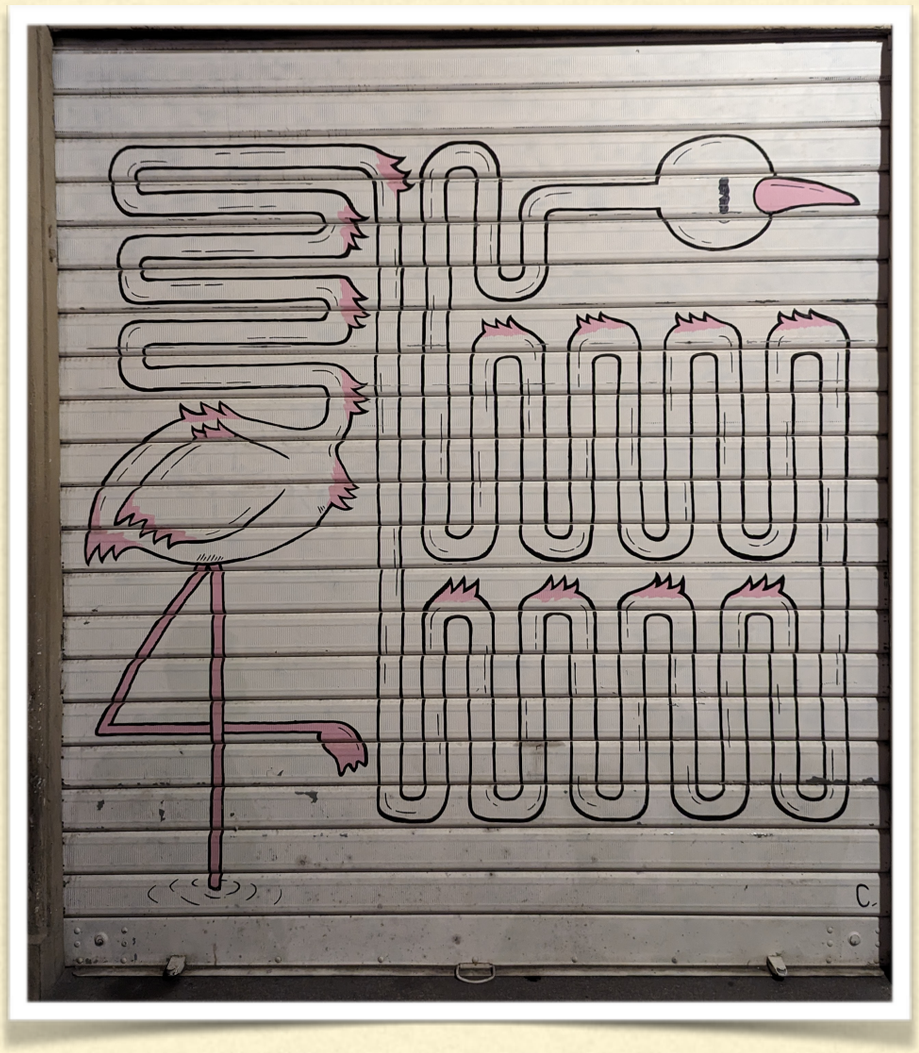
  [Homrich, Simmons-Duffin, Vieira '22]

- Understand the origin of the degeneracies for the horizontal trajectories

- Operatorial formulation for the degeneracy

- Find the $\mathcal{N} = 4$ Odderon intercept for all coupling

Thank you for your attention!



Tateo's snake-bird (?)
Street art, Bologna 2023(?)