Conformal Field Theory and Integrable Models

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Critical points of coupled Potts models and correlated percolation

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Based on :

G. Delfino, J. Viti, On three-point connectivity in two-dimensional percolation, J. Phys. A 44 (2011) 032001

G. Delfino, J. Viti, Potts q-color field theory and scaling random cluster model, Nucl. Phys. B 852 (2011) 149

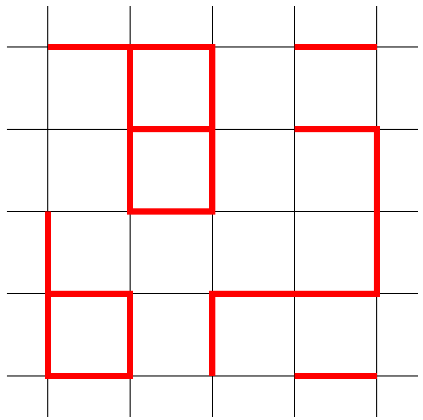
G. Delfino, M. Picco, R. Santachiara, J. Viti, Spin clusters and conformal field theory, J. Stat. Mech. (2013) P11011

N. Lamsen, Y. Diouane, G. Delfino, Critical points of coupled Potts models and correlated percolation, J. Stat. Mech. (2023) 013203

Outline

- percolation as basic example in critical phenomena
- ferromagnetism and percolation
- universality classes of critical clusters
- theoretical difficulties
- recent results from CFT

Random percolation

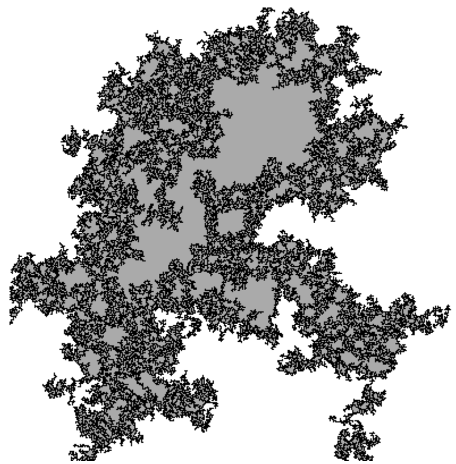


bond present with prob. p

weight $= p^{\#\,\text{bonds}}(1-p)^{\#\,\text{absent bonds}}$

for $p > p_c$ there is a probability $P > 0$ that a site belongs to a cluster extending to infinity

$P \sim (p - p_c)^{\beta}\,, \quad p \to p_c^{+}$

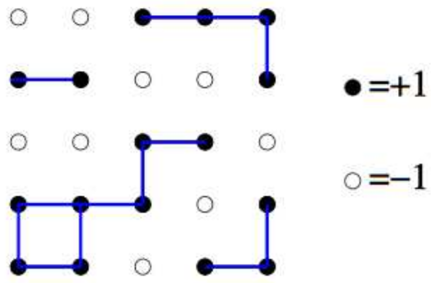


β related to fractal dimension

Ferromagnetism as correlated percolation

$$\mathcal{H}_{\text{Ising}} = -K \sum_{\langle i,j \rangle} s_i s_j \,, \qquad s_i = \pm 1$$

- Fisher '67: spontaneous magnetization below T_c produced by percolation of spin clusters; requirements:



1. percolation occurs at T_c
2. percolative and magnetic exponents coincide

- problem: spin clusters found to percolate away from T_c in 3D Ising [Muller-Krumbhaar '74]

- way out: Fortuin-Kasteleyn (FK) clusters percolate at T_c in any dimension and yield magnetic exponents [Coniglio,Klein '80]

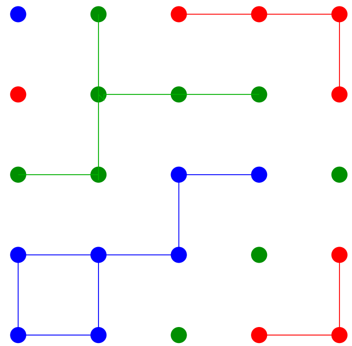
Percolation and the r-state Potts model

Potts model : $\mathcal{H} = -K \sum_{\langle i,j \rangle} \delta_{s_i, s_j}$, $\quad s(x) = 1, \ldots, r$

symmetry: S_r, permutations of r "colors"

$$Z = \sum_{\{s(x)\}} e^{-\mathcal{H}} = \text{[Fortuin, Kasteleyn '69]} \qquad p = 1 - e^{-K}$$

$$\propto \sum_{\text{bond configs}} p^{\#\,\text{bonds}} (1-p)^{\#\,\text{absent bonds}}\, r^{\#\,\text{clusters}}$$



- each cluster can take r colors
- realizes analytic continuation in r
- random percolation for $r \to 1$, correlated percolation otherwise (Ising for $r = 2$)

Spin and FK clusters in the 2D Ising model

- in $d = 2$ spin clusters also percolate at T_c [Coniglio et al '77] with new fractal dimension [Sykes, Gaunt '76]

- unified RG picture for Ising spin and FK clusters [Coniglio, Klein '80]:

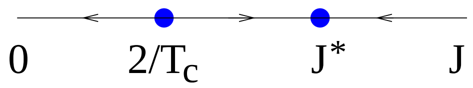
– couple Ising to auxiliary $r \to 1$ Potts

$$\mathcal{H} = \mathcal{H}_{Ising} - J \sum_{\langle ij \rangle} t_i t_j \left(\delta_{s_i, s_j} - 1 \right), \qquad t_i = 0, 1, \qquad s_i = 1, \ldots, r$$

$$p \equiv 1 - e^{-J}$$

$$\begin{aligned} Z &= \sum_{\{t_i\}} \sum_{\{s_i\}} e^{-\mathcal{H}} = \\ &= \sum_{\{t_i\}} e^{-\mathcal{H}_{Ising}} r^{\#\, vacancies} \sum_{\{bonds\, between\, +\, spins\}} r^{\#\, clusters}\, p^{\#\, bonds}\, (1-p)^{\#\, absent\, bonds} \end{aligned}$$

– approximate RG at T_c yields two fixed points for J:



$J = 2/T_c$: fixed point for FK clusters

$J = J^*$: fixed point for spin clusters ($p = 1$)

– correspond to critical Ising and tricritical $r \to 1$ Potts ($c = 1/2$ for both) cluster size $\sim$ (linear extension)^D

$D = 2 - X_s$ fractal dimension

X_s spin field scaling dimension

$$\mathrm{CFT}: \qquad D = \begin{cases} 15/8 = 1.87.. \ \text{Ising FK clusters} \\ 187/96 = 1.94.. \ \text{Ising spin clusters} \\ 91/48 = 1.89.. \ \text{random percolation clusters} \end{cases}$$

Conjectured generalization to q-state Potts clusters

- couple q-state Potts to auxiliary $r \to 1$ Potts:

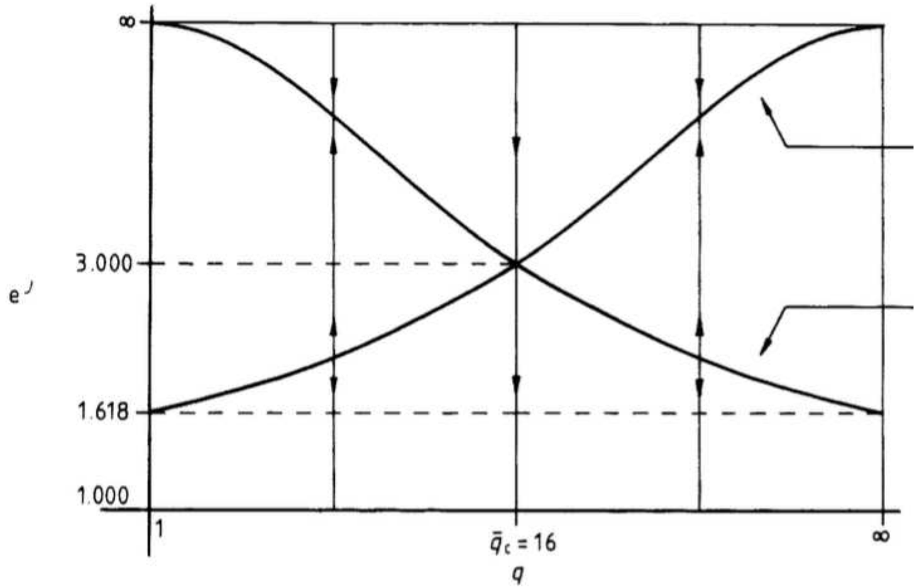
$$\mathcal{H}_{q,r} = -J_1 \sum_{\langle i,j \rangle} \delta_{s_{i,1},s_{j,1}} - J_2 \sum_{\langle i,j \rangle} \delta_{s_{i,2},s_{j,2}} - J \sum_{\langle i,j \rangle} \delta_{s_{i,1}s_{j,1}} \delta_{s_{i,2},s_{j,2}} \,, \qquad s_{i,1} = 1, \ldots, q; \;\; s_{i,2} = 1, \ldots, r$$

bond between spin of same color with probability $p = 1 - e^{-J}$

- approximate RG analysis

[Coniglio,Peruggi '82]

conjecture: spin branch and FK branch coalesce and terminate at $q = 4$



- further conjecture [Vanderzande '92]: **FK and spin branches** related to critical and tricritical Potts, **analytical continuation of each other** $\to$ formula for fractal dimension of spin clusters in good agreement with simulations $\to$ conjecture accepted

Potts clusters and CFT

- fundamental percolative observables are the connectivities

$P_n(x_1,\ldots,x_n) =$ prob. $x_1, ..., x_n$ in same cluster

related to correlators of Potts spins σ_α, $\alpha = 1,\ldots,q$:

$$P_2 = \frac{\langle\sigma_\alpha\sigma_\alpha\rangle}{q-1}, \qquad P_3 = \frac{\langle\sigma_\alpha\sigma_\alpha\sigma_\alpha\rangle}{(q-1)(q-2)}, \qquad \cdots$$

- P_2 determined by scaling dimension X_σ, known for FK clusters from lattice Coulomb gas [Nienhuis '82], and corresponding to [Dotsenko,Fateev '84] $X_\sigma = X_{1/2,0}$

$$X_{m,n} = \frac{[(t+1)m-tn]^2-1}{2t(t+1)} \qquad c = 1-\frac{6}{t(t+1)} \qquad \sqrt{q} = 2\sin\frac{\pi(t-1)}{2(t+1)}$$

- σ_α nondegenerate for q generic $\Rightarrow$ no differential eqs for connectivities !

becomes degenerate on a boundary $\rightarrow$ crossing probabilities [Cardy '92,Watts '96, ...,Smirnov '01,....]

Three-point connectivity of FK clusters [GD,Viti '11]

$$P_3 = R\sqrt{P_2(x_1,x_2)P_2(x_1,x_3)P_2(x_2,x_3)} \qquad R \text{ universal}$$

color symmetry factorizes at three-point level $\Rightarrow$

$$R = \sqrt{2}\, C_{X_\sigma X_\sigma X_\sigma}$$

$C_{X_\sigma X_\sigma X_\sigma}$ OPE coefficient of colorless theory = analytic continuation of minimal model OPE coefficients

OPE coefficients in $c<1$ CFT [Al.Zamolodchikov '05; Kostov,Petkova '06]:

$$
\begin{aligned}
C_{\Delta_1,\Delta_2,\Delta_3} &= \frac{A\,\Upsilon(2\beta-\beta^{-1}+a_1+a_2+a_3)}{[\Upsilon(2a_1+\beta)\Upsilon(2a_1+2\beta-\beta^{-1})]^{\frac{1}{2}}}\times \\
&\times \frac{\Upsilon(a_1+a_2-a_3+\beta)\Upsilon(a_2+a_3-a_1+\beta)\Upsilon(a_3+a_1-a_2+\beta)}{[\Upsilon(2a_2+\beta)\Upsilon(2a_2+2\beta-\beta^{-1})\Upsilon(2a_3+\beta)\Upsilon(2a_3+2\beta-\beta^{-1})]^{\frac{1}{2}}}
\end{aligned}
$$

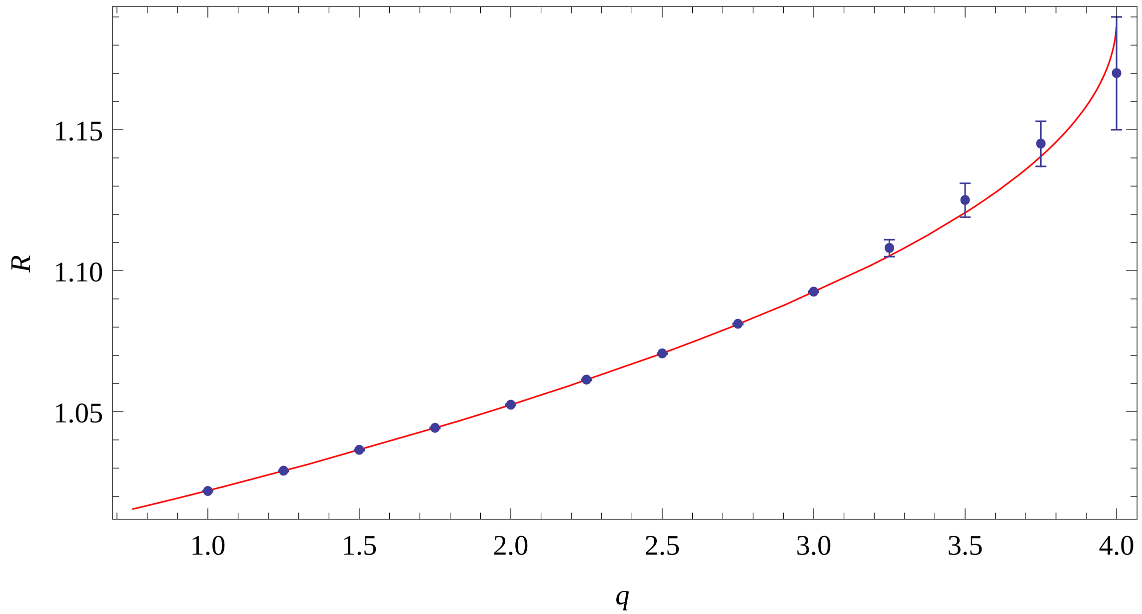
$$c=1-\frac{6}{t(t+1)} \qquad \beta=\sqrt{t/(t+1)} \qquad \Delta_i=a_i(a_i+\beta-\beta^{-1})$$

$$A=\frac{\beta^{\beta^{-2}-\beta^2-1}[\gamma(\beta^2)\gamma(\beta^{-2}-1)]^{1/2}}{\Upsilon(\beta)} \qquad \gamma(x)\equiv\frac{\Gamma(x)}{\Gamma(1-x)}$$

$$\Upsilon(x)=\exp\left\{\int_0^\infty\frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2e^{-t}-\frac{\sinh^2\left[\left(\frac{Q}{2}-x\right)\frac{t}{2}\right]}{\sinh\frac{\beta t}{2}\sinh\frac{t}{2\beta}}\right]\right\} \qquad Q=\beta+\beta^{-1}$$

analytic continuation of minimal model OPE coefficients, up to special cases where finite ("mysterious") numbers are obtained instead of zero

Monte Carlo verification [Ziff,Simmons,Kleban '11] ($q = 1$);
[Picco,Santachiara,Viti,GD '13] (q generic)

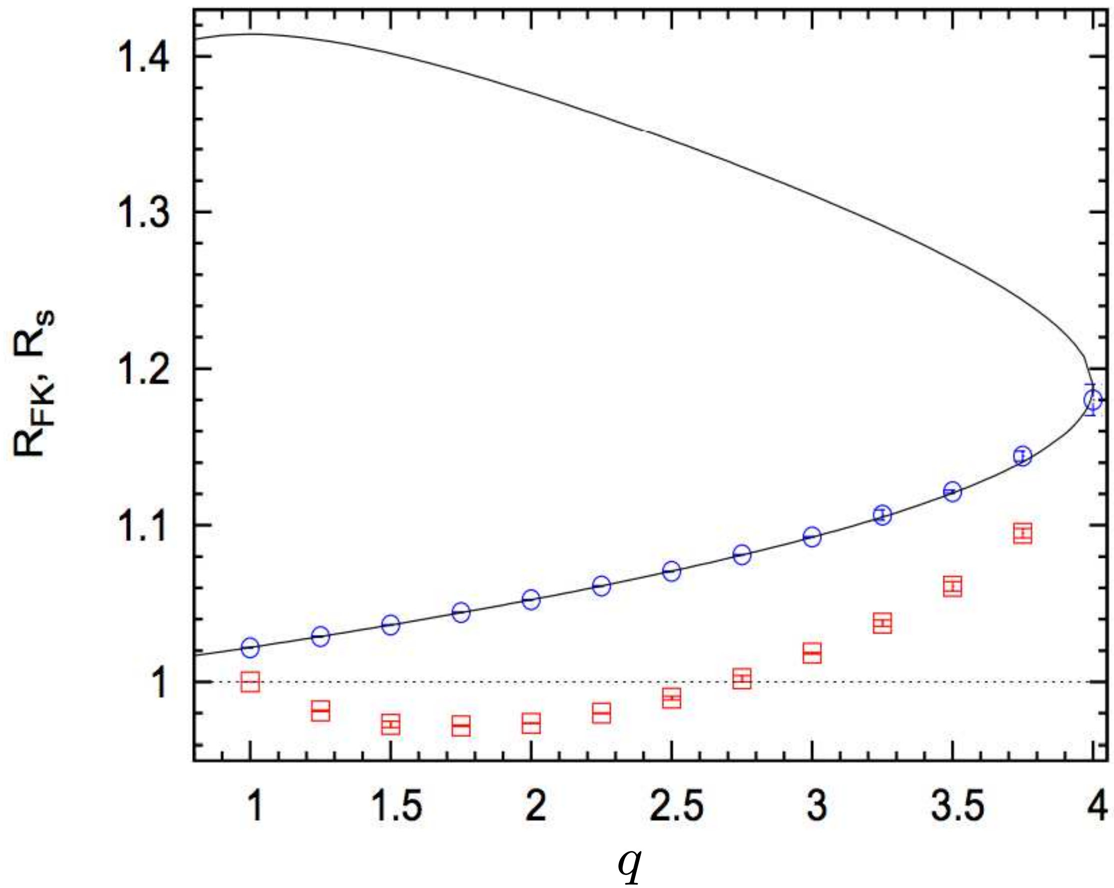


- $q = 2$ sheds light on Zamolodchikov's "mysterious numbers"

- no color symmetry factorization for $P_{n>3}$ [GD,Viti '11]

P_4 studied by semi-analytic conformal bootstrap $\rightarrow$ Ribault's talk
$\rightarrow$ 26 digit verification of P_3 [Nivesvivat, Ribault '21]

Spin clusters from FK clusters? [GD,Picco,Santachiara,Viti '13]

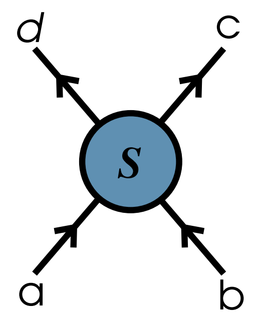


the conjecture that the properties of spin clusters can be obtained from those of FK clusters by analytic continuation is not generally true. Where does it fail?

Exact search for RG fixed points [GD '13]

- Euclidean field theory in 2D $\longleftrightarrow$ relativistic quantum field theory in (1+1)D $\longrightarrow$ particles

- at criticality ∞-dimensional conformal symmetry makes **scattering completely elastic**



- center of mass energy only relativistic invariant, dimensionful $\Rightarrow$ **constant amplitudes** by scale invariance

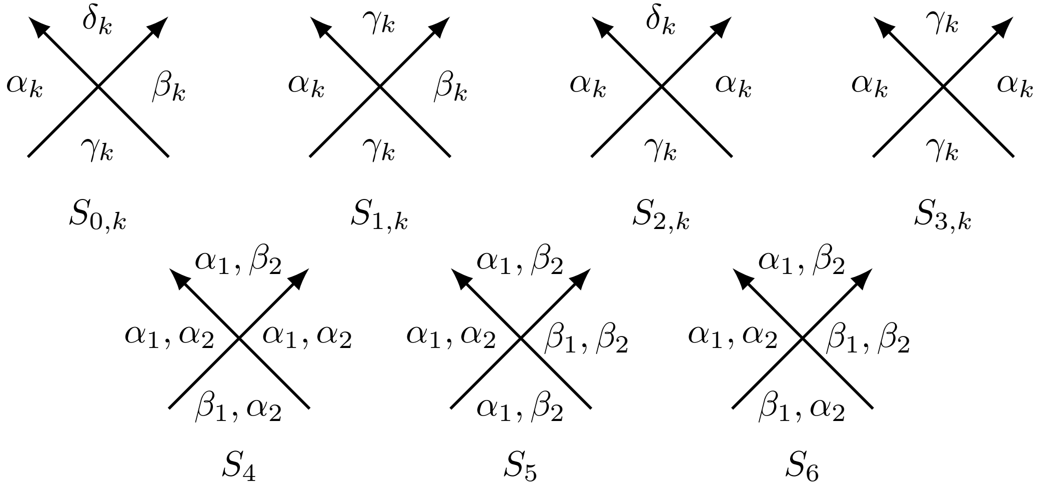
crossing: $S_{ab}^{cd} = [S_{ad}^{cb}]^*$

unitarity: $\sum_{ef} S_{ab}^{ef} [S_{ef}^{cd}]^* = \delta_{ac}\delta_{bd}$

- solutions are RG fixed points with the symmetry implemented by the particle basis

- used for quenched disorder [GD '17], liquid crystals [GD,Diouane, Lamsen '21], ...

Critical points of coupled q-state and r-state Potts models [Lamsen,Diouane,GD '23]



particles: $\alpha_k \uparrow \beta_k, \quad k = 1, 2$

$\alpha_1 \neq \beta_1 = 1, \ldots, q$

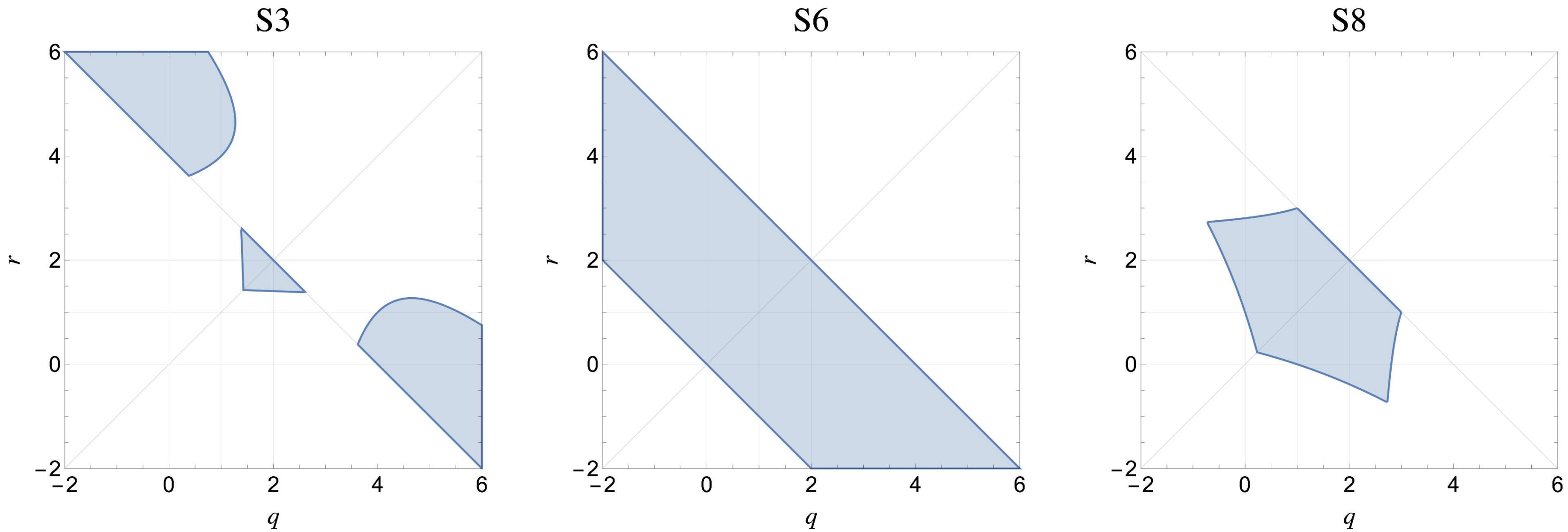
$\alpha_2 \neq \beta_2 = 1, \ldots, r$

$$S_{0,k} = S^*_{0,k} \equiv \rho_{0,k}, \qquad S_{1,k} = S^*_{2,k} \equiv \rho_{1,k} e^{i\phi_k}, \qquad S_{3,k} = S^*_{3,k} \equiv \rho_{3,k},$$

$$S_4 = S^*_5 \equiv \rho_4 e^{i\theta}, \qquad S_6 = S^*_6 \equiv \rho_6$$

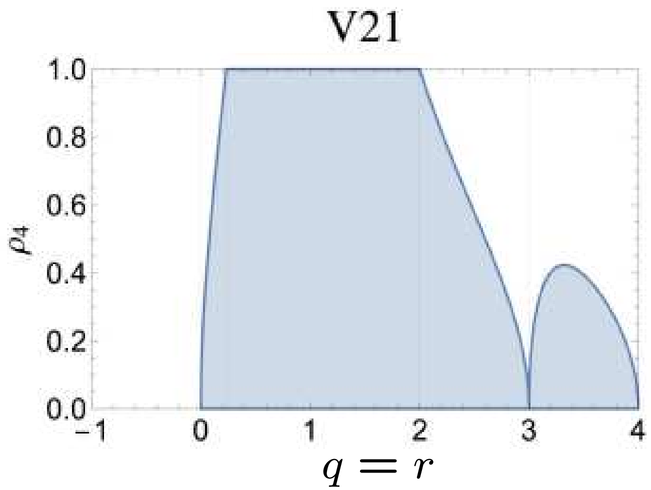
$$0=(\text{q-4})\rho^2_{0,1} + 2\rho_{1,1}\rho_{0,1}\cos\phi_1, \qquad 0 = (r-4)\rho^2_{0,2} + 2\rho_{1,2}\rho_{0,2}\cos\phi_2,$$
$$1 = (q-3)\rho^2_{0,1} + \rho^2_{1,1}, \qquad 0 = (q-3)\rho^2_{1,1} + 2\rho_{1,1}\rho_{3,1}\cos\phi_1 + (r-1)\rho^2_4,$$
$$1 = (r-3)\rho^2_{0,2} + \rho^2_{1,2}, \qquad 0 = (r-3)\rho^2_{1,2} + 2\rho_{1,2}\rho_{3,2}\cos\phi_2 + (q-1)\rho^2_4,$$
$$1 = (q-2)\rho^2_{1,1} + \rho^2_{3,1} + (r-1)\rho^2_4, \qquad 1 = (r-2)\rho^2_{1,2} + \rho^2_{3,2} + (q-1)\rho^2_4,$$
$$0 = \rho_4\left[\rho_{3,2}e^{i\theta} + \rho_{3,1}e^{-i\theta} + (q-2)\rho_{1,1}e^{-i(\theta+\phi_1)} + (r-2)\rho_{1,2}e^{i(\theta+\phi_2)}\right],$$
$$1 = \rho^2_4 + \rho^2_6, \qquad 0 = 2\rho_4\rho_6\cos\theta$$

solutions (= RG fixed points) exist in domains of q-r plane, e.g.



- equations very restrictive for q,r integers > 1
- for coupled ferromagnets only $q = r = 2$ (Ashkin-Teller) $\rightarrow$ lines of fixed points
- V21 yields coupled antiferromagnets

studied on the lattice [Au-Yang,Perk '92; Martins,Nienhuis '98; Fendley,Jacobsen '08; Vernier,Jacobsen,Saleur '14]



$r \to 1:$ we solve exactly the RG problem studied approximately in [Coniglio,Peruggi '82]

• for $r = 1 + \epsilon$ the range $q \in [2, 4]$ is covered only piecewisely $\Rightarrow$ no general analytical continuation from FK to spin clusters is possible

• this explains, in particular, the numerically observed failure of the analytic continuation conjecture for P_3

• the conjectured formula for the spin cluster fractal dimension, so far consistent with simulations, can be exact if it can be derived directly *at* $r = 1$

Conclusion

- different types of clusters can be simultaneously critical in 2D spin models

- study of these universality classes made difficult by need of subtle analytical continuations

- recent progress for connectivities by CFT methods beyond minimal model technology

- scattering framework allows exact search for critical points and sheds light on old conjectures