# Structure Constants of Short Operators in $N = 4$ Super Yang-Mills

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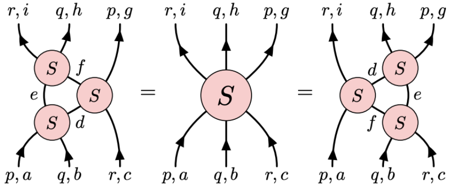
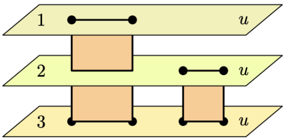
# Introduction

- Integrability: powerful tool to solve 1+1d models
  - Spin chains, vertex models, sigma models, PDEs
  - Mathematical structures: Bethe ansatz, Quantum groups, Algebraic curve, Inverse scattering... (Nikla's talk)
- 4d Quantum Field Theories: ubiquitous in physics
  - Complicated: perturbation theory or Monte Carlo
  - Exact results ?
  - Integrability ? Same as 2d ?



# Planar $\mathcal{N} = 4$ Super Yang-Mills

- Not only Superconformal $PSU(2,2|4)$: Integrable !
- 2pt functions: spectrum of anomalous dimensions: $\Delta = \Delta_0 + \gamma(g)$
  - Weak coupling: integrable quantum Spin chain [Minahan, Zarembo]
  - Strong coupling: integrable (semi)classical Sigma model [Bena, Polchinski, Roiban]
  - Asymptotic finite coupling $su(2|2)^2$ Bethe Ansatz [Beisert, Eden, Staudacher]
  - Finite size corrections: TBA [Bombardelli, Fioravanti, Tateo, Arutyunov,...] $\Rightarrow$ Y/T-System $\Rightarrow$ Quantum Spectral Curve [Gromov, Kazakov, Leurent, Volin]
- 3pt functions: Structure Constants $C_{123}$
  - Integrable bootstrap: Hexagonalization [Basso, Komatsu, Vieira]
  - Finite size corrections ? Mismatch at 5-loops [Geogourdis, Gonçalves]
  - Conjectured solution using QSC elements

# Overview

1. Planar $N = 4$ Super Yang-Mills in a nutshell
2. Spectrum: T-system and QSC
3. Dressing hexagons and checks

# Planar $N = 4$ Super Yang-Mills

- Simplest gauge theory in 4d
- Field content $\boxed{A_\mu, \varphi^I, \psi^a, \bar{\psi}_a}$ → Adjoint rep. of $SU(N)$
- Single trace operators:

$$\boxed{O(x) = \mathrm{Tr}[D^{j_1}(\phi^I(x))^{j_2}...(\psi^a(x))^{j_n}...]}$$

- Planar limit ⇒ integrable

$$\boxed{g = \frac{\sqrt{\lambda}}{4\pi}, \quad \lambda = g_{YM}^2 N, \quad g_{YM} \to 0, \quad N \to \infty}$$

- Conformal invariance: only need $(\Delta, C_{123})$

# Global symmetry group + integrability

- CFT + SUSY $\Rightarrow$ $psu(2,2|4)$: similar to $gl(4|4)$
  - Bosonic part $SU(2,2) \times SU(4)$ + fermionic part
  - Cartan charges: $(\Delta; S_1, S_2; J_1, J_2, J_3)$
- Spectrum of Anomalous dimensions $\Delta = \Delta_0 + \gamma(g)$
  - Operators mapped to spin chain states [Minahan, Zarembo]

Vacuum: $\mathrm{Tr}[Z^L]$ Excited states: $\mathrm{Tr}[Z^L X^M], \mathrm{Tr}[Z^L D^M]$...

  - Complex scalars: $Z$ and $X$
  - $su(2)$ and $sl(2)$ sectors
  - $\gamma(g)$ mapped to energy $E$ of integrable model

# Our mantra

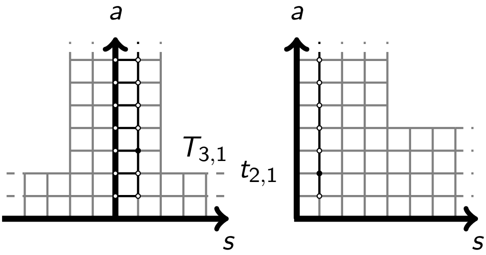
Algebra (symmetry) + Model input (analiticity) $\Rightarrow$ Spectrum

# T-System

- Full finite size spectrum of $\mathcal{N} = 4$ SYM: commuting transfer matrices $\mathbb{T}_{a,s}$ satisfying Hirota equation:

$$\mathbb{T}^{+}_{a,s}\mathbb{T}^{-}_{a,s} = \mathbb{T}_{a+1,s}\mathbb{T}_{a-1,s} + \mathbb{T}_{a,s+1}\mathbb{T}_{a,s-1}$$

  - Notation: $\mathbb{T}^{[+n]}_{a,s} = \mathbb{T}_{a,s}\left(u + n\frac{i}{2}\right)$
- Algebra: boundary conditions
  - Hooks defining representations of $psu(2,2|4)$
  - Non compact T-hook and compact L-hook



# T-system

- Model input: analiticity conditions
  - Large gauge invariance $\mathbb{T}_{a,s} \to g_1^{[a+s]} g_2^{[a-s]} g_3^{[s-a]} g_4^{[-a-s]} \mathbb{T}_{a,s}$
  - Black-magic gauge $\mathbf{T}$: small Zhukovsky cuts and

$$\mathbf{T}_{a-1,0}, \mathbf{T}_{a,\pm 1}, \mathbf{T}_{a+1,2} \text{ have no cuts inside the strip } |Im(u)| < \frac{a}{2}$$

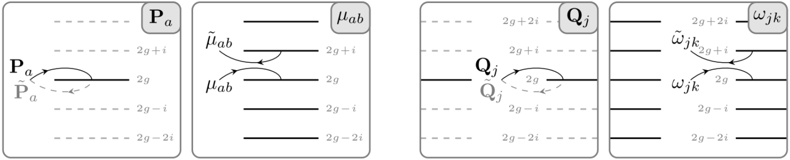
  - Complicated cut structure consistent with Type IIB classical strings in $AdS^5 \times S_5$ [Gromov, Vieira]
- Contain Y-system and TBA since

$$Y_{a,s} = \frac{\mathbf{T}_{a,s+1}\mathbf{T}_{a,s-1}}{\mathbf{T}_{a+1,s}\mathbf{T}_{a-1,s}}$$

# Quantum Spectral Curve

- Integrability of T-system: Wronskian solution
- Only need $4+4$ elementary Q-functions that satisfies the monodromy properties:

$$\tilde{P}_a = \mu_{ab} P^b, \qquad \tilde{Q}_i = \omega_{ij} Q^j$$




- $\mathrm{P}\mu$ and $\mathrm{Q}\omega$ systems: Riemann-Hilbert problems linked via the central $Q$-function

$$Q^+_{a|i} - Q^-_{a|i} = P_a Q_i$$

# QSC - Asymptotics

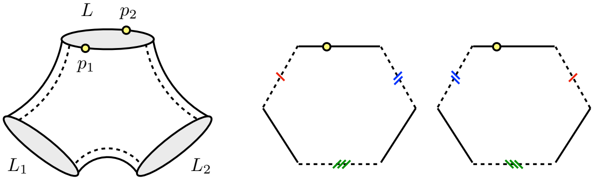
- Large $u$ asymptotics $\Rightarrow$ charges of state !

$$P_a \sim A_a u^{-\tilde{M}_a}, \quad P^a \sim A^a u^{\tilde{M}_a - 1}, \quad Q_i \sim B_i u^{\hat{M}_i - 1}, \quad Q^i \sim B^i u^{-\hat{M}_i}$$

$$\tilde{M}_a = \left\{ \frac{J_1 + J_2 - J_3 + 2}{2}, \frac{J_1 - J_2 + J_3}{2}, \ldots \right\}$$
$$\hat{M}_i = \left\{ \frac{\Delta - S_1 - S_2 + 2}{2}, \frac{\Delta + S_1 + S_2}{2}, \ldots \right\}$$

- Given operator charges, can obtain $\Delta$
- Any local, some non-local (cusped Wilson-lines) operators, BFKL, Hagedorn temperature... [Ekhammar, Gromov, Kazakov, Levkovich Maslyuk, Minahan,...]
- 11 loops $\gamma(g)$ for Konishi $\mathrm{Tr}[D^2 Z^2]$ [Marlboe, Volin]

# Hexagons and Structure Constants

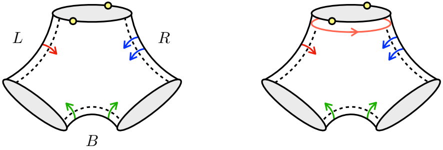



- Cut open the spin-chain/worldsheet ⇒ Hexagons: building blocks of correlators ⇒ Integrable bootstrap à la Castro-Alvaredo [Basso, Komatsu, Vieira]
- Large operators + large bridge ⇒ asymptotic factorization, $su(2|2)^2$ S-matrix

$$C_{123} = \frac{1}{N_G} \sum_{\alpha \cup \bar{\alpha} = \{1,\dots,M\}} (-1)^{|\bar{\alpha}|} \prod_{j \in \bar{\alpha}} e^{i p_j \ell_R} \prod_{\substack{j<k \\ j \in \bar{\alpha}, k \in \alpha}} S\left(u_i, u_j\right) h(\alpha) h(\bar{\alpha})$$

# Hexagons and Structure Constants

- Short operators ? Wrapping corrections: insert complete basis of states at the seams ⇒ mirror magnons




- Special configurations: 2 BPS, 1 non protected $sl(2)$ ⇒ Study Lüscher corrections $(e^{-\mathbf{E}L})$ and double wrapping and extrapolate

$$C^{\circ\circ\bullet} \sim \langle [Z_1^{L_1}][Z_2^{L_2}][D^S Z^L] \rangle$$

- Gluing back the hexagons:

$$C^{\circ\circ\bullet} = N \times \sum\!\!\!\!\!\!\int_L \times \sum\!\!\!\!\!\!\int_R \times \sum\!\!\!\!\!\!\int_B e^{-\ell_L \mathbf{E}_L - \ell_R \mathbf{E}_R - \ell_B \mathbf{E}_B} \, |H|^2$$

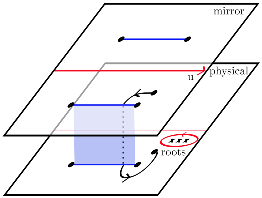
# Hexagons and Structure Constants

- Integrals given by:

$$\not{\sum} = \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{i=1}^{N} \sum_{a_i=1}^{\infty} \int_C \frac{du_i}{2\pi} \mu_{a_i}(u_i) \prod_{i<j}^{N} p_{a_i a_j}(u_i, u_j)$$

- Measure has double poles $\mu^2(u)$!

$$p_{ab}(u,v) = k_{ab}^{++} k_{ab}^{+-} k_{ab}^{-+} k_{ab}^{--}, \quad k_{ab}^{\pm\pm} = \frac{x^{[\pm a]} - y^{[\pm b]}}{x^{[\pm a]} y^{[\pm b]} - 1}$$




- Integration countour: along real axis + encircle Bethe roots
- Shift poles:
  $p_{ab}(u,v) \to p_{ab}(u+i0, v-i0)$
- Equivalent to sum over partitions

[Komatsu, Kostov, Serban]

# Spectrum enters the game

- Probability weights factorizes

$$|H|^2 = \prod_{i,j,k=1}^{N_{L,R,B}} \frac{W^{L}_{a_i}(u_i)\, W^{R}_{b_j}(v_j)\, W^{B}_{c_k}(w_k)}{p_{a_i b_j}(u_i, v_j)}$$

- Weights are full full $psu(2,2|4)$ Transfer matrices $\Rightarrow$ wrapping uplift symmetry

$$W_a^{R/L}(u) = e^{\frac{1}{2} L \boldsymbol{E}_a(u)} \frac{\mathbf{T}_{a,1}(u)}{\mathbf{T}^{+/-}_{a,0}(u)}, \quad W_a^{B}(u) = e^{-\frac{1}{2} L \boldsymbol{E}_a(u)} \mathbf{t}_{a,1}(u)$$

- Bottom channel: compact rep + meromorphic $\Rightarrow$ no magnon transport !

# Checks

- Asymptotic limit: $\mathbf{T}_{a,1}$ after analytical continuation and $L \to \infty$ reduces to $2|2$ transfer matrices

$$\mathbf{T}_{a,1} \to \frac{\mathbb{T}_a^{2|2}(u)}{h(u)} + \mathbb{T}_a^{2|2}(u^{2\gamma})h(u^{2\gamma})$$

  - Proof using generating function from [Kazakov, Leurent, Volin]
  - OK for L and R channels
- Bottom channel check: focus on ratio

$$R(\ell_B) = C^{\circ\circ\bullet} / \lim_{\ell_B \to \infty} C^{\circ\circ\bullet}$$

  - Single sum of mirror integrals with 1, 2, 3... magnons

$$R = 1 + \sum_{a=1}^{\infty} \int \frac{du}{2\pi} \mu_a(u) e^{-\ell_B \mathbf{E}_a(u)} W_a^B(u) + \dots$$

# Weak coupling

- $N$-particle terms appears at $N(\ell_B + N)$ loops at the earliest, but first wrapping correction is at $L + \ell_B + 2$ loops
  - Konishi $\mathrm{Tr}[D^2 Z^2]$ with $\ell_B = 1$: can test wrapping at 5 loops with 1 particle
- Solve weight in terms of QSC $Q$ functions

$$\mathbf{t}_{a,1}(u) = -\sum_{j=1}^{4} \mathbf{Q}_j(u + ia/2)\tilde{\mathbf{Q}}^j(u - ia/2)$$

- Obtain $Q$ functions numerically with the solver from [Marboe, Volin] (or [Gromov, Hegedus, Julius, Sokolova] if you're on a rush !)
- Integration by residues reproduces 5-loop result from [Georgoudis, Gonçalves]

$$\frac{\delta R}{g^{10}} = 972\zeta_3 - 2700\zeta_5 + 5355\zeta_7 - 2376\zeta_3\zeta_5 - 1512\zeta_9$$

# Strong coupling

- Q functions reduces to characters depending on string quasi-momenta $\left\{e^{ip_1}, e^{ip_2}, e^{ip_3}, e^{ip_4} \mid e^{ip_{\hat{1}}}, e^{ip_{\hat{2}}}, e^{ip_{\hat{3}}}, e^{ip_{\hat{4}}}\right\}$ and solving classical Hirota equation
- Sum exponentiate and can be done via Clustering [Komatsu, Kostov, Serban]

$$\log R_{\rm st} = \int_{U^+} \frac{du(x)}{2\pi} \int_0^{\xi} \frac{dq}{q} \log\left[\sum_{a=0}^{\infty} q^a \mathbf{t}_{a,1}(x)\right]$$

  - Generating function for 4|4 characters [Gromov, Kazakov, Tsuboi]

$$\sum_{a=0}^{\infty} q^a\, \mathbf{t}_{a,1} = \prod_{j=1}^{4} \frac{1 - q\, e^{ip_j(x)}}{1 - q\, e^{ip_{\hat{j}}(x)}}$$

- Integration is straightforward and agrees with string theory [Komatsu]

$$\log R_{\rm st} = \int_{U^+} \frac{du(x)}{2\pi} \sum_{j=1}^{4} (\mathrm{Li}_2(\xi e^{ip_j}) - \mathrm{Li}_2(\xi e^{ip_{\hat{j}}}))$$

# Conclusion

- QSC/T-system: powerful mathematical tool to study multiple observables beyond spectrum
  - Insights from classical and quantum integrability
  - Origins of integrability ?
- Uplift from asymptotic to full superconformal symmetry
- Extend to other configurations and sectors
  - $SO(6)$ excited operator [Alday, Hansen, Silva]
  - Spinning hexagons [Bercini, Gonçalves, Homrich, Vieira]
- SoV representation ? [Bercini, Homrich, Vieira], [Kostov, Lefundes, Levkovich-Maslyuk, Serban, AKS]
- Generalise to other theories ? ABJM ?

# Thanks for your attention!