On the origin of the correspondence between integrable models and differential equations.

A possible explanation of the ODE/IM correspondence

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A quick recap of the ODE/IM correspondence

- Consider the ODE (Schroedinger) $-\frac{d^2}{dx^2}\psi(x) + \left(\frac{l(l+1)}{x^2} + x^{2M}\right)\psi(x) = E\psi(x)$
- A solution $y(x) \sim x^{-M/2} \exp\left(-\frac{x^{M+1}}{M+1}\right)$ as $x \to +\infty$
- Two solutions $\chi_+ \sim x^{l+1}\,, \quad \chi_- \sim x^{-l}$ as $x \to 0$

Define connection coefficients $Q_\pm(E)$: $y(x) = Q_+(E)\chi_-(x) + Q_-(E)\chi_+(x)$

$Q_\pm(E) \sim W[y, \chi_\pm]$ are highest weight state eigenvalues of Q-operators (Q-functions) of CFT minimal models Dorey,Tateo; Bazhanov, Lukyanov, Zamolodchikov '98

$$c = 1 - 6(\beta - \beta^{-1})^2\,, \quad \Delta = \left(\frac{p}{\beta^2}\right)^2 + \frac{c-1}{24}$$
$$M = 1/\beta^2 - 1\,, \quad l = \frac{2p}{\beta^2} - \frac{1}{2}$$

- $Q_{\pm}(E)$ are entire functions of E
- They satisfy QQ-system, functional relations

$Q_+(Ee^{\frac{2i\pi}{M+1}})Q_-(E) - Q_+(E)Q_-(Ee^{\frac{2i\pi}{M+1}}) = const$

- Defining other solutions of the ODE $y_{\pm 1}(x) \sim y(x^{\mp\frac{i\pi}{M+1}}, E^{\pm\frac{2i\pi}{M+1}})$

$W[y_{-1}, y_1] = T(E)$, transfer matrix, entire function of E

- Functional relation TQ-system

$T(E)Q_{\pm}(E) = e^{\mp\frac{i\pi(2l+1)}{2M+2}} Q_{\pm}(Ee^{\frac{2i\pi}{M+1}}) + e^{\pm\frac{i\pi(2l+1)}{2M+2}} Q_{\pm}(Ee^{-\frac{2i\pi}{M+1}})$

• Generalisation to off-critical case: PDEs

$$(\partial_w + V(w,\bar{w}))\Psi = (\partial_{\bar{w}} + \bar{V}(w,\bar{w}))\Psi = 0\,,$$

V, $\bar{V}$ 2x2 matrices: a Lax pair.

One considers V and $\bar{V}$ depending on a field $\hat{\eta}$ such that the compatibility condition

$$\partial_w \bar{V} - \partial_{\bar{w}} V + [V, \bar{V}] = 0$$

implies for $\hat{\eta}$

$$\partial_w \partial_{\bar{w}} \hat{\eta} = 2 \sinh 2\hat{\eta}\,,$$

i.e. the classical sinh-Gordon equation.

For a suitable solution of sinh-Gordon equation connection coefficients between different vector solutions Ψ are Q-functions of sine-Gordon model

Gaiotto-Moore-Neitzke '08,'09; Lukyanov, Zamolodchikov '10

Plan: give an explanation for ODE/IM

Why does ODE/IM appear? To answer this question, we reverse the arrow. We start from quantum integrable field theories: we suppose Baxter's TQ-relations (T is the eigenvalue of the transfer matrix)

$$T(\theta)Q_{\pm}(\theta) = \phi_1(\theta)Q_{\pm}(\theta + i\gamma) + \phi_2(\theta)Q_{\pm}(\theta - i\gamma)\,,$$

with $T, Q_{\pm}$ entire (state dependent) functions of θ and ϕ_i given functions (for CFT $E = e^{\frac{2\theta M}{M+1}}, \gamma = \pi/M$).

When $\theta = \theta_n^+$ ($\theta = \theta_n^-$) zero of Q_+ (or Q_-), a TQ-relation implies Bethe equations

$$\phi_1(\theta_n^{\pm})Q_{\pm}(\theta_n^{\pm} + i\gamma) + \phi_2(\theta_n^{\pm})Q_{\pm}(\theta_n^{\pm} - i\gamma) = 0\,.$$

We want to associate to a state of a quantum integrable model a classical

model: two PDEs (Lax pair). Tool: A Marchenko-like equation Marchenko '55

We will discuss the example of sine-Gordon model in the vacuum, but the discussion can be made more general.

Functional relations

Sine-Gordon model on a cylinder

$$\mathcal{L} = \frac{1}{16\pi}\left[(\partial_t\varphi)^2 - (\partial_x\varphi)^2\right] + 2\mu\cos\beta\varphi\,, \quad \varphi(x+R,t) = \varphi(x,t)$$

Different k-vacua: $\varphi \to \varphi + 2\pi/\beta \Rightarrow |\Psi_k\rangle \to e^{2\pi i k}|\Psi_k\rangle$. Vacuum eigenvalues of local conserved charges appear in asymptotic expansion at $|\mathrm{Re}\theta| \to +\infty$ of Q-functions $Q_\pm(\theta)$ ($\pm$ sign of k). Properties of $Q_\pm$:

- Entire quasi-periodic functions: $Q_\pm(\theta + i\tau) = e^{\pm i\pi\left(l+\frac{1}{2}\right)}Q_\pm(\theta)$, $l = 2|k| - 1/2$, quasi-period $\tau = \pi/(1-\beta^2)$
- TQ-system: usual form

$$T(\theta)Q_\pm(\theta) = Q_\pm\left(\theta + \frac{i\pi\beta^2}{1-\beta^2}\right) + Q_\pm\left(\theta - \frac{i\pi\beta^2}{1-\beta^2}\right)$$

  By using quasi-periodicity one gets 'universal' shifts

$$T(\theta)Q_\pm(\theta) = e^{\mp i\pi\left(l+\frac{1}{2}\right)}Q_\pm(\theta + i\pi) + e^{\pm i\pi\left(l+\frac{1}{2}\right)}Q_\pm(\theta - i\pi)$$

- Asymptotics $\ln Q_\pm(\theta + i\tau/2) \simeq -w_0 e^{\theta} - \bar{w}_0 e^{-\theta}$, $\quad w_0 = -\dfrac{MR}{4\cos\frac{\pi\beta^2}{2(1-\beta^2)}}$
- Extensions: Homogeneous sine-Gordon model (many masses)

From functional relations to integral equations

- $Q_\pm$ can be found as solutions of an integral equation

$$Q_\pm\left(\theta + i\frac{\pi}{2}\right) = q_\pm(\theta) \pm \int_{-\infty}^{+\infty} \frac{d\theta'}{4\pi} \tanh\frac{\theta-\theta'}{2} T\left(\theta' + i\frac{\tau}{2}\right) e^{-w_0(e^\theta+e^{\theta'})-\bar{w}_0(e^{-\theta}+e^{-\theta'})}$$

$$\cdot e^{\pm(\theta-\theta')l} Q_\pm\left(\theta' + i\frac{\tau}{2}\right) , \quad q_\pm(\theta) = C_\pm e^{\pm\frac{i\pi}{4}\pm\left(\theta+\frac{i\pi}{2}\right)l} e^{-w_0 e^\theta - \bar{w}_0 e^{-\theta}}$$

- The TQ-system holds due to the property (of the kernel):

$$\lim_{\epsilon\to 0^+}\left[\tanh\left(x+\frac{i\pi}{2}-i\epsilon\right) - \tanh\left(x-\frac{i\pi}{2}+i\epsilon\right)\right] = 2\pi i\delta(x) , \quad x\in\mathbb{R} .$$

- $q_\pm(\theta)$ take into account the asymptotics and satisfy

$$e^{\mp i\pi\left(l+\frac{1}{2}\right)} q_\pm(\theta - i\pi) + e^{\pm i\pi\left(l+\frac{1}{2}\right)} q_\pm(\theta + i\pi) = 0$$

  They are zero modes of the shift operator.

- Problem: integral equation ill defined since

$$T\left(\theta + i\frac{\tau}{2}\right) \sim \exp(ae^\theta + \bar{a}e^{-\theta})$$

A solution

- Define the functions $X_\pm(\theta)$: $q_\pm(\theta)X_\pm(\theta) = Q_\pm(\theta + i\tau/2)$

$$X_\pm(\theta) = 1 \pm \int_{-\infty}^{+\infty} \frac{d\theta'}{4\pi} \tanh\frac{\theta-\theta'}{2} T\left(\theta' + i\frac{\tau}{2}\right) e^{-2w_0 e^{\theta'} - 2\bar{w}_0 e^{-\theta'}} X_\pm(\theta')$$

- Make $w_0, \bar{w}_0$ dynamical: $w_0 \to iw'$, $\bar{w}_0 \to -i\bar{w}'$, $X_\pm(\theta) \to X_\pm(w', \bar{w}'|\theta)$
- The transfer matrix T stays the same
- Integral equation satisfied by $X_\pm(w', \bar{w}'|\theta)$, $\lambda = e^\theta$:

$$X_\pm(w', \bar{w}'|\theta) = 1 \pm \int_0^{+\infty} \frac{d\lambda'}{4\pi\lambda'} \frac{\lambda-\lambda'}{\lambda+\lambda'} T(\lambda' e^{\frac{i\tau}{2}}) e^{-2iw'\lambda' + 2i\frac{\bar{w}'}{\lambda'}} X_\pm(w', \bar{w}'|\theta')$$

We expect that in the limit (non trivial) $w' \to -iw_0$,

$$X_\pm(w', \bar{w}'|\theta) \sim \frac{1}{q_\pm(\theta)} Q_\pm(\theta + i\tau/2)$$

Getting a Marchenko-like equation

- Let us define the Fourier transform of $X_\pm - 1$ (with 'active' role for w')

$$K_\pm(w', \xi; \bar{w}') = \int_{-\infty - i\epsilon}^{+\infty - i\epsilon} d\lambda e^{i(\xi - w')\lambda} [X_\pm(w', \bar{w}'|\theta) - 1] \,.$$

- Let us take the Fourier transform of the integral equation for $X_\pm$. We get

$$K_\pm(w', \xi; \bar{w}') \pm F(w' + \xi; \bar{w}') \pm \int_{w'}^{+\infty} \frac{d\xi'}{2\pi} K_\pm(w', \xi'; \bar{w}') F(\xi' + \xi; \bar{w}') = 0 \,, \quad \xi > w' \,,$$

with $F(x; \bar{w}') = i \int_0^{+\infty} d\lambda' e^{-ix\lambda' + 2i\frac{\bar{w}'}{\lambda'}} T(\lambda' e^{i\frac{\tau}{2}})$.

- This has the structure of a Marchenko equation appearing in quantum inverse scattering (from scattering data and bound states to Schroedinger). However for usual Marchenko $F(x) = \int_{-\infty}^{+\infty} d\lambda e^{-ix\lambda}(S(\lambda) - 1) + \sum_n S(\lambda_n)$: S=S-matrix, λ_n bound states
- In our construction scattering data and bound states are encoded in T, vacuum eigenvalue of the transfer matrix of a quantum integrable model.

From Marchenko-like to Schroedinger

- Define Jost wave function $\psi_\pm(w', \bar{w}'|\theta) = e^{-iw'\lambda + i\frac{\bar{w}'}{\lambda}} X_\pm(w', \bar{w}'|\theta)$

$$X_\pm(w', \bar{w}'|\theta) - 1 = \int_{w'}^{+\infty} \frac{d\xi}{2\pi} e^{-i(\xi - w')\lambda} K_\pm(w', \xi; \bar{w}') \,, \quad \lambda = e^\theta$$

- Differentiate (twice) and use our Marchenko-like equation: we get

$$\frac{\partial^2}{\partial w'^2}\psi_\pm(w', \bar{w}'|\theta) + e^{2\theta}\psi_\pm(w', \bar{w}'|\theta) = u_\pm(w'; \bar{w}')\psi_\pm(w', \bar{w}'|\theta) \,,$$

with potentials

$$u_\pm(w'; \bar{w}') = -2\frac{d}{dw'}\frac{K_\pm(w', w'; \bar{w}')}{2\pi} \,.$$

- Explicit solution of Marchenko equation gives access to the potential and to a wave function. The potential is

$$u_\pm(w'; \bar{w}') = \mp\partial_{w'^2}\hat{\eta} + (\partial_{w'}\hat{\eta})^2 \,, \quad \hat{\eta} = \ln\det(1 + \hat{V}) - \ln\det(1 - \hat{V})$$

$$V(\theta, \theta') = \frac{T\left(\theta + i\frac{\tau}{2}\right)}{4\pi}\frac{e^{-2iw'e^\theta + 2i\bar{w}'e^{-\theta}}}{\cosh\frac{\theta - \theta'}{2}}$$

Wave function and first Lax

- The wave function is $\psi_\pm(w',\bar{w}'|\theta)=X_\pm(w',\bar{w}'|\theta)e^{-iw'\lambda+i\frac{\bar{w}'}{\lambda}}$,

$$X_\pm(w',\bar{w}'|\theta)=-2\mp\int\frac{d\theta'}{4\pi}e^{\frac{\theta-\theta'}{2}}V(\theta,\theta')X_\pm(w',\bar{w}'|\theta')$$

  and satisfies the '$T\psi$-system' (extension of TQ-system)

$$T\left(\theta+i\frac{\tau}{2}\right)\psi_\pm(w',\bar{w}'|\theta)=\mp i\psi_\pm(w',\bar{w}'|\theta+i\pi)\pm i\psi_\pm(w',\bar{w}'|\theta-i\pi)\,.$$

- To summarise, we have obtained two Schroedinger equations

$$\frac{\partial^2}{\partial w'^2}\psi_\pm(w',\bar{w}'|\theta)+e^{2\theta}\psi_\pm(w',\bar{w}'|\theta)=u_\pm(w';\bar{w}')\psi_\pm(w',\bar{w}'|\theta)\,,$$

- Introduce ($w=iw'$) $D_{\hat{\eta}}=\partial_w+\frac{1}{2}\partial_w\hat{\eta}\,\sigma^3-e^{\theta+\hat{\eta}}\sigma^+-e^{\theta-\hat{\eta}}\sigma^-$.

$$\mathbf{D}=\begin{pmatrix}D_{\hat{\eta}} & 0\\ 0 & D_{-\hat{\eta}}\end{pmatrix}\,,\quad \Psi=\begin{pmatrix}e^{\frac{\theta+\hat{\eta}}{2}}\psi_+\\ e^{-\frac{\theta+\hat{\eta}}{2}}(\partial_w+\partial_w\hat{\eta})\psi_+\\ e^{\frac{\theta-\hat{\eta}}{2}}\psi_-\\ e^{-\frac{\theta-\hat{\eta}}{2}}(\partial_w-\partial_w\hat{\eta})\psi_-\end{pmatrix}$$

- The first order matrix equation $\mathbf{D}\Psi=0$ is equivalent to Schroedinger equations in w'.

Second Lax

- A differential equation in $\bar{w}'$ is defined by using the Fourier transform

$$K_{\pm}^{bis}(\bar{w}', \xi; w') = \int_{-\infty - i\epsilon}^{+\infty - i\epsilon} d\lambda^{-1} e^{i(\xi + \bar{w}')\lambda^{-1}} [X_{\pm}(w', \bar{w}'|\theta) - 1] ,$$

(with active role for $\bar{w}'$) on the equation for $X_{\pm}$. Following the Marchenko procedure, we end up with the 'conjugate' differential equation

$$-\frac{\partial^2}{\partial \bar{w}'^2} \psi_{\pm}^{bis}(w', \bar{w}'|\theta) + \bar{u}_{\mp}(w', \bar{w}')\psi_{\pm}^{bis}(w', \bar{w}'|\theta) = e^{-2\theta} \psi_{\pm}^{bis}(w', \bar{w}'|\theta)$$

for

$$\psi_{\pm}^{bis}(w', \bar{w}'|\theta) = e^{-iw'\lambda + i\bar{w}'\lambda^{-1}} \left[1 + \int_{-\bar{w}'}^{+\infty} \frac{d\xi}{2\pi} e^{-i(\xi + \bar{w}')\lambda^{-1}} K_{\pm}^{bis}(\bar{w}', \xi; w') \right] .$$

- Introduce ($\bar{w} = -i\bar{w}'$) $\bar{D}_{\hat{\eta}} = \partial_{\bar{w}} - \frac{1}{2}\partial_{\bar{w}}\hat{\eta}\,\sigma^3 - e^{-\theta + \hat{\eta}}\sigma^- - e^{-\theta - \hat{\eta}}\sigma^+$ and

$$\bar{\mathbf{D}} = \begin{pmatrix} \bar{D}_{\hat{\eta}} & 0 \\ 0 & \bar{D}_{-\hat{\eta}} \end{pmatrix} , \quad \Psi^{bis} = \begin{pmatrix} e^{\frac{\theta - \hat{\eta}}{2}} (\partial_{\bar{w}} + \partial_{\bar{w}}\hat{\eta})\psi_{-}^{bis} \\ e^{-\frac{\theta - \hat{\eta}}{2}} \psi_{-}^{bis} \\ e^{\frac{\theta + \hat{\eta}}{2}} (\partial_{\bar{w}} - \partial_{\bar{w}}\hat{\eta})\psi_{+}^{bis} \\ e^{-\frac{\theta + \hat{\eta}}{2}} \psi_{+}^{bis} \end{pmatrix}$$

- The first order matrix equation $\bar{\mathbf{D}}\Psi^{bis} = 0$ is equivalent to Schroedinger equations in $\bar{w}'$.

The classical model

- Let us compare the two vectors Ψ and Ψ^{bis}.
- By examining the solutions we constructed we find that $\psi_{\pm}^{bis}(w', \bar{w}'|\theta) = \psi_{\pm}(w', \bar{w}'|\theta) e^{\pm\hat{\eta}(w,\bar{w})}$.
- On the four-vectors this connection implies $\Psi = -e^{\theta}\Psi^{bis}$. Then, we can write $\mathbf{D}\Psi = \bar{\mathbf{D}}\Psi = 0$: from this relations we get that $[\mathbf{D}, \bar{\mathbf{D}}]\Psi = 0$, which means for $\hat{\eta}$
$$\partial_w \partial_{\bar{w}} \hat{\eta} = 2 \sinh 2\hat{\eta} \,,$$
i.e. that $\hat{\eta}$ satisfies the classical sinh-Gordon equation.
- The two Lax problems $\mathbf{D}\Psi = \bar{\mathbf{D}}\Psi = 0$ coincide with the starting point of usual ODE/IM construction (Lukyanov and Zamolodchikov). We have completed our inverse construction.

Conformal limit

- Potentials $u_{\pm}(w', \bar{w}')$ of Schroedinger equations are complicated functions (Fredholm determinants)
- Simplifications occur in the conformal limit, when masses $(w_0) \to 0$, $\bar{w}' \to 0$ and w' scales as

$$\frac{dw'}{dx} = \sqrt{p(x)} e^{-\theta} \quad \theta \to +\infty$$

with $p(x) = x^{2M} - E$, $M = 1/\beta^2 - 1$ (θ 'rapidity').

- Then, the new wave function $\psi^{cft}(x) = \psi_+(w')p(x)^{-\frac{1}{4}}$ satisfies the ODE

$$-\frac{d^2}{dx^2}\psi^{cft}(x) + \left(p(x) + \frac{l(l+1)}{x^2}\right)\psi^{cft}(x) = 0$$

which is ODE considered by Dorey and Tateo and Bazhanov, Lukyanov, Zamolodchikov.

A special case

- If $\beta^2 = 2/3, l = 0$, functional relations imply $T = 1$
  TQ-system reduces to

$$Q_\pm(\theta) = \mp i Q_\pm(\theta + i\pi) \pm i Q_\pm(\theta - i\pi)$$

- Now $\hat{\eta} = \ln\det(1 + \hat{V}) - \ln\det(1 - \hat{V})$, with

$$V(\theta, \theta') = \frac{e^{-2iw' e^{\theta} + 2i\bar{w}' e^{-\theta}}}{4\pi \cosh \frac{\theta - \theta'}{2}}$$

- The field $\hat{\eta}$ depends only on $t = 4\sqrt{w'\bar{w}'}$, $w' = \frac{t}{4} e^{i\varphi}$ and the sinh-Gordon equation $\partial_w \partial_{\bar{w}} \hat{\eta} = 2 \sinh 2\hat{\eta}$ reduces to the Painlevè III_3 equation:

$$\frac{1}{t}\frac{d}{dt}\left(t \frac{d}{dt}\hat{\eta}(t)\right) = \frac{1}{2}\sinh 2\hat{\eta}(t)$$

  $\Rightarrow$ connection to Painlevè theory.

Summary and Perspectives

- We have given a possible explanation for the occurrence of the ODE/IM correspondence. The idea is that the *TQ*-functional relation is equivalent to a an equation with the form of a Marchenko equation. From this Marchenko-like equation one gets Schroedinger equations by a standard procedure.
- We have proved this in the case of vacuum eigenvalues of $\hat{T}$, $\hat{Q}$ for (Homogeneous) sine-Gordon model. Possible extension: construction of (the at present unknown) Schroedinger equations corresponding to excited states of sine-Gordon.
- More in general: *TQ*-relations are common in quantum integrable models (they are equivalent to Bethe Ansatz), They can be written for instance for spin chains, which means that in principle we can derive Schroedinger equations corresponding to generic states of a spin chain.

Symmetries

We found the differential equation

$$\frac{\partial^2}{\partial w'^2}\psi_\pm(w',\bar{w}'|\theta) + e^{2\theta}\psi_\pm(w',\bar{w}'|\theta) = u_\pm(w';\bar{w}')\psi_\pm(w',\bar{w}'|\theta)\,,$$

with potentials

$$u_\pm(w';\bar{w}') = \mp\partial_{w'^2}\hat{\eta} + (\partial_{w'}\hat{\eta})^2\ , \quad \hat{\eta} = \ln\det(1+\hat{V}) - \ln\det(1-\hat{V})$$

$$V(\theta,\theta') = \frac{T\left(\theta + i\frac{\tau}{2}\right)}{4\pi}\,\frac{e^{-2iw'e^{\theta}+2i\bar{w}'e^{-\theta}}}{\cosh\frac{\theta-\theta'}{2}}$$

The differential operator has two symmetries:

- $\hat{\Pi}$-symmetry: $\theta \to \theta - i\pi$
- From $T(\theta+i\tau) = T(\theta)$, $\hat{\eta}(w'e^{i\tau}, \bar{w}'e^{-i\tau}) = \hat{\eta}(w',\bar{w}')$, one gets $\hat{\Omega}$-symmetry (or Symanzik rotation), $w' \to w'e^{i\tau}, \theta \to \theta - i\tau + i\pi$

They act non trivially on the wave functions

$$(\hat{\Pi}\psi_\pm)(w',\bar{w}'|\theta) = \psi_\pm(w',\bar{w}'|\theta - i\pi)\,,$$

$$(\hat{\Omega}\psi_\pm)(w',\bar{w}'|\theta) = \psi_\pm(w',\bar{w}'|\theta + i\pi)\,.$$

Back to quantum (usual path)

- As in usual ODE/IM, in the classical model we constructed we find Q functions of (Homogeneous) sine-Gordon as connection coefficients between different solutions.
- In the Wick rotated new variable $w = iw'$, when $w \to w_0$ the potentials

$$u_{\pm} \simeq -l(l \pm 1)/(w - w_0)^2$$

- We have solutions (Frobenius) that when $w \to w_0$

$$f_{+}^{(-l)}(w', \bar{w}') \simeq (w - w_0(\vec{c}))^{-l}, \quad f_{+}^{(l+1)}(w', \bar{w}') \simeq (w - w_0(\vec{c}))^{l+1},$$
$$f_{-}^{(l)}(w', \bar{w}') \simeq (w - w_0(\vec{c}))^{l}, \quad f_{-}^{(-l+1)}(w', \bar{w}') \simeq (w - w_0(\vec{c}))^{-l+1}.$$

In terms of f we expand $\psi_{\pm}$

$$\psi_{+}(w', \bar{w}'|\theta) = -e^{\theta(l+1)} Q_{-}(\hat{\theta}) f_{+}^{(l+1)}(w', \bar{w}') + e^{-\theta l} Q_{+}(\hat{\theta}) f_{+}^{(-l)}(w', \bar{w}')$$
$$\psi_{-}(w', \bar{w}'|\theta) = e^{\theta l} Q_{-}(\hat{\theta}) f_{-}^{(l)}(w', \bar{w}') - e^{-\theta(l-1)} Q_{+}(\hat{\theta}) f_{-}^{(-l+1)}(w', \bar{w}')$$

- Connection coefficients contain Q-functions of the quantum model:

$$\lim_{w \to w_0} (w - w_0)^{\pm l} \psi_{\pm}(w', \bar{w}'|\theta) = D_{\pm} e^{\mp\theta l} Q_{\pm}\left(\hat{\theta} = \theta + i\frac{\tau}{2}\right).$$

- The wave functions $\psi_{\pm}(w', \bar{w}'|\theta)$ depend only on $t, \theta + i\varphi$. Since they satisfy differential equations in t, φ, they satisfy also differential equations in t, θ. This means that $Q_{\pm}(\theta) = \psi_{\pm}(t = 4w_0|\theta)$ satisfy differential equations (in θ).

$$\frac{d^2 Q_{\pm}(\theta)}{d\theta^2} + \tanh(\theta \pm \hat{\eta}_0) \left[-\frac{dQ_{\pm}(\theta)}{d\theta} \mp 2w_0 \hat{\eta}_0' Q_{\pm}(\theta) \right] - 4w_0^2 (\hat{\eta}_0')^2 Q_{\pm}(\theta) +$$
$$+ \quad 2w_0^2 [\cosh 2\theta + \cosh 2\hat{\eta}_0] Q_{\pm}(\theta) = 0 \, ,$$

where $\hat{\eta}_0 = \hat{\eta}(t = 4w_0)$.

The occurrence of a differential equation for $Q_{\pm}$ reminds the XXZ spin chain at a particular point.

Interesting application to XXZ spin chain with anisotropy $\Delta = -1/2$ and $2n+1$ sites.

$f_{\pm}(u) = (\sin u)^{2n+1} Q_{\pm}^{XXZ}(u)$, satisfy the simple functional relations $f_{\pm}(u+2\pi/3) + f_{\pm}(u-2\pi/3) + f_{\pm}(u) = 0$, which, after $\theta = 3iu/2$, $f_{\pm} = e^{\pm 3iu/4} Q_{\pm}$, maps into our TQ-system in case $\beta^2 = 2/3, l = 0$.

In our framework, starting from the solutions of the differential equation $\psi_{\pm}$ and taking the limit $t = 4|w| \to 0$, the functions $f_{\pm}$ are found from $\psi_{\pm}(t, \varphi | u) = e^{\mp 3iu/4} t^{2n+1} f_{\pm}\left(u + \frac{2}{3}\varphi\right) + \ldots$.

Indeed, as a consequence of differential equations satisfied by $\psi_{\pm}$, $f_{\pm}$ satisfy the ODEs

$$\frac{d^2 f_{\pm}}{du^2} - 6n \cot(3u + 2\varphi) \frac{df_{\pm}}{du} + (c_{\pm} - 9n^2) f_{\pm} = 0 \, ,$$

which uniquely characterize them. They coincide with ODEs found by direct solution of functional equations for $f_{\pm}$ by Stroganov in 2003.