Exact solvability of loop models

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Overview

Solvable 2d CFTs and degenerate fields

Degenerate field: a primary field whose operator product with any primary field gives rise to only finitely many primary fields.

Degenerate fields $\left\{V^d_{\langle r,s\rangle}\right\}_{r,s\in\mathbb{N}^*} = \left\{V^d_{\langle 1,1\rangle}, V^d_{\langle 2,1\rangle}, V^d_{\langle 1,2\rangle}, \cdots\right\}$

Some operator products:

$$V^d_{\langle 2,1\rangle} V^d_{\langle r,s\rangle} \sim \sum_{\pm} V^d_{\langle r\pm 1,s\rangle} \quad , \quad V^d_{\langle 1,2\rangle} V^d_{\langle r,s\rangle} \sim \sum_{\pm} V^d_{\langle r,s\pm 1\rangle}$$

$$V^d_{\langle 2,1\rangle} V_P \sim \sum_{\pm} V_{P\pm\frac{\beta}{2}} \quad , \quad V^d_{\langle 1,2\rangle} V_P \sim \sum_{\pm} V_{P\pm\frac{1}{2\beta}}$$

- $V_P =$ generic primary field of dimension $\Delta = \frac{c-1}{24} + P^2 \in \mathbb{C}$
- $V^d_{\langle r,s\rangle}$ has dimension $\Delta_{(r,s)} = \frac{c-1}{24} + P^2_{(r,s)}$ with $P_{(r,s)} = \frac{1}{2}(r\beta - s\beta^{-1})$
- Central charge $c = 13 - 6\beta^2 - 6\beta^{-2} \in \mathbb{C}$

$\rightarrow$ building the spectrums of **minimal models**, solving **Liouville theory**

Fields with nonzero conformal spin

Consider primary fields $V_{(r,s)}$ whose left and right dimensions may differ:

$$(\Delta, \bar{\Delta}) = (\Delta_{(r,s)}, \Delta_{(r,-s)}) \quad \Longrightarrow \quad S = \bar{\Delta} - \Delta = rs$$

where now a priori $r, s \in \mathbb{C}$. Assumptions:

- $V^d_{\langle 1,2\rangle}$ exists (but not $V^d_{\langle 2,1\rangle}$!)
- Spins are integer $S \in \mathbb{Z}$

Operator product $V^d_{\langle 1,2\rangle} V_{(r,s)} \sim \sum_\pm V_{(r,s\pm 1)} \implies r \in \frac{1}{2}\mathbb{Z}\ ,\ rs \in \mathbb{Z}$

Two types of fields:

- Diagonal fields $r = 0$, $s = 2\beta P \in \mathbb{C}$ i.e. $V_P = V_{(0,2\beta P)}$
- Non-diagonal fields $r \in \frac{1}{2}\mathbb{N}^*\ ,\ s \in \frac{1}{r}\mathbb{Z}$

$\rightarrow$ Elementary rederivation of $O(n)$ model spectrum

[di Francesco, Saleur, Zuber 1987]

Diagonal three-point structure constants

Crossing symmetry of 4pt functions with $V^d_{\langle 2,1\rangle}$ or $V^d_{\langle 1,2\rangle}$

$$\implies \textbf{Shift equations: } \frac{C_{P_1+\beta,P_2,P_3}}{C_{P_1,P_2,P_3}} = \text{known} \ , \ \frac{C_{P_1+\beta^{-1},P_2,P_3}}{C_{P_1,P_2,P_3}} = \text{known}$$

for 3pt structure constants $C_{P_1,P_2,P_3} \propto \langle V_{P_1} V_{P_2} V_{P_3} \rangle$

$$\implies C_{P_1,P_2,P_3} = \prod_{\epsilon_1,\epsilon_2,\epsilon_3=\pm} \Gamma_\beta^{-1}\left(\tfrac{\beta+\beta^{-1}}{2} + \textstyle\sum_i \epsilon_i P_i\right)$$

using the double Gamma function Γ_β, which obeys

$$\Gamma_\beta(x+\beta) = \sqrt{2\pi}\frac{\beta^{\beta x-\frac12}}{\Gamma(\beta x)}\Gamma_\beta(x) \quad , \quad \Gamma_\beta(x+\beta^{-1}) = \sqrt{2\pi}\frac{\beta^{-\beta^{-1}x+\frac12}}{\Gamma(\beta^{-1}x)}\Gamma_\beta(x)$$

$\to$ solution of Liouville theory [Teschner 1995]

General three-point structure constants

$C_{(r_1,s_1)(r_2,s_2)(r_3,s_3)}$ with $r_i \in \frac{1}{2}\mathbb{N}^*, s_i \in \frac{1}{r}\mathbb{Z}$?

- $V^d_{\langle 1,2\rangle}$ shift equations $\implies s_i \to s_i + 2$ [Estienne, Ikhlef 2015]
- Reduces to C_{P_1,P_2,P_3} if $r_1 = r_2 = r_3 = 0$ (even though no $V^d_{\langle 2,1\rangle}$!) [Delfino, Viti 2010]

Reference structure constant = one nice solution (not unique)

$$\boxed{C^{\text{ref}}_{(r_1,s_1)(r_2,s_2)(r_3,s_3)} = \prod_{\epsilon_1,\epsilon_2,\epsilon_3=\pm} \Gamma_\beta^{-1}\left(\frac{\beta+\beta^{-1}}{2} + \frac{\beta}{2}\left|\sum_i \epsilon_i r_i\right| + \frac{\beta^{-1}}{2}\sum_i \epsilon_i s_i\right)}$$

Conjecture: $\frac{C}{C^{ref}}$ *is **polynomial** in* $n, w(P_i)$

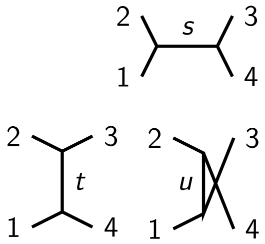
- "loop weights" $w(P) = 2\cos(2\pi\beta P)$
- "weight of contractible loops" $n = w(P_{(1,1)}) = -2\cos(\pi\beta^2)$

Crossing symmetry equations

For a four-point function $\left\langle \prod_{i=1}^4 V_{(r_i,s_i)} \right\rangle$:

$$\sum_{k\in\mathbb{Z}} D^{(s)}_{P+k\beta^{-1}} \mathcal{G}^{(s)}_{P+k\beta^{-1}} + \sum_{r=1}^{\infty} \sum_{s\in\frac{1}{r}\mathbb{Z}} D^{(s)}_{(r,s)} \mathcal{G}^{(s)}_{(r,s)}$$

$$= \sum_{r=1}^{\infty} \sum_{s\in\frac{1}{r}\mathbb{Z}} D^{(t)}_{(r,s)} \mathcal{G}^{(t)}_{(r,s)} = \sum_{r=1}^{\infty} \sum_{s\in\frac{1}{r}\mathbb{Z}} D^{(u)}_{(r,s)} \mathcal{G}^{(u)}_{(r,s)}$$



- $\mathcal{G}^{(s)}_P, \mathcal{G}^{(s,t,u)}_{(r,s)}$ = known 4pt conformal blocks (logarithmic if $s \in \mathbb{Z}$ [Nivesvivat, Ribault 2020])
- $D^{(s)}_P, D^{(s,t,u)}_{(r,s)}$ = unknown 4pt structure constants
- $\dim(\text{space of solutions}) = \sum_{i=1}^4 r_i^2$ [Grans-Samuelsson, Jacobsen, Nivesvivat, Ribault, Saleur 2023]

Numerical results for four-point structure constants

Reference 4pt structure constant assembled from 3pt:

$$D = D^{\text{ref}} d = C^{\text{ref}} C^{\text{ref}} d$$

Conjecture $\implies$ d is polynomial in $n, w(P_i)$

Numerically solving crossing symmetry
$\to$ extrapolating exact expressions for d
in ~ 20 cases including $\left\langle \prod_{i=1}^4 V_{P_i} \right\rangle$, $\left\langle V^4_{(\frac12,0)} \right\rangle$, $\left\langle V_{(\frac32,0)} V_{(1,1)} V_{(1,0)} V_{(\frac12,0)} \right\rangle$

Case of $\left\langle \prod_{i=1}^4 V_{P_i} \right\rangle$

1 diagonal field in each channel $\to$ 8 variables $n, w_1, w_2, w_3, w_4, w_s, w_t, w_u$

$$d^{(s,t,u)}_{\text{diag}} = 1$$

$$d^{(s)}_{(1,0)} = w_t + w_u \quad , \quad d^{(t)}_{(1,0)} = w_s + w_u \quad , \quad d^{(u)}_{(1,0)} = w_s + w_t$$

$$d^{(s)}_{(1,1)} = w_u - w_t \quad , \quad d^{(t)}_{(1,1)} = w_u - w_s \quad , \quad d^{(u)}_{(1,1)} = w_s - w_t$$

$$2d^{(s)}_{(2,0)} = -(n-2)(w_t+w_u)(w_1+w_2)(w_4+w_3) - (n+2)(w_t-w_u)(w_1-w_2)(w_4-w_3) + (n^2-4)\left[w_t^2+w_u^2+2w_s-4\right]$$

$$2d^{(s)}_{(2,\frac{1}{2})} = n^2(w_u^2-w_t^2) + n(w_t-w_u)(w_1+w_2)(w_4+w_3) + n(w_t+w_u)(w_1-w_2)(w_4-w_3)$$

$$2d^{(s)}_{(2,1)} = -(n+2)(w_t+w_u)(w_1+w_2)(w_4+w_3) - (n-2)(w_t-w_u)(w_1-w_2)(w_4-w_3) + (n^2-4)\left[w_t^2+w_u^2-2w_s-4\right]$$

Case of $\left\langle V^4_{(\frac{1}{2},0)} \right\rangle$

1 diagonal field in the s-channel $\rightarrow$ 2 variables n, w_s

$$d^{(t)}_{(1,0)} = 1 \quad , \quad d^{(t)}_{(1,1)} = -1$$

$$2d^{(t)}_{(2,0)} = (n-2)\left[w_s(n+2) - 8\right]$$

$$2d^{(t)}_{(2,\frac{1}{2})} = -n(w_s n - 4) \quad , \quad 2d^{(t)}_{(2,1)} = w_s(n^2 - 4)$$

$$3d^{(t)}_{(3,0)} = n^2(n-2)^2\left[w_s^2(n+2)^2 - 4w_s(n+2) + n^2 + 8\right]$$

$$3d^{(t)}_{(3,\frac{1}{3})} = -(n-1)^2(n+1) \times \left[(n^2-3)(n+1)w_s^2 - 2(n+1)(2n-3)w_s - 2(n-2)(n^2+4n+1)\right]$$

$$3d^{(t)}_{(3,\frac{2}{3})} = (n^2-3)(n^2-1)\left[w_s^2(n^2-1) - 2w_s(2n-1) - 2(n+1)(n-2)\right]$$

$$3d^{(t)}_{(3,1)} = -(n^2-4)^2\left[w_s^2 n^2 - 4w_s n + (n+2)^2\right]$$

Case of $\left\langle V_{(\frac{3}{2},0)} V_{(1,1)} V_{(1,0)} V_{(\frac{1}{2},0)} \right\rangle$

a solution with no diagonal field $\rightarrow$ 1 variable n

$$d^{(s)}_{(\frac{1}{2},0)} = 0 \quad , \quad 3d^{(s)}_{(\frac{3}{2},0)} = n+2 \quad , \quad 3d^{(s)}_{(\frac{3}{2},\frac{2}{3})} = -2(n-1)$$

$$5d^{(s)}_{(\frac{5}{2},\frac{2}{5})} = 4\cos\left(\tfrac{\pi}{10}\right)\left[-n^4+4n^2-2+2\cos\left(\tfrac{\pi}{5}\right)(n+1)(n^2-3)\right]$$

$$5d^{(s)}_{(\frac{5}{2},\frac{4}{5})} = 4\cos\left(\tfrac{3\pi}{10}\right)\left[n^4-4n^2+2+2\cos\left(\tfrac{2\pi}{5}\right)(n+1)(n^2-3)\right]$$

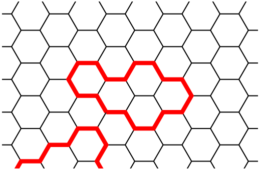
$$d^{(t)}_{(1,0)} = d^{(t)}_{(1,1)} = d^{(t)}_{(2,0)} = d^{(t)}_{(3,0)} = 0$$

$$2d^{(t)}_{(2,\frac{1}{2})} = \sqrt{2}n^2 \quad , \quad d^{(t)}_{(2,1)} = n^2-4$$

$$3d^{(t)}_{(3,\frac{1}{3})} = (n-4)(n-1)(n+1)^2(n^2-3)$$

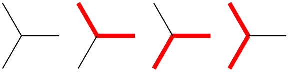
$$3d^{(t)}_{(3,\frac{2}{3})} = \sqrt{3}(n^2-1)^2(n+2)(n-3) \quad , \quad 3d^{(t)}_{(3,1)} = 2n^2(n^2-4)^2$$

4-vertex loop model on a honeycomb lattice



loop weight n

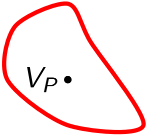
Vertex weights:



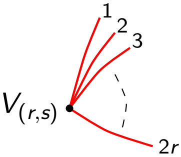
Critical coupling:

$$K_c = \frac{1}{\sqrt{2+\sqrt{2-n}}}$$

Primary fields:



$w(P)$

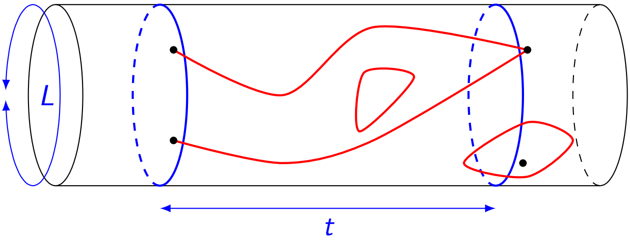


$\exp \frac{i}{2} s \sum_{k=1}^{2r} \theta_k$

$Z = \sum_{\text{loop configurations}} \text{weight}$

[Nienhuis 1982] (model), [GJNRS 2023] ($V_{(r,s)}$)

Lattice 4pt function



Size of lattice: $L \sim 5$, $t \sim 500$

$$Z(L,t,K) = \left\langle out \right| T^t \left| in \right\rangle = \sum_{(r,s)|2r\leq L} A_{(r,s)}(L,K)\Lambda_{(r,s)}(L,K)^t$$

- $\Lambda_{(r,s)}(L,K) =$ eigenvalue of the transfer matrix T
- $r,s =$ determined by diagonalizing e.g. the angular momentum
- $A_{(r,s)}(L,K) =$ amplitude

Lattice amplitudes vs CFT 4pt structure constants

Naive expectation: agreement in the critical limit:

$$\lim_{L\to\infty} A_{(r,s)}(L, K_c) \sim D_{(r,s)}$$

Observed agreement **off-criticality** for dependence on w:

$$\forall L \geq 2r \,, \forall K \,, \qquad \frac{A_{(r,s)}(L,K,w)}{A_{(r,s)}(L,K,w')} = \frac{d_{(r,s)}(w)}{d_{(r,s)}(w')}$$

based on numerical lattice results for $\left\langle V^4_{(\frac12,0)} \right\rangle$ and $\left\langle \prod_{i=1}^4 V_{P_i} \right\rangle$

Earlier hints of L-independence in loop models:
[Jacobsen, Saleur 2018][He, Grans-Samuelsson, Jacobsen, Saleur 2020]

Conclusion

Bootstrap side:

- Strong evidence that $D = D^{\text{ref}} \times d_{(r,s)}$ with $d_{(r,s)}$ polynomial
- Critical loop models are likely solvable
- Still far from solved: factorize 4pt structure constants into 3pt?

Lattice side:

- Polynomial factors $d_{(r,s)}$ appear off-criticality
- Algebraic determination of $d_{(r,s)}$ from the Temperley–Lieb algebra?
- Is this a step towards an exact solution?