From exact WKB analysis to instanton counting at strong coupling

Pietro Longhi
Uppsala University

10th Bologna Workshop on Conformal Field Theory and Integrable Models

This talk is based on joint work with **I.Coman** and **J.Teschner** about instanton partition functions of 4d $\mathcal{N} = 2$ QFT.

Our main goal is to define and compute these away from weak-coupling, where localization techniques based on Lagrangian descriptions cease to apply.

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Outline:

1. Exact results in $4d$ $\mathcal{N} = 2$ gauge theory
2. Quantum curves
3. τ-functions and instantons
4. Weak/strong coupling connection coefficients and the global picture

1. Exact results in $4d$ $\mathcal{N} = 2$ gauge theory

4d $\mathcal{N} = 2$ Yang-Mills

The $\mathcal{N} = 2$ Yang-Mills Lagrangian ($\tau = \theta/2\pi + 4\pi i/g^2$ and $G = SU(2)$)

$$\begin{aligned}\mathcal{L} &= \frac{1}{8\pi}\mathrm{Im}\Big(\int d^2\theta\, \tau\, W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta}\ 2\tau\, \Phi^\dagger e^{-2V}\Phi\Big)\\ &= \frac{1}{g^2}\mathrm{Tr}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g^2\frac{\theta}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - \frac{1}{2}[\phi^\dagger,\phi]^2\\ &\qquad - i\ \lambda\sigma^\mu D_\mu\bar{\lambda} - i\ \bar{\psi}\bar{\sigma}^\mu D_\mu\psi - i\sqrt{2}[\lambda,\psi]\phi^\dagger - i\sqrt{2}[\bar{\lambda},\bar{\psi}]\phi\Big)\end{aligned}$$

is a supersymmetric extension of Yang-Mills-Higgs models, with (adjoint) Higgs potential

$$U = -\frac{1}{2g^2}\mathrm{Tr}\left([\phi^\dagger,\phi]^2\right)$$

Classical vacua are defined by $[\phi^\dagger, \phi] = 0$ and come in families parameterized by $\phi \in \mathfrak{t}$ valued in a Cartan subalgebra of $\mathfrak{g}$.

The classical expectation value $\phi \sim a\,\sigma_3$ induces a spontaneous breaking of $SU(2) \to U(1)$. The low energy theory is free Abelian gauge theory.

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At the quantum level the IR theory is interacting. The moduli space of 'Coulomb' vacua $\mathcal{B}$ is not lifted, and the gauge-invariant order parameter is

$$u = \frac{1}{2}\langle \mathrm{Tr}\phi^2 \rangle \, .$$

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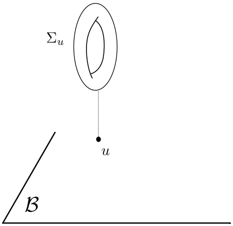
The $U(1)$ low energy effective action is governed by the prepotential $\mathcal{F}$

$$\mathcal{L} = \frac{1}{8\pi}\mathrm{Im}\Big(\int d^2\theta\, \mathcal{F}''(\Phi)\, W^\alpha W_\alpha + 2\int d^2\theta d^2\bar{\theta}\, \mathcal{F}'(\Phi)\Phi^\dagger \Big)$$

with: $$\mathcal{F} = \mathcal{F}_{\text{pert.}} + \mathcal{F}_{\text{instanton}} = \frac{i}{2\pi} a^2 \ln\frac{a^2}{\Lambda^2} + \sum_{k=1}^\infty \mathcal{F}_k \left(\frac{\Lambda}{a}\right)^{4k} a^2$$

A geometric proposal for $\mathcal{F}$ in terms of elliptic curves Σ with differential λ.

[Seiberg Witten]



Dictionary

$$a(u) := \frac{1}{\pi}\oint_\alpha \lambda \qquad a_D(u) := \frac{1}{\pi}\oint_\beta \lambda$$

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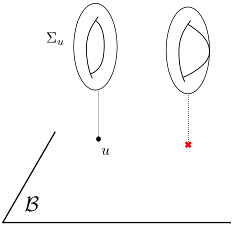

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Singularities on $\mathcal{B}$:
When a cycle pinches, the corresponding combination of a, a_D vanishes.
If $\mathcal{F}$ diverges the IR description is not valid. This is due to new massless d.o.f.

Singularities from massless BPS particles

Yang-Mills-Higgs models have finite-energy particle states with ['t Hooft, Polyakov]

$$\text{mass} \quad M(u) \qquad \text{charge} \quad \gamma = (e, m)\,.$$

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4d $\mathcal{N} = 2$ supersymmetry has a central extension $\{Q^A_\alpha, Q^B_\beta\} \sim \epsilon_{\alpha\beta}\epsilon^{AB} Z$

- Linear in (e, m) [Olive Witten]

$$Z_{(e,m)}(u) \sim \int d^3x\, \partial_j \left[\left(\frac{1}{g^2}\, F^{0j} + \frac{\tau}{4\pi}\tilde{F}^{0j}\right) a^\dagger\right] \sim a_\infty(e + \tau \cdot m)$$

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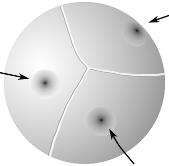
- The central charge is a holomorphic function $Z_\gamma(u) = \frac{1}{\pi} \oint_\gamma \lambda$.
- At singularities BPS states become massless $M(u) = |Z_\gamma(u)| \to 0$.

Light degrees of freedom on the Coulomb branch



The Seiberg-Witten solution has 3 singularities on $\mathcal{B}$:

- One at weak coupling, where $\mathcal{F}$ has the expansion shown previously $\rightsquigarrow$ d.o.f. of $SU(2)$ Yang-Mills with light **W-bosons** $Z_{\gamma_1+\gamma_2} \approx 0$
- Two at strong coupling, where $\mathcal{F}$ has a rather different kind of expansion $\rightsquigarrow$ d.o.f. of 'dual' $U(1)$ QED with light **monopole** $Z_{\gamma_1} \approx 0$ or **dyon** $Z_{\gamma_2} \approx 0$

[Figure from Lerche 9611190]

Instanton counting

The Seiberg-Witten solution was conjectural, but instanton corrections at weak coupling were later confirmed by direct computation in QFT

- Compute k-instanton contributions $\mathcal{F}_k$ by considering a $G \times T^2$-equivariant integral over the moduli space $\widetilde{\mathcal{M}}_k$ [Losev Nekrasov Shatashvili] [Moore Nekrasov Shatashvili]
- Result obtained by localization, reducing to a sum over fixed points labeled by colored partitions $(Y_1, \ldots, Y_N)$
- With T^2 equivariant parameters specialized to $\epsilon_1 = -\epsilon_2 = \hbar$ [Nekrasov]

$$Z_{\text{inst}}(a, \hbar; q) = \sum_{Y_1, Y_2} q^{|Y_1|+|Y_2|} \prod_{i,j} \frac{a + \hbar(Y_{1,i} - Y_{2,j} + j - i)}{a + \hbar(j - i)}$$

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Then

$$\lim_{\hbar \to 0} \ln Z_{\text{inst}}(a, \hbar; q) = \frac{1}{\hbar^2} \mathcal{F}_{\text{inst}}(a, \Lambda)$$

Remarks on instanton counting:

- Z_{inst} recovers the Seiberg Witten prepotential, but also contains much more information: $\mathcal{F}_{\text{inst}}$ is only the leading term in the $\hbar$ expansion.
- Limitation in the range of validity: relying on the Lagrangian description ($SU(2)$ Yang-Mills) recovers only the weak-coupling expansion of $\mathcal{F}_{\text{inst}}$.

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Questions motivating our work:

- What about instanton expansions $\mathcal{F}_{D,\text{inst}}$ near strong coupling singularities? Do they also admit $\hbar$ deformations?
- No UV Lagrangian description amenable to localization is available for the light d.o.f. at the monopole and dyon points. How can they be computed?
- Related in topological strings: how to define $Z_{\text{top}} \sim Z_{\text{inst}}$ away from large volume - large B-field limit?

2. From curve quantization to instantons

Class S theories

A large class of superconformal (and asymptotically free) 4d $\mathcal{N} = 2$ QFTs can be engineered by partially twisted compactifications of 6d $(2,0)$ QFT on a Riemann surface C [Gaiotto] [Gaiotto Moore Neitzke] [...]

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The quantum moduli space of vacua of a class S theory on $\mathbb{R}^3 \times S^1_R$ encodes both Coulomb moduli and electric-magnetic Wilson lines on S^1_R [Seiberg Witten]

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$$T^{2r} \to \mathcal{M}_H \to \mathcal{B}\,.$$

$\mathcal{M}_H$ is defined by the reduction of instanton equations on C

$$F + R^2[\varphi, \bar{\varphi}] = 0\,, \qquad \bar{\partial}_A \varphi = 0\,,$$

where A is a $\mathfrak{g}$ connection over C and $\varphi \in H^0(\mathfrak{g}_{\mathbb{C}} \otimes K)$.

$T^{2r} \to \mathcal{M}_H \to \mathcal{B}$ can be viewed as an integrable system [Hitchin].

- The spectral curve is a covering of C in T^*C

$$\Sigma : \; \det(\lambda - \varphi) = 0\,,$$

  determined by $u = \{\mathrm{Tr}\varphi^k\} \in \mathcal{B}$.

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$$d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0 \quad \text{for} \quad \mathcal{A} = \frac{R}{\zeta}\varphi + A + R\bar{\zeta}\bar{\varphi}$$

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Then Σ encodes the small $\hbar$ leading WKB asymptotics for $(d + \mathcal{A})\chi = 0$.

Opers

At leading order in $\hbar$ the linear system $(d+\mathcal{A})\chi = 0$ is equivalent to an N-th order ODE (here $\mathfrak{g} = A_{N-1}$)

$$\left[(\hbar\partial_x)^N + \sum_{i=2}^{N} \mathrm{Tr}\varphi^{\mathrm{i}} (\hbar\partial_x)^{N-i}\right]\psi(x) = 0\,.$$

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To retain all information about $\mathcal{A}$ one needs to go beyond leading order in $\hbar$. In general, this leads to opers with **apparent singularities**. [Coman L Teschner]

Emergence of apparent singularities

To illustrate this point we return to our main example. For Yang-Mills theory $C = \mathbb{P}^1$ and $\mathcal{A} \in \mathfrak{sl}_2(\mathbb{C})$

$$\mathcal{A} = \frac{1}{\hbar} \begin{pmatrix} \mathcal{A}_0 & \mathcal{A}_+ \\ \mathcal{A}_- & -\mathcal{A}_0 \end{pmatrix} = \frac{1}{\hbar}\varphi + A + O(\hbar)$$

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Applying a gauge transformation defined by

$$h = \begin{pmatrix} \mathcal{A}_-^{-1/2} & 0 \\ 0 & \mathcal{A}_-^{1/2} \end{pmatrix} \begin{pmatrix} 1 & \frac{\hbar}{2}\mathcal{A}_-'/\mathcal{A}_- + \mathcal{A}_0 \\ 0 & 1 \end{pmatrix}$$

takes the connection to oper form

$$h^{-1} \cdot (\partial_x - \mathcal{A}) \cdot h = \partial_x - \frac{1}{\hbar} \begin{pmatrix} 0 & q(x,\hbar) \\ 1 & 0 \end{pmatrix}$$

$$q(x,\hbar) = \underbrace{\mathcal{A}_0^2 + \mathcal{A}_+\mathcal{A}_-}_{\frac{1}{2}\mathrm{Tr}\varphi^2} - \hbar\left(\mathcal{A}_0' - \frac{\mathcal{A}_0\mathcal{A}_-'}{\mathcal{A}_-}\right) + \hbar^2\left(\frac{3}{4}\left(\frac{\mathcal{A}_-'}{\mathcal{A}_-}\right)^2 - \frac{\mathcal{A}_-''}{2\mathcal{A}_-}\right)$$

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The $\hbar$ corrections have singularities at $\mathcal{A}_- = 0$. (eigenvectors of $\mathcal{A}$ do as well)

Quantum curve for $SU(2)$ Yang-Mills

The 'quantum curve' is then

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For $SU(2)$ Yang-Mills the $\hbar$-deformed Schrödinger potential is

$$q(x, \hbar) = \frac{\Lambda^2}{x^3} + \frac{U}{x^2} + \frac{\Lambda^2}{x} - \hbar \frac{u(2x - u)}{x^2(x - u)} v + \hbar^2 \frac{3}{4(x - u)^2}$$

where $U \in \mathcal{B}$ parametrizes a Coulomb vacuum, u is the position of the apparent singularity. v is a dependent parameter determined by $v^2 = \frac{\Lambda^2}{u^3} + \frac{U}{u^2} + \frac{\Lambda^2}{u}$.

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$$\left[\hbar^2 \partial_x^2 - q(x,\hbar)\right] \psi(x) = 0\,.$$

For $SU(2)$ Yang-Mills the $\hbar$-deformed Schrödinger potential is

$$q(x,\hbar) = \frac{\Lambda^2}{x^3} + \frac{U}{x^2} + \frac{\Lambda^2}{x} - \hbar \frac{u(2x-u)}{x^2(x-u)} v + \hbar^2 \frac{3}{4(x-u)^2}$$

where $U \in \mathcal{B}$ parametrizes a Coulomb vacuum, u is the position of the apparent singularity. v is a dependent parameter determined by $v^2 = \frac{\Lambda^2}{u^3} + \frac{U}{u^2} + \frac{\Lambda^2}{u}$.

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Apparent singularities arise naturally when describing $SL_2\mathbb{C}$ flat connections though 2nd order ODEs. They encode next-to-leading order $\hbar$ corrections to $\mathcal{A}$, providing a complete parametrization of $\mathcal{M}_H$ in a neighbourhood of $\mathcal{M}_{\text{oper}}$.

Isomonodromy

Viewing $\mathcal{M}_H$ as a moduli space of flat connections $\mathcal{A}$, a local parametrization is given in terms of **monodromy data**

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While apparent singularities contribute nothing to **local** monodromy at $x = u$, they contribute $\hbar$-corrections to **global** monodromies.

- Σ has $\mathrm{rk}\ H_1(\Sigma) = 2$ independent cycles, therefore $\dim_{\mathbb{C}} \mathcal{M}_H = 2$
- but $q(x, \hbar)$ depends on 3 parameters: (u, v, Λ)
- Therefore $\mu(u, v, \Lambda)$ is over-parameterized: there must be a 1-parameter family of 'isomonodromic deformations'

Isomonodromic deformations of $SU(2)$ YM quantum curve are described by a non-autonomous Hamiltonian system (Painlevé III_3 / r-sine-Gordon)

$$\partial_r u = \frac{\partial H}{\partial v} \qquad \partial_r v = -\frac{\partial H}{\partial u} \qquad H = \frac{v^2}{2r} - r\,\cos u$$

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where in this case $\mathbf{t} = r$ and $H_i = H$.

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NB: It is important to realize that normalization of τ is ambiguous

$$\tau \sim f(\mu) \cdot \tau$$

3. τ-functions and instantons

Tau function and instantons

The relevance of τ to 4d $\mathcal{N} = 2$ gauge theory lies in the relation [Gamayun Iorgov Lisovyy]

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Rmk: These relations can be explained by string theory (in hindsight)

- $\tau \sim Z_{\text{inst},D}$ is related to free-fermion partition functions on Σ [Nekrasov] [Aganagic Dijkgraaf Klemm Marino Vafa] [Nekrasov Okounkov] [...]
- String dualities further predict that $Z_{\text{ff}}(\Sigma)$ should admit a (Fourier-type) decomposition with coefficients Z_{top}. [Dijkgraaf Hollands Sulkowski Vafa]
- Free fermion representations of conformal blocks are also related to Z_{inst} by 2d-4d correspondences [Alday Gaiotto Tachiwaka] [Wyllard] [...]

Expansions of $\tau_{\mathrm{P}_{III}}$ – part 1

It was shown by [Gavrylenko Lisovyy] that near $\Lambda \approx 0$ there exist **coordinates** $\mu = (\sigma, \eta)$ such that

$$\tau^{(w)}(\sigma, \eta; \Lambda) = \sum_{n \in \mathbb{Z}} e^{4\pi \mathrm{i}\, n\, \eta}\, \mathcal{N}^{(w)}(\sigma + n)\, \mathcal{Z}^{(w)}(\sigma + n, \Lambda)$$

where

$$\mathcal{N}^{(w)}(\sigma) = \prod_{s=\pm} \frac{1}{G(1 + 2s\sigma)}, \qquad \mathcal{Z}^{(w)}(\sigma, \Lambda) = \Lambda^{4\sigma^2}\left(1 + \sum_{k=1}^{\infty} \mathcal{Z}_k^{(w)}(\sigma)\Lambda^{4k}\right),$$

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In particular $\mathcal{Z}_k^{(w)}(\sigma)$ admit explicit descriptions in terms of sums over pairs of Young diagrams (Y_1, Y_2), reproducing $\mathcal{Z}^{(w)} \sim Z_{\text{inst}}$ of [Nekrasov].

Expansions of $\tau_{\mathrm{P}_{III}}$ – part 2

On the other hand when $\Lambda \to \infty$ another, rather different, expansion of τ was conjectured by [Its Lysovyy Tykhyy] in another set of **coordinates** $\mu = (\nu, \rho)$

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$$(\sigma, \eta) \to \mathcal{N}^{(w)}, \mathcal{Z}^{(w)} \qquad (\nu, \rho) \to \mathcal{N}^{(s)}, \mathcal{Z}^{(s)}$$

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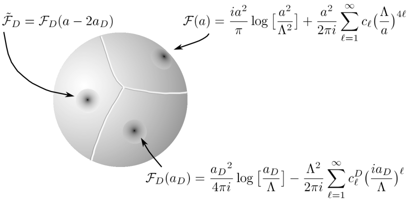
- $\mathcal{Z}^{(w)}$ is a series in Λ, while $\mathcal{Z}^{(s)}$ in Λ^{-1}.

QFT interpretation [Its Lysovyy Tykhyy] [Bonelli Lisovyy Maruyoshi Sciarappa Tanzini] [...]



	weak	strong (new!)
Λ	small	large
Z_{pert}	$\mathcal{N}^{(w)}(\sigma, \Lambda)$	$\mathcal{N}^{(s)}(\nu, \Lambda)$
Z_{inst}	$\mathcal{Z}^{(w)}(\sigma, \Lambda)$	$\mathcal{Z}^{(s)}(\nu, \Lambda)$
$Z_\gamma \approx 0$	W-bosons	monopole / dyon
$Z_{\text{pert}} \sim G(\cdot, \Lambda)^{-\Omega}$	$\Omega = -2$	$\Omega = 1$
?	(σ, η)	(ν, ρ)

Relation between weak and strong coupling expansions

Just like $\mathcal{Z}^{(w)}$ matches with Z_{inst}, the match between $\mathcal{Z}^{(s)}$ and $\mathcal{F}_D(a_D)$ in the $\hbar \to 0$ limit suggests that this can be taken as a definition of the instanton partition function at **strong coupling**.

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Both $\mathcal{Z}^{(w/s)}$ are obtained from the τ function, but there are differences:

- The expansion of τ that defines $\mathcal{Z}$ is performed in two different sets of monodromy coordinates (σ, η) and (ν, ρ) related by

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- Tau functions $\tau^{(w/s)}$ are not identical due to the normalization ambiguity

$$\tau^{(w)} = \chi(\mu)\cdot\tau^{(s)}$$

4. Weak/strong coupling connection coefficients and the global picture

Geometrization of instanton partition functions

In [Coman PL Teschner] we formulate a proposal that explains:

- why (σ, η) and (ν, ρ) are distinguished coordinates at weak/strong coupling
- why they are related in this particular way
- how the relative normalization factor $\chi(\mu) = \tau^{(w)}/\tau^{(s)}$ arises

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Main results (valid for any theory of class $S[A_1]$)

- There is a natural definition of quantum curve, and of isomonodromic τ.
- We define a decomposition of $\mathcal{M}_H = \{\mathcal{R}_\alpha\}_\alpha$ with a canonical choice of monodromy coordinates in each region

$$(x_\alpha, y_\alpha): \ \mathcal{R}_\alpha \to (\mathbb{C}^*)^{2r}$$

- We determine relations among coordinates of any two patches, and provide the connection coefficient for τ

$$(x_\alpha, y_\alpha) \to (x_\beta, y_\beta) \qquad \tau^{(\beta)} = \chi^{(\beta\alpha)} \, \tau^{(\alpha)} \, .$$

- In each region we obtain a **geometric definition** of $Z^{(\alpha)}_{\text{inst}}$ by series decomposition of $\tau^{(\alpha)}$ w.r.t. the chosen coordinates. In agreement with localization at weak coupling, new predictions for all other regions.

Coordinate charts

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Definition for 2nd order ODEs: given the (classical) quadratic differential $q(x)$, the network $\mathcal{W}(U, \vartheta)$ consists of critical leaves of the horizontal foliation

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Coincides with the **Stokes graph** of exact WKB analysis of Schrödinger's equation with [see Ito's talk]

$$V(x) - E = q(x) \qquad \arg\hbar = \vartheta$$

For the quantum curve of $SU(2)$ YM, the appropriate potential $q(x, \hbar)$ is determined by by the choice between limits $r \to 0$ or $r \to \infty$

- At **weak coupling** the spectral network produces **Fenchel-Nielsen** coordinates $(\mathcal{U}, \mathcal{V})$
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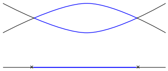
In both cases, coordinates correspond to Borel-resummed **Voros symbols** of the ODE. [Iwaki Nakanishi] [Allegretti]

$$\left(-\hbar^2 \partial_x^2 + q(x,\hbar)\right) \psi(x) = 0 \qquad \psi^{(a)}(x) = \exp\left(\frac{1}{\hbar}\int^x y^{(a)}(x',\hbar)\, dx\right)$$

$$V_\gamma := \mathscr{B}\left[\exp\left(\frac{1}{\hbar}\int_{\wp(\gamma)} y^{(a)}_{\mathsf{odd}}(x',\hbar)\, dx\right)\right]$$



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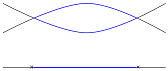

What changes from FN to FG is, essentially, the type of Stokes graph:

$$V_\gamma$$

$\swarrow$ $\searrow$

weak coupling $(\mathcal{U}, \mathcal{V})$	$(\mathcal{X}, \mathcal{Y})$ strong coupling

Juggling between Fock-Goncharov and Fenchel-Nielsen

Spectral Networks compute FN/FG coordinates as Voros symbols.

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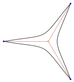
FG coordinate patches: 1-1 with **triangulations** dual to the Stokes graph



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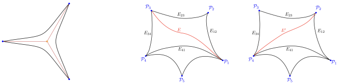

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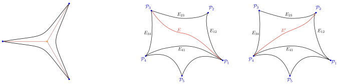

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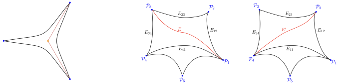
FG coordinate patches: 1-1 with **triangulations** dual to the Stokes graph



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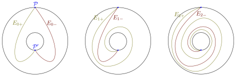
Novelty: finitely many flips take FG $\to$ FG, but an **infinite** sequence of flips (a.k.a. the 'juggle') takes **Fock-Goncharov to Fenchel-Nielsen** coordinates



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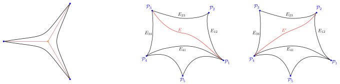

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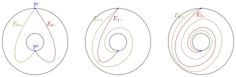

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We prove: $(\mathcal{U} = e^{2\pi i\sigma}, \ \mathcal{V} = i\, e^{2\pi i\eta})$ and $(\mathcal{X} = -e^{-8\pi i\rho + 2\pi\nu}, \mathcal{Y} = -e^{-2\pi\nu})$ and that $(\mathcal{U}, \mathcal{V}) \leftrightarrow (\mathcal{X}, \mathcal{Y})$ coincides with $(\sigma, \eta) \leftrightarrow (\rho, \nu)$ from [Its Lisovyy Tykhyy].

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Given the known relation $(\sigma, \eta) \leftrightarrow (\nu, \rho)$, one could set up the difference equation. But it is hard to solve.

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Solution:

$$\chi_{\mathsf{flip}}(x,x') = \exp\left(2\pi i x x' + \frac{1}{2\pi i}\mathrm{Li}_2(1-e^{2\pi i x'})\right)$$

Change of normalization for τ under flip

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The **difference generating function** of the change of coordinates between two neighbouring patches (flip) is also the **relative normalization** of τ.

Normalized τ and Z_{inst} across moduli space: a proposal

A system of **charts** $\{\mathcal{R}_\alpha\}$ over $\mathcal{M}_H$ is defined by Stokes graphs.
In $\mathcal{R}_\alpha$ a distinguished set of **coordinates** (x_α, y_α) (FG/FN or Voros).

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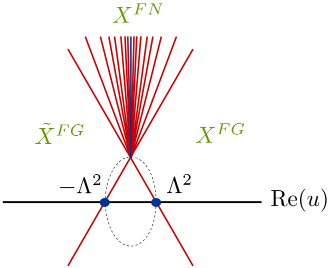
This data provides a definition of instanton partition function in each patch

$$\tau^{(\alpha)}(x_\alpha, y_\alpha, \Lambda) \quad \to \quad Z_{\text{inst}}^{(\alpha)}(x_\alpha, \Lambda)\,.$$

Chart system and BPS spectrum

Charts $\{\mathcal{R}_\alpha\}_\alpha$ are regions where the Stokes graph is regular. Degenerations due to Stokes automorphisms (flips) of (x, y) happen along lines in $\mathcal{B}$



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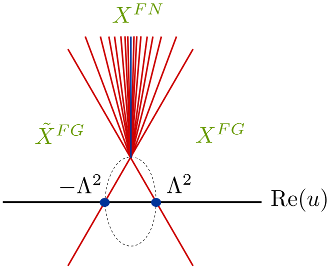

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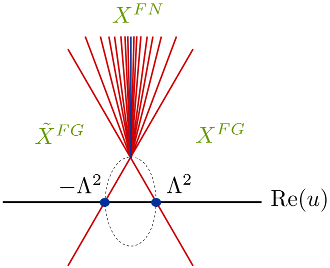

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Indeed Stokes automorphisms are 1-1 with **BPS states** of the 4d theory

$$\text{'Stokes' lines in } \mathcal{B} \ : \{\arg Z_\gamma = \vartheta \ \& \ \Omega(\gamma, u) \neq 0\}\,.$$

The BPS spectrum governs the chart system and the definition of Z_{inst}

5. Conclusions

Summary

4d $\mathcal{N} = 2$ QFTs of class S are naturally associated to quantum curves arising from quantization of Hitchin spectral curves, with apparent singularitites.

Exact WKB analysis defines a system of charts & coordinates over $\mathcal{M}_H$. The global structure is governed by the BPS spectrum and its wall-crossing.

Coordinate transformations across charts are described by a known universal function χ_{flip}. The same function describes renormalization of τ.

Taking a Fourier transform of appropriately normalized $\tau^{(\alpha)}$ with respect to local coordinates (x_α, y_α) yields a definition of $Z_{\mathsf{inst}}(x_\alpha)$.

Agreement with a Lagrangian description where available (weak coupling). In all other patches the definition is new. A field theoretic interpretation likely involves the identification of local degrees of freedom within $\mathcal{R}_\alpha$.

Outlook

Generalizations beyond A_1 theories presents new features with interesting implications for/from integrability:

- Higher order ODEs still governed by TBAs [Hollands Neitzke] [Fioravanti Poghossian Poghossian] [Ito Marino Shu] [Ito Kondo Shu] [...], however new 'wild regions' are present with dense jumps in ϑ [Galakhov PL Mainiero Moore Neitzke].

- 5d $\mathcal{N} = 1$ QFT on S^1 also feature an integrable SW structure [Nekrasov]. Exact WKB analysis of q-difference equations, still largely undeveloped, plays a central role [Banerjee PL Romo] [Alim Saha Teschner Tulli] [Alim Hollands Tulli] [Grassi Hao Neitzke] [Del Monte PL]. Quantum periods governed by TBAs via qDE/IM [Frenkel Koroteev Zeitlin], conjecturally related to doubly periodic monopoles [Cherkis] and multiplicative Hitchin systems [Elliott Pestun].