Inverse Scattering from Spectral Curves

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I. Introduction

Overview

Goal: Understand how to obtain inverse scattering from the infinite-length limit of spectral curves.

Overview:

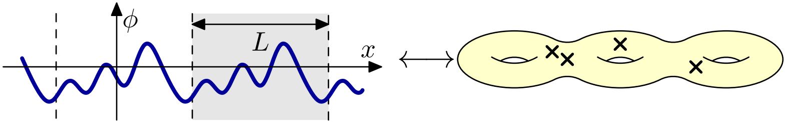
- motivation
- review: spectral curves and inverse scattering in KdV
- intuition: explicit wave train and soliton solution in KdV
- finite-length extrapolation for KdV
- continuous Heisenberg model

Solving 1D Integrable Models

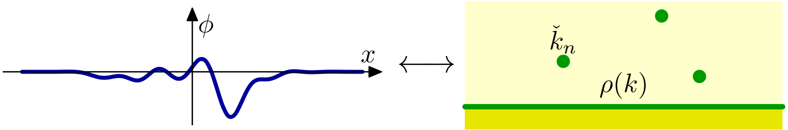
Integrable models solved by efficient methods.

Boundary conditions relevant for fields and chains.
Relevant equations often embody these boundary conditions. Main cases:

finite domain
open/**closed** boundaries
spectral curve method



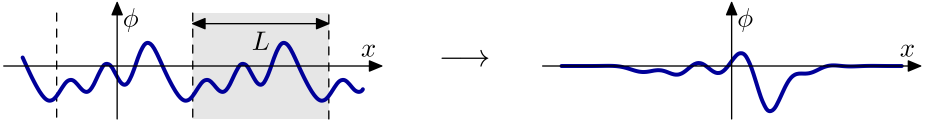
infinite domain
asymptotic boundaries
inverse scattering method

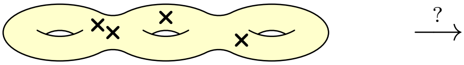


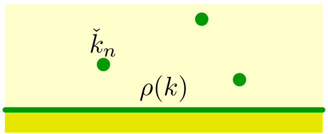
Both methods based on an auxiliary linear problem (ALP) specified by Lax connection.

Infinite-Length Limit

Can relate different boundary conditions: open/closed to asymptotic.





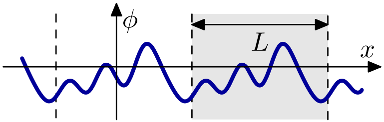


Questions:

- Asymptotic boundary conditions for spectral curves:
  Which families of spectral curves have proper infinite-length limits?
- How do integrable structures transform into each other?
  Spectral curve and inverse scattering technically rather different!
- Understood how to obtain solitons from periodic solution (1970's: Matveev, Its)

Spectral Curves

Represent periodic states as spectral curve with divisor:



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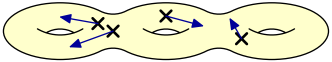


State encoded as:

- complex curve: conserved charges of state (space and time-invariant)
- dynamical divisor: phase degrees of freedom (depends on space and time)
- space and time evolution: linear on Jacobian of curve

Equivalent information on both sides, transformation non-trivial:

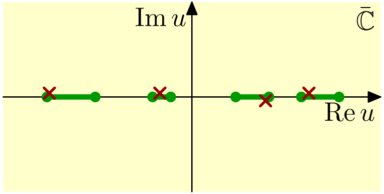
- state $\to$ spectral curve: parallel transport, eigenvalue problem
- construction of spectral curve: complex analysis $(\ldots)$
- spectral curve $\to$ state: expansion at distinguished point

Hyper-Elliptic Curves

Consider simpler class of states (finite gap): restrict to curves with finite genus g.

Hyper-elliptic curve

$$\left(\frac{\mathrm{d}q}{\mathrm{d}u}\right)^2 = \frac{P_{\approx g}(u)^2}{Q_{\approx 2g}(u)}$$



two sheets connected by $g+1$ branch cuts

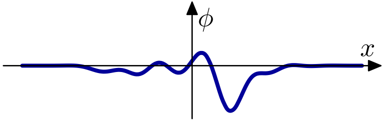
Interpretation of Cuts:

- n-th cut represents n-th periodic mode
- location of cut fixed by n (periodicity)
- size of cut $\sim$ amplitude (action variable)
- marked point on cut $\sim$ phase (angle variable)

Altogether: spectral curve acts as adjusted non-linear Fourier decomposition.

Inverse Scattering

Represent asymptotic states as reflection function plus bound state poles.



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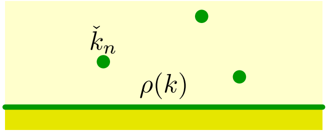


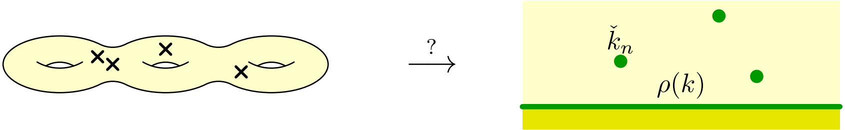
Scattering problem using of state as potential $\to$ scattering data:

- reflection function probes amplitude and phase of state at continuous momentum
- zeros of transmission function in upper half-plane indicate bound states: solitons
- scattering data has linear time evolution
- original state can be reconstructed from scattering data by GLM integral equation

Altogether: scattering transformation acts as Fourier transformations plus solitons.

Motivation

Goal: Understand infinite-length limit of spectral curves:



Issues:

- Easy to specify scattering data; then solve concrete GLM integral equation.
- Elaborate to set up spectral curve with divisor; need to solve abstract RH problem.
- How to obtain two classes of objects in the infinite-length limit?
- What determines allowable momenta of solitons (imaginary axis)?
- Why are solitons very simple while reflective potentials difficult?
- Do we need spectral curves with infinite genus needed?
- How to impose asymptotic limit in spectral curve and divisor?

II. Korteweg-de Vries Equation: Spectral Curve and Inverse Scattering

Korteweg-de Vries Equation

Choose KdV equation as a simple model where above questions can be addressed

$$\dot{\phi} = 6\phi\phi' - \phi'''.$$

Integrable structures encoded in Lax connection ($u \in \bar{\mathbb{C}}$ is spectral parameter)

$$\frac{\partial}{\partial x}\begin{pmatrix}\psi(u;x)\\ \psi'(u;x)\end{pmatrix} = \begin{pmatrix}0 & 1\\ \phi(x)-u & 0\end{pmatrix}\begin{pmatrix}\psi(u;x)\\ \psi'(u;x)\end{pmatrix}, \qquad \psi''(u;x) = \bigl(\phi(x)-u\bigr)\psi(u;x).$$

Here: second-order linear differential equation for auxiliary function $\psi(u;x)$.

Infinitely many local conserved charges

$$Q = \int \mathrm{d}x\,\tfrac{1}{2}\phi, \qquad P = \int \mathrm{d}x\,\tfrac{1}{2}\phi^2, \qquad E = \int \mathrm{d}x\,\bigl[\tfrac{1}{2}\phi'^2 + \phi^3\bigr], \qquad \ldots$$

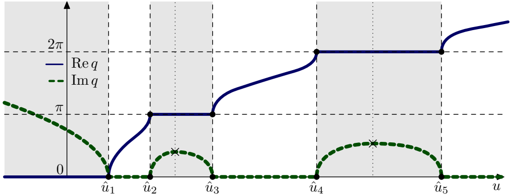
Galilei-invariant momentum $\tilde{P} = P - 2Q^2/L$, energy $\tilde{E} = E - 12PQ/L + 16Q^3/L^2$.

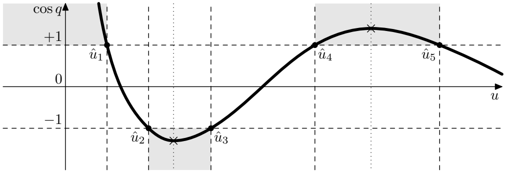
Spectral Curves

Spectral curve $\mathrm{d}q/\mathrm{d}u$ defined by monodromy eigenvalue problem:

$$\psi(u; x+L) = \exp\big(\mathring{\imath} q(u)\big)\psi(u; x).$$

Monodromy trace is $2\cos q(u)$ and branch points are where $\cos q(u) = \pm 1$.
Sketch of quasi-momentum function $q(u)$ and $\cos q(u)$:

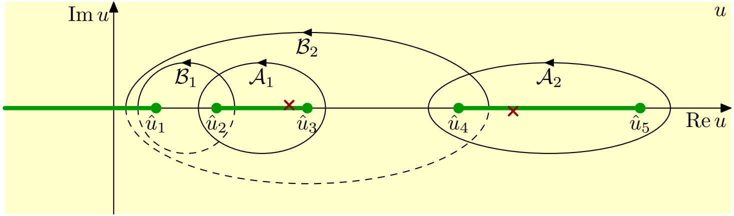




• Cuts are forbidden zones for (real) cosine function. • One cut stretches to $u = -\infty$.

Cuts and Divisor

A,B-period integrals describe moduli of spectral curve:



$$\oint_{\mathcal{A}_j} \mathrm{d}q = 0,$$

$$\frac{1}{2\pi} \oint_{\mathcal{B}_j} \mathrm{d}q = n_j,$$

$$\frac{1}{\mathring{\imath}\pi} \oint_{\mathcal{A}_j} u \, \mathrm{d}q = I_j.$$

- mode number n_j determines location; • action variable I_j determines size of cut.

Dynamical divisor is set of poles $\{\tilde{u}_j(x)\}$ of $\psi'(u;x)/\psi(u;x)$:

- one pole on each cut; one pole fixed at $u = \infty$;
- pole oscillates n_j times over one period of x;
- divisor specified by point on Jacobian: linear space and time dependence.

Inverse Scattering

Scattering Data consists of:

- reflection coefficient function $\rho(k) \in \mathbb{C}$ for $k \in \mathbb{R}$;
  satisfies momentum space reality $\rho(k)^* = \rho(-k^*)$ and is bounded: $|\rho(k)| < 1$.
- bound state momenta $\check{k}_n \in \mathring{\imath}\mathbb{R}^+$ and associated dynamical variables $\mu_n \in \mathbb{R}^+$.

KdV solved by GLM integral equation with scattering data in $N(w)$

$$\Psi(x,y) = N(x+y) + \int_{-\infty}^{x} \mathrm{d}z\, \Psi(x,z)\, N(z+y), \qquad \phi(x) = 2\partial_x\bigl(\Psi(x,x)\bigr)$$

Transmission function from scattering dispersion formula

$$\tau(k) = \left[\prod\nolimits_j \frac{k+\check{k}_j}{k-\check{k}_j}\right] \exp\left[\frac{1}{2\pi\mathring{\imath}} \int \frac{\mathrm{d}k'}{k'-k-\mathring{\imath}0} \log\bigl(1-|\rho(k')|^2\bigr)\right].$$

Expansion at $k=\infty$ describes local conserved charges.

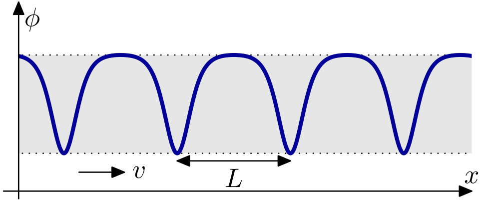
III. Wave Trains and Solitons in KdV

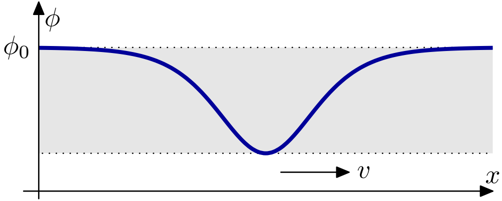
Travelling Wave States

Gain intuition from class of states with fixed shape and velocity: $\phi(x,t) = \phi(x - vt)$.

Explicit solutions: Periodic wave train (cnoidal wave) and soliton with asymptotic decay

$$\phi(x) = \begin{cases} -\frac{1}{6}v - \frac{2}{3}\alpha^2(1-2m) - 2\alpha^2 m\,\mathrm{cn}(\alpha x + \beta, m)^2 & (0 < m < 1) \\ -\frac{1}{6}v + \frac{2}{3}\alpha^2 - 2\alpha^2\,\mathrm{sech}(\alpha x + \beta)^2 \end{cases}$$





Periodicity length $L = 2\,\mathrm{K}(m)/\alpha$ among moduli of solution.

Straight-forward to take infinite-length limit: $m \to 1$ with α, β, v fixed.

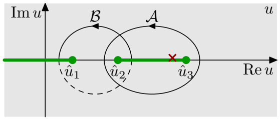
Elliptic Curve

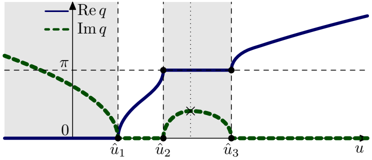
How does infinite-length limit act on spectral curve?

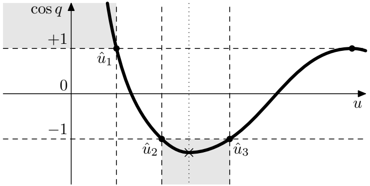
Auxiliary function $\psi(u;x)$ determined by matching of singularities:

$$\psi = \exp\!\Big(\tfrac{\mathring{\imath}}{2}(\alpha x+\beta)q/\mathrm{K}\Big)\,\frac{\mathrm{tn}(\alpha x+\beta+z)}{\mathrm{tn}(\alpha x+\beta)\,\mathrm{tn}(z)}\,,\qquad \begin{aligned} u &= -\tfrac{1}{6}v+\tfrac{1}{3}\alpha^2(m-2)-\alpha^2\,\mathrm{cs}(z)^2,\\ q &= 2\mathring{\imath}\mathrm{K}\big[\mathrm{zn}(z)+\mathrm{cs}(z)\,\mathrm{dn}(z)\big].\end{aligned}$$

Sketch of cuts and quasi-momentum function:



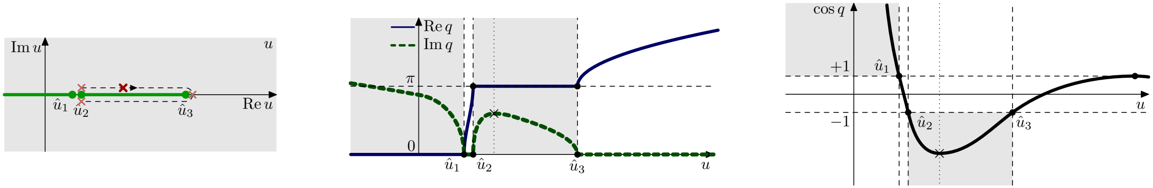




Divisor pole oscillates (in space and time) back and forth on cut.

Infinite-Length Limit of Curve

Sketch of curve and quasi-momentum at $m \approx 1$:



Exponentially small separation of cuts, spectral curve pinched:

$$\hat{u}_2 - \hat{u}_1 = \alpha^2(1-m) = 16\alpha^2 \mathrm{e}^{-\alpha L} + \ldots .$$

Singular point given by soliton momentum: $\hat{u}_{1,2} \to -\frac{1}{6} - \frac{1}{3}\alpha^2 = -(\operatorname{Im} \check{k})^2$.

Divisor: • at $x \to \mp\infty$ pole resides almost fixed at singular point;
• near soliton location $x = x_0$ pole moves back and forth to other end of branch cut.

IV. Finite-Length Extrapolation

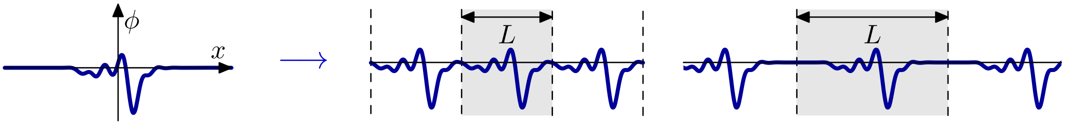
Procedure

Some further thoughts:

- Genus-g curves yield g-soliton states upon complete degeneration.
- How to obtain non-trivial continuum in reflection function $\rho(k)$?
- How to set up family of spectral curves with appropriate infinite-length limit?

Procedure:

- Start with generic asymptotic state described by scattering data $\rho(k)$, $\{\check{k}_j\}$.
- Periodically identify region of interest with adjustable length L.
- Analyse spectral curve for resulting family of states.



Approximation

Key Insight: Same field $\phi(x)$, auxiliary function $\psi(u,x)$ compatible for both situations!

Consider Lax transport with spectral parameter $u = k^2$

$$W(k;b,a) := \overleftarrow{\mathrm{P}}\left[\exp \int_a^b \mathrm{d}x\ \mathcal{A}(k;x)\right].$$

Defines Lax monodromy $T(k)$ and auxiliary scattering matrix $S(k)$

$$T = W(x+L,x), \qquad S = \lim_{a,b\to\mp\infty} \operatorname{diag}(\mathrm{e}^{\mathring{\imath}kb},\mathrm{e}^{-\mathring{\imath}kb})\,W(b,a)\ \operatorname{diag}(\mathrm{e}^{-\mathring{\imath}ka},\mathrm{e}^{\mathring{\imath}ka}).$$

If asymptotic limit exists, can approximate finite Lax transport (over region of interest)

$$W(b,a) \approx \operatorname{diag}(\mathrm{e}^{-\mathring{\imath}kb},\mathrm{e}^{\mathring{\imath}kb})\,S\ \operatorname{diag}(\mathrm{e}^{\mathring{\imath}ka},\mathrm{e}^{-\mathring{\imath}ka}).$$

Note: Patching discontinuous! Expect small glitches.

Reconstruction

Form of scattering matrix in inverse scattering

$$S(k) = \begin{pmatrix} 1/\tau(k) & \rho(k)/\tau(k) \\ \rho(-k)/\tau(-k) & 1/\tau(-k) \end{pmatrix}.$$

Transmission function $\tau(k)$ determined through reflection function $\rho(k)$.
Recover Lax transport:

$$W(b,a) \sim \begin{pmatrix} \mathrm{e}^{-\mathring{\imath}k(b-a)}/\tau(k) & \mathrm{e}^{-\mathring{\imath}k(b+a)}\rho(k)/\tau(k) \\ \mathrm{e}^{\mathring{\imath}k(b+a)}\rho(-k)/\tau(-k) & \mathrm{e}^{\mathring{\imath}k(b-a)}/\tau(-k) \end{pmatrix}.$$

Further steps:

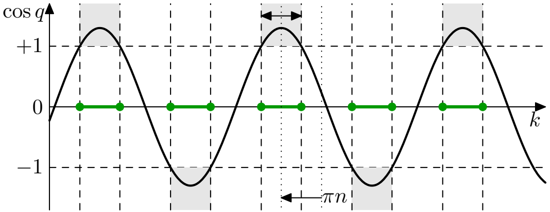
- Determine branch cuts for real k: continuum cuts.
- Determine divisor for real k: continuum divisor.
- Determine branch cuts for imaginary k: solitons.
- Figure out divisor for solitons.

Continuum Cuts

Determine cuts on real axis as forbidden zones for $\cos q$:

$$\cos q(k) = \tfrac{1}{2}\operatorname{tr} T(k) \approx \frac{e^{-ikL}}{2\tau(k)} + \frac{e^{ikL}}{2\tau(-k)} = \frac{1}{|\tau(k)|}\cos\bigl(kL + \arg\tau(k)\bigr).$$

Note: kL induces fast oscillation while $\tau(k)$ is slow; approximate as constant τ_n.



array of small cuts:

- size: $\Delta k_n = (2/L)\arccos|\tau_n|$.
- position: $k_n = (\pi n - \arg\tau_n)/L$.

note:

- unitarity: $\Delta k_n = (2/L)\arcsin|\rho_n|$.
- full filling $\Delta k_n = \pi/L$ for $|\rho| = 1$.

Action variable for cut $I_n = -(2\pi n/L^2)\log(1 - |\rho_n|^2)$.

Continuum Divisor

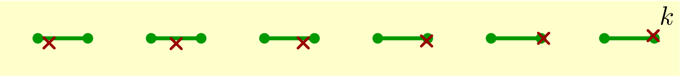
Divisor poles where monodromy eigenvector aligned with $(1, 1)$. Resulting condition:

$$\sin\bigl(\tilde{k}L + \arg\tau(\tilde{k})\bigr) \approx -|\rho(\tilde{k})| \sin\bigl(\tilde{k}(L + 2x) - \arg\rho(\tilde{k}) + \arg\tau(\tilde{k})\bigr).$$

Note: difficulty to interpret x of monodromy: near region of interest vs. asymptotic.

One solution $\tilde{k}_n$ on each cut:

- effective phase $\sigma_n := 2\pi n x/L - \arg\rho_n$ determines position around cut (non-linear).
- one period in dynamical reflection phase $\arg\rho_n$ yields one cycle around cut.
- one period in x yields n cycles around cut.
- effective phase shifts by $\sigma_{n+1} - \sigma_n \approx 2\pi x/L$ between adjacent cuts.



Soliton Cuts

Cuts for solitons are along positive imaginary k-axis. **Problem:**

- Exponents $\exp(\mathring{i}k*)$ converge or diverge exponentially fast.
- Errors in approximations may get unduly attenuated.
- Only $1/\tau(k)$ well-defined and holomorphic on upper half-plane.

Branch points determined by $\cos q(k) = \pm 1$:

- function typically large and dominated by $1/\tau(k)$;
- contribution by $1/\tau(-k)$ typically small.

$$\cos q(k) \approx \frac{\mathrm{e}^{-\mathring{i}kL}}{2\tau(k)} = \pm 1.$$

Soliton intermissions at poles $\check{k}_n$ of $\tau(k)$; width Δk determined by residue

$$k_n \approx \check{k}_n, \qquad \Delta k_n \approx -4 \operatorname{res} \tau(\check{k}_n) \exp(\mathring{i}\check{k}_n L).$$

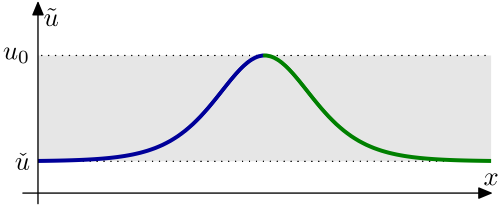
Action variable for cut between two poles $I_n = (4\mathring{i}L/3\pi)(\check{k}_n^3 - \check{k}_{n+1}^3)$.

Soliton Divisor

Unfortunately, the dynamical data μ_n describing the soliton positions are suppressed by the regularisation an therefore not even encoded into the scattering matrix.

Not easy to reconstruct. Consider instead divisor of soliton states:

Spatial dependence of divisor for single soliton: soliton shape in u-space

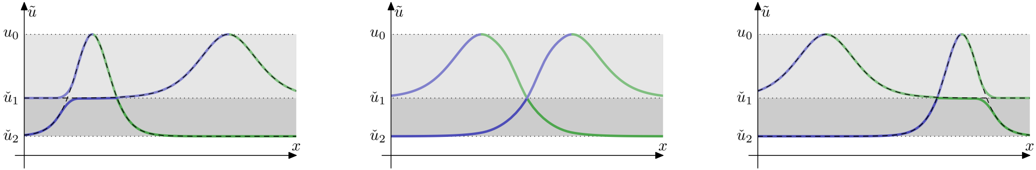


How to encode positions for N solitons on N consecutive branch cuts?

- Note: $1 \leq n \leq N$ full cycles around n-th cut!

Soliton Divisor

Consider explicit two-soliton state. Determine divisor.
Find superposition of two soliton shapes:



Amazing function:

- repulsion for two poles on same side of cuts.
- crossing for two poles on opposite sides of cuts (at singularity).
- Constraints: $1 \leq n \leq N$ full cycles for n-th cut. Precisely 1 pole per cut.

Anyway: • Divisor can be in arbitrary position depending on relative positions.

Limit of Curves

Summary:

A family of curves with $L \to \infty$ has a proper asymptotic limit if:

- the action variables of the first N cuts scale as L and the dynamical divisor has a proper $L \to \infty$ limit.
- the action and angle variables of the remaining cuts must behave as

$$I_n \sim -\frac{2\pi n}{L^2} \log\bigl(1 - |\rho(2\pi n/L)|\bigr), \qquad \sigma_n \sim \frac{2\pi n x}{L} - \arg \rho(2\pi n/L).$$

- Scattering data follows from limiting values as described earlier.

Notes:

- Solitons require 1 cut each.
- Continuum implies an array of small cuts.
- Can work with finite but linearly growing number of cuts.

V. Continuous Heisenberg Model

Continuous Heisenberg Model

Can analogously consider infinite-length limit for Continuous Heisenberg Model:

- Spectral curve and inverse scattering methods work analogously.
- Many details different that need adjustments in construction.

Relevant (interrelated) differences:

- Spectral curve has vertical cuts.
- No forbidden zones for $\cos q$.
- Reflection coefficient unbounded.
- Soliton poles can be anywhere on positive half plane.
- Solitons can form breathers.

In a Nutshell:

- Array of small vertical cuts of length $\sim \operatorname{arsinh} \rho_n$.
- Two (or $2k$) exponentially similar long cuts make up a soliton (breather).

VI. Summary and Outlook

Infinite-Length Limit of Spectral Curves

understood how to prepare suitable family of spectral curves

understood how solitons and continuum arise.

discussed divisor where feasible.

Open Questions:

can the soliton divisor be made more concrete?

other more elaborate models?

derive GLM equations for inverse scattering from infinite-length limit.