

Thermodynamics, transport, and fluctuations in the sine-Gordon model

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Hungarian Research Network

Budapest University of Technology and Economics

joint work with

[Botond Nagy](#) and Gábor Takács

[arXiv:2305.15474](#)

Giuseppe Del Vecchio Del Vecchio , Alvis Bastianello, Benjamin Doyon

[arXiv:2305.10495](#)

Bologna Workshop of CFT & Integrable Models

6 September 2023



The sine-Gordon model

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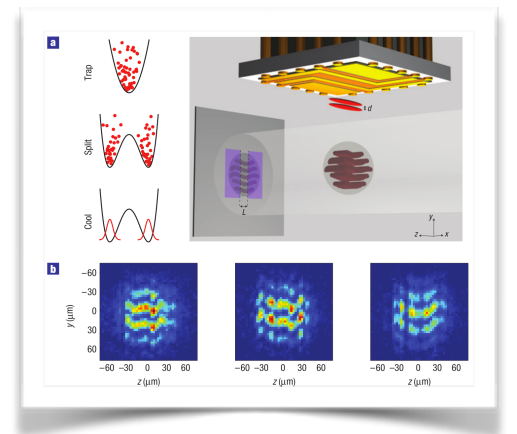
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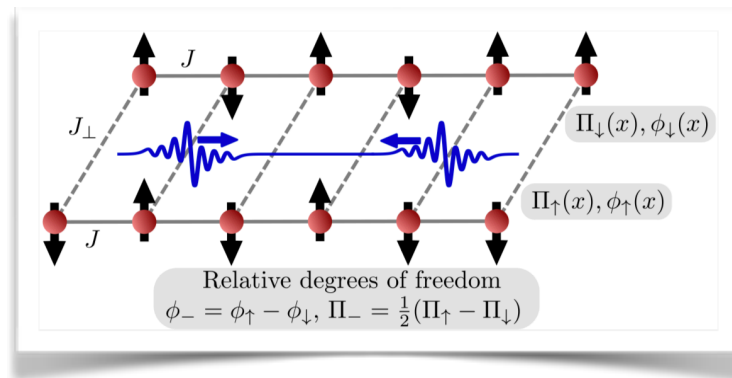
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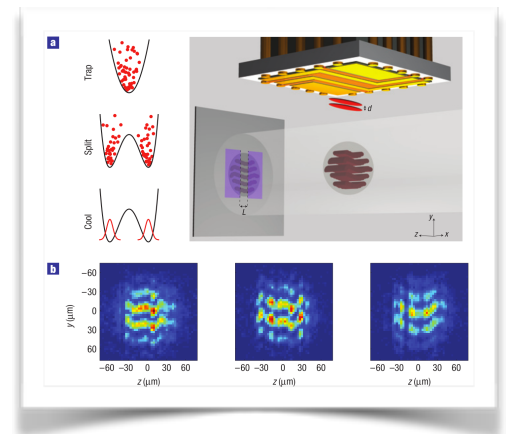
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E. Wybo et al. PRB **106** 075102 (2022)

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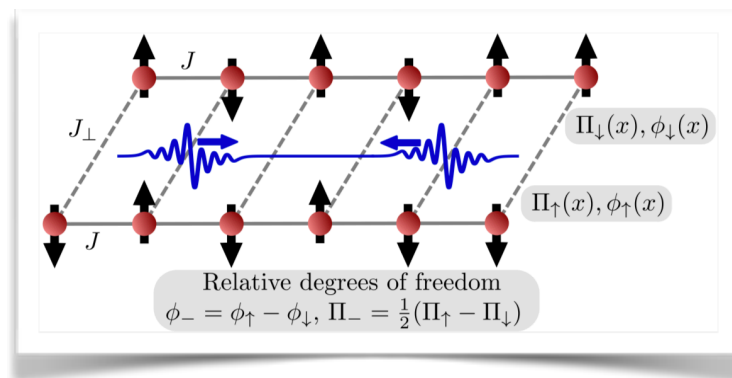
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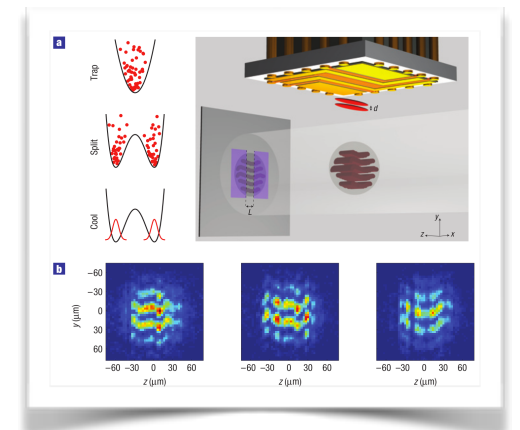
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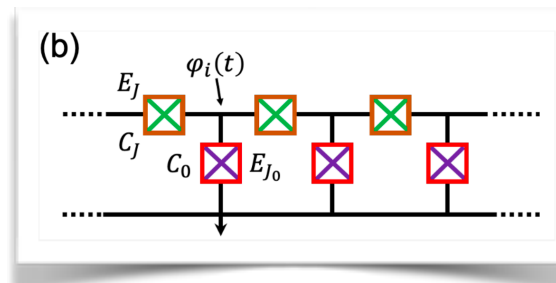


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- quantum circuits of Josephson junctions



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A. Roy et al., Nucl. Phys. B **968**, 115445 (2021)

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Nonlinear integral equations (NLIE) for all couplings
but only for canonical thermal states (~~chemical potential~~, GGE)

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First step: B. Bertini, L. Piroli, M.K., PRB '19

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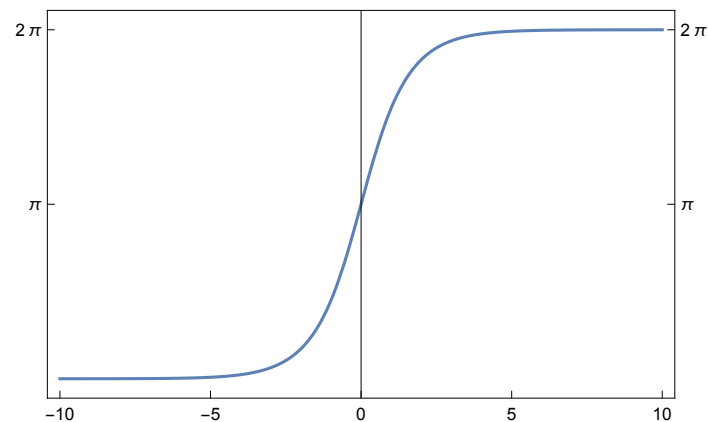
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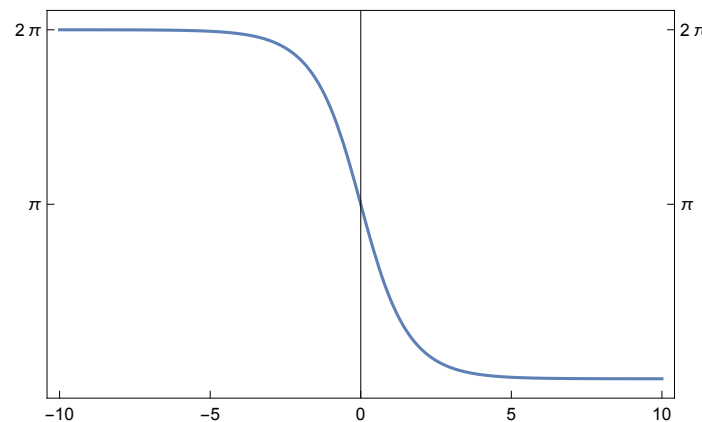
$$\varphi = n \frac{2\pi}{\beta}, \quad n \in \mathbb{Z}$$

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soliton

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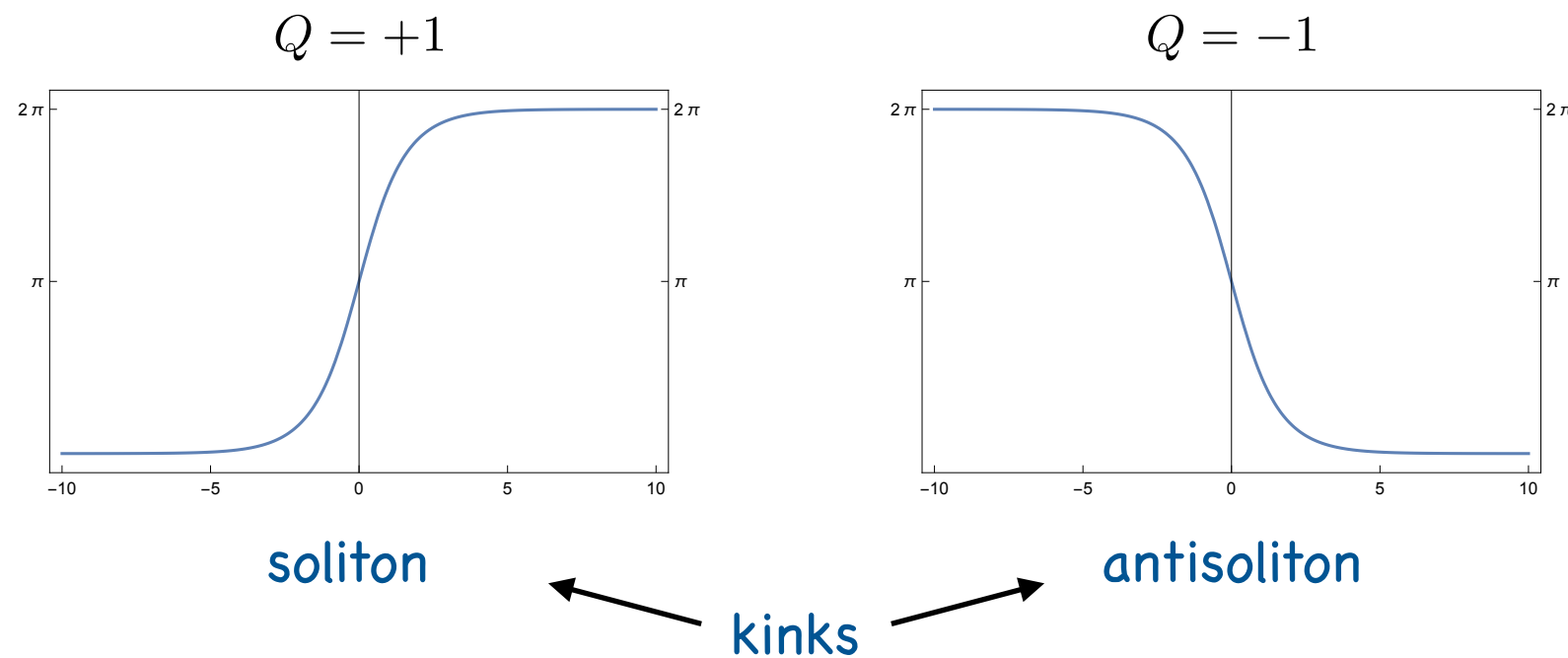
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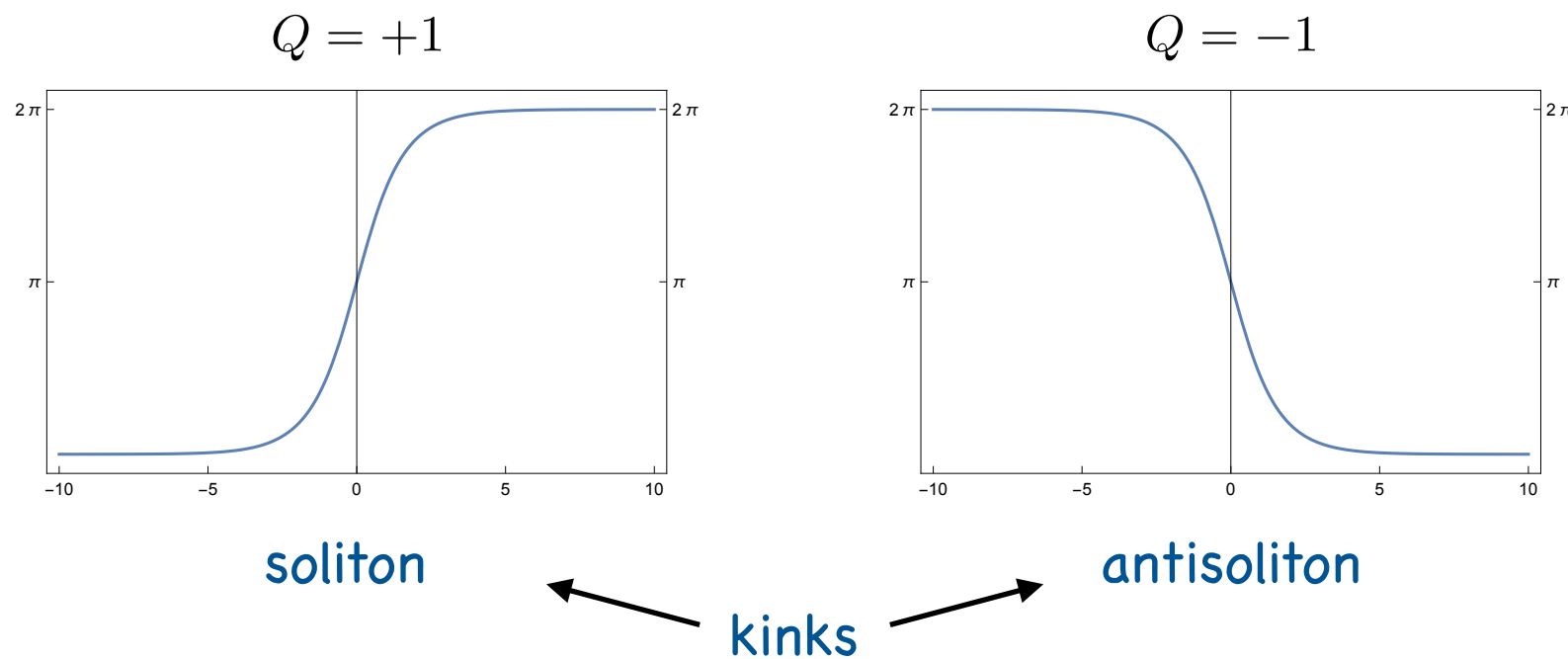


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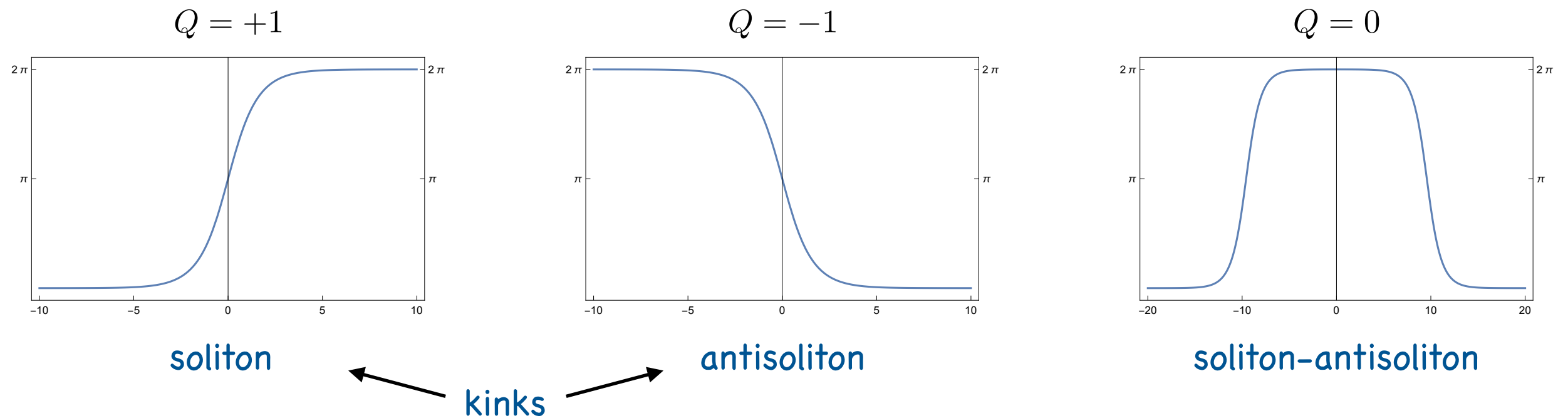
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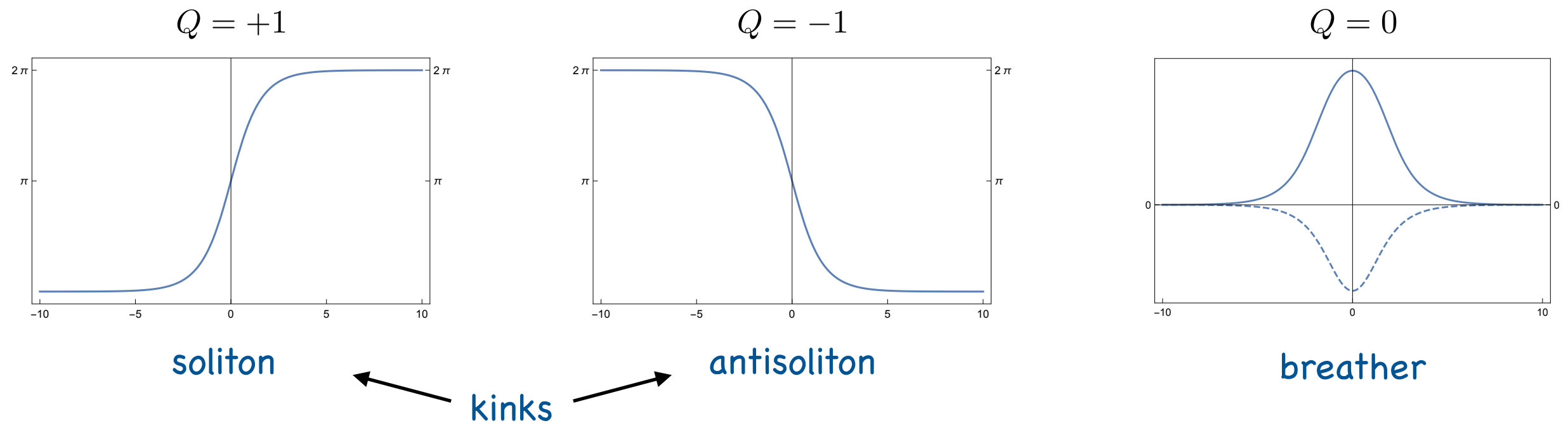
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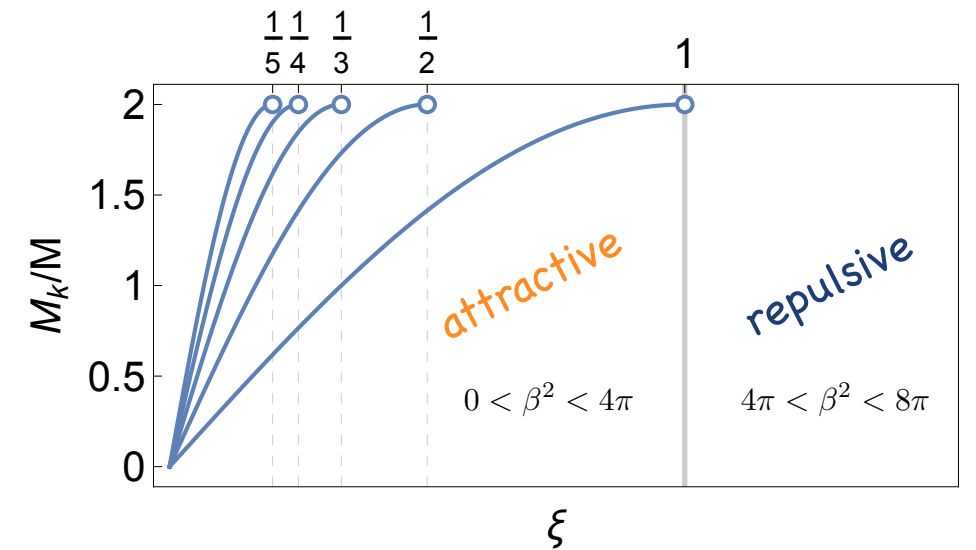
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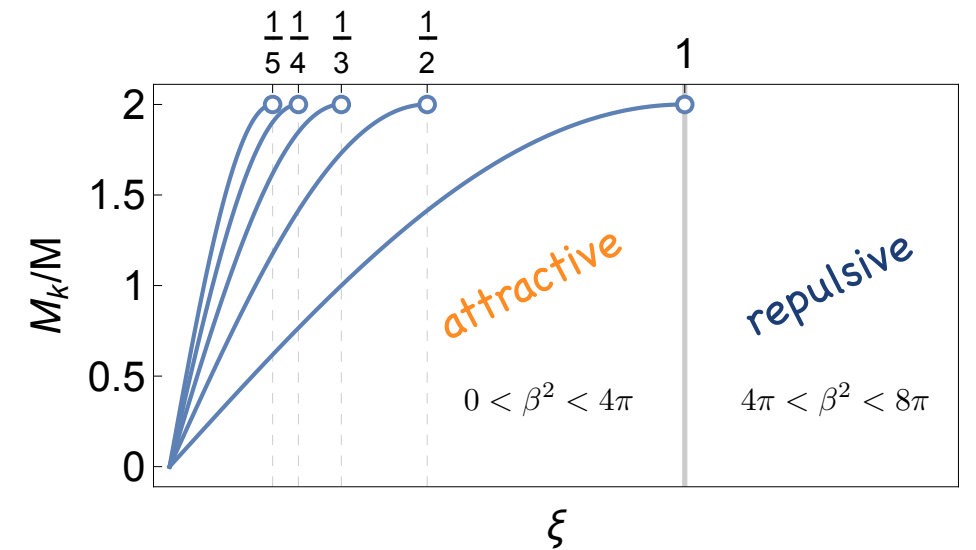
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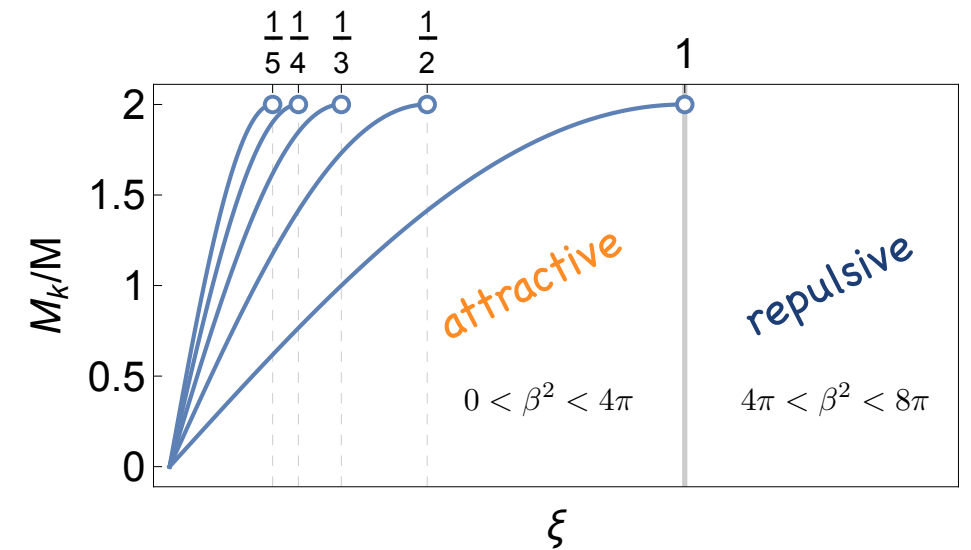
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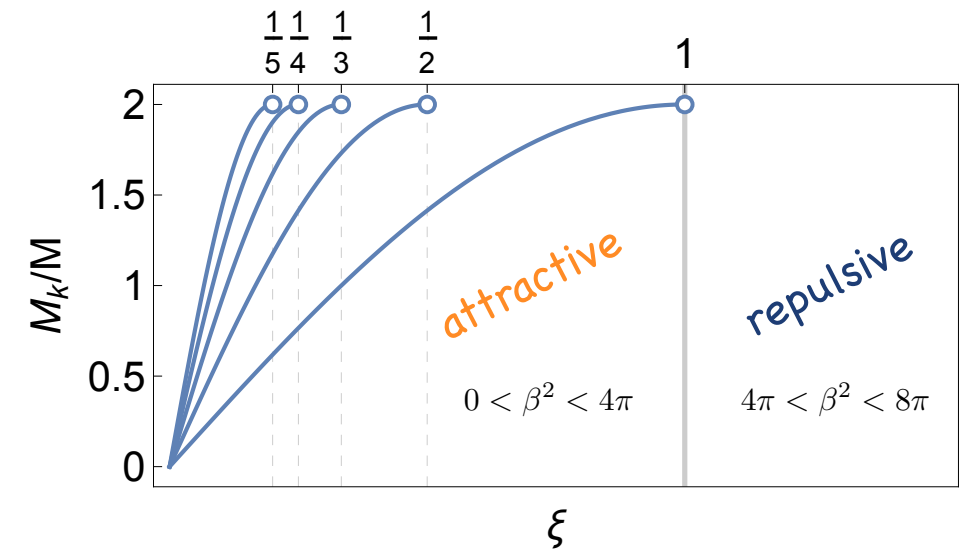
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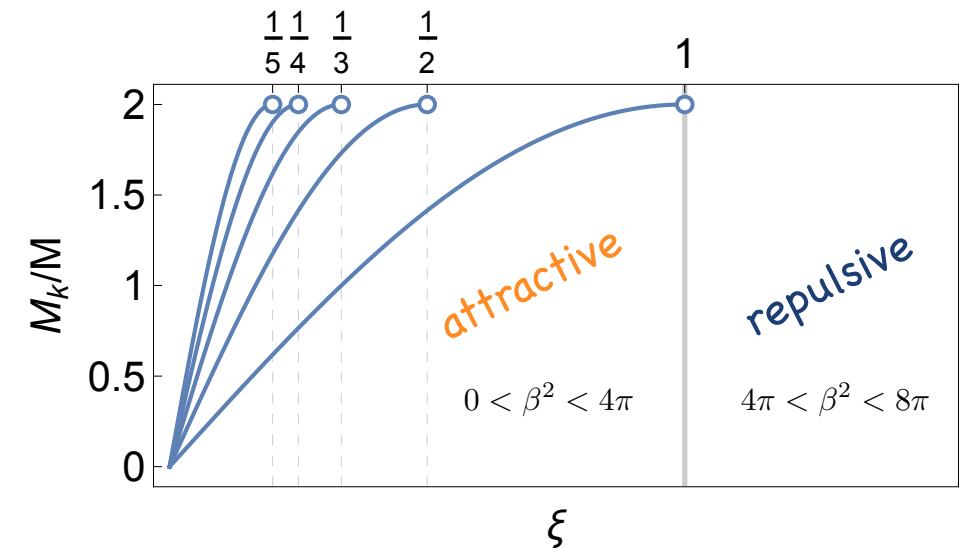
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$$S = \begin{pmatrix} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \\ \hline S_0 & 0 & 0 & 0 \\ 0 & S_R & S_T & 0 \\ 0 & S_T & S_R & 0 \\ \hline 0 & 0 & 0 & S_0 \end{pmatrix} \begin{matrix} |++\rangle \\ |+-\rangle \\ |-+\rangle \\ |--\rangle \end{matrix}$$

$$|S_T(\theta)|^2 + |S_R(\theta)|^2 = 1$$



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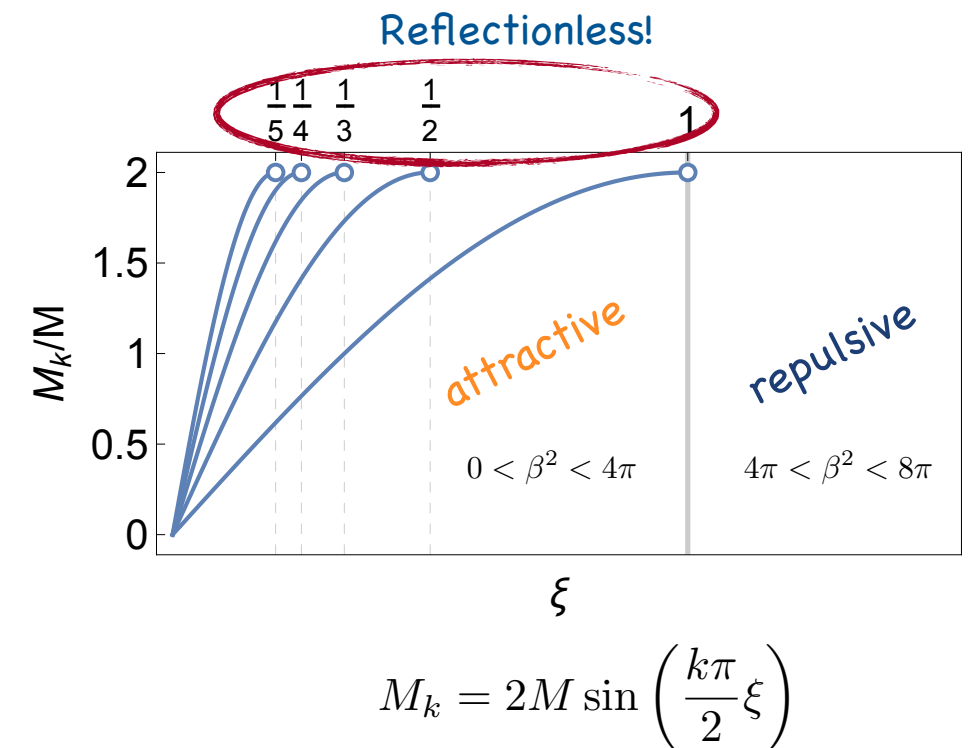
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Exact S-matrix

Zamolodchikov² '79

$$S_{++}^{++}(\theta) = S_{--}^{--}(\theta) = S_0(\theta), \quad S_{+-}^{+-}(\theta) = S_{-+}^{-+}(\theta) = S_T(\theta)S_0(\theta), \quad S_{+ -}^{-+}(\theta) = S_{- +}^{+-}(\theta) = S_R(\theta)S_0(\theta),$$
$$S_T(\theta) = \frac{\sinh\left(\frac{\theta}{\xi}\right)}{\sinh\left(\frac{i\pi - \theta}{\xi}\right)}, \quad S_R(\theta) = \frac{i \sin\left(\frac{\pi}{\xi}\right)}{\sinh\left(\frac{i\pi - \theta}{\xi}\right)}, \quad S_0(\theta) = -\exp\left(i \int_{-\infty}^{\infty} \frac{dt}{t} \frac{\sinh\left(\frac{t\pi}{2}(\xi - 1)\right)}{2 \sinh\left(\frac{\pi\xi t}{2}\right) \cosh\left(\frac{\pi t}{2}\right)} e^{i\theta t}\right)$$

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Building block: $[a]_{\theta} = \frac{\sinh \theta + i \sin \pi a}{\sinh \theta - i \sin \pi a} = -e^{i\delta_a(\theta)}$

$$S_{\pm, B_n}(\theta) = \prod_{a \in P_n} [a]_{\theta} \quad P_n = \left\{ \frac{1-n\xi}{2}, \frac{1-(n-2)\xi}{2}, \dots, \frac{1+(n-2)\xi}{2} \right\}$$

$$S_{B_n, B_m}(\theta) = \prod_{a \in P_{nm}} [a]_{\theta},$$

$$P_{nm} = \left\{ \frac{(n+m)\xi}{2}, \frac{(n+m-2)\xi}{2}, \frac{(n+m-2)\xi}{2}, \dots, \frac{(|n-m|+2)\xi}{2}, \frac{(|n-m|+2)\xi}{2}, \frac{|n-m|\xi}{2} \right\}$$

Nested Bethe Ansatz equations

G. Fehér, G. Takács, NPB '11

$$e^{iM_{B_i} L \sinh \theta_{B_i}^{(j)}} \prod_{\substack{k=1 \\ (k,l) \neq (i,j)}}^{n_B} \prod_{l=1}^{N_{B_k}} S_{B_i, B_k} \left(\theta_{B_i}^{(j)} - \theta_{B_k}^{(l)} \right) \prod_{k=1}^{N_s} S_{\pm, B_i} \left(\theta_{B_i}^{(j)} - \theta_k \right) = 1, \quad i = 1, \dots, n_B, \quad j = 1, \dots, N_{B_i},$$

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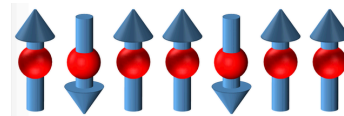
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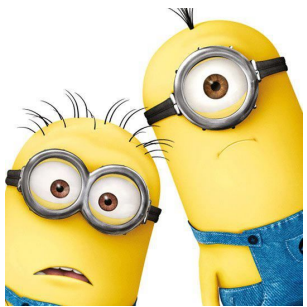
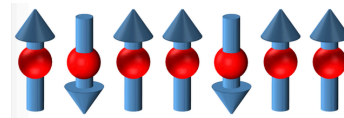
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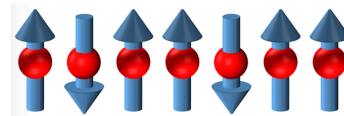
$$e^{iM_{B_i} L \sinh \theta_{B_i}^{(j)}} \prod_{\substack{k=1 \\ (k,l) \neq (i,j)}}^{n_B} \prod_{l=1}^{N_{B_k}} S_{B_i, B_k} \left(\theta_{B_i}^{(j)} - \theta_{B_k}^{(l)} \right) \prod_{k=1}^{N_s} S_{\pm, B_i} \left(\theta_{B_i}^{(j)} - \theta_k \right) = 1, \quad i = 1, \dots, n_B, \quad j = 1, \dots, N_{B_i},$$

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Eigenvalue of the transfer matrix

Magnons: "waves of charge flips" over the all-soliton state

$$Q = N_s - 2N_m$$



auxiliary particles having zero (bare) momentum and energy but nonzero charge



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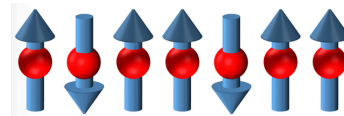
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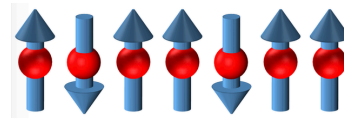
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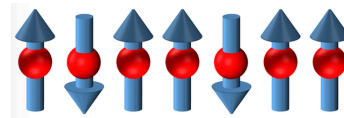
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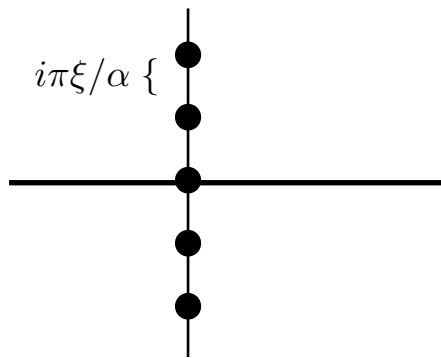
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Kinks, breathers, magnons.

Thermodynamic limit: strings

String hypothesis:

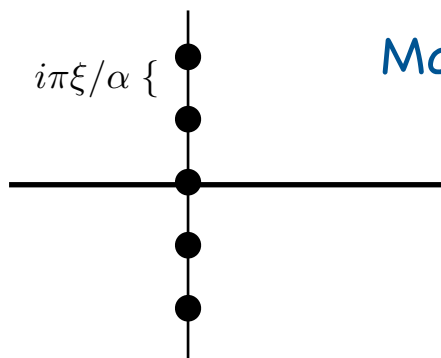
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Magnetic BAEs coincide with the BAEs of the gapless XXZ spin chain!

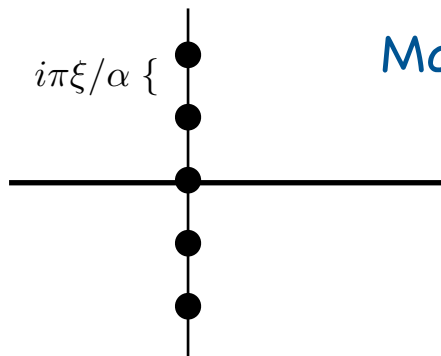


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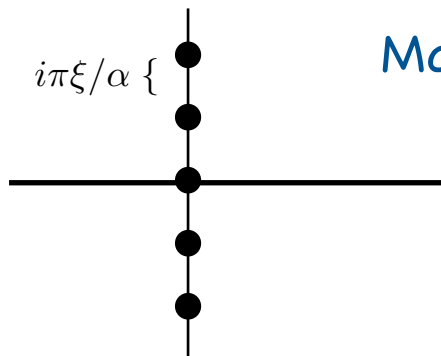


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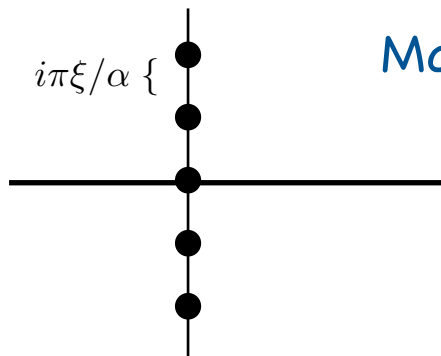


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Takahashi's book

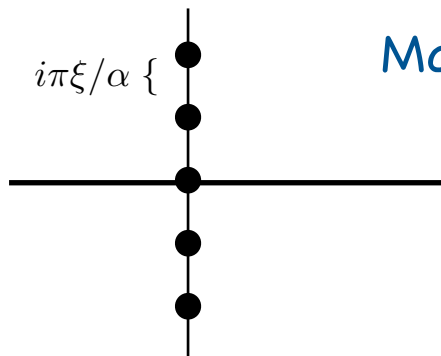


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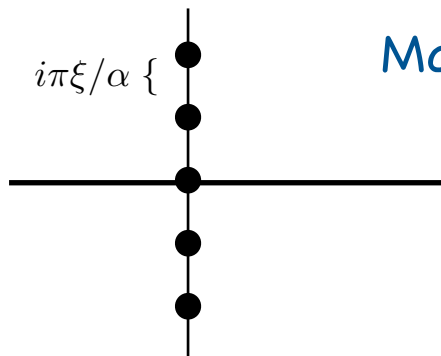
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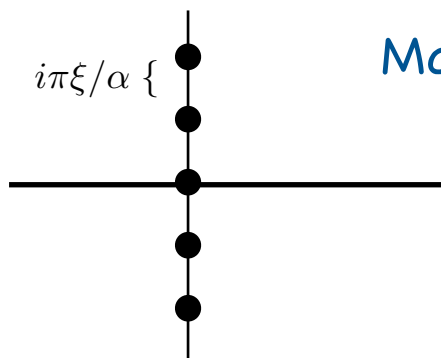
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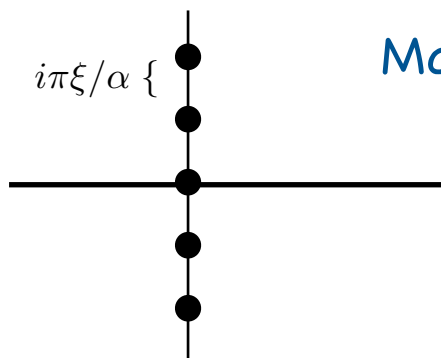
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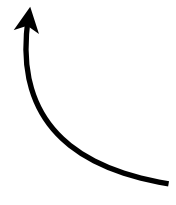
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Partially decoupled equations

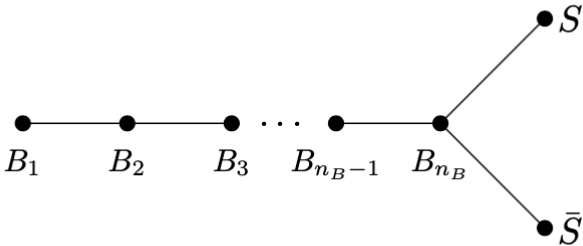
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Reflectionless points:

A. Zamolodchikov, PLB '91



Excitations	Labels	w	q	η	$\sigma^{(1)}$	$\sigma^{(2)}$
Breathers	$B_i, i = 1, \dots, n_B$	$M_{B_i} \cosh(\theta)/T$	0	+1	+1	+1
Soliton	S	$M \cosh(\theta)/T - \mu/T$	+1	+1	+1	+1
Antisoliton	$\bar{S}$	$M \cosh(\theta)/T + \mu/T$	-1	+1	+1	+1

$$\Phi_\alpha(\theta) = \frac{1}{\xi \cosh(\theta/\xi)}$$

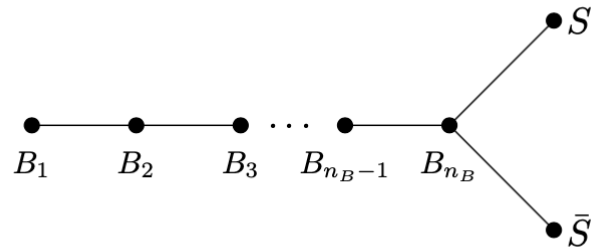
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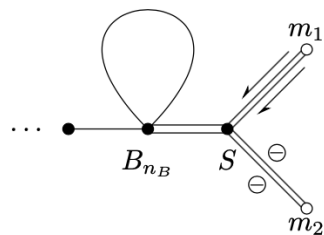
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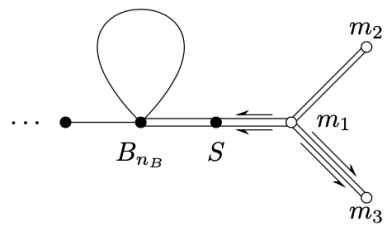
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1 magnonic level: $\xi = \frac{1}{n_B + 1/p}$

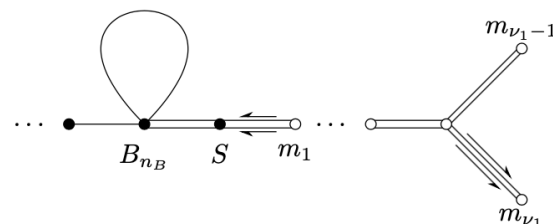
($\xi \in \mathbb{Z}$: B. Bertini, L. Piroli, M.K., PRB '19)



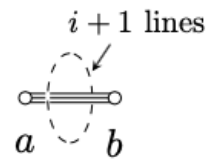
(a) $\nu_1 = 2$.



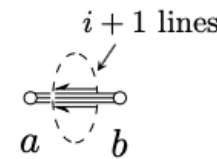
(b) $\nu_1 = 3$.



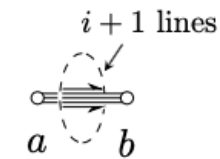
(c) $\nu_1 > 3$.



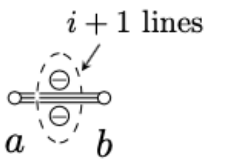
$$\tilde{K}_{ab} = \tilde{\Phi}_{p_i}(t)$$



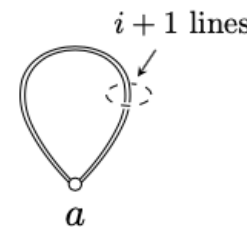
$$-\tilde{K}_{ab} = \tilde{K}_{ba} = \tilde{\Phi}_{p_i}(t)$$



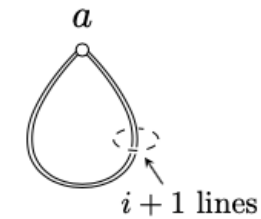
$$\tilde{K}_{ab} = -\tilde{K}_{ba} = \tilde{\Phi}_{p_i}(t)$$



$$\tilde{K}_{ab} = -\tilde{\Phi}_{p_i}(t)$$



$$\tilde{K}_{aa} = \tilde{\Phi}_{\text{self}}^{(i)}(t)$$



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$$\tilde{\Phi}_{p_i}(t) = \frac{1}{2 \cosh(p_i \frac{\pi}{2} \frac{\xi}{\alpha} t)}$$

$$\tilde{\Phi}_{\text{self}}^{(i)}(t) = \frac{\cosh((p_i - p_{i+1}) \frac{\pi}{2} \frac{\xi}{\alpha} t)}{2 \cosh(p_i \frac{\pi}{2} \frac{\xi}{\alpha} t) \cosh(p_{i+1} \frac{\pi}{2} \frac{\xi}{\alpha} t)}$$

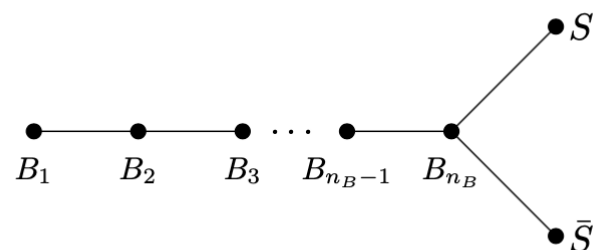
$$p_0 = \alpha, \quad p_1 = 1, \quad p_2 = 1/\nu_2$$

Partially decoupled equations

$$\epsilon_a = w_a + \sum_b K_{ab} * \left(\sigma_b^{(1)} \epsilon_b - \sigma_b^{(2)} w_b + \log(1 + e^{-\epsilon_b}) \right)$$

Reflectionless points:

A. Zamolodchikov, PLB '91



Excitations	Labels	w	q	η	$\sigma^{(1)}$	$\sigma^{(2)}$
Breathers	$B_i, i = 1, \dots, n_B$	$M_{B_i} \cosh(\theta)/T$	0	+1	+1	+1
Soliton	S	$M \cosh(\theta)/T - \mu/T$	+1	+1	+1	+1
Antisoliton	$\bar{S}$	$M \cosh(\theta)/T + \mu/T$	-1	+1	+1	+1

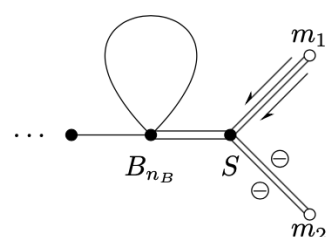
$$\Phi_\alpha(\theta) = \frac{1}{\xi \cosh(\theta/\xi)}$$

$$\tilde{\Phi}_\alpha(t) = \frac{1}{2 \cosh(\frac{\pi}{2} \xi t)}$$

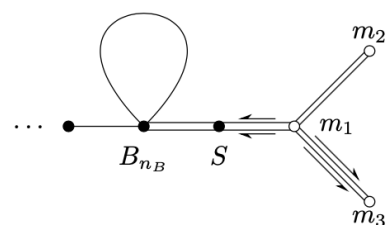
1 magnonic level:

$$\xi = \frac{1}{n_B + 1/p}$$

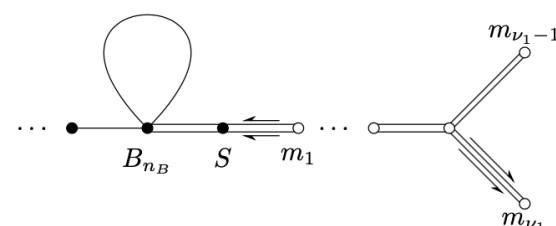
($\xi \in \mathbb{Z}$: B. Bertini, L. Piroli, M.K., PRB '19)



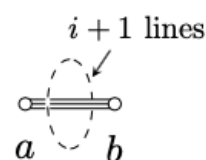
(a) $\nu_1 = 2$.



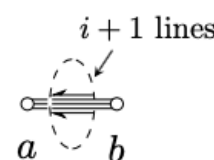
(b) $\nu_1 = 3$.



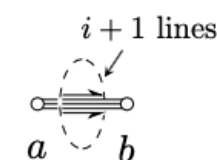
(c) $\nu_1 > 3$.



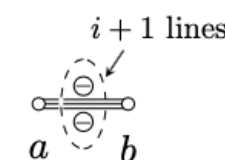
$$\tilde{K}_{ab} = \tilde{\Phi}_{p_i}(t)$$



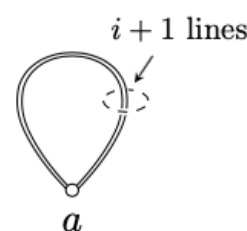
$$-\tilde{K}_{ab} = \tilde{K}_{ba} = \tilde{\Phi}_{p_i}(t)$$



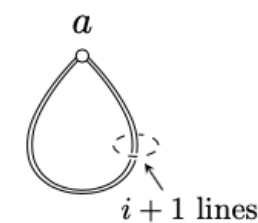
$$\tilde{K}_{ab} = -\tilde{K}_{ba} = \tilde{\Phi}_{p_i}(t)$$



$$\tilde{K}_{ab} = -\tilde{\Phi}_{p_i}(t)$$



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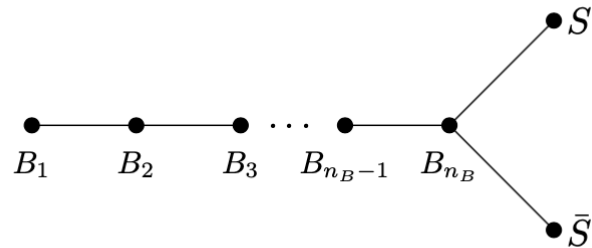
$$p_0 = \alpha, p_1 = 1, p_2 = 1/\nu_2$$

Partially decoupled equations

$$\epsilon_a = w_a + \sum_b K_{ab} * \left(\sigma_b^{(1)} \epsilon_b - \sigma_b^{(2)} w_b + \log(1 + e^{-\epsilon_b}) \right)$$

Reflectionless points:

A. Zamolodchikov, PLB '91



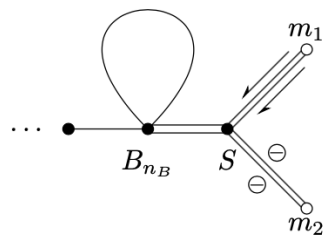
Excitations	Labels	w	q	η	$\sigma^{(1)}$	$\sigma^{(2)}$
Breathers	$B_i, i = 1, \dots, n_B$	$M_{B_i} \cosh(\theta)/T$	0	+1	+1	+1
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$$\Phi_\alpha(\theta) = \frac{1}{\xi \cosh(\theta/\xi)}$$

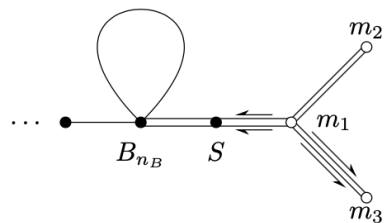
$$\tilde{\Phi}_\alpha(t) = \frac{1}{2 \cosh(\frac{\pi}{2} \xi t)}$$

1 magnonic level: $\xi = \frac{1}{n_B + 1/p}$

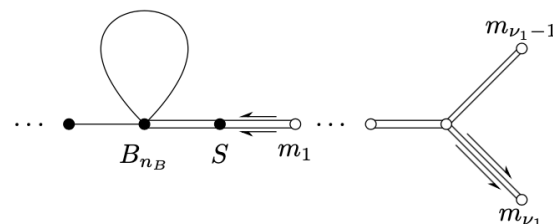
($\xi \in \mathbb{Z}$: B. Bertini, L. Piroli, M.K., PRB '19)



(a) $\nu_1 = 2$.

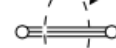


(b) $\nu_1 = 3$.

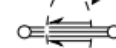


(c) $\nu_1 > 3$.

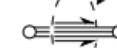
$i + 1$ lines



$i + 1$ lines



$i + 1$ lines



$i + 1$ lines



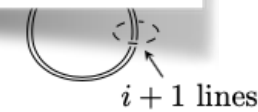
Excitations	Labels	w	q	η	v	$\sigma^{(1)}$	$\sigma^{(2)}$
Breathers	$B_i, i = 1, \dots, n_B$	$M_{B_i} \cosh(\theta)/T$	0	+1	0	+1	+1
Soliton	S	$M \cosh(\theta)/T$	+1	+1	0	0	0
First level intermediate magnons	$m_j, j = 1, \dots, \nu_1 - 2$	0	-2 · j	-1	+1	+1	0
First level next-to-last magnon	$m_{\nu_1-1}, (j = \nu_1 - 1)$	$\nu_1 \cdot \mu/T$	-2 · j	-1	+1	+1	0
First level last magnon	m_{ν_1}	$\nu_1 \cdot \mu/T$	-2	+1	-1	0	0

$\tilde{\Phi}_{p_i}(t)$

$$\tilde{K}_{ab} = -\tilde{\Phi}_{p_i}(t)$$



$$\tilde{K}_{aa} = \tilde{\Phi}_{\text{self}}^{(i)}(t)$$



$$\tilde{K}_{aa} = -\tilde{\Phi}_{\text{self}}^{(i)}(t)$$

$$\tilde{\Phi}_{p_i}(t) = \frac{1}{2 \cosh(p_i \frac{\pi}{2} \frac{\xi}{\alpha} t)}$$

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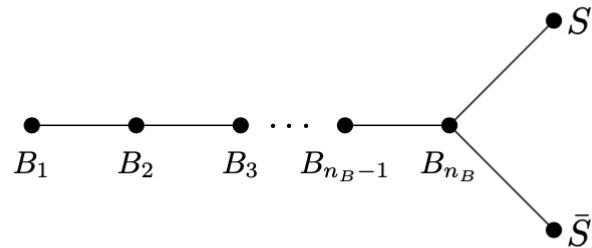
$$p_0 = \alpha, p_1 = 1, p_2 = 1/\nu_2$$

Partially decoupled equations

$$\epsilon_a = w_a + \sum_b K_{ab} * \left(\sigma_b^{(1)} \epsilon_b - \sigma_b^{(2)} w_b + \log(1 + e^{-\epsilon_b}) \right)$$

Reflectionless points:

A. Zamolodchikov, PLB '91



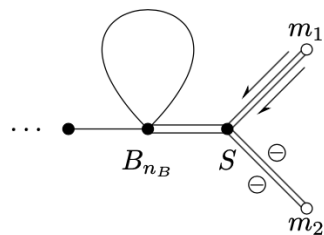
Excitations	Labels	w	q	η	$\sigma^{(1)}$	$\sigma^{(2)}$
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Antisoliton	$\bar{S}$	$M \cosh(\theta)/T + \mu/T$	-1	+1	+1	+1

$$\Phi_\alpha(\theta) = \frac{1}{\xi \cosh(\theta/\xi)}$$

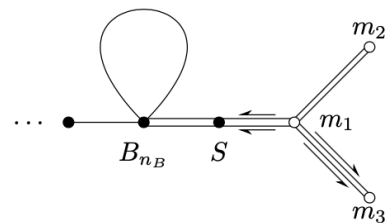
$$\tilde{\Phi}_\alpha(t) = \frac{1}{2 \cosh(\frac{\pi}{2} \xi t)}$$

1 magnonic level: $\xi = \frac{1}{n_B + 1/p}$

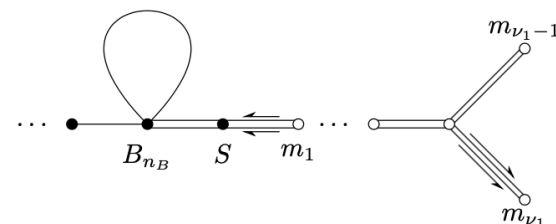
($\xi \in \mathbb{Z}$: B. Bertini, L. Piroli, M.K., PRB '19)



(a) $\nu_1 = 2$.



(b) $\nu_1 = 3$.



(c) $\nu_1 > 3$.

$i + 1$ lines

$i + 1$ lines

$i + 1$ lines

$i + 1$ lines

Excitations	Labels	w	q	η	v	$\sigma^{(1)}$	$\sigma^{(2)}$
Breathers	$B_i, i = 1, \dots, n_B$	$M_{B_i} \cosh(\theta)/T$	0	+1	0	+1	+1
Soliton	S	$M \cosh(\theta)/T$	+1	+1	0	0	0
First level intermediate magnons	$m_j, j = 1, \dots, \nu_1 - 2$	0	$-2 \cdot j$	-1	+1	+1	0
First level next-to-last magnon	$m_{\nu_1-1}, (j = \nu_1 - 1)$	$\nu_1 \cdot \mu/T$	$-2 \cdot j$	-1	+1	+1	0
First level last magnon	m_{ν_1}	$\nu_1 \cdot \mu/T$	-2	+1	-1	0	0

$$\tilde{K}_{aa} = \tilde{\Phi}_{\text{self}}^{(i)}(t)$$

$$\tilde{K}_{aa} = -\tilde{\Phi}_{\text{self}}^{(i)}(t)$$

$$\tilde{\Phi}_{p_i}(t) = \frac{1}{2 \cosh(p_i \frac{\pi}{2} \frac{\xi}{\alpha} t)}$$

$$\tilde{\Phi}_{\text{self}}^{(i)}(t) = \frac{\cosh((p_i - p_{i+1}) \frac{\pi}{2} \frac{\xi}{\alpha} t)}{2 \cosh(p_i \frac{\pi}{2} \frac{\xi}{\alpha} t) \cosh(p_{i+1} \frac{\pi}{2} \frac{\xi}{\alpha} t)}$$

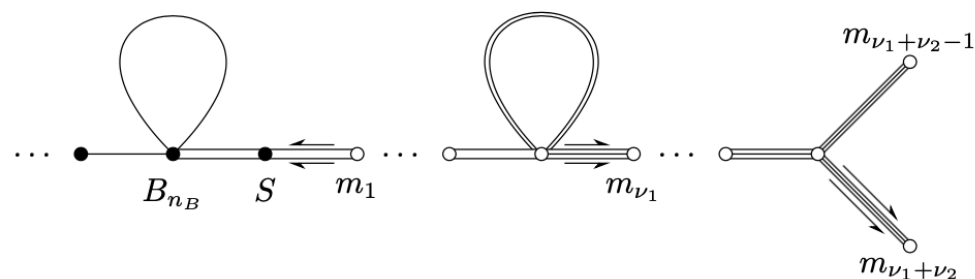
$$\tilde{K}_{ab} = -\tilde{\Phi}_{p_i}(t)$$

$$p_0 = \alpha, p_1 = 1, p_2 = 1/\nu_2$$

Partially decoupled equations

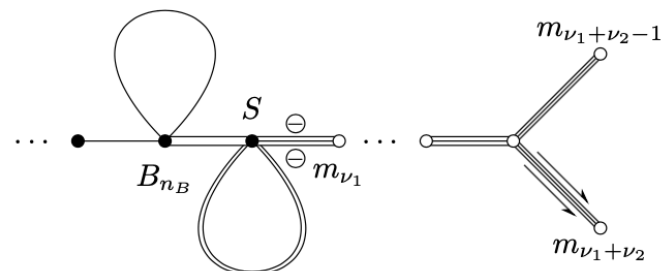
$$\epsilon_a = w_a + \sum_b K_{ab} * \left(\sigma_b^{(1)} \epsilon_b - \sigma_b^{(2)} w_b + \log(1 + e^{-\epsilon_b}) \right)$$

2 magnonic levels:

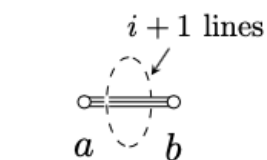


(b) $\nu_1 > 2, \nu_2 > 2$.

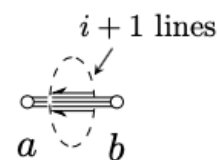
$$\xi = \frac{1}{n_B + \frac{1}{\nu_1 + \frac{1}{\nu_2}}}$$



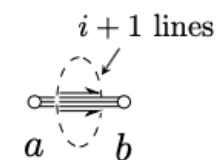
(b) $\nu_1 = 1, \nu_2 > 2$.



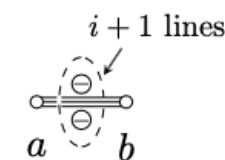
$$\tilde{K}_{ab} = \tilde{\Phi}_{p_i}(t)$$



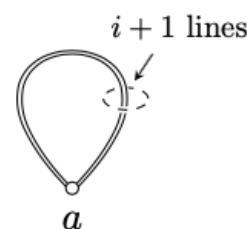
$$-\tilde{K}_{ab} = \tilde{K}_{ba} = \tilde{\Phi}_{p_i}(t)$$



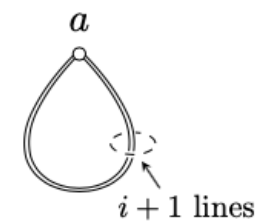
$$\tilde{K}_{ab} = -\tilde{K}_{ba} = \tilde{\Phi}_{p_i}(t)$$



$$\tilde{K}_{ab} = -\tilde{\Phi}_{p_i}(t)$$



$$\tilde{K}_{aa} = \tilde{\Phi}_{\text{self}}^{(i)}(t)$$



$$\tilde{K}_{aa} = -\tilde{\Phi}_{\text{self}}^{(i)}(t)$$

$$\tilde{\Phi}_{p_i}(t) = \frac{1}{2 \cosh\left(p_i \frac{\pi}{2} \frac{\xi}{\alpha} t\right)}$$

$$\tilde{\Phi}_{\text{self}}^{(i)}(t) = \frac{\cosh\left((p_i - p_{i+1}) \frac{\pi}{2} \frac{\xi}{\alpha} t\right)}{2 \cosh\left(p_i \frac{\pi}{2} \frac{\xi}{\alpha} t\right) \cosh\left(p_{i+1} \frac{\pi}{2} \frac{\xi}{\alpha} t\right)}$$

$$p_0 = \alpha, \quad p_1 = 1, \quad p_2 = 1/\nu_2$$

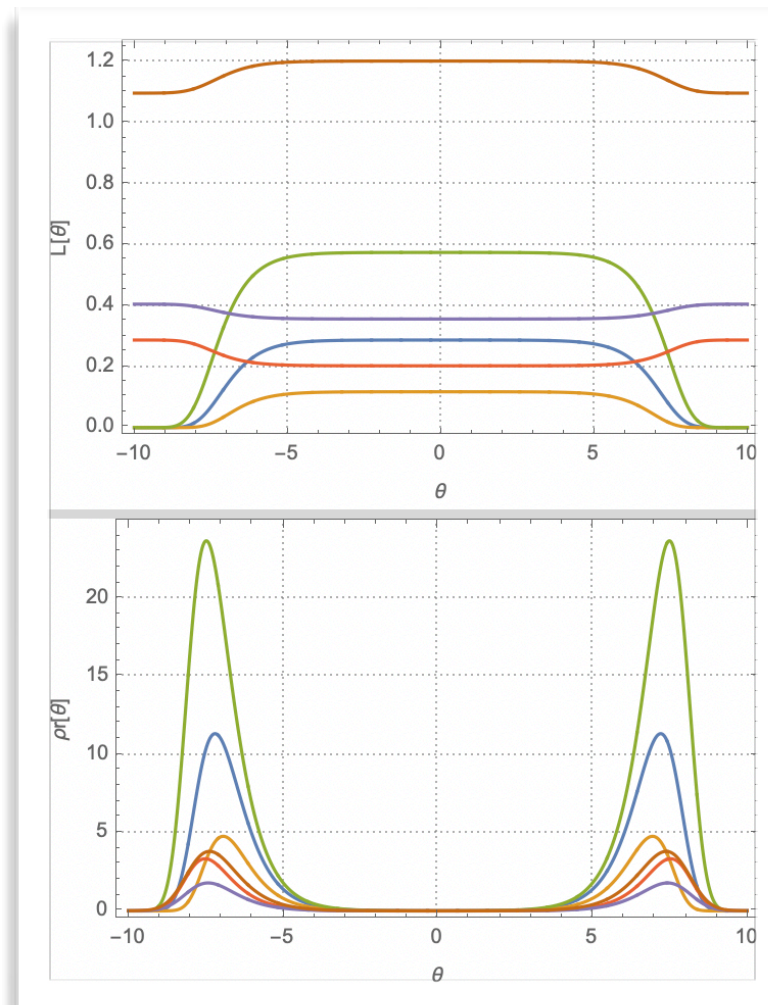
Check: high T limit

Check: high T limit

1 magnonic level: $\xi = \frac{1}{n_B + 1/p}$

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1 magnonic level: $\xi = \frac{1}{n_B + 1/p}$



Check: high T limit

1 magnonic level:

$$\xi = \frac{1}{n_B + 1/p}$$

$$\bar{y}_{B_1}^2 = 1 + \bar{y}_{B_2},$$

$$\bar{y}_{B_l}^2 = (1 + \bar{y}_{B_{l-1}})(1 + \bar{y}_{B_{l+1}}),$$

$$\bar{y}_{B_l}^2 = (1 + \bar{y}_{B_{l-1}})(1 + \bar{y}_{B_l})(1 + \bar{y}_S), \quad l = n_B$$

$$\bar{y}_S^2 = (1 + \bar{y}_{B_l})/(1 + \bar{y}_{m_1}), \quad l = n_B$$

$$\bar{y}_{m_1}^2 = (1 + \bar{y}_S^{-1})(1 + \bar{y}_2),$$

$$\bar{y}_{m_k}^2 = (1 + \bar{y}_{m_{k-1}})(1 + \bar{y}_{m_{k+1}}),$$

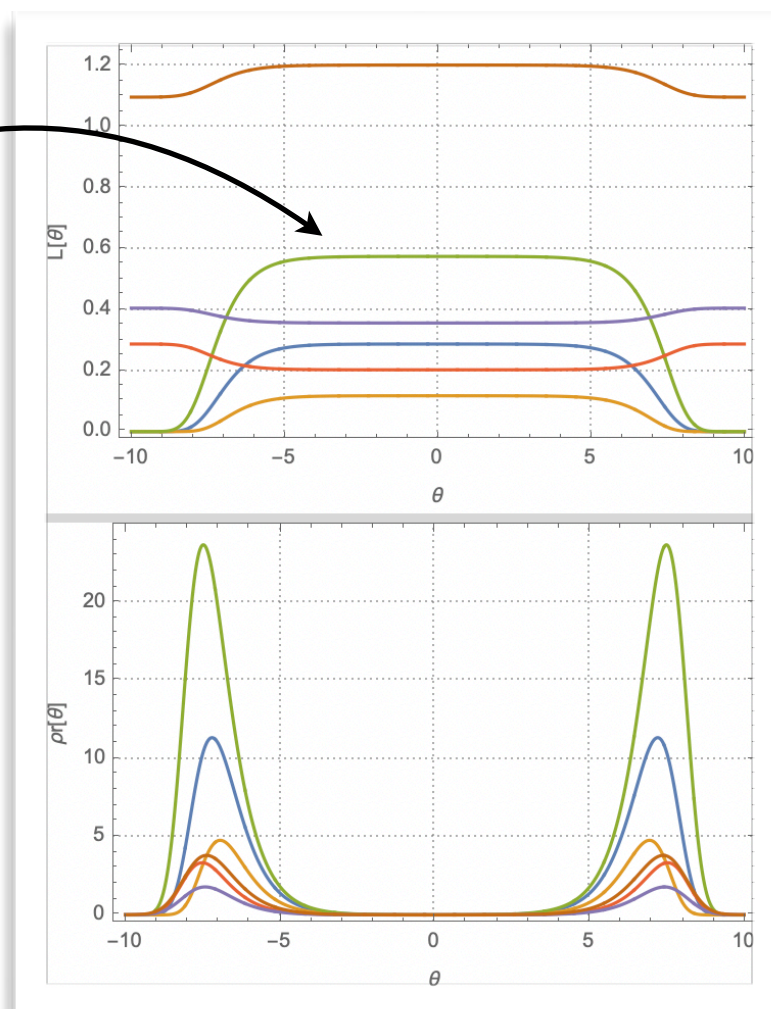
$$\bar{y}_{m_{p-2}}^2 = (1 + \bar{y}_{m_{p-3}})(1 + \bar{y}_{m_{p-1}})(1 + \bar{y}_{m_p}^{-1}),$$

$$\bar{y}_{m_{p-1}}^2 = e^{2p\mu/T}(1 + \bar{y}_{m_{p-2}}),$$

$$\bar{y}_{m_p}^2 = e^{2p\mu/T}(1 + \bar{y}_{m_{p-2}})^{-1}$$

Pleateau values

$$\bar{y}_a = e^{\bar{\epsilon}_a}$$



Check: high T limit

1 magnonic level:

$$\xi = \frac{1}{n_B + 1/p}$$

$$\bar{y}_{B_1}^2 = 1 + \bar{y}_{B_2},$$

$$\bar{y}_{B_l}^2 = (1 + \bar{y}_{B_{l-1}})(1 + \bar{y}_{B_{l+1}}),$$

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$$\bar{y}_{m_{p-1}}^2 = e^{2p\mu/T}(1 + \bar{y}_{m_{p-2}}),$$

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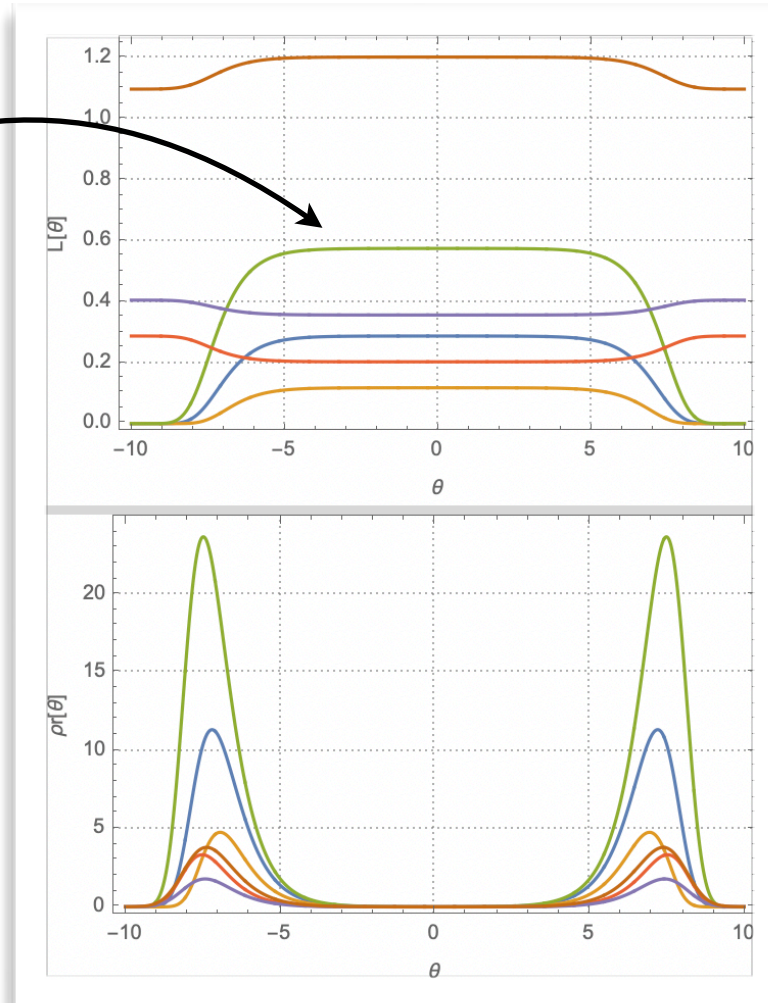


$$\bar{y}_{B_k} = (k + 1)^2 - 1,$$

$$\bar{y}_S = \left[\left(\frac{n_B + 2}{n_B + 1} \right)^2 - 1 \right]^{-1},$$

$$\bar{y}_{m_k} = \left(k + \frac{n_B + 2}{n_B + 1} \right)^2 - 1, \quad k = 1, \dots, p - 2,$$

$$\bar{y}_{m_{p-1}} = \bar{y}_{m_p}^{-1} = p - \frac{n_B}{n_B + 1}$$



Pleateau values

$$\bar{y}_a = e^{\bar{\epsilon}_a}$$

Check: high T limit

1 magnonic level:

$$\xi = \frac{1}{n_B + 1/p}$$

$$\bar{y}_{B_1}^2 = 1 + \bar{y}_{B_2},$$

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$$\bar{y}_{m_{p-2}}^2 = (1 + \bar{y}_{m_{p-3}})(1 + \bar{y}_{m_{p-1}})(1 + \bar{y}_{m_p}^{-1}),$$

$$\bar{y}_{m_{p-1}}^2 = e^{2p\mu/T}(1 + \bar{y}_{m_{p-2}}),$$

$$\bar{y}_{m_p}^2 = e^{2p\mu/T}(1 + \bar{y}_{m_{p-2}})^{-1}$$

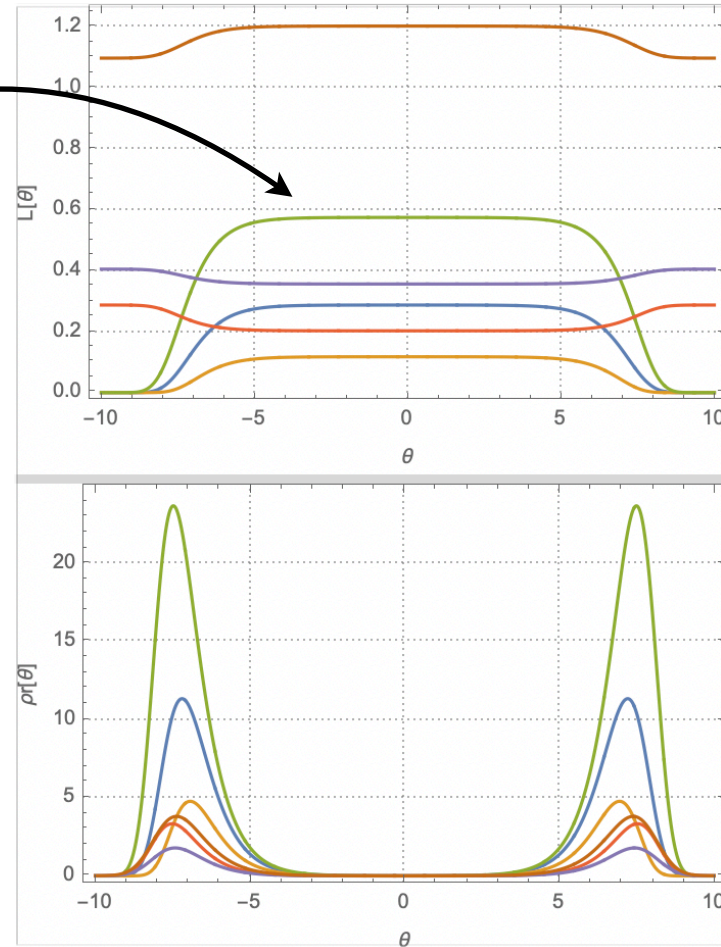


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$$\bar{y}_S = \left[\left(\frac{n_B + 2}{n_B + 1} \right)^2 - 1 \right]^{-1},$$

$$\bar{y}_{m_k} = \left(k + \frac{n_B + 2}{n_B + 1} \right)^2 - 1, \quad k = 1, \dots, p-2,$$

$$\bar{y}_{m_{p-1}} = \bar{y}_{m_p}^{-1} = p - \frac{n_B}{n_B + 1}$$



Pleateau values

$$\bar{y}_a = e^{\bar{\epsilon}_a}$$

$$\tilde{y}_{m_1}^2 = 1 + \tilde{y}_{m_2},$$

$$\tilde{y}_{m_k}^2 = (1 + \tilde{y}_{m_{k-1}})(1 + \tilde{y}_{m_{k+1}}),$$

$$\tilde{y}_{m_{p-2}}^2 = (1 + \tilde{y}_{m_{p-3}})(1 + \tilde{y}_{m_{p-1}})(1 + \tilde{y}_{m_p}^{-1}),$$

$$\tilde{y}_{m_{p-1}}^2 = e^{2p\mu/T}(1 + \tilde{y}_{m_{p-2}}),$$

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Check: high T limit

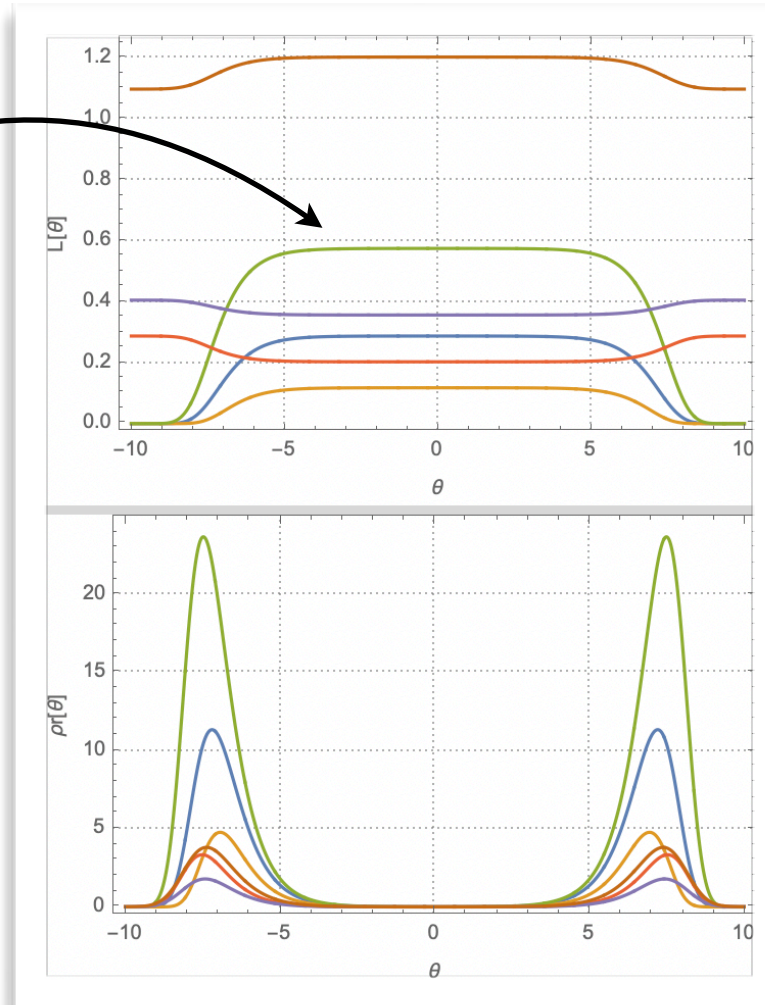
1 magnonic level:

$$\xi = \frac{1}{n_B + 1/p}$$

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$$\begin{aligned}\bar{y}_{B_k} &= (k+1)^2 - 1, \\ \bar{y}_S &= \left[\left(\frac{n_B + 2}{n_B + 1} \right)^2 - 1 \right]^{-1}, \\ \bar{y}_{m_k} &= \left(k + \frac{n_B + 2}{n_B + 1} \right)^2 - 1, \quad k = 1, \dots, p-2, \\ \bar{y}_{m_{p-1}} &= \bar{y}_{m_p}^{-1} = p - \frac{n_B}{n_B + 1}\end{aligned}$$



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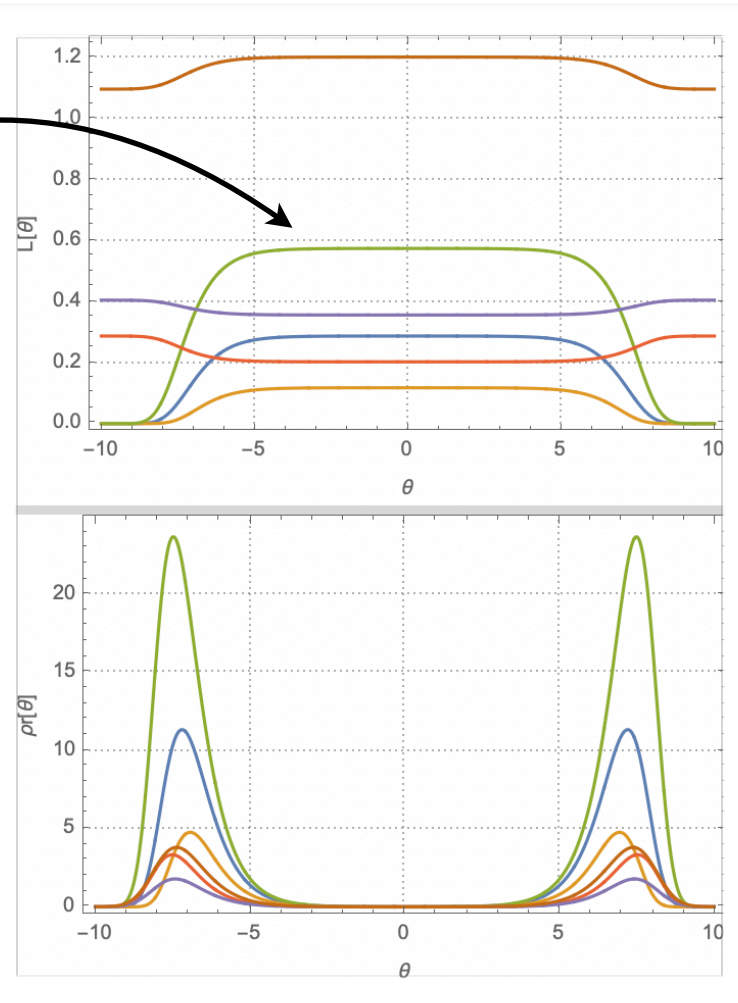
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$$f = T^2 \left(\sum_{k=1}^{n_B} L(\bar{\vartheta}_{B_k}) + L(\bar{\vartheta}_S) - \sum_{j=1}^{n_m} \eta_{m_j} \left[L(\bar{\vartheta}_{m_j}) - L(\tilde{\vartheta}_{m_j}) \right] \right) = \frac{\pi^2 T^2}{6}$$

Rogers dilogarithm

$$L(x) = -\frac{1}{2} \int_0^x dy \left(\frac{\log y}{1-y} + \frac{\log(1-y)}{y} \right)$$

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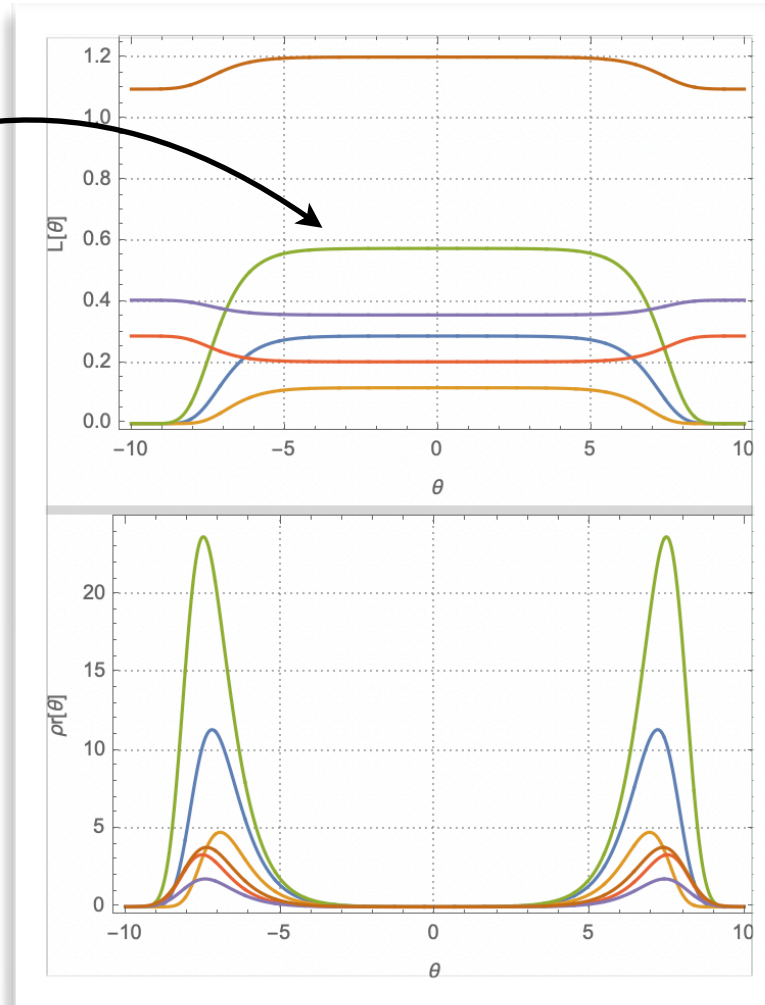
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Cross-check: NLIE

Nonlinear integral equation (Destri-De Vega)

C. Destri, H. J. de Vega, NPB '95

$$Z(\theta) = \frac{M}{T} \sinh \theta - i \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \Phi_0(\theta - \theta' - i\varepsilon) \log \left(1 + e^{iZ(\theta' + i\varepsilon)} \right) + i \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \Phi_0(\theta - \theta' + i\varepsilon) \log \left(1 + e^{-iZ(\theta' - i\varepsilon)} \right)$$

$$\frac{f}{T} = -2 \operatorname{Im} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} M \sinh(\theta + i\varepsilon) \log \left(1 + e^{iZ(\theta + i\varepsilon)} \right)$$

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Does not rely on the string hypothesis! No magnons, breathers do not appear, only the kinks!

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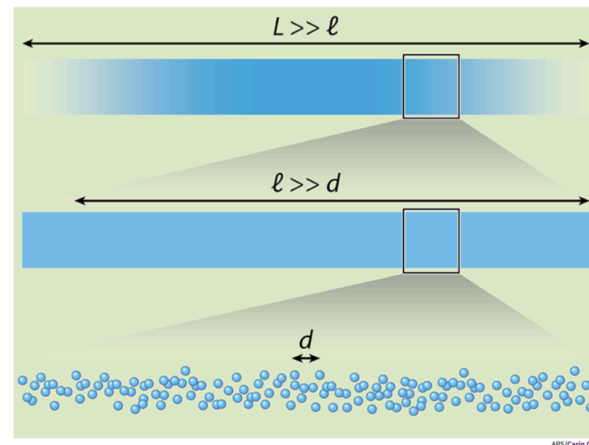
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	DdV	TBA	DdV	TBA	DdV	TBA
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10	-5.22194300631	-5.22194300532	-5.19859206357	-5.19859206333	-5.10437699605	-5.10437749589
5	-2.59068229582	-2.59068229533	-2.55324704632	-2.55324704620	-2.47030920896	-2.47030944780
2	-0.98474359521	-0.98474359502	-0.92684324811	-0.92684324807	-0.87159594008	-0.87159602002
1	-0.41783483396	-0.41783483387	-0.35869323482	-0.35869323480	-0.33218987940	-0.33218990576
0.5	-0.11948181719	-0.11948181716	-0.08847607829	-0.08847607828	-0.08339026939	-0.08339027349
0.2	-0.00386984298	-0.00386984298	-0.00258436645	-0.00258436645	-0.00256472781	-0.00256472782

Generalised HydroDynamics



De Nardis et al., JSM '22

Separation of scales



APS/Carlini Galin

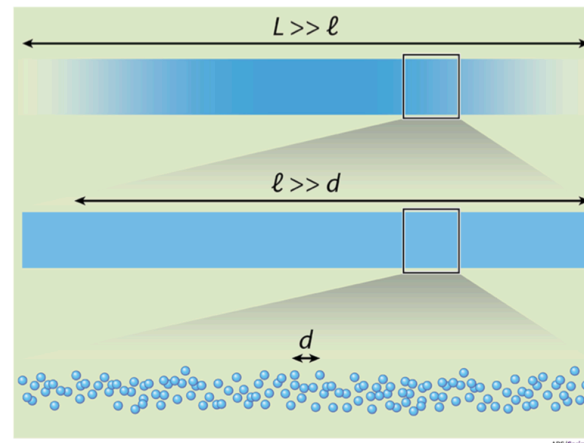
I. Bouchoule, J. Dubail, JSM '22

Bertini et al. '16
Castro-Alvaredo et al. '16

Generalised HydroDynamics



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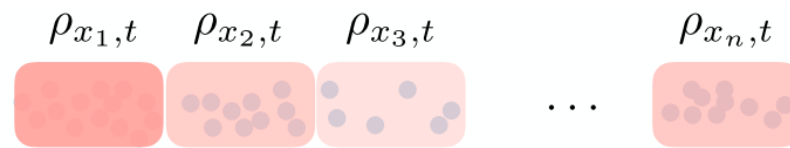
Euler scale

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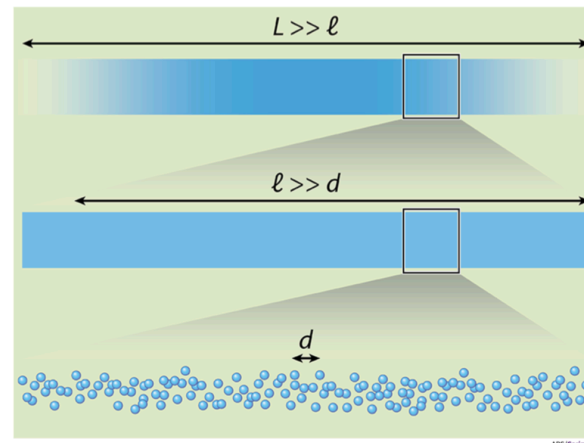
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Generalised HydroDynamics



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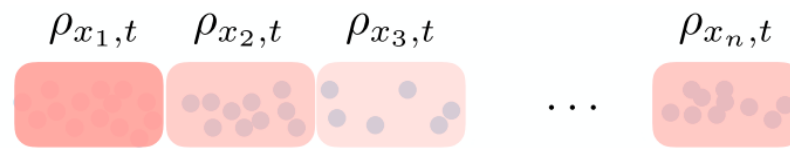
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Hydro: transport of conserved quantities

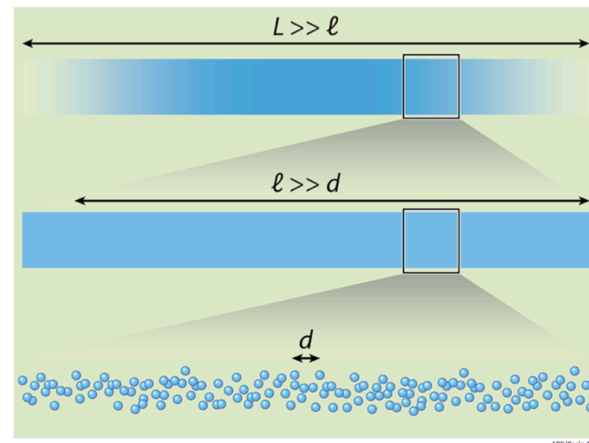
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Generalised HydroDynamics

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De Nardis et al., JSM '22



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I. Bouchoule, J. Dubail, JSM '22

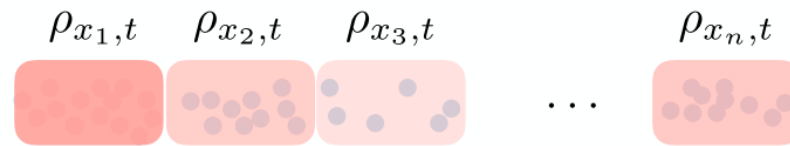
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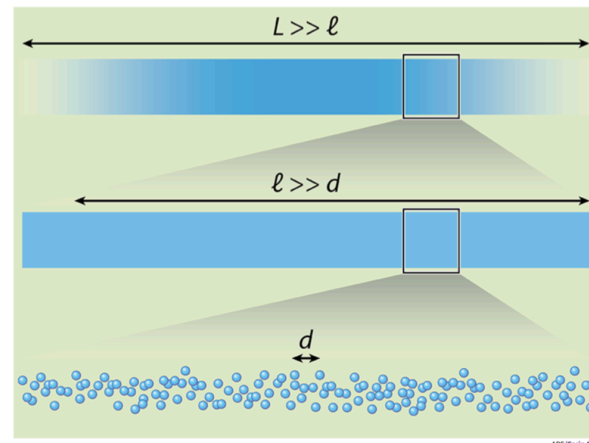
M. Borsi et al., PRX '20
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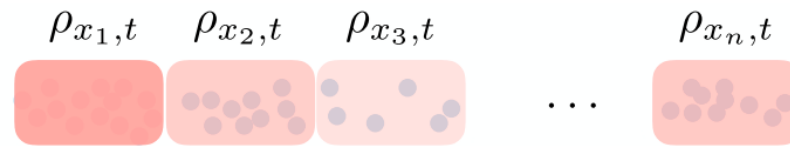
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GHD equation

$$\partial_t \rho^r(\theta; x, t) + \partial_x [v^{\text{eff}}(\theta; x, t) \rho^r(\theta; x, t)] = 0$$

Generalised HydroDynamics

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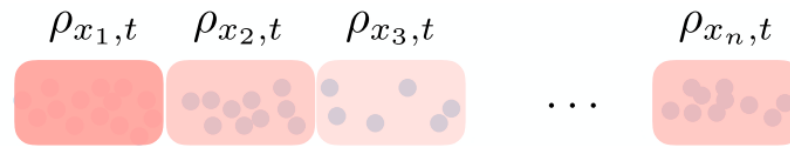
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Generalised HydroDynamics

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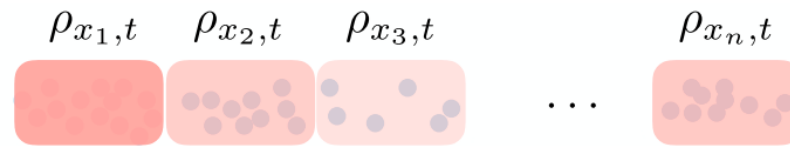
$$q_a^{(k), \text{dr}}(\theta) = \frac{\partial \epsilon_a(\theta)}{\partial \beta^k}$$

Dressing operation

$$\eta_a q_a^{\text{dr}} = q_a + \sum_b K_{ab} * \left[\left(\sigma_b^{(1)} - \vartheta_b \right) \eta_b q_b^{\text{dr}} - \sigma_b^{(2)} q_b \right]$$

Generalised HydroDynamics

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Generalised HydroDynamics

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Effective velocity

$$v_a^{\text{eff}}(\theta) = \frac{(\partial_\theta e_a)^{\text{dr}}(\theta)}{(\partial_\theta p_a)^{\text{dr}}(\theta)} = \frac{(\partial_\theta e_a)^{\text{dr}}(\theta)}{2\pi \rho_a^{\text{tot}}(\theta)}$$

Generalised HydroDynamics

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Generalised HydroDynamics

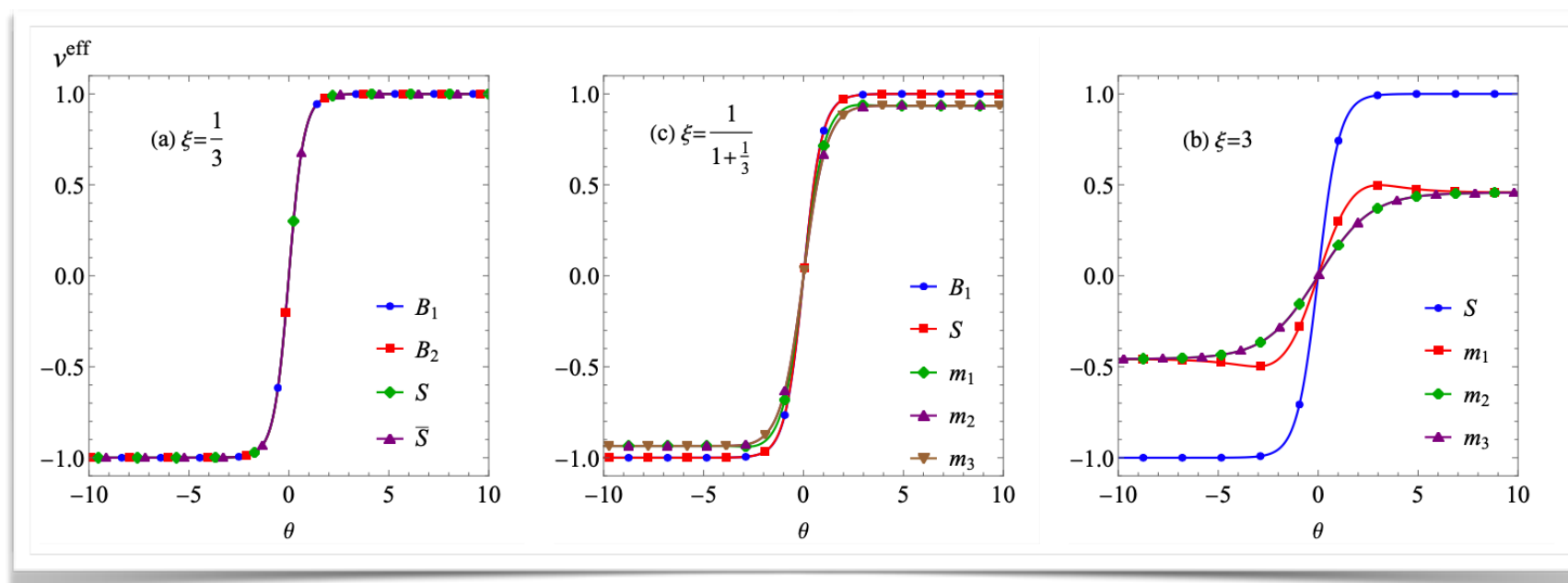
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Generalised HydroDynamics

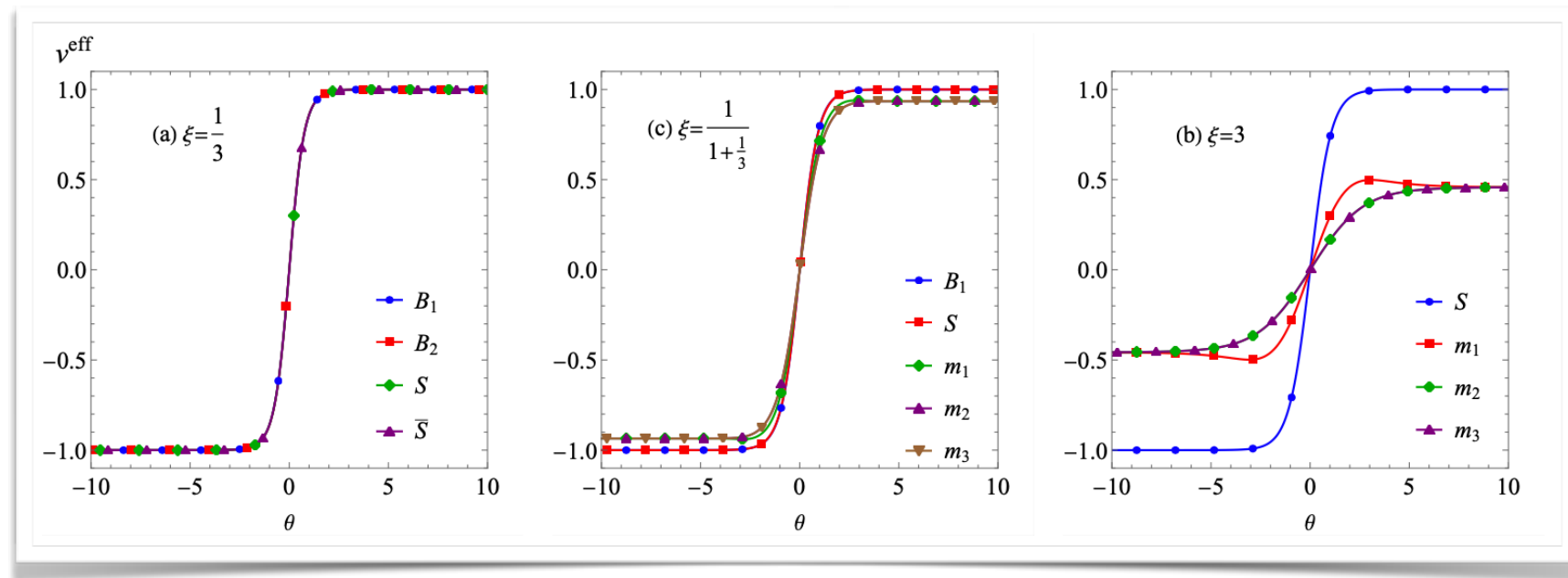
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$$v_a^{\text{eff}}(\theta) = \frac{(\partial_\theta e_a)^{\text{dr}}(\theta)}{(\partial_\theta p_a)^{\text{dr}}(\theta)} = \frac{(\partial_\theta e_a)^{\text{dr}}(\theta)}{2\pi \rho_a^{\text{tot}}(\theta)}$$

$$\rho_a^{\text{tot}} = \left(\frac{p'_a}{2\pi} \right)^{\text{dr}}$$

$$T/M = 1$$



Note:

- The magnons can have nonzero dressed momentum and energy!

Generalised HydroDynamics

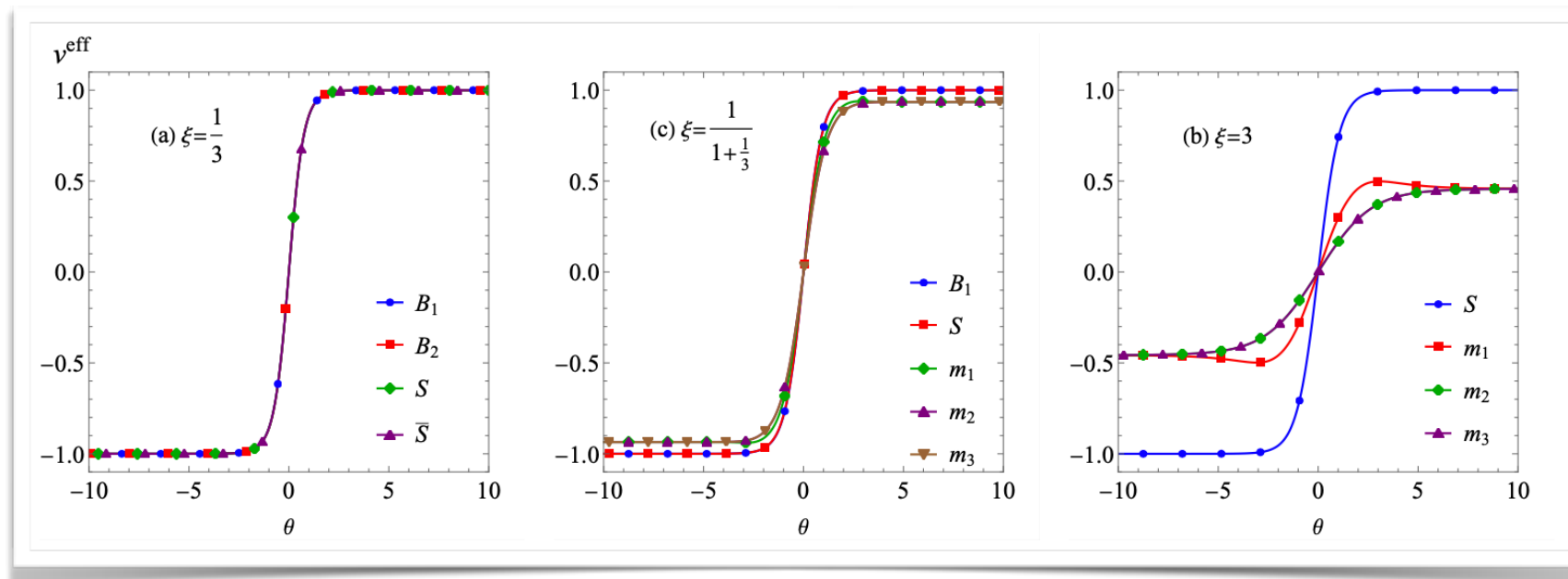
$$\partial_t \rho^r(\theta; x, t) + \partial_x [v^{\text{eff}}(\theta; x, t) \rho^r(\theta; x, t)] = 0$$

Effective velocity

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$$\rho_a^{\text{tot}} = \left(\frac{p'_a}{2\pi} \right)^{\text{dr}}$$

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Note:

- The magnons can have nonzero dressed momentum and energy!
- The breathers can have nonzero dressed topological charge.

Generalised HydroDynamics

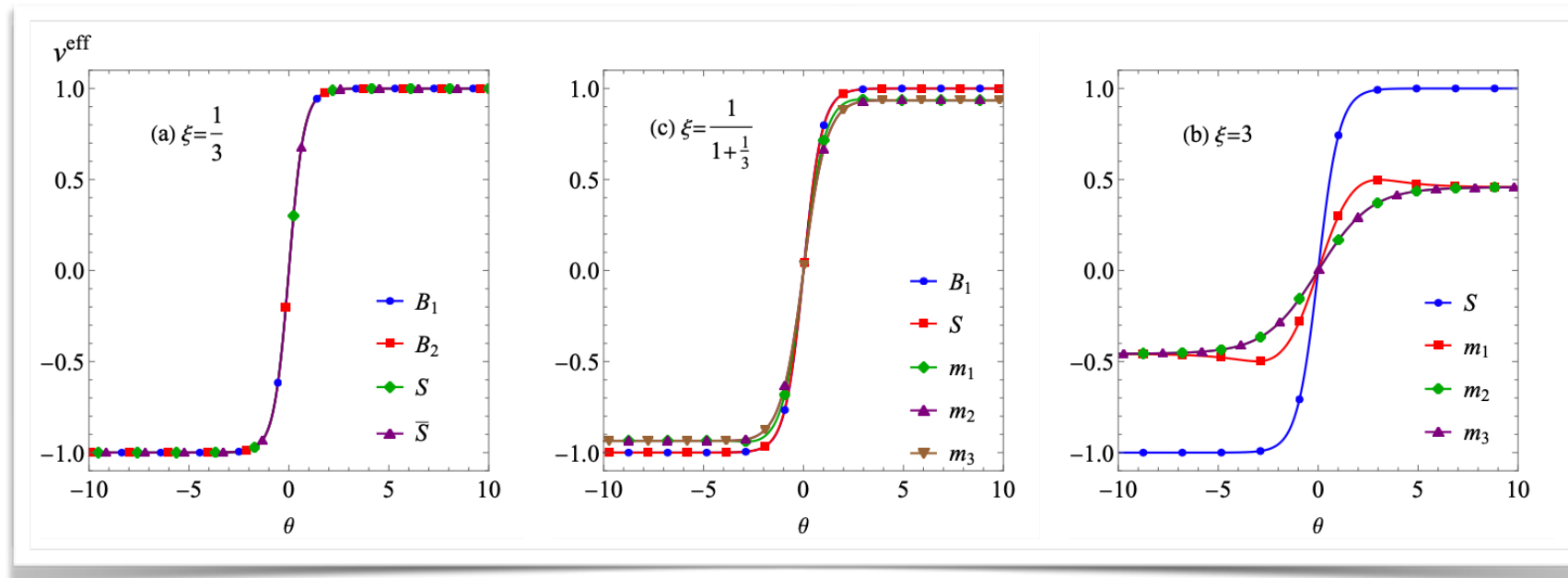
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Note:

- The magnons can have nonzero dressed momentum and energy!
- The breathers can have nonzero dressed topological charge.
- At zero chemical potential, only the last two magnons have nonzero dressed charge!

Generalised HydroDynamics

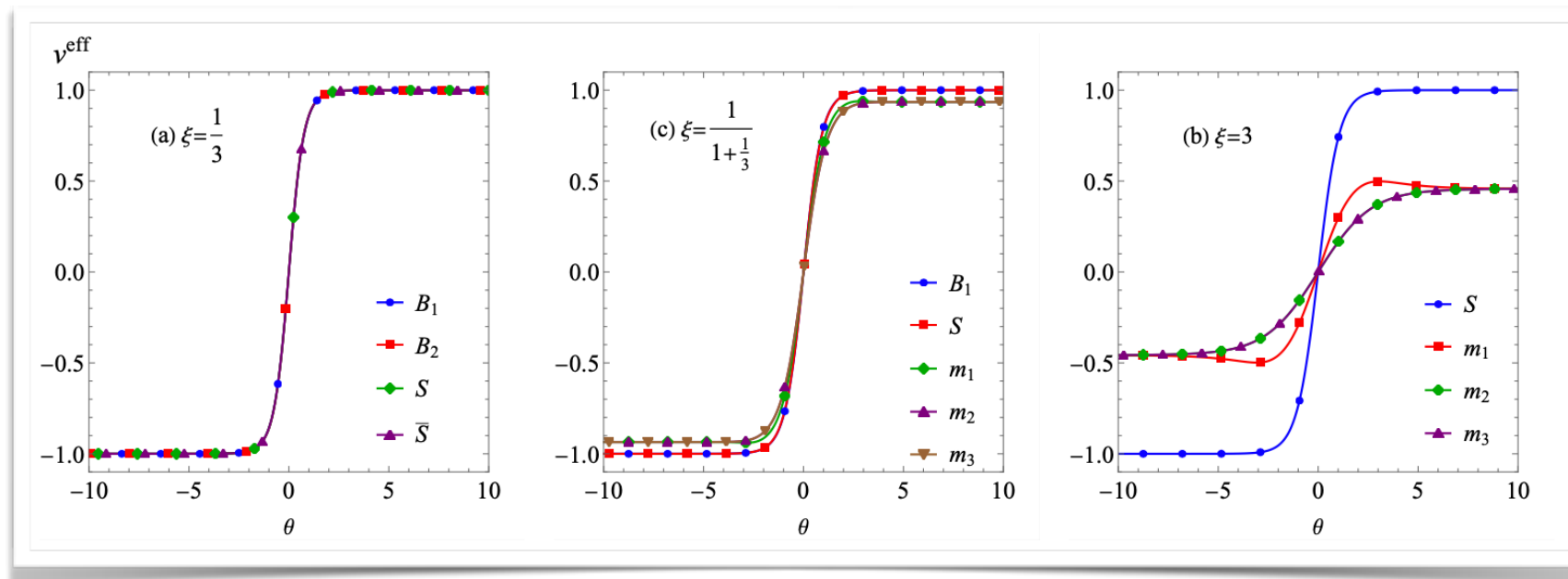
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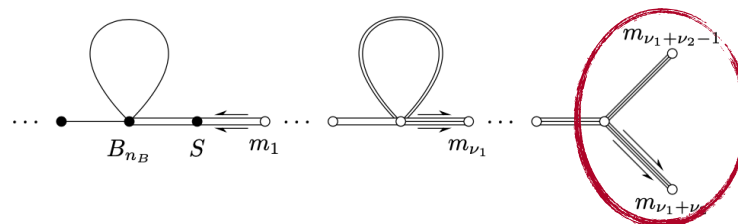
$$\rho_a^{\text{tot}} = \left(\frac{p'_a}{2\pi} \right)^{\text{dr}}$$

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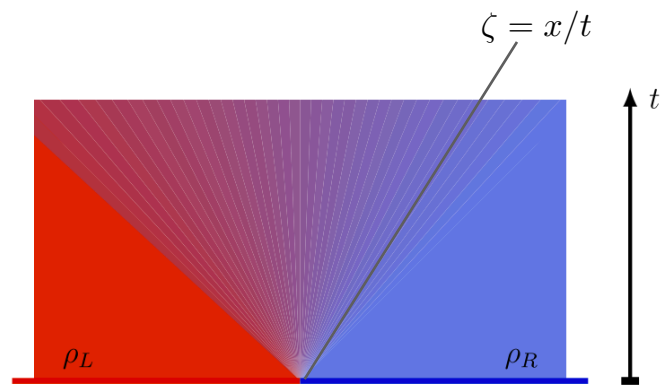


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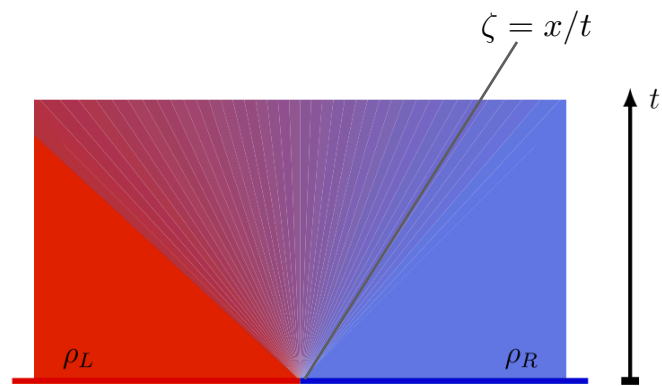
Partitioning protocol



Bertini et al., RMP '21

$$w_a = \begin{cases} w_{a,L} &= \sum_b \beta_L^b Q_a^{(b)}, & x < 0, \\ w_{a,R} &= \sum_b \beta_R^b Q_a^{(b)}, & x > 0. \end{cases}$$

Partitioning protocol

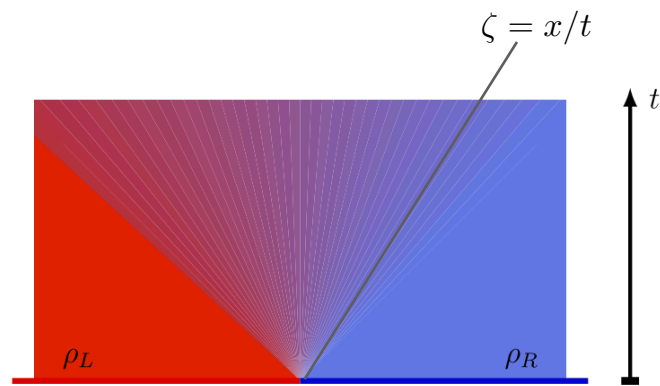


Bertini et al., RMP '21

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$$\vartheta_a(\zeta, \theta) = \Theta(v_a^{\text{eff}}(\zeta, \theta) - \zeta) \vartheta_{a,L}(\theta) + \Theta(\zeta - v_a^{\text{eff}}(\zeta, \theta)) \vartheta_{a,R}(\theta)$$

Partitioning protocol



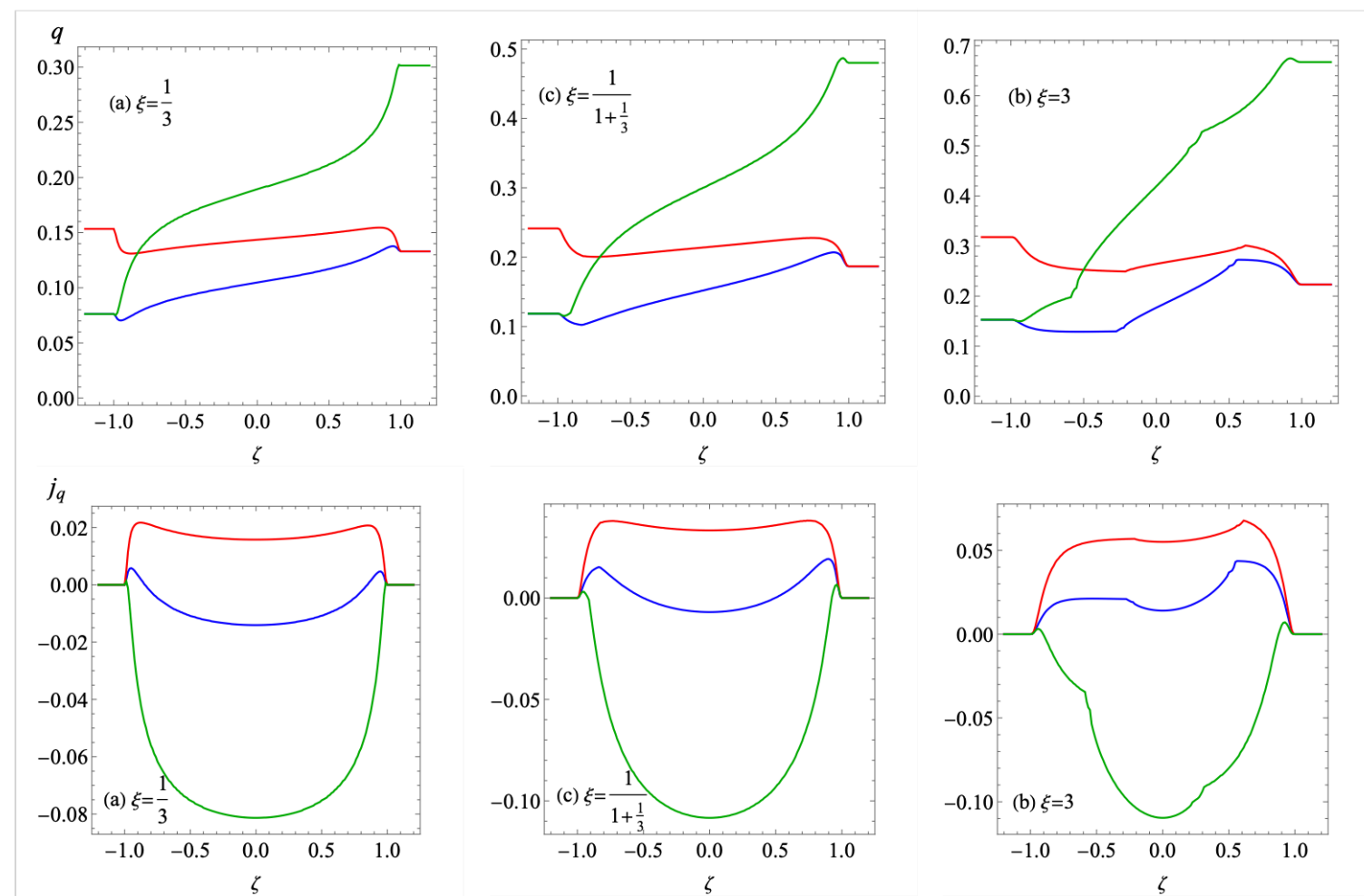
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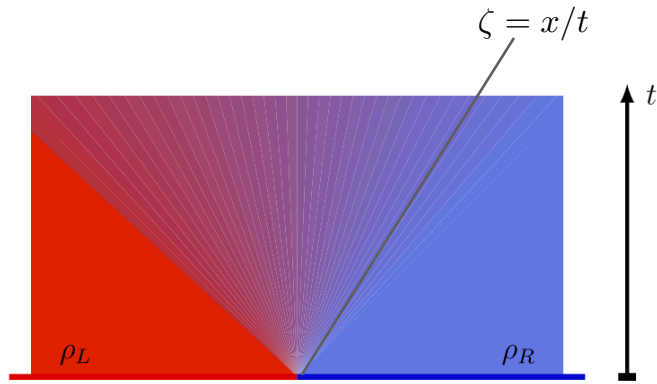
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Charge and current density profiles

$$\zeta = \frac{x}{t}$$



Partitioning protocol



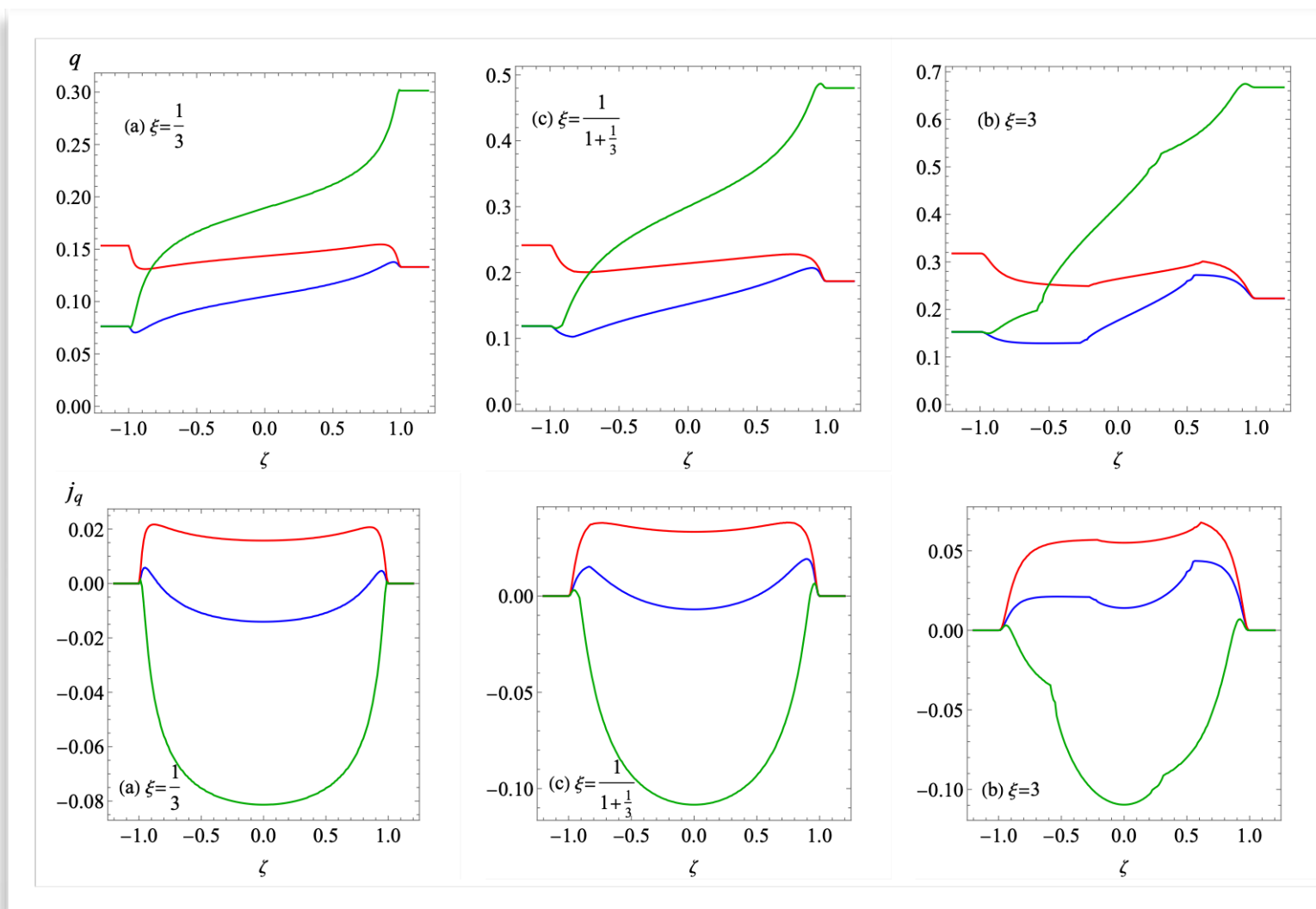
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Charge and current density profiles

$$\zeta = \frac{x}{t}$$



Cusps at maximal velocities of various excitations.

See L. Piroli et al., PRB '17

Drude weight

$$\text{Re } \sigma_{ik}(\omega) = 2\pi \tilde{D}_{ik} \delta(\omega) + \sigma_{ik}^{\text{reg}}(\omega)$$

$$D_{ik} = \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} dt \int dx \langle j^{(i)}(x, t) j^{(k)}(0, 0) \rangle^c$$

B. Doyon, H. Spohn '17
E. Ilievski, J. De Nardis '17

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GHD:

Drude weight

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B. Doyon, H. Spohn '17
E. Ilievski, J. De Nardis '17

GHD:

Partitioning protocol with infinitesimal bias:

$$D_q = \left. \frac{\partial}{\partial \delta \mu} \int d\zeta \, j(\zeta) \right|_{\delta \mu = 0}$$

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B. Doyon, H. Spohn '17
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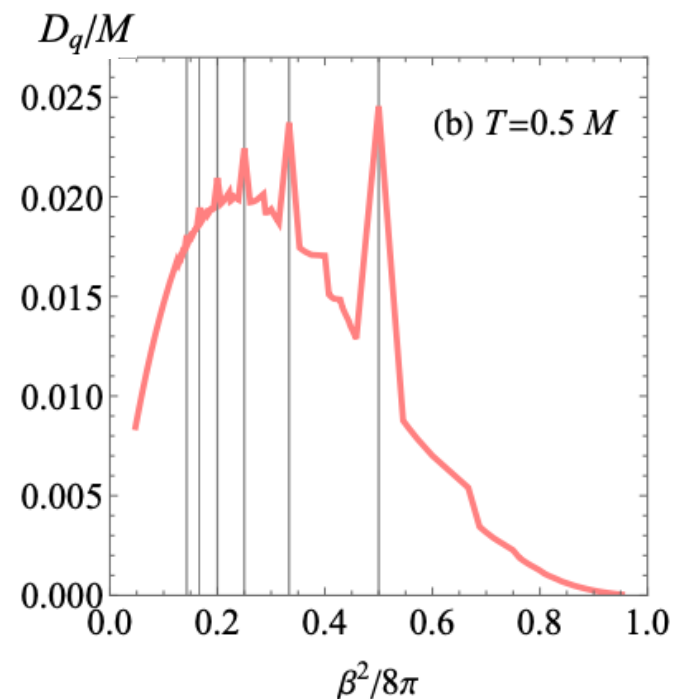
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B. Doyon, H. Spohn '17
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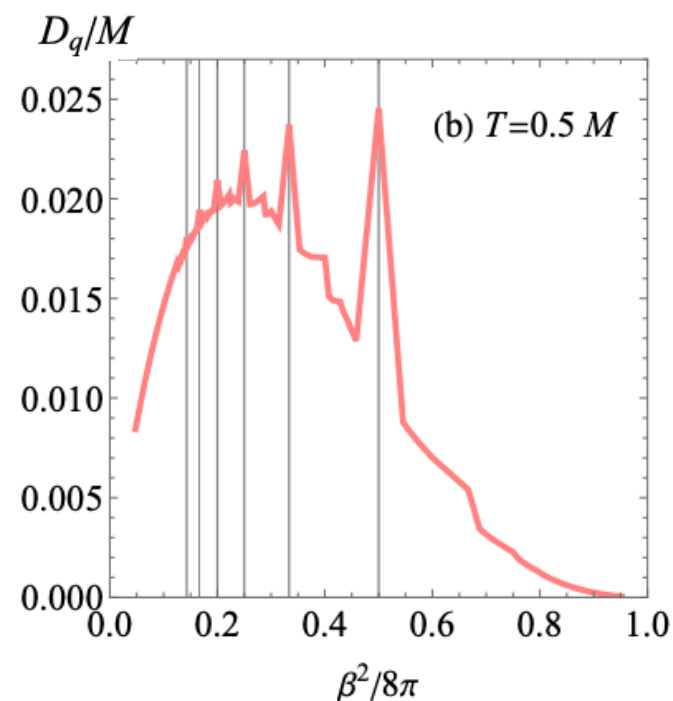
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Fractal (popcorn) dependence on the coupling!

Drude weight

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B. Doyon, H. Spohn '17
E. Ilievski, J. De Nardis '17

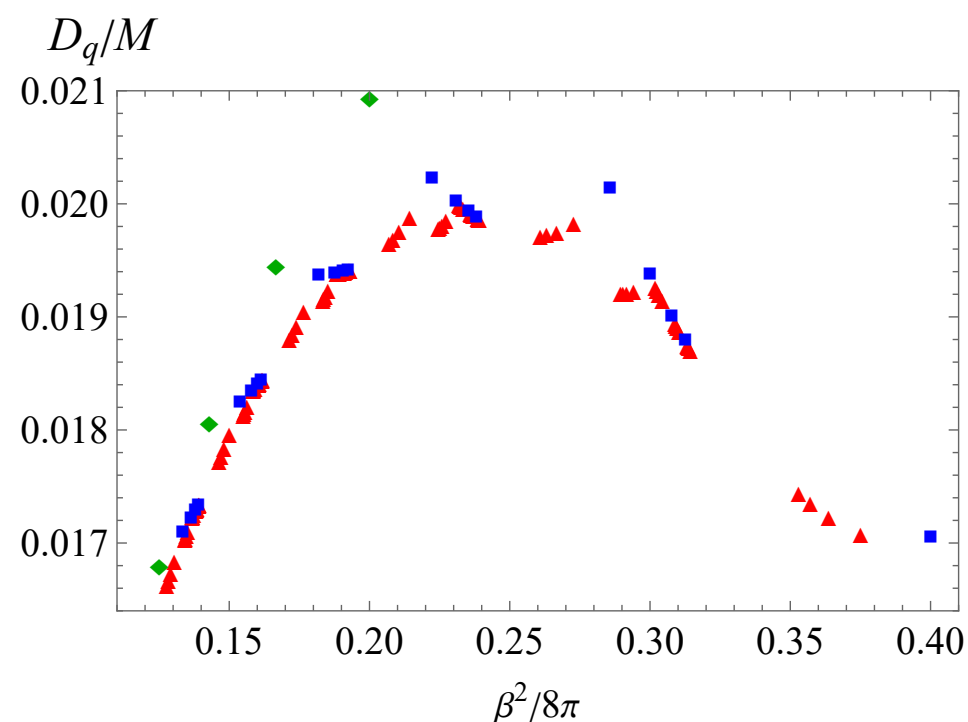
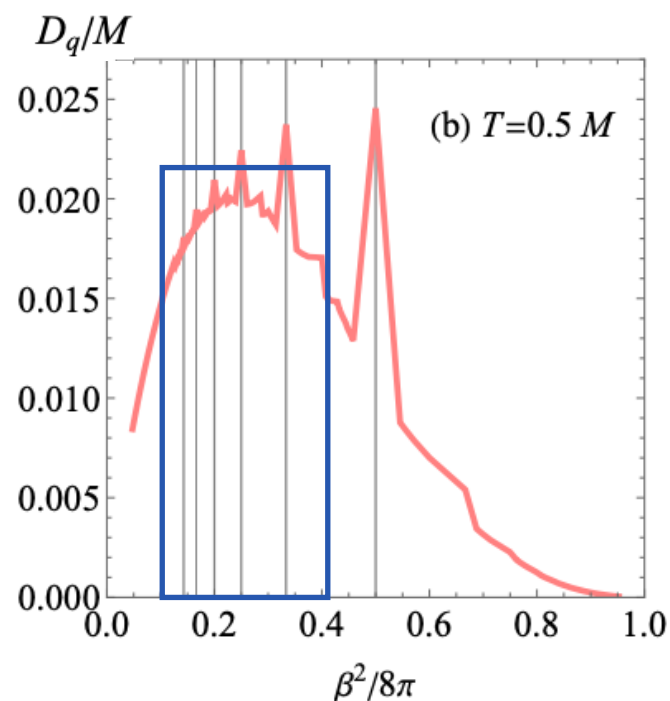
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B. Doyon, H. Spohn '17
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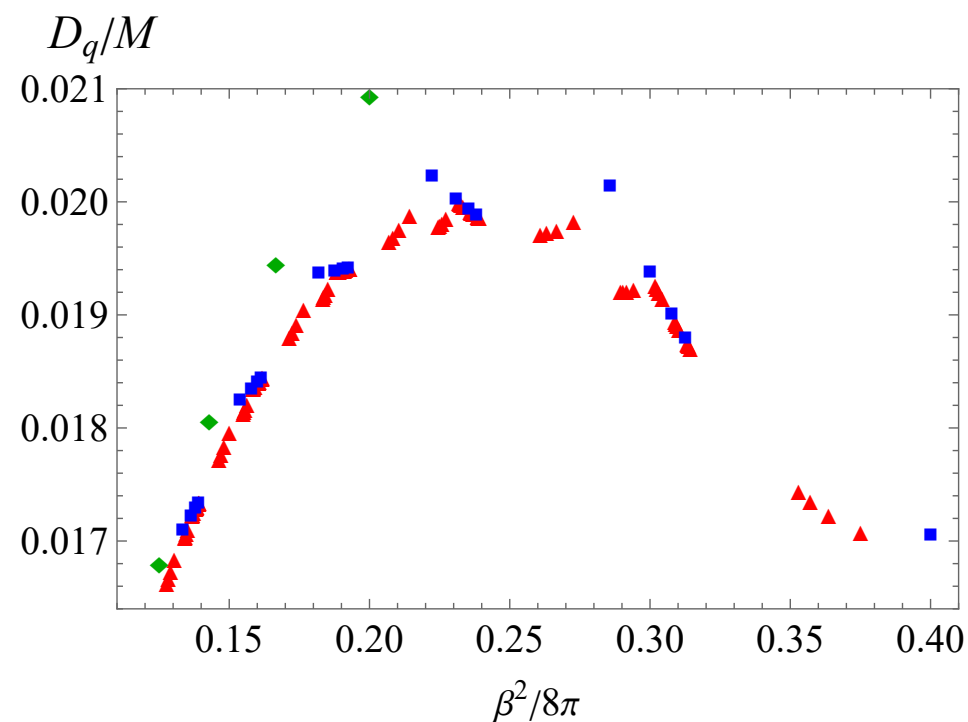
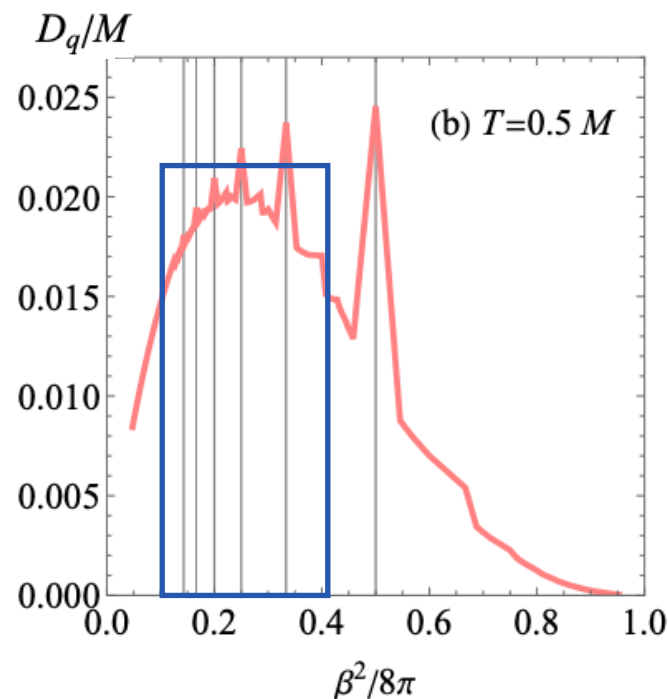
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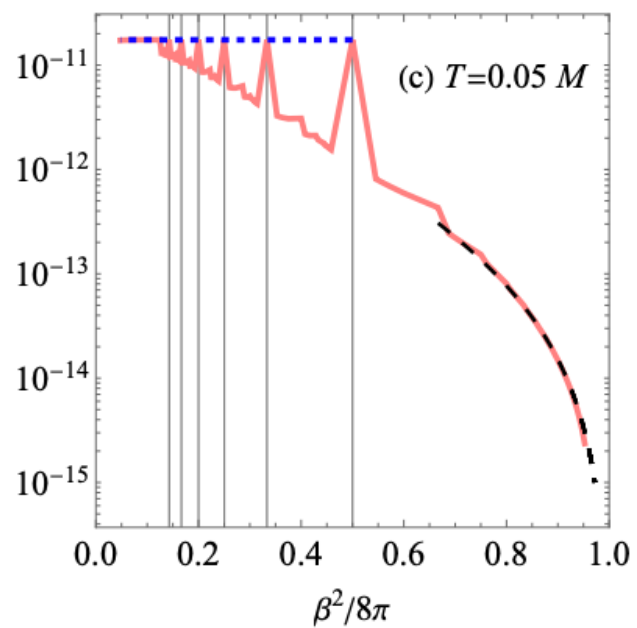
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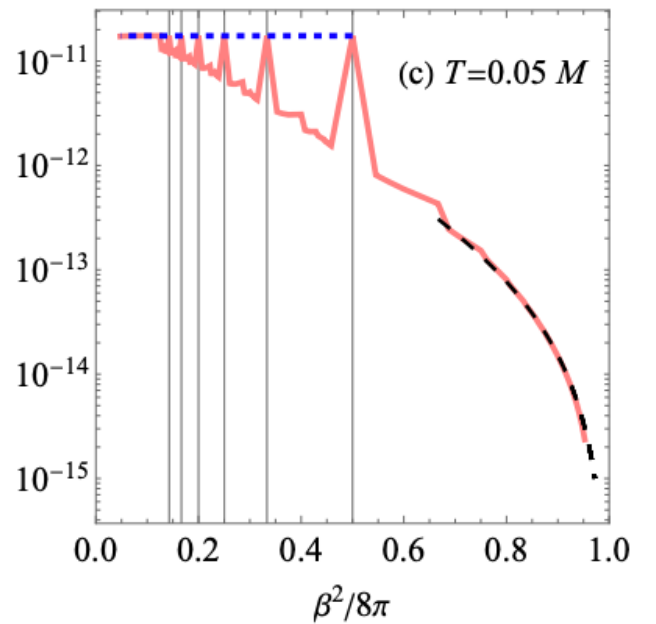
XXZ:
T. Prosen, E. Ilievski '13
E. Ilievski, J. De Nardis '17

Drude weight, low T



$$D_q = \sum_a \int d\theta \rho_a^{\text{tot}}(\theta) \vartheta_a(\theta) [1 - \vartheta_a(\theta)] [v_a^{\text{eff}}(\theta) q_a^{\text{dr}}(\theta)]^2$$

Drude weight, low T

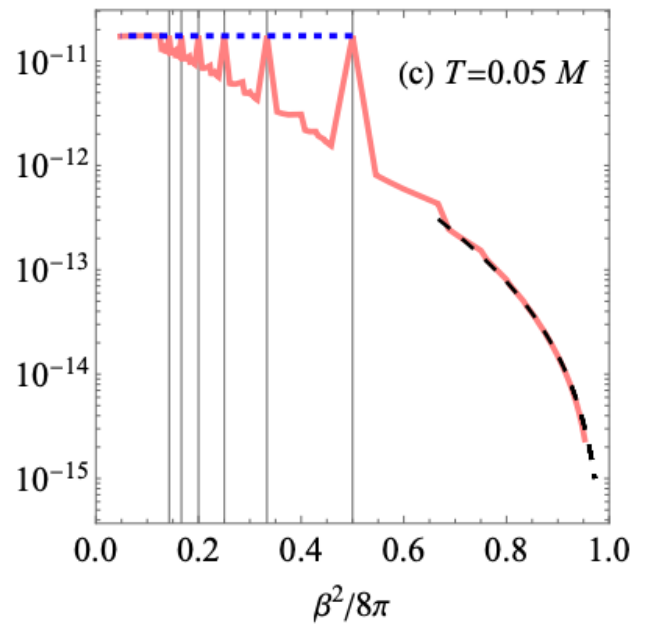


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Reflectionless points: the source terms dominate, dressing can be neglected

$$D_q^{\text{low-}T} = 2 \int \frac{d\theta}{2\pi} \frac{M \cosh \theta e^{-M \cosh \theta / T}}{(1 + e^{-M \cosh \theta / T})^2} \tanh^2 \theta$$

Drude weight, low T



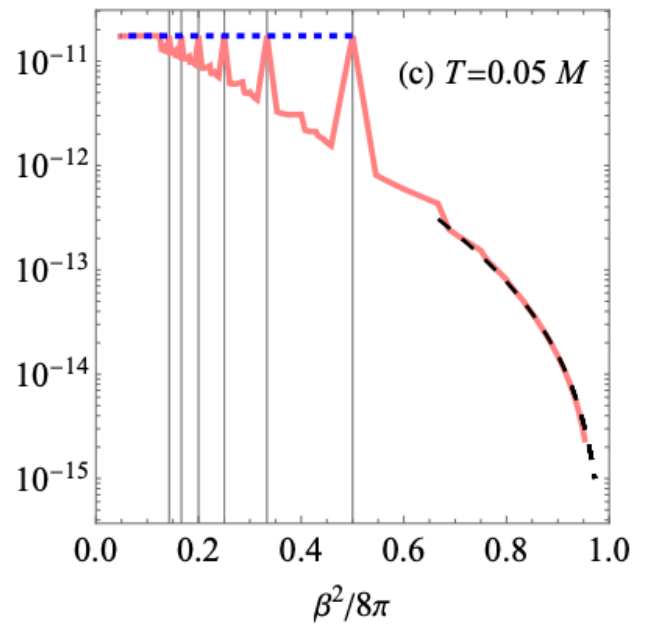
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Repulsive regime (1 level, $\xi \in \mathbb{Z}$): magnonic fillings are constant, only last two magnons contribute

Drude weight, low T



$$D_q = \sum_a \int d\theta \rho_a^{\text{tot}}(\theta) \vartheta_a(\theta) [1 - \vartheta_a(\theta)] [v_a^{\text{eff}}(\theta) q_a^{\text{dr}}(\theta)]^2$$

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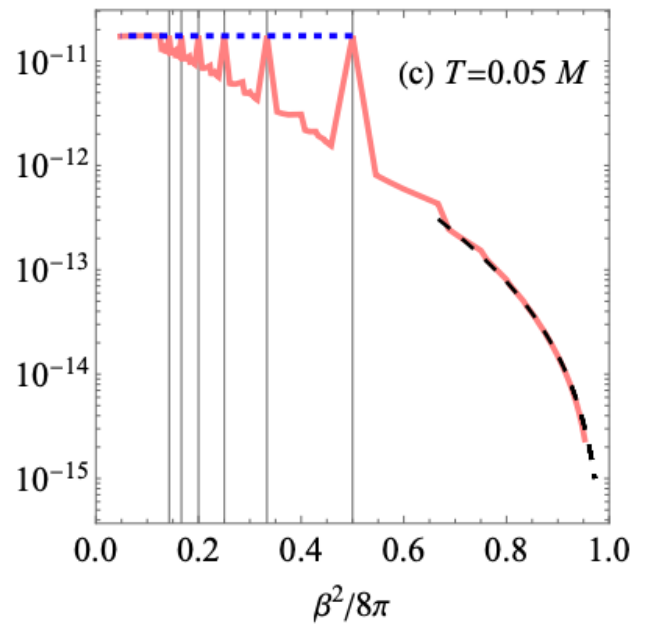
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ρ_a^{tot} can be calculated

B. Bertini, L. Piroli, M.K., PRB '19

Drude weight, low T



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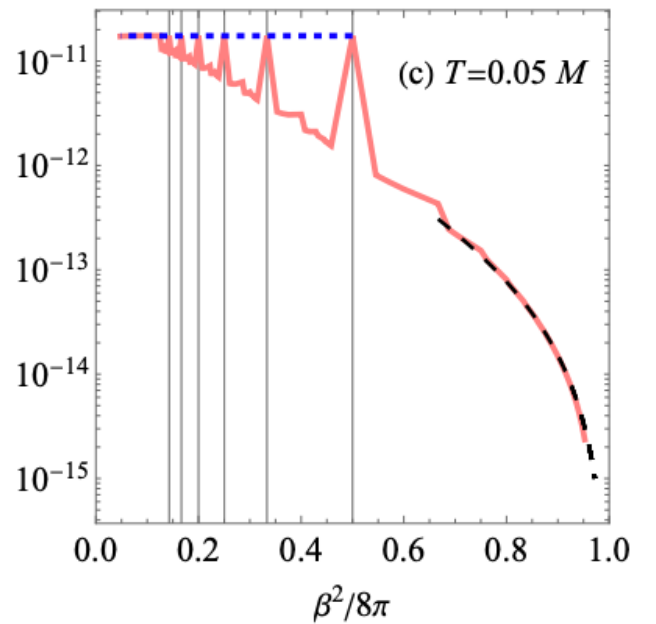
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B. Bertini, L. Piroli, M.K., PRB '19

$$D_q^{\text{low-}T} = e^{-M/T} \frac{\sqrt{2} T^{5/2}}{\pi^{3/2} M^{3/2}} \frac{2\pi/\xi - \sin(2\pi/\xi)}{\xi \sin^2(\pi/\xi)}$$

Drude weight, low T



$$D_q = \sum_a \int d\theta \rho_a^{\text{tot}}(\theta) \vartheta_a(\theta) [1 - \vartheta_a(\theta)] [v_a^{\text{eff}}(\theta) q_a^{\text{dr}}(\theta)]^2$$

Reflectionless points: the source terms dominate, dressing can be neglected

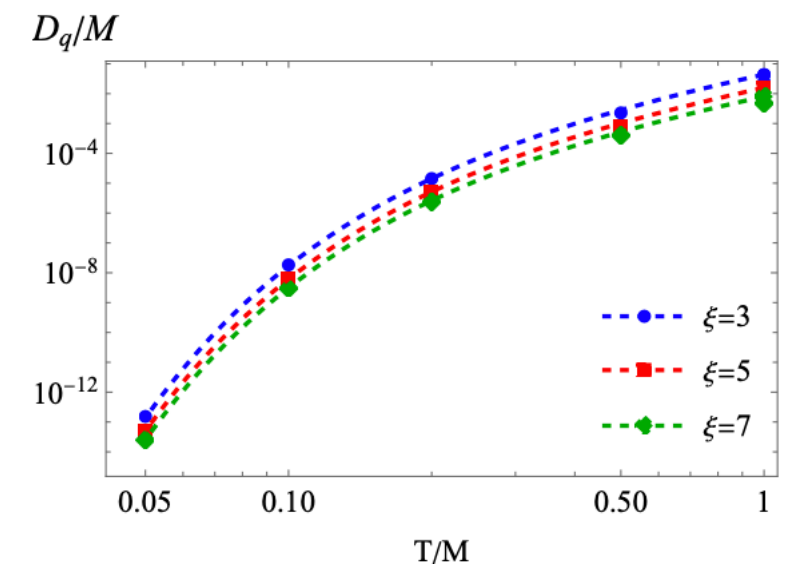
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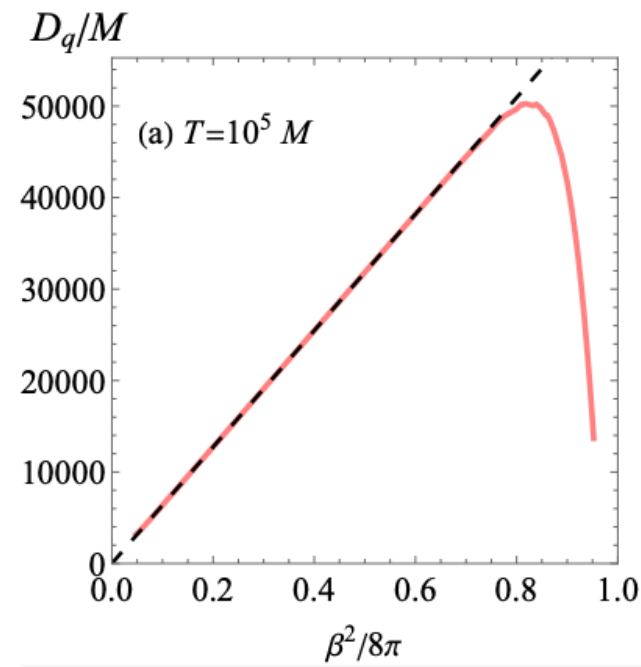
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B. Bertini, L. Piroli, M.K., PRB '19

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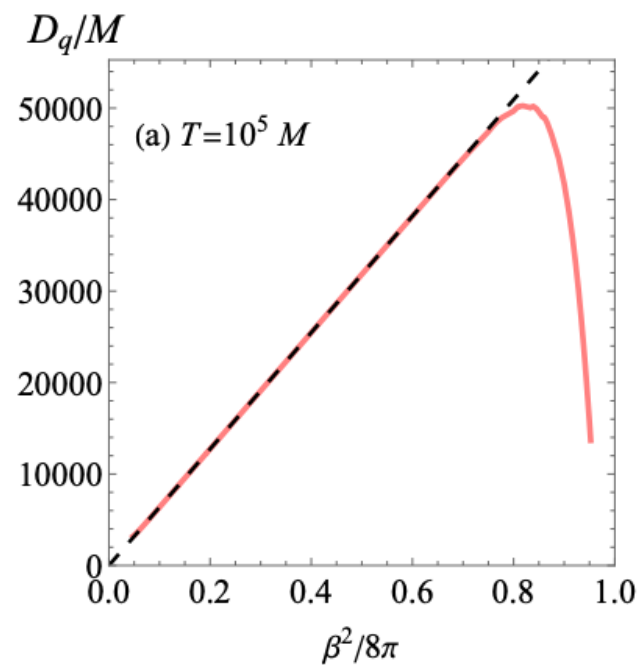


Drude weight, high T



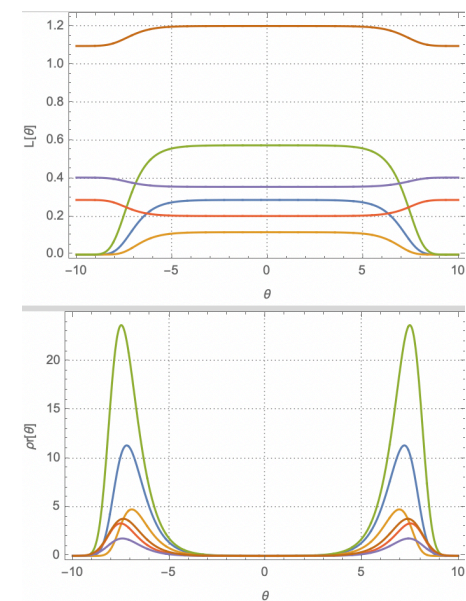
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Drude weight, high T

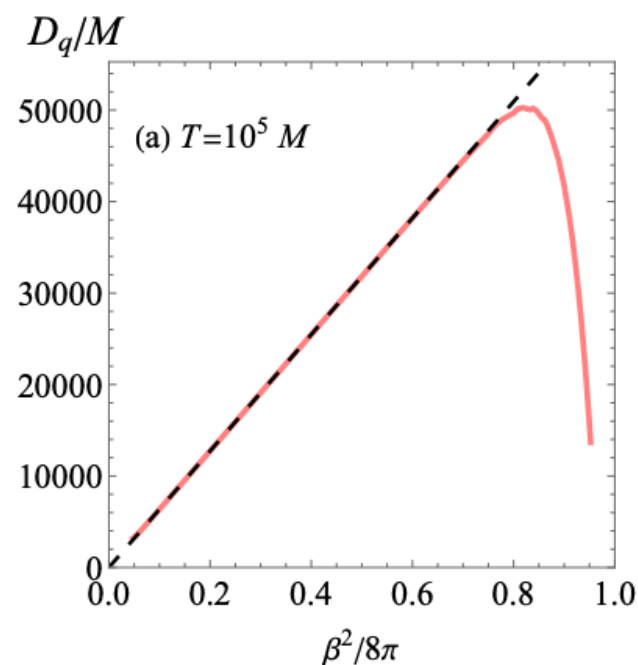


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$$\xi = \frac{1}{n_B + 1/p}$$

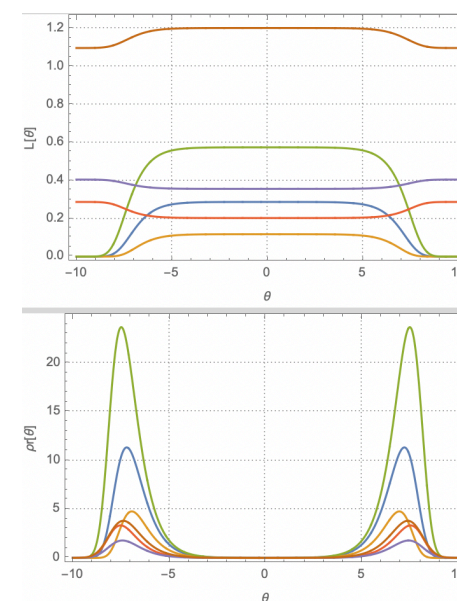


Drude weight, high T

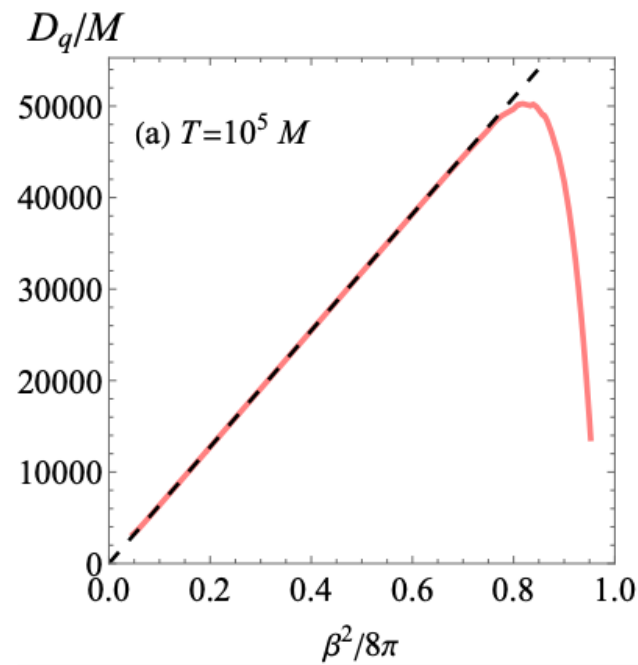


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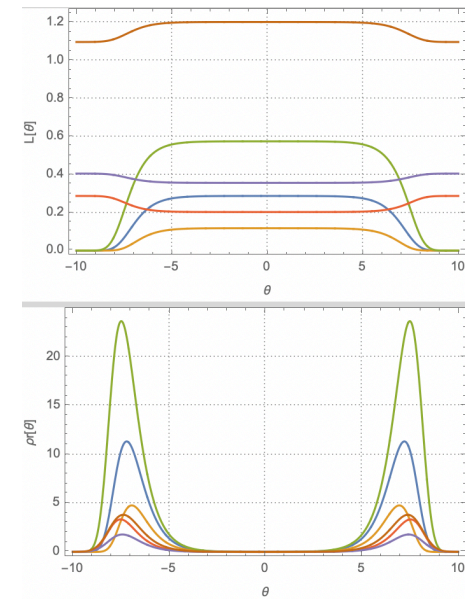


Drude weight, high T

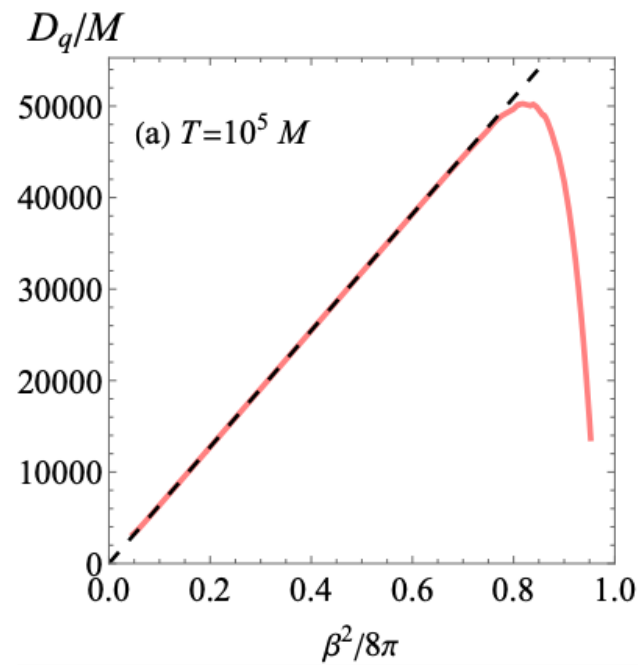


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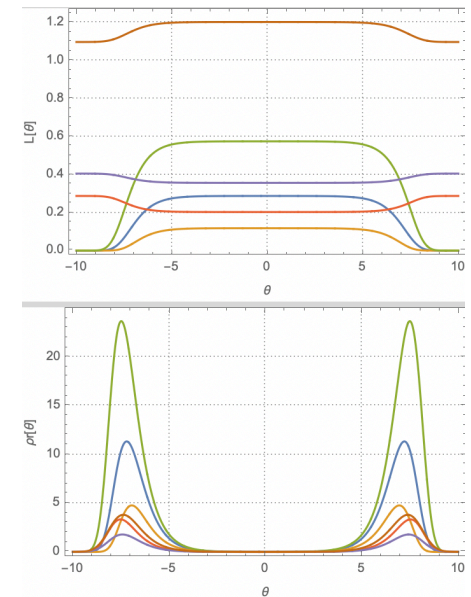
Drude weight, high T



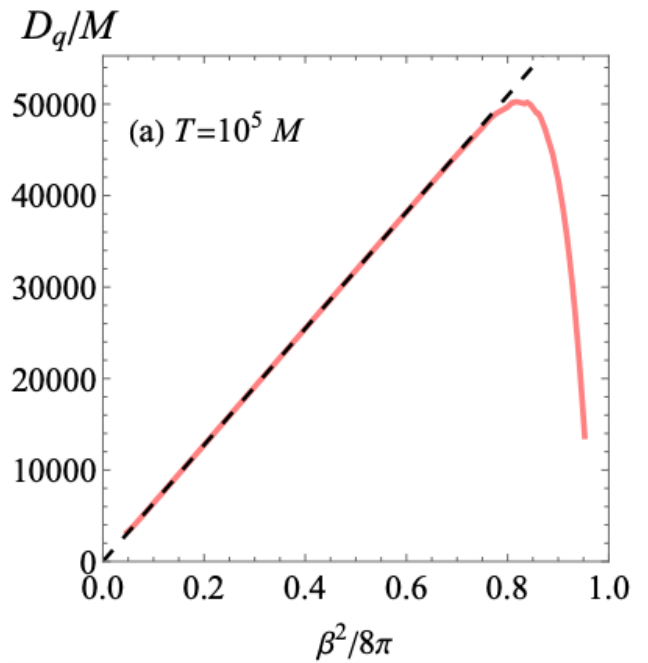
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$$\rho_a^{\text{r}}(\theta) = \frac{T}{2\pi} \epsilon'_a(\theta) \vartheta_a(\theta)$$

$$\xi = \frac{1}{n_B + 1/p}$$



Drude weight, high T

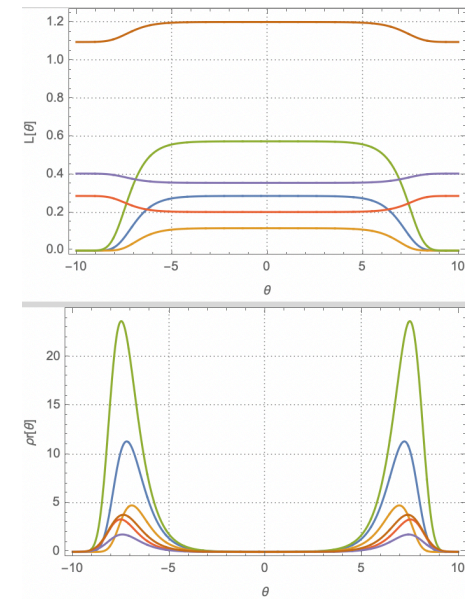


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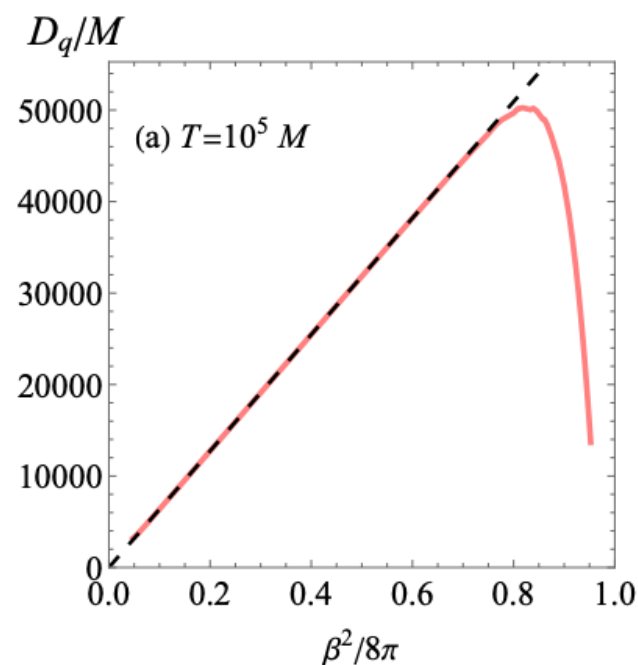
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$$\xi = \frac{1}{n_B + 1/p}$$

$$D_q^{\text{high-T}} = \frac{T}{\pi} p^2 2 \int_0^\infty d\theta \vartheta'_{p-1}(\theta) = \frac{2T}{\pi} p^2 (\tilde{\vartheta}_{p-1} - \bar{\vartheta}_{p-1}) = \frac{2T}{\pi} \frac{\xi}{\xi + 1} = \frac{2T}{\pi} \frac{\beta^2}{8\pi}$$



Drude weight, high T

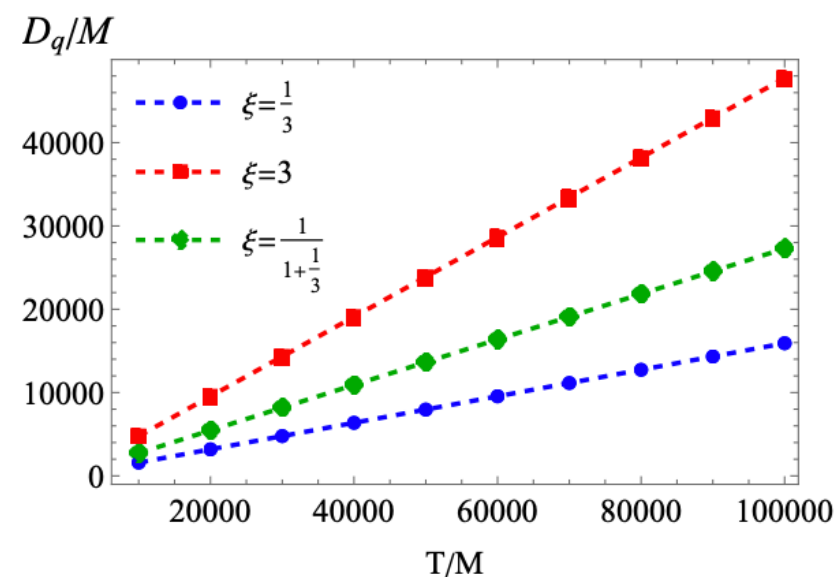
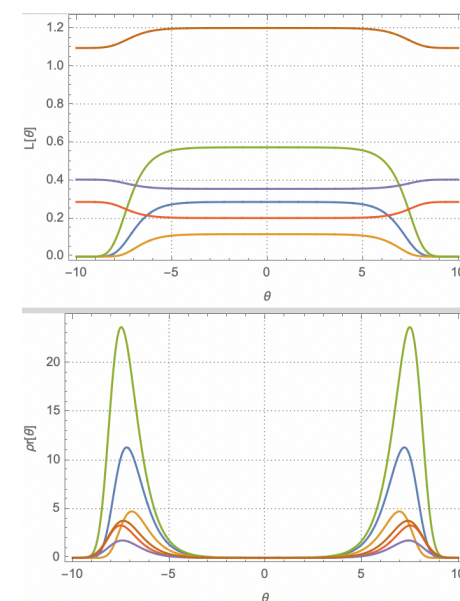


$$D_q = \sum_a \int d\theta \rho_a^{\text{tot}}(\theta) \vartheta_a(\theta) [1 - \vartheta_a(\theta)] \left[\underset{1}{v_a^{\text{eff}}(\theta)} \underset{p}{q_a^{\text{dr}}(\theta)} \right]^2$$

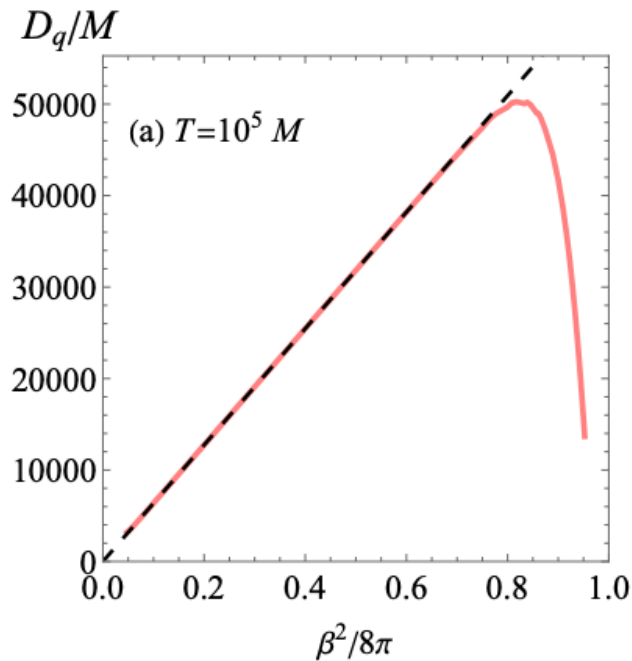
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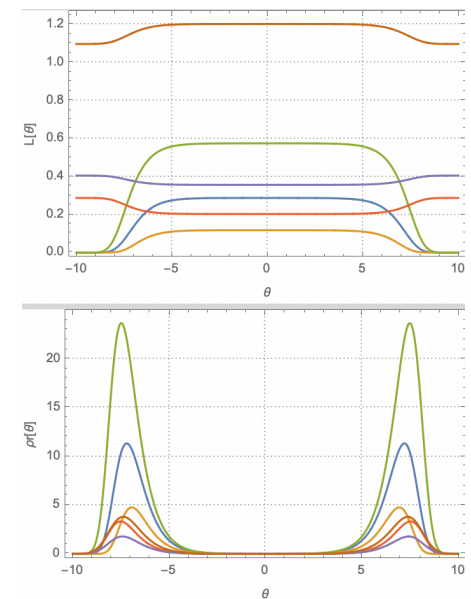


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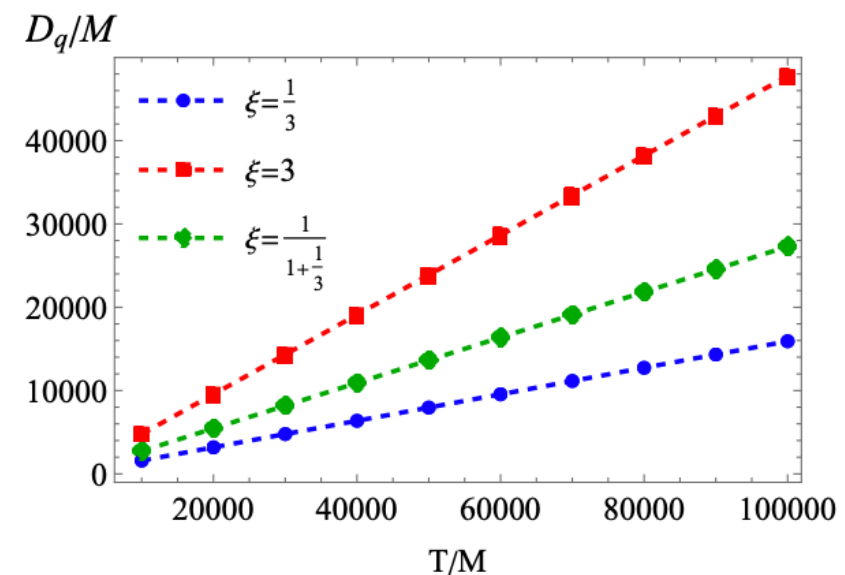
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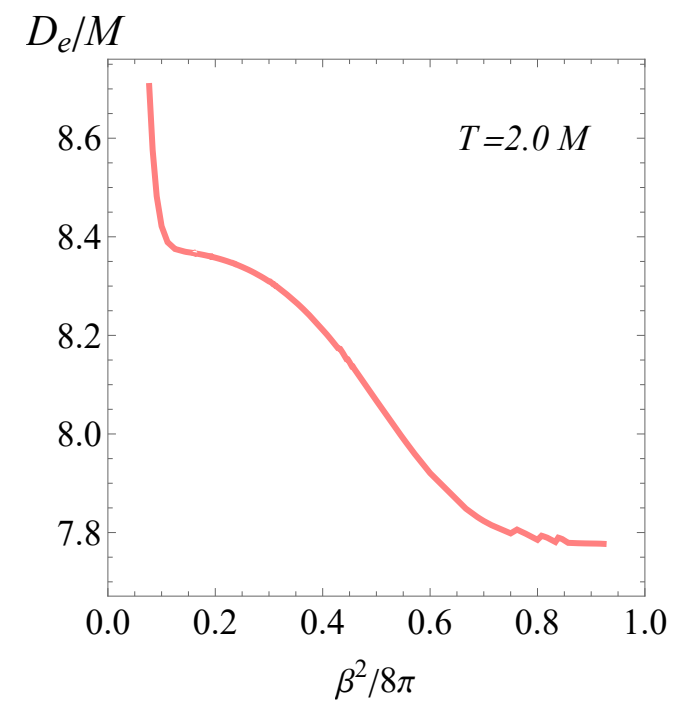
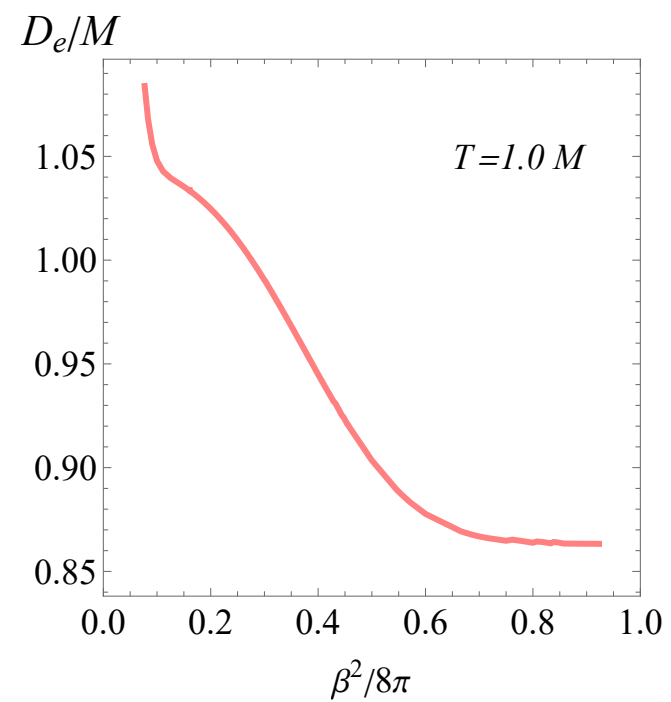
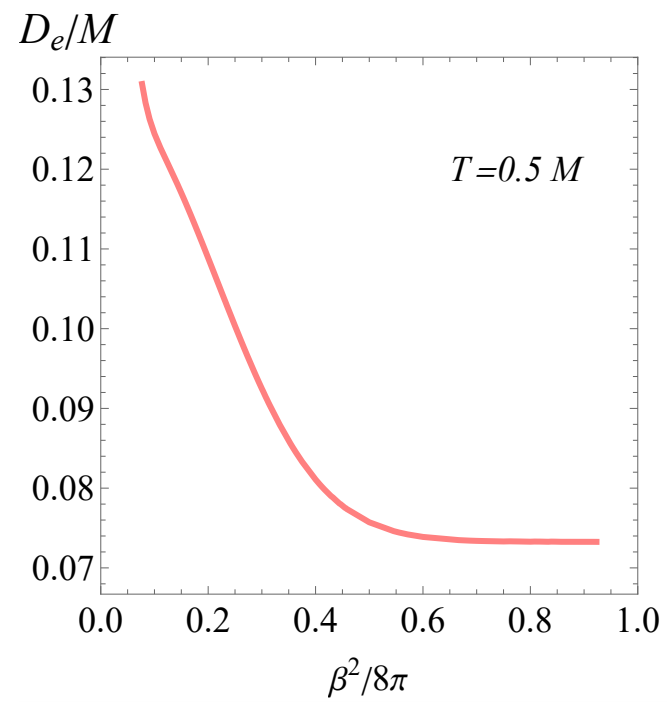
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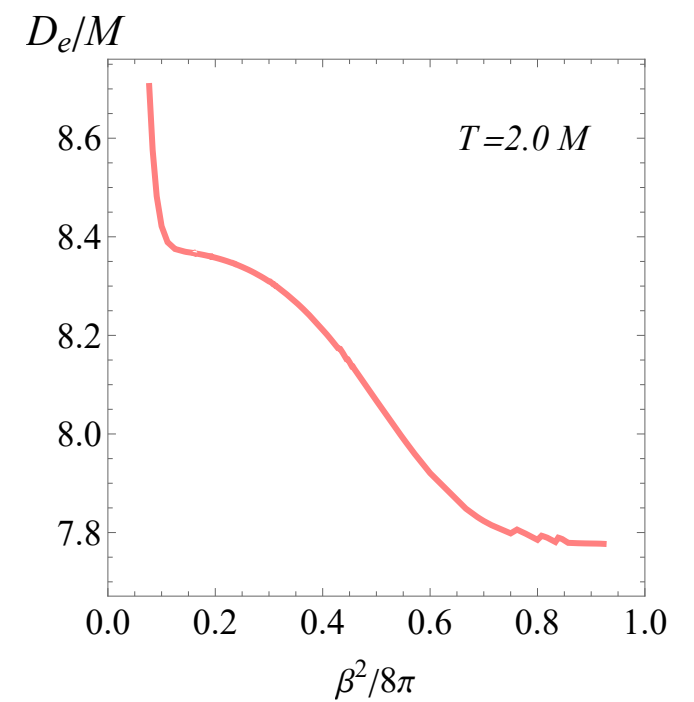
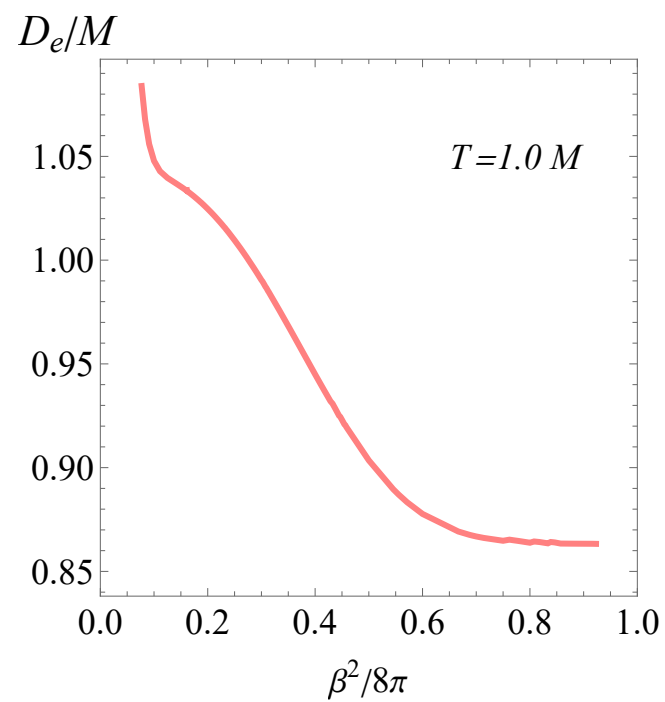
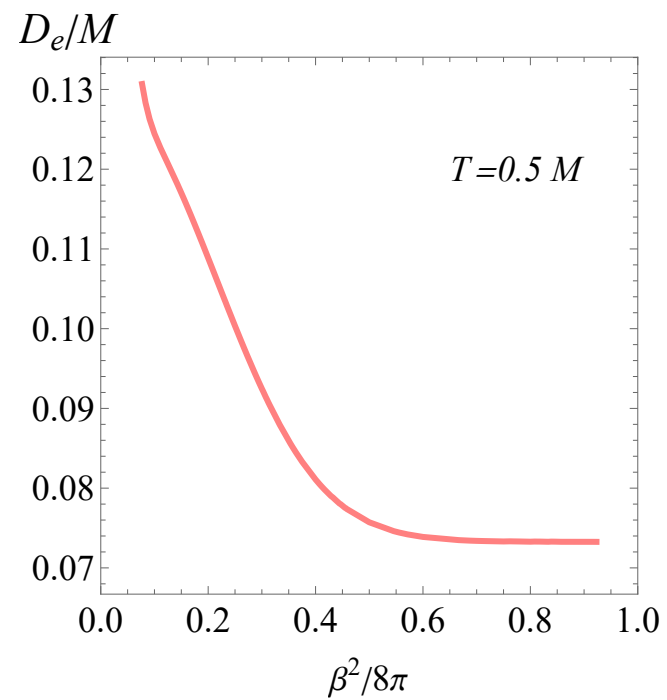
The $T \rightarrow \infty$ and $\xi \rightarrow \infty$ limits do not commute!



Energy Drude weight

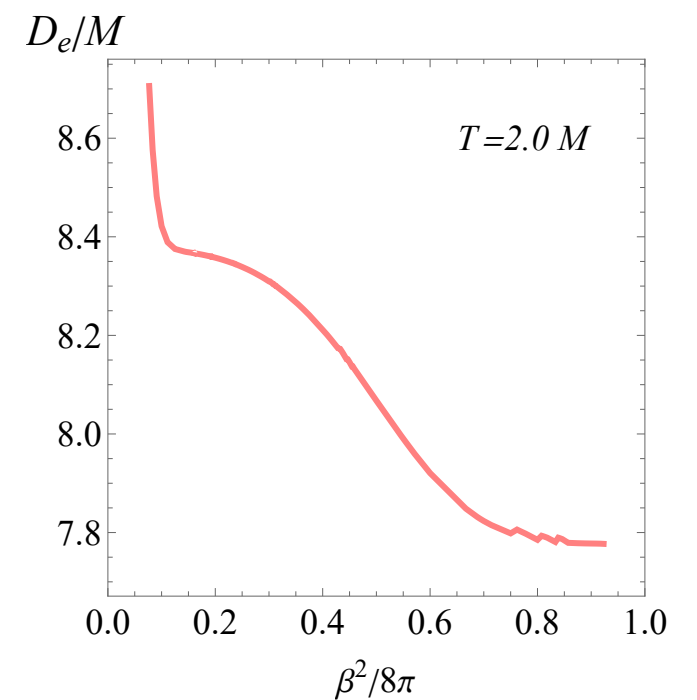
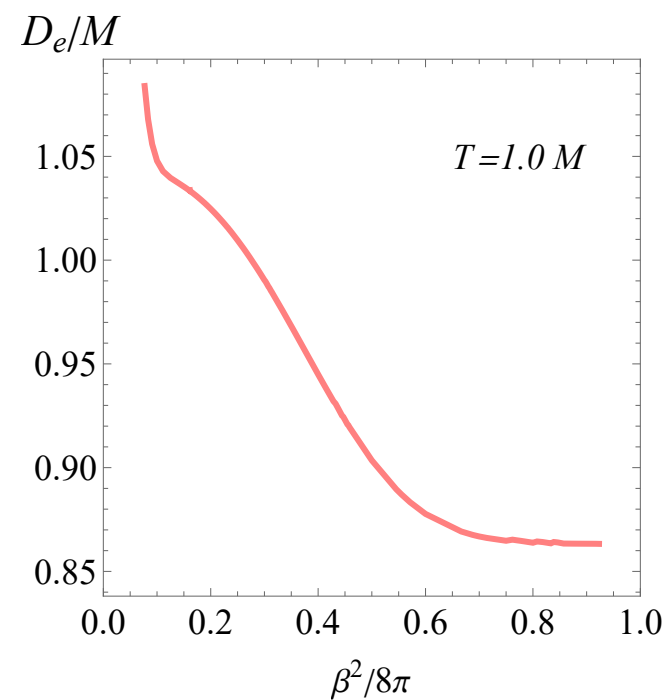
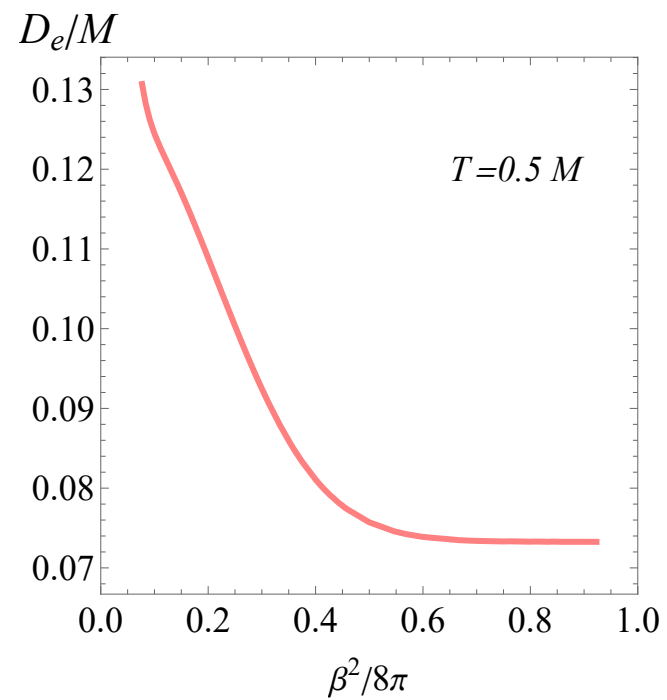


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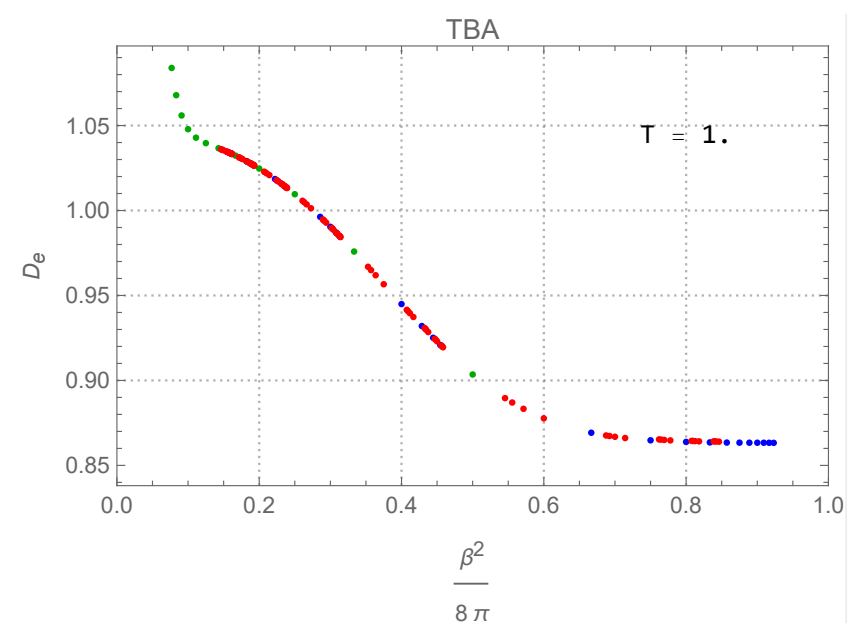


The energy Drude weight is continuous.

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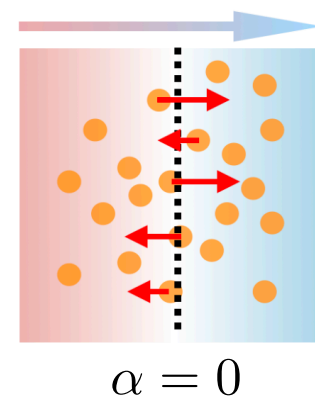
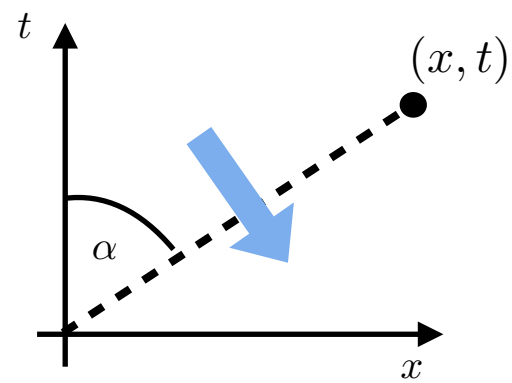


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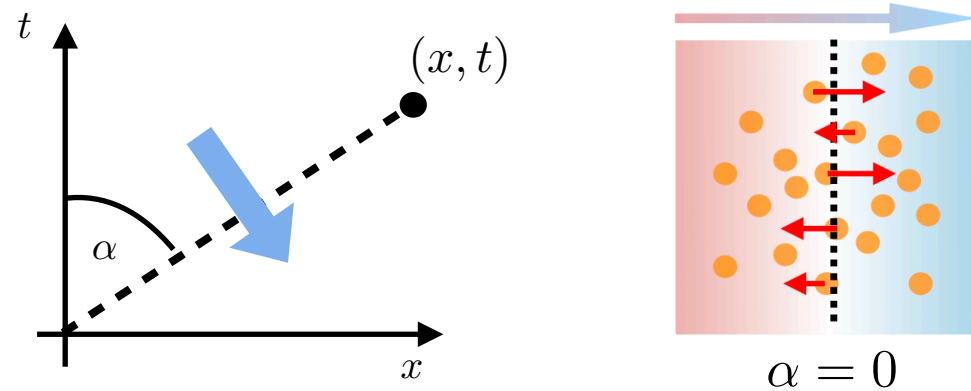
Ballistic Fluctuation Theory

J. Myers et al., '20; Doyon, Myers '20



Ballistic Fluctuation Theory

J. Myers et al., '20; Doyon, Myers '20



$$\left\langle e^{\lambda \int dt j(x,t) - dx q(x,t)} \right\rangle \asymp e^{\ell F_{\alpha}(\lambda)}$$

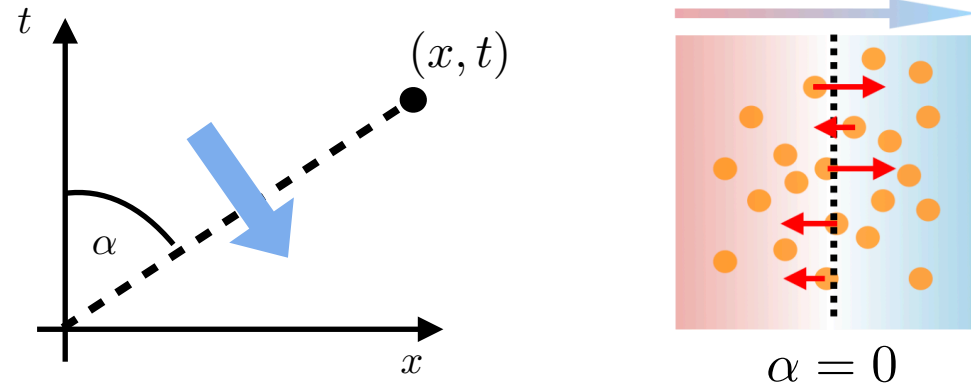
$F_{\alpha}(\lambda)$: Full Counting Statistics (FCS), scaled cumulant generating function

$$\ell = \sqrt{x^2 + t^2}$$

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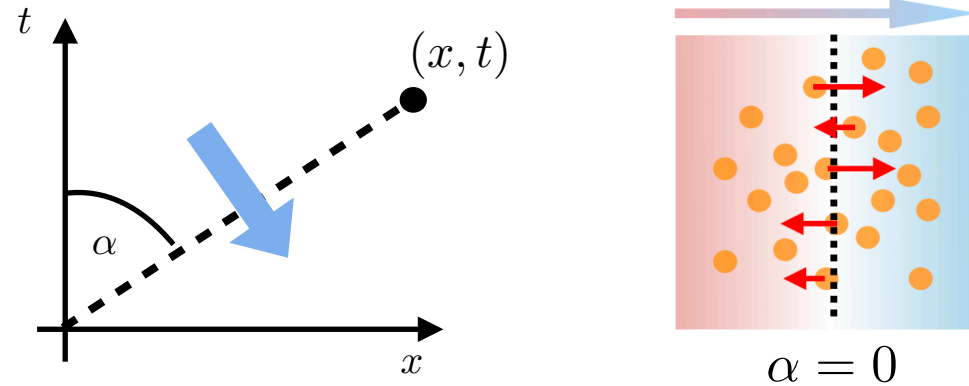
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sine-Gordon: fluctuations of the phase difference

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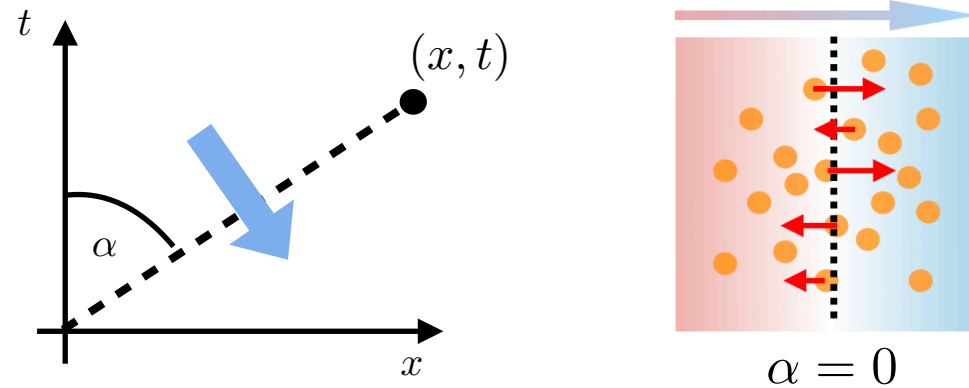
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$$c_n(\alpha) = \frac{1}{\ell} \left\langle \left(\frac{\beta}{2\pi} \Delta \varphi(x,t) \right)^n \right\rangle_c = \left. \frac{d^n F_{\alpha}(\lambda)}{d\lambda^n} \right|_{\lambda=0}$$

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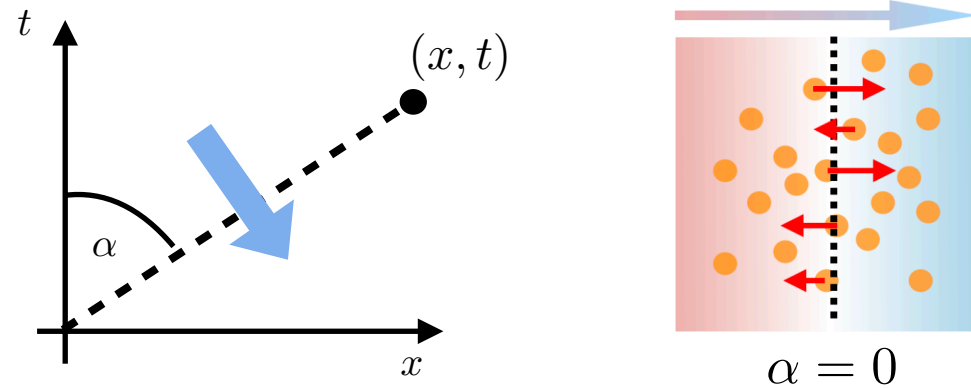
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Phase Fluctuations in sG

G. Del Vecchio², MK, A. Bastianello, B. Doyon, arXiv:2305.10495

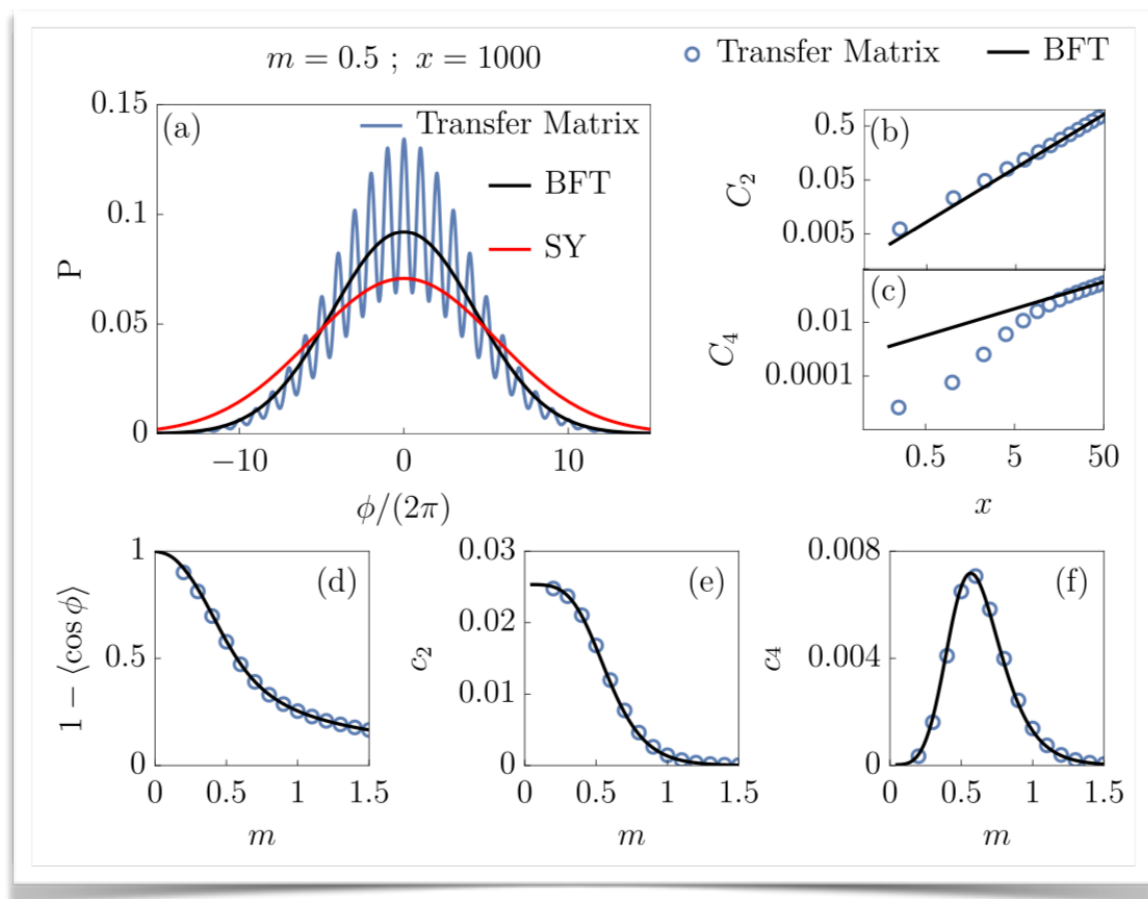
Classical limit (see the next talk by Alvise)

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Equal time ($\alpha = \pi/2$)

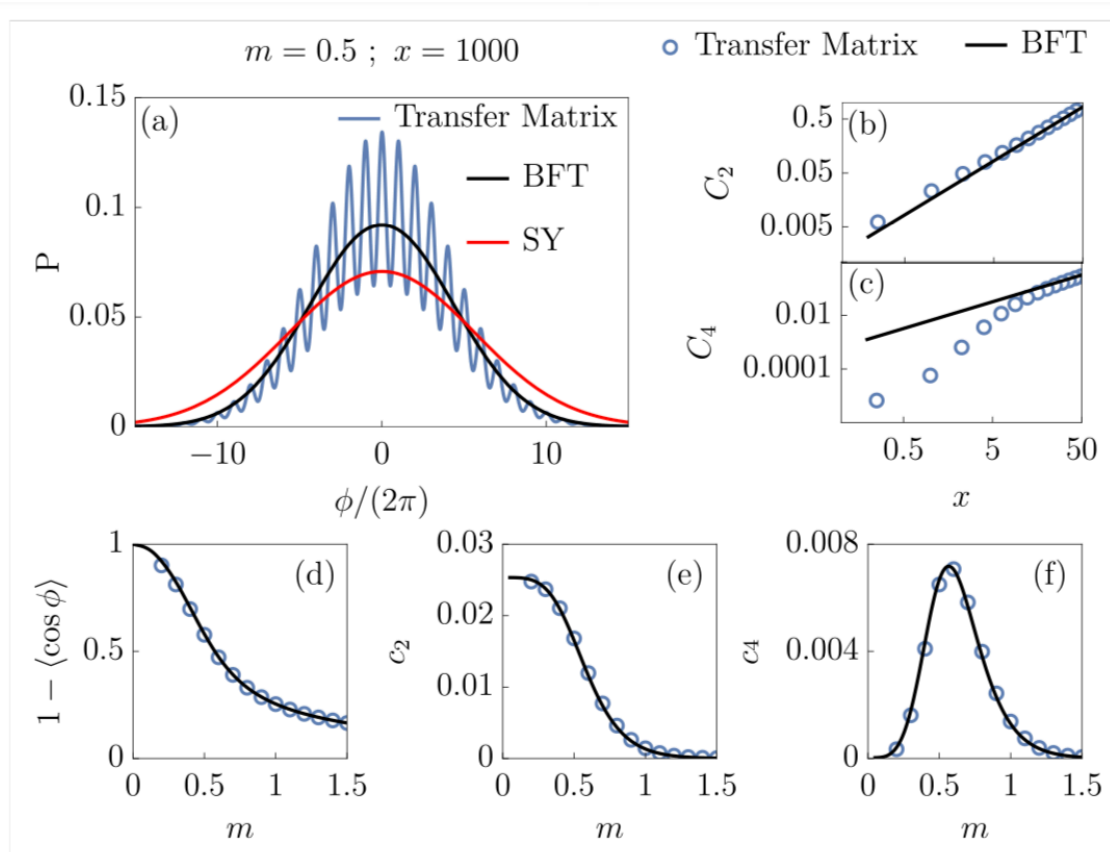


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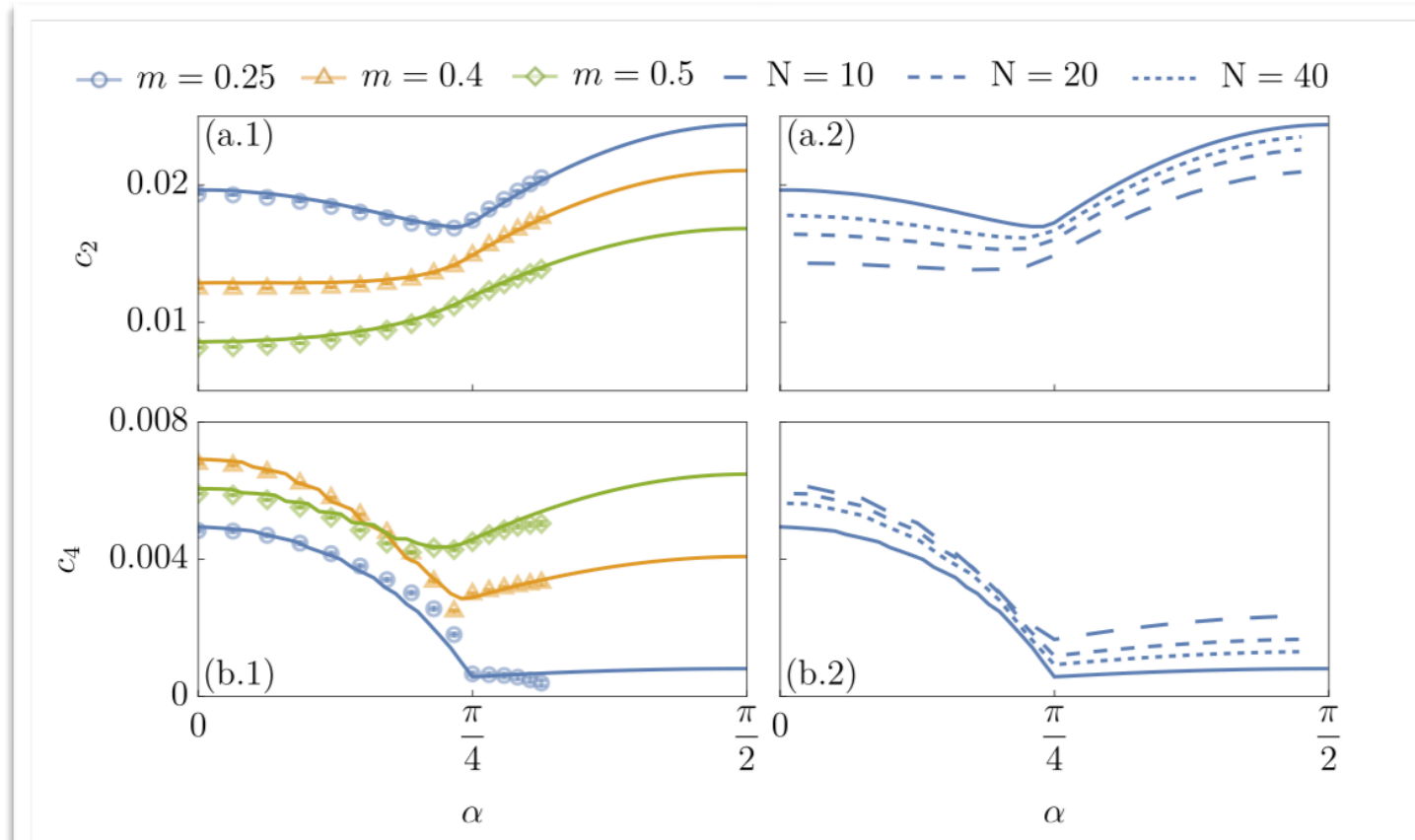
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Space-time direction (α) dependence



Phase Fluctuations in sG

Generic coupling
(in progress)

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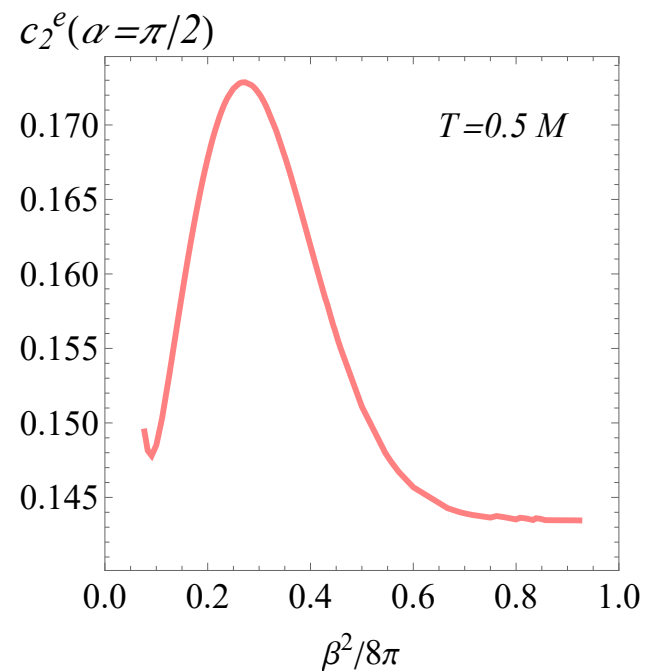
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Energy

$$\alpha = \pi/2$$



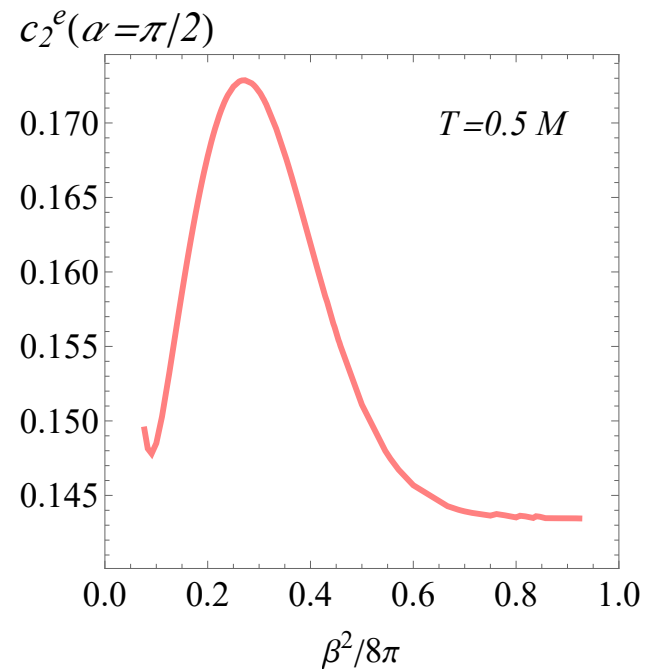
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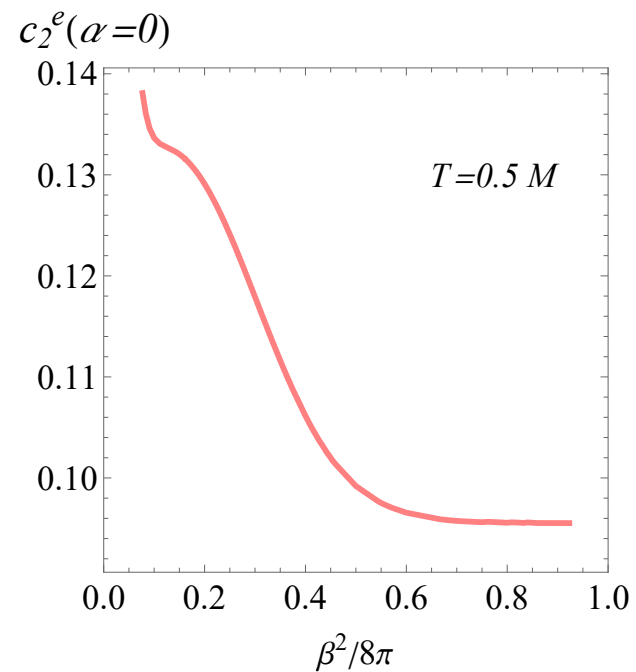
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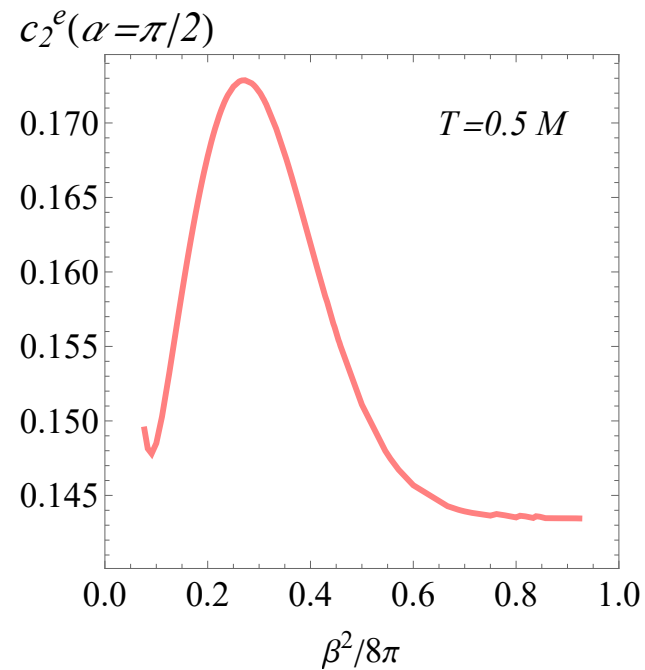


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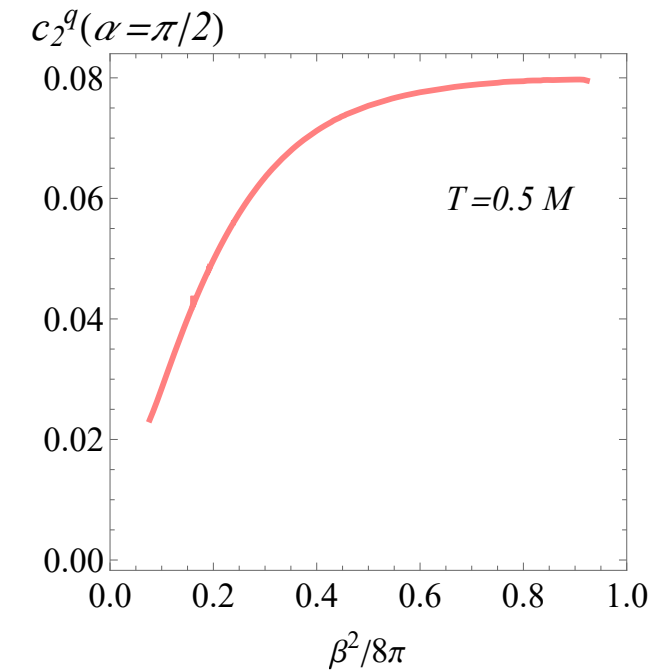
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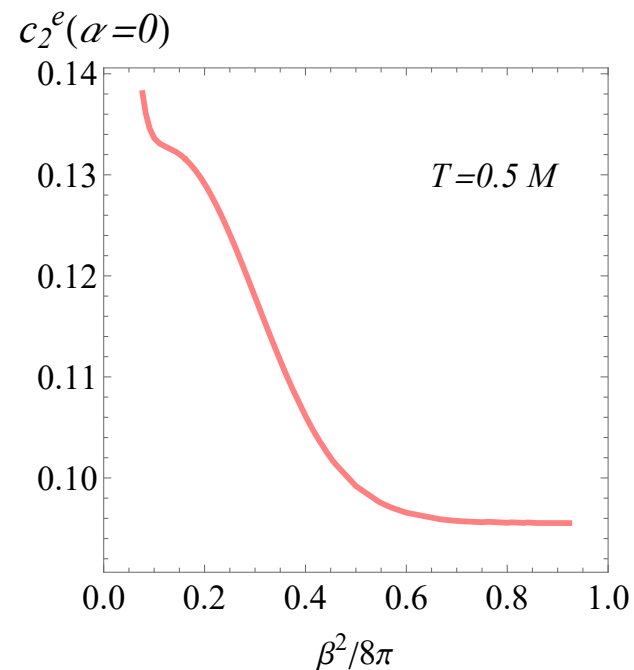


$\alpha = \pi/2$

Topological charge



$\alpha = 0$

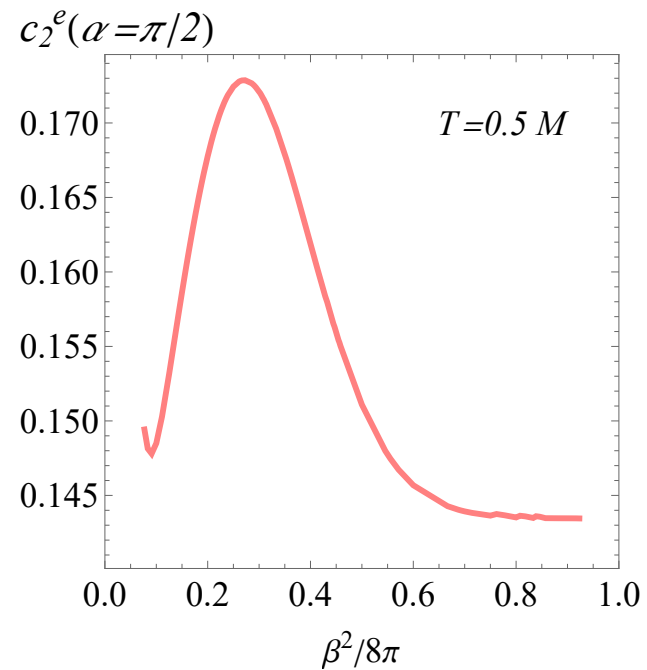


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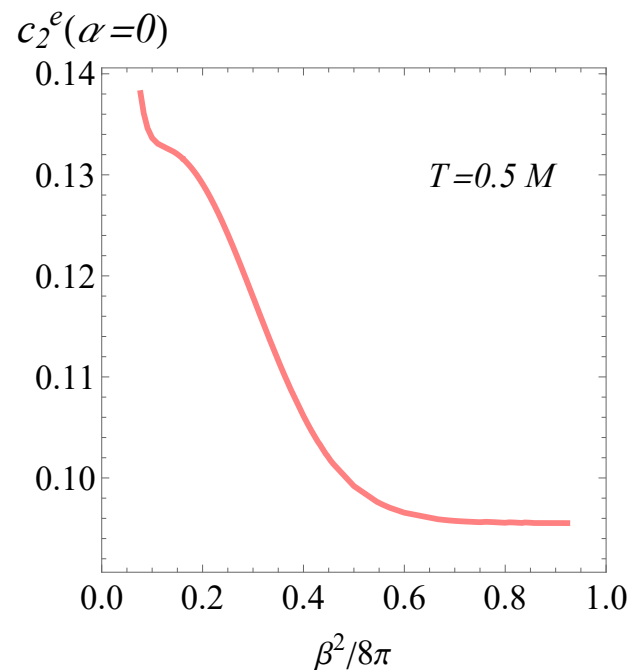
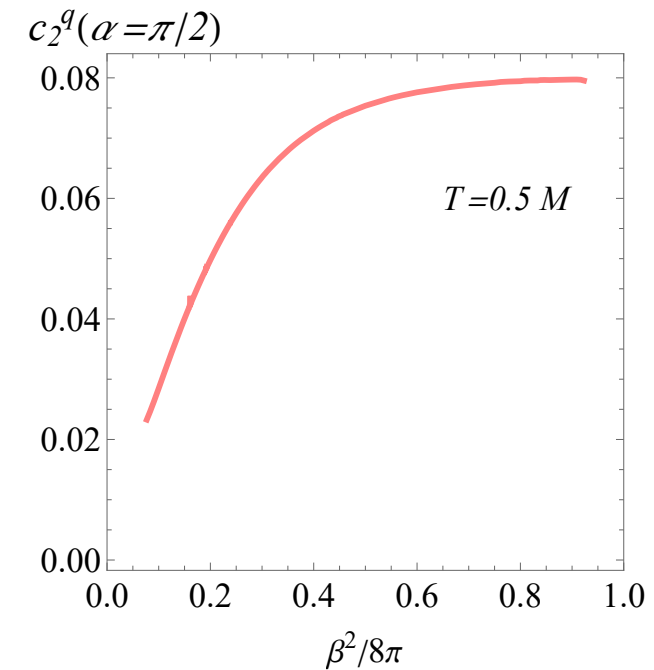
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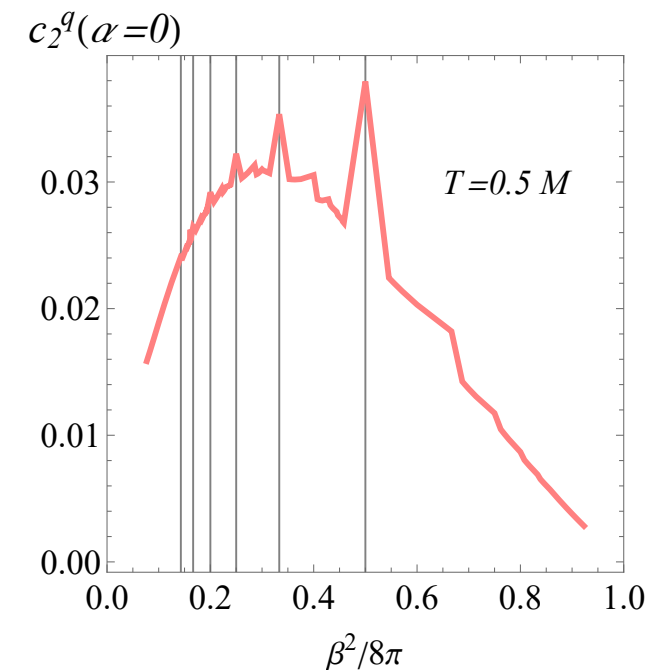


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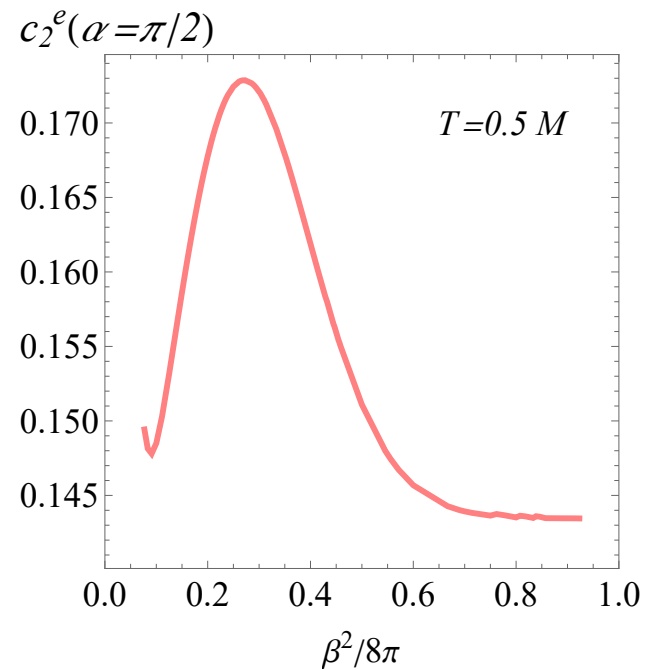


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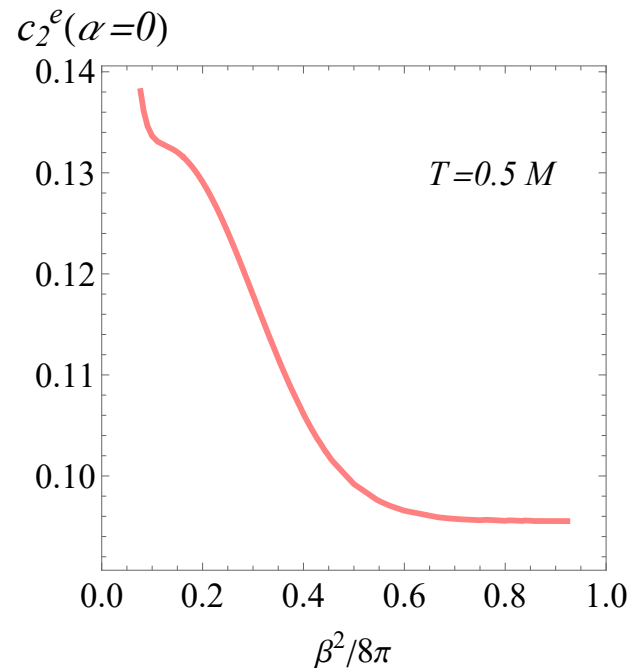
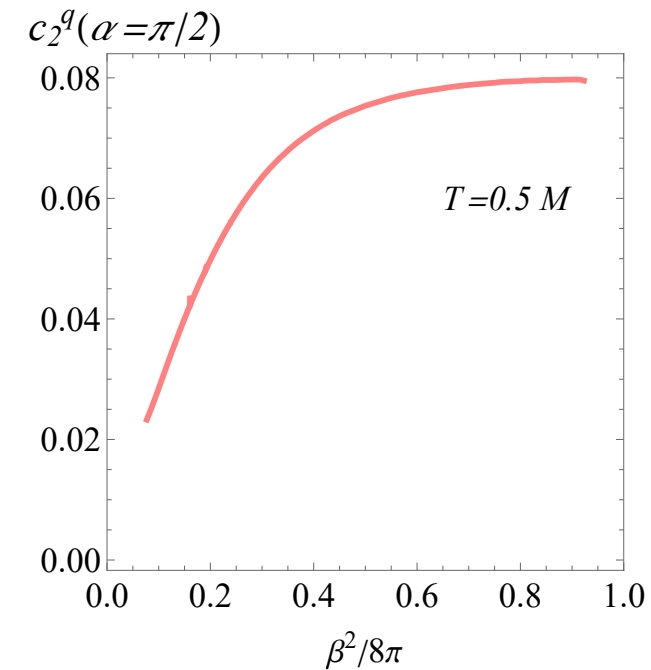
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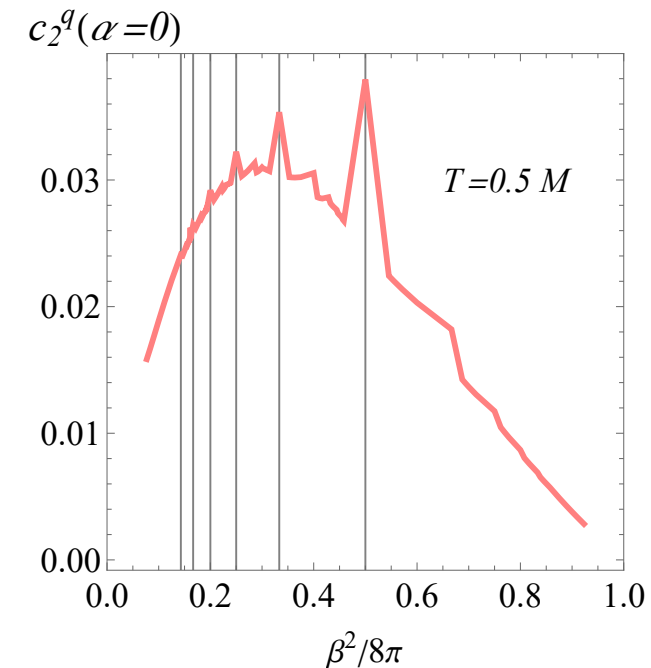


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Fractal (popcorn)!

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Thank you for your attention!

