Gas of Bethe ansatz wavepackets and an ab initio derivation of generalised hydrodynamics

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Introduction

- The dynamics of integrable models at larges scales of space and time is well described by generalised hydrodynamics (GHD) [Castro-Alvaredo, BD, Yoshimura, 2016; Bertini, Collura, De Nardis, Fagotti 2016]

$$\partial_t \rho_{\rm p}(x,t;\theta) + \partial_x \left[v^{\rm eff}(x,t;\theta)\rho_{\rm p}(x,t;\theta)\right] = 0.$$

where $$v^{\rm eff}(\theta) = \theta + \int {\rm d}\alpha\, \varphi(\theta-\alpha)\, \rho_{\rm p}(\alpha)\, (v^{\rm eff}(\alpha) - v^{\rm eff}(\theta))$$

- Holds for classical and quantum models, particles, spins, and continuous fields; confirmed experimentally in cold atom gases [Schemmer et al 2019; Møller et al 2021; Malvania et al 2021].
- **Rigorous proofs** that this equation emerges from the microscopic dynamics in hard-rod models [Boldrighini, Dobrushin, Sukhov 1983; Ferrari, Franceschini, Grevino, Spohn 2022], in cellular automata [Croydon Sasada 2020], in gases of solitons [El, Kamchatnov, Pavlov, Tovbis 2003-2022].
- In all quantum models the only known derivation until now is based on the **assumption of local entropy maximisation**. As usual for hydrodynamic equations more generally.

⇒ We provide a derivation in the quantum delta-interaction Bose gas (Lieb-Liniger model) from its Schrödinger equation.

1. The problem: inhomogeneous dynamics of many-body integrable systems

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As an example consider the Lieb-Liniger model, which describes point-like interactions of Galilean invariant Bose gases. Its Hamiltonian is

$$H = -\sum_{i=1}^{N} \frac{1}{2} \frac{\partial^2}{\partial x_i^2} + \sum_{(i \neq j)=1}^{N} \frac{c}{2} \delta(x_i - x_j)$$

1. The problem: inhomogeneous dynamics of many-body integrable systems

It admits infinitely many local conserved quantities Q_i:

$$\hat{Q}_a = \int \mathrm{d}x \, \hat{q}_a(x).$$

with conservation laws

$$\partial_t \hat{q}_a + \partial_x \hat{j}_a = 0.$$

1. The problem: inhomogeneous dynamics of many-body integrable systems

For instance, the number of particle $\hat{Q}_0$, the momentum $\hat{Q}_1$, the energy $H = \hat{Q}_2/2$, and the first non-trivial one $\hat{Q}_3$, with densities

$$\hat{q}_0(x) = \sum_i \delta(x - \hat{x}_i)$$

$$\hat{q}_1(x) = \mathrm{He} \sum_i \delta(x - \hat{x}_i)\hat{p}_i$$

$$\hat{q}_2(x) = \mathrm{He} \sum_i \delta(x - \hat{x}_i)\big(\hat{p}_i^2 + \sum_{j\neq i} c\delta(\hat{x}_i - \hat{x}_j)\big)$$

$$\hat{q}_3(x) = \mathrm{He} \sum_i \delta(x - \hat{x}_i)\big(\hat{p}_i^3 + \sum_{j\neq i} 3c\delta(\hat{x}_i - \hat{x}_j)\hat{p}_i\big)$$

$$\dots$$

1. The problem: inhomogeneous dynamics of many-body integrable systems

We can describe theoretically the problem as follows. The initial state is an equilibrium state with inhomogeneous external fields

$$\langle \hat{A} \rangle = \frac{\mathrm{Tr}\,(\hat{\varrho}\hat{A})}{\mathrm{Tr}\,\hat{\varrho}}, \quad \hat{\varrho} = \exp\left[-\sum_a \int \mathrm{d}x\,\beta_a(x)\hat{q}_a(x)\right]$$

with a slowly varying fields $\beta_a(x) = \bar{\beta}_a(x/L)$, L large.

For instance, one realises in cold atom experiments the initial state

$$\exp\left[-\beta\left(\hat{H} + \int \mathrm{d}x\,V(x)\hat{q}_0(x)\right)\right]$$

Then the evolution occurs with the original Hamiltonian

$$\langle \hat{A}(x,t)\rangle = \langle e^{\mathrm{i}\hat{H}t}\hat{A}(x)e^{-\mathrm{i}\hat{H}t}\rangle$$

1. The problem: inhomogeneous dynamics of many-body integrable systems

We are interested in the finite-density Euler scaling limit

$$\lim_{L\to\infty} \langle \hat{A}(x,t) \rangle, \quad N/L = \nu, \quad x = L\bar{x},\, t = L\bar{t}$$

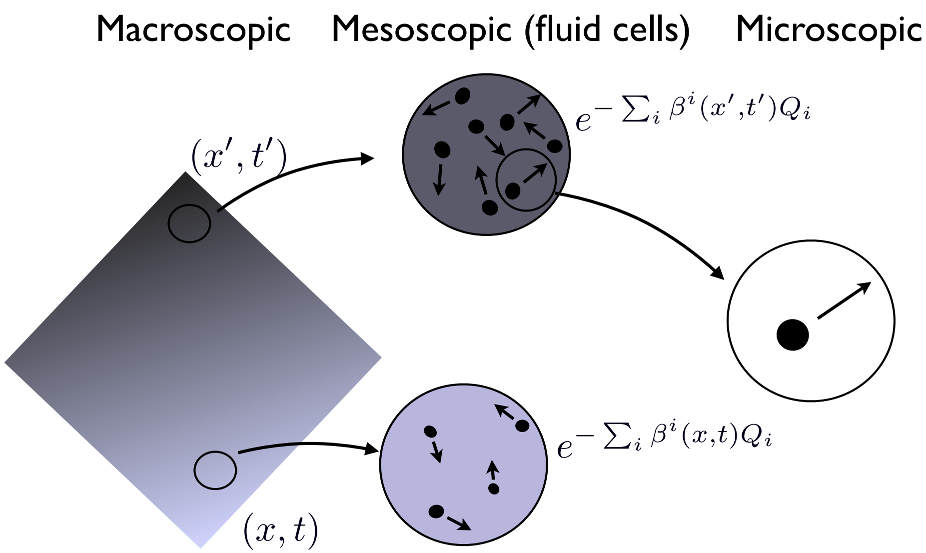
Even if the model is integrable, this remains a formidable task with standard methods.

2. Hydrodynamics

2. Hydrodynamics

One may use the **hydrodynamic principle**: the assumption that locally, in every fluid cell, entropy maximises with respect to the available conserved quantities. Thus locally we have generalised Gibbs ensembles (GGEs) [reviews Polkovnikov, Sengupta, Silva, Vengalattore 2011; Eisert, Friesdorf, Gogolin 2015; Essler, Fagotti 2016; Vidmar, Rigol 2016; Ilievski, Medenjak, Prosen, Zadnik 2016]

$$\text{at fluid cell } x,t: \quad \exp\Big[-\sum_a \beta_a(x,t)\hat{Q}_a\Big]$$



2. Hydrodynamics

- All densities and currents are approximated by

$$\langle \hat{q}_a(x,t)\rangle \approx \langle \hat{q}_a\rangle_{\mathrm{GGE}(x,t)} =: \mathtt{q}_a(x,t), \quad \langle \hat{j}_a(x,t)\rangle \approx \langle \hat{j}_a\rangle_{\mathrm{GGE}(x,t)} =: \mathtt{j}_a(x,t)$$

- The microscopic conservation equations are recast into “mesoscopic” equations

$$\partial_t \mathtt{q}_a + \partial_x \mathtt{j}_a = 0$$

- Using the Bethe ansatz in finite volume L and taking the limit $L, N \to \infty$ (or other techniques), one evaluates the GGE averages. The crucial result is the currents (GGE equations of state) [reviews Borsi, Pozsgay, Pristyák 2021, Cubero, Yoshimura, Spohn 2021]

$$\mathtt{q}_a = \int \mathrm{d}\theta\, \theta^a \rho_{\mathrm{p}}(\theta), \quad \mathtt{j}_a = \int \mathrm{d}\theta\, \theta^a v^{\mathrm{eff}}(\theta)\rho_{\mathrm{p}}(\theta)$$

- Considering this for all a’s, this is the GHD equation.

3. Interpretation

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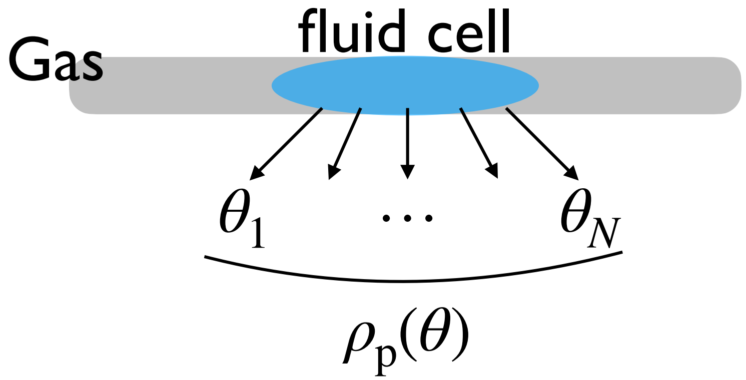
There is a simple physical interpretation of the Bethe root density $\rho_{\rm p}(\theta)$:



This is based on **elastic scattering**.

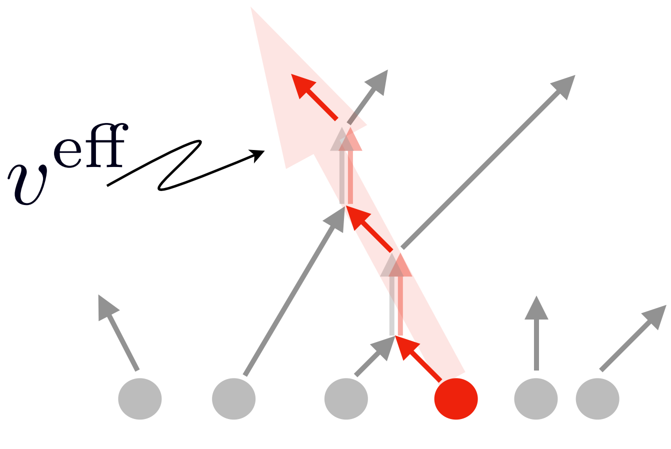
3. Interpretation

There is a simple **classical kinetic interpretation** to the effective velocity:

$$v^{\mathrm{eff}}(\theta) = \theta + \int \mathrm{d}\alpha\, \varphi(\theta-\alpha)\, \rho_{\mathrm{p}}(\alpha)\, (v^{\mathrm{eff}}(\alpha) - v^{\mathrm{eff}}(\theta))$$

where

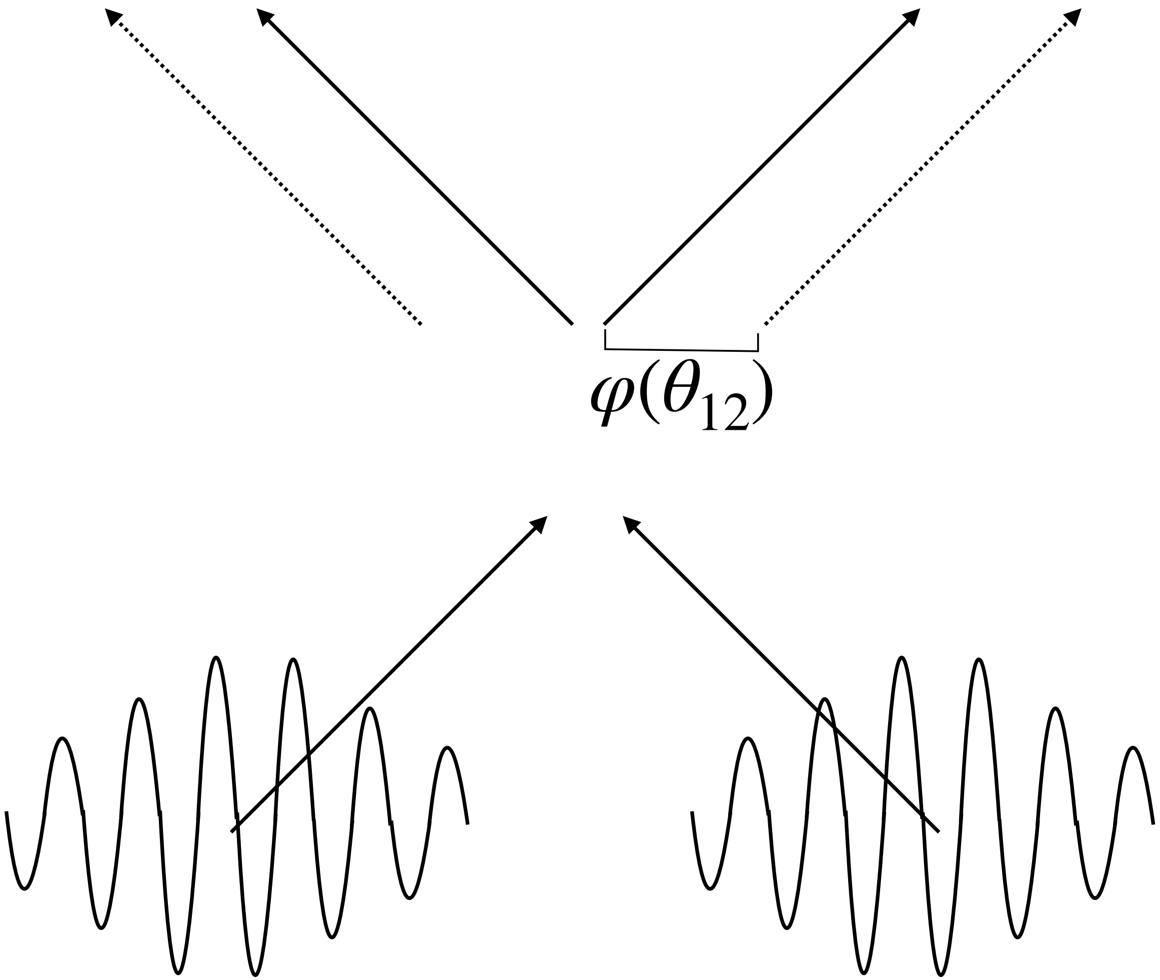
$$\varphi(\theta) = \partial_\theta \phi(\theta) = \frac{2c}{\theta^2 + c^2}$$



This is based on **factorised scattering**.

3. Interpretation

The function $\varphi(\theta)$ is the semiclassical scattering shift, seen in scattering of wave packets [see e.g. Bouchoule, Dubail 2022]:



4. Microscopic origin: slowly varying amplitudes

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We would like to show that, in the **macrocanonical ensemble** and under $x = L\bar{x}$, $t = L\bar{t}$,

$$\lim_{L\to\infty} \frac{\mathrm{Tr}\,(\hat{\varrho}\,e^{\mathrm{i}\hat{H}t}\hat{q}_a(x)e^{-\mathrm{i}\hat{H}t})}{\mathrm{Tr}\,\hat{\varrho}} = \int \mathrm{d}\theta\,\theta^a \rho_{\mathrm{p}}(\bar{x},\bar{t};\theta)$$

for all a, where $\rho_{\mathrm{p}}(\bar{x},\bar{t};\theta)$ satisfies GHD.

Two problems:

- $\hat{q}_a(x)$ are complicated in general
- $\mathrm{Tr}\,e^{\mathrm{i}\hat{H}t}\hat{\varrho}e^{\mathrm{i}\hat{H}t}\cdots$ is difficult to evaluate

4. Microscopic origin: slowly varying amplitudes

Reminder on the Bethe wavefunction...

With $\theta_1 > \theta_2 > \ldots > \theta_N$ this is a basis for the Hilbert space,

$$\Psi_{\boldsymbol{\theta}}(\boldsymbol{x}) = s(\boldsymbol{x}) \sum_{\sigma \in S_N} (-1)^{|\sigma|} e^{\mathrm{i}\Phi_{\boldsymbol{\theta}}(\boldsymbol{x}_\sigma)}$$

where $s(\boldsymbol{x}) = \prod_{i<j} \operatorname{sgn}(x_{ij})$ and

$$\Phi_{\boldsymbol{\theta}}(\boldsymbol{x}) = \boldsymbol{\theta} \cdot \boldsymbol{x} + \frac{1}{2} \sum_{i<j} \phi(\theta_{ij}) \operatorname{sgn}(x_{ij}), \quad \phi(\theta) = 2 \mathrm{Arctan}\, \frac{\theta}{c}$$

"Scattering eigenfunctions" on the line, that is

$$\hat{H} \Psi_{\boldsymbol{\theta}}(\boldsymbol{x}) = \frac{\boldsymbol{\theta}^2}{2} \Psi_{\boldsymbol{\theta}}(\boldsymbol{x}).$$

Note: **no need to quantise the momenta**...

4. Microscopic origin: slowly varying amplitudes

Define the **empirical density operator** $\hat{\rho}(x,\theta) = \sum_i \delta(x - \hat{x}_i)\delta(\theta - \hat{\theta}_i)$:

$$\hat{\rho}(x,\theta)\Psi_{\boldsymbol{\theta}}(\boldsymbol{x}) = s(\boldsymbol{x}) \sum_{\sigma \in S_N} (-1)^{|\sigma|} \delta(x - x_{\sigma(i)})\delta(\theta - \theta_i)\, e^{\mathrm{i}\Phi_{\boldsymbol{\theta}}(\boldsymbol{x}_\sigma)}$$

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1. Representation of local densities: we can show that for all $a = 0,1,2,\ldots$

$$\hat{q}_a(x) = \mathrm{He}\int \mathrm{d}\theta\,\theta^a\hat{\rho}(x,\theta)$$

are local conserved densities.

So for the GHD equation, we only have to show that

$$\lim_{L\to\infty}\frac{\mathrm{Tr}\,(\hat{\varrho}\,e^{\mathrm{i}\hat{H}t}\hat{\rho}(x)e^{-\mathrm{i}\hat{H}t})}{\mathrm{Tr}\hat{\varrho}} = \rho_{\mathrm{p}}(\bar{x},\bar{t};\theta)$$

4. Microscopic origin: slowly varying amplitudes

We may write

$$\hat{\varrho} = \exp\Big[-\mathrm{He}\,\hat{\rho}[\beta]\Big]$$

where

$$\hat{\rho}[\beta] = \int \mathrm{d}x\mathrm{d}\theta\, \beta(x,\theta)\hat{\rho}(x,\theta), \qquad \beta(x,\theta) = \sum_a \beta_a(x)\theta^a$$

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2. "Macroscopic classicalisation": the macroscopic operators $\hat{\rho}[f]$ are essentially classical

$$[\hat{\rho}[f], \hat{\rho}[g]] = O(1), \quad [\hat{\rho}[f], \hat{\rho}[f]^\dagger] = O(1)$$

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As a consequence,

$$\langle\boldsymbol{\theta}|\hat{\varrho}|\boldsymbol{\theta}\rangle = c_N \int \mathrm{d}^N x\,\Psi_{\boldsymbol{\theta}}(x)^\dagger \exp\big(-\mathrm{He}\,\hat{\rho}[\beta]\big)\,\Psi_{\boldsymbol{\theta}}(\boldsymbol{x}) = c_N\,||\Psi_A||^2$$

where we are left with **slowly-varying amplitude modulations of the Bethe ansatz wave functions**:

$$\Psi_A(\boldsymbol{x}) = s(\boldsymbol{x}) \sum_{\sigma\in S_N} (-1)^{|\sigma|} A(\boldsymbol{x}_\sigma, \boldsymbol{\theta})\, e^{\mathrm{i}\Phi_{\boldsymbol{\theta}}(\boldsymbol{x}_\sigma)}, \quad A(\boldsymbol{x},\boldsymbol{\theta}) = \prod_i e^{-\beta(x_i,\theta_i)/2}$$

4. Microscopic origin: slowly varying amplitudes

So the macrocanonical partition function is the "total mass" of a measure on **classical phase space**

$$Z_A = \mathrm{Tr}\hat{\varrho} = \sum_{N=0}^{\infty} c_N \int_{\theta_i > \theta_{i+1}} \mathrm{d}^N\theta \int \mathrm{d}^N x \, \Psi_A^*(\boldsymbol{x})\Psi_A(\boldsymbol{x})$$

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Strategy:

- Evolve the amplitude modulation

$$e^{-\mathrm{i}\hat{H}t}\Psi_A(\boldsymbol{x}) = \Psi_{A_t}(\boldsymbol{x})$$

- Perturb to get the density

$$A_t^{\gamma}(\boldsymbol{x},\boldsymbol{\theta}) = A_t(\boldsymbol{x},\boldsymbol{\theta})e^{\sum_i \bar{\gamma}(x_i/L,\theta_i)/2}, \quad \rho_{\mathrm{p}}(\bar{x},\bar{t};\theta) = \frac{\delta}{\delta\bar{\gamma}(\bar{x},\theta)} \log Z_{A_t^{\gamma}}$$

5. Microscopic origin: gas of wave packets

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Consider the "**Bethe-Fourier transform**"

$$A(\boldsymbol{x},\boldsymbol{\theta}) = \int \mathrm{d}^N\alpha\, A^{\mathrm{BF}}(\boldsymbol{\alpha},\boldsymbol{\theta}) e^{\mathrm{i}\boldsymbol{y}^{\boldsymbol{\theta}}(\boldsymbol{x})\cdot\boldsymbol{\alpha}}$$

which combines the **Fourier transform with the Bethe transform**

$$y_i^{\boldsymbol{\theta}}(\boldsymbol{x}) = \frac{\partial \Phi_{\boldsymbol{\theta}}(\boldsymbol{x})}{\partial \theta_i} = x_i + \frac{1}{2}\sum_j \varphi(\theta_{ij})\,\mathrm{sgn}(x_{ij})$$

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Slowly varying $\Rightarrow$ $A^{\mathrm{BF}}(\boldsymbol{\alpha},\boldsymbol{\theta})$ supported on small values $\alpha_i = O(L^{-1})$. So

$$\Phi_{\boldsymbol{\theta}+\boldsymbol{\alpha}}(\boldsymbol{x}) = \Phi_{\boldsymbol{\theta}}(\boldsymbol{x}) + \boldsymbol{y}^{\boldsymbol{\theta}}(\boldsymbol{x})\cdot\boldsymbol{\alpha} + O(1)$$

and

$$A(\boldsymbol{x},\boldsymbol{\theta}) e^{\mathrm{i}\Phi_{\boldsymbol{\theta}}(\boldsymbol{x}_\sigma)} \asymp \int \mathrm{d}^N\alpha\, A^{\mathrm{BF}}(\boldsymbol{\alpha},\boldsymbol{\theta}) e^{\mathrm{i}\Phi_{\boldsymbol{\theta}+\boldsymbol{\alpha}}(\boldsymbol{x})}$$

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3. Eigenstate decomposition of amplitude modulations (coefficient is $\asymp$ to what is written)

$$\Psi_A = \int \mathrm{d}^N\alpha \, A^{\mathrm{BF}}(\boldsymbol{\alpha}, \theta)\Psi_{\boldsymbol{\theta}+\boldsymbol{\alpha}}$$

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Therefore Lieb-Liniger evolution is simple:

$$\begin{aligned} e^{-\mathrm{i}Ht}\Psi_A &= \int \mathrm{d}^N\alpha\, A^{\mathrm{BF}}(\boldsymbol{\alpha}, \boldsymbol{\theta})e^{-\mathrm{i}(\boldsymbol{\theta}+\boldsymbol{\alpha})^2t/2}\Psi_{\boldsymbol{\theta}+\boldsymbol{\alpha}} \\ &= e^{-\mathrm{i}\boldsymbol{\theta}^2t/2}\int \mathrm{d}^N\alpha\, A^{\mathrm{BF}}(\boldsymbol{\alpha}, \boldsymbol{\theta})e^{-\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{\alpha}\, t}\;\Psi_{\boldsymbol{\theta}+\boldsymbol{\alpha}} \end{aligned}$$

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3. Eigenstate decomposition of amplitude modulations (coefficient is $\asymp$ to what is written)

$$\Psi_A = \int \mathrm{d}^N\alpha \, A^{\mathrm{BF}}(\boldsymbol{\alpha}, \theta) \Psi_{\boldsymbol{\theta}+\boldsymbol{\alpha}}$$

So the **BF transform evolves simply** for all macroscopic times $t = L\bar{t} = O(L)$

$$A_t^{\mathrm{BF}}(\boldsymbol{\alpha}, \boldsymbol{\theta}) = A^{\mathrm{BF}}(\boldsymbol{\alpha}, \boldsymbol{\theta}) e^{-\mathrm{i}\boldsymbol{\theta}\cdot\boldsymbol{\alpha}\, t}$$

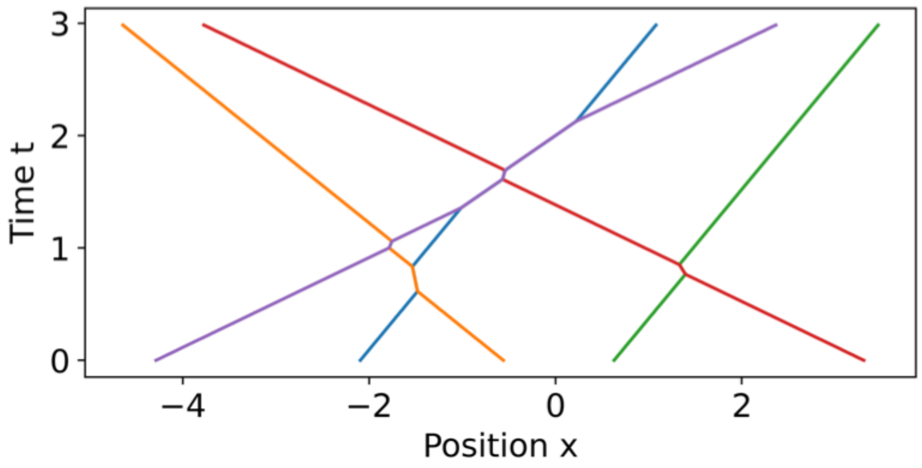
Transforming back to real space, we have **nonlinear trajectories for the coordinates of the amplitude**, $A_t(\boldsymbol{x}, \theta) \asymp A(\boldsymbol{x}(-t), \theta)$ with

$$\boxed{y_i + \theta_i t = x_i(t) + \frac{1}{2} \sum_j \varphi(\theta_{ij}) \, \mathrm{sgn}(x_{ij}(t))}$$

5. Microscopic origin: gas of wave packets

$$y_i + \theta_i t = x_i(t) + \frac{1}{2}\sum_j \varphi(\theta_{ij})\,\mathrm{sgn}(x_{ij}(t))$$

Gas of interacting wave packets which automatically implements the kinetic picture:



5. Microscopic origin: gas of wave packets

Taking the time derivative

$$\theta_i = \dot{x}_i + \sum_j \varphi(\theta_{ij})\delta(x_{ij})(\dot{x}_i - \dot{x}_j)$$

This is **solved using the effective velocity functional**

$$\dot{x}_i = v^{\text{eff}}_{[\rho]}(x_i, \theta_i)$$

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4. GHD equation for the wave packets' phase-space density

$$\rho(\bar{x}, \bar{t}; \theta) = L^{-1} \sum_i \delta(\bar{x} - x_i(t)/L)\delta(\theta - \theta_i)$$

satisfies the GHD equation

$$\partial_{\bar{t}}\rho + \partial_{\bar{x}}(v^{\rm eff}\rho) = 0$$

5. Microscopic origin: gas of wave packets

Finally, by mean-field argument of the explicit integrals defining Z_A (interpreted as a **signed measure on classical phase space**) one argues that

$$\frac{\delta}{\delta\bar{\gamma}(\bar{x},\theta)} \log Z_{A_t^\gamma} \stackrel{L\to\infty}{\to} \rho(\bar{x},\bar{t};\theta)$$

6. New integrable models

[BD, Hübner, Yoshimura in preparation]

6. New integrable models

The classical mechanics of wave packets: the Lieb-Liniger phase $\Phi_{\boldsymbol{\theta}}(\boldsymbol{x})$ is a generating function for the canonical transformation to scattering coordinates

$$y_i = \frac{\partial \Phi_{\boldsymbol{\theta}}(\boldsymbol{x})}{\partial \theta_i}$$

$$p_i = \frac{\partial \Phi_{\boldsymbol{\theta}}(\boldsymbol{x})}{\partial x_i}$$

$$\{y_i, \theta_j\} = \delta_{ij} \qquad \Leftrightarrow \qquad \{x_i, p_j\} = \delta_{ij}$$

$$H = \frac{\boldsymbol{\theta}^2}{2} = \frac{\boldsymbol{p}^2}{2} + V(\boldsymbol{x}, \boldsymbol{p})$$

The model is **Liouville integrable**, has **elastic, factorised scattering** and has **local interaction** (the potential $V(\boldsymbol{x}, \boldsymbol{p})$ depends on both $\boldsymbol{x}$ and $\boldsymbol{p}$ in a complicated way...).

6. New integrable models

- This works for "any" $\Phi_{\boldsymbol{\theta}}(\boldsymbol{x})$ of the Bethe ansatz form.
- We can show that the free energy of these models is given by the (classical) TBA.
- We can evaluate e.g. the real momentum distribution

$$\mathrm{distribution}(p) = n(\theta(p))$$

  in terms of the occupation function and of the inverse of the Dressed momentum.
- These occur as $T\bar{T}$ deformations of free particles.

Conclusion

Conclusion

We have:

- Derived the GHD equations from the microscopic evolution in LL model
- Explained the kinetic picture in terms of a gas of wave packets

What more can be done with the gas of wave packets?

- Introduce external force (we think we know how...)
- Study correlations and fluctuations (macroscopic fluctuations?)
- Derive the diffusive terms, perhaps higher-order hydrodynamics (dispersion, all orders?)
- Apply similar ideas to quantum chains