ODE/IM correspondence: Wall-Crossing of TBA equations

Katsushi Ito (Tokyo Institute of Technology)

10th Bologna Workshop on Conformal Field Theory and Integrable Models
@University of Bologna
September 6, 2023

arXiv:1811.04812 (w/ Marcos Mariño, Hongfei Shu)
arXiv:1910.09406 (w/ H. Shu)
arXiv:2004.09856, 2104.13680(w/ Takayasu Kondo, Kohei Kuroda, H. Shu)
arXiv:2111.11047 (w/ T. Kondo, H. Shu)
arXiv:2305.03283, 2206.08024 (w/ Mingshuo Zhu)

Introduction

WKB analysis of the linear problem for the modfied ATFT equations

exact WKB periods and TBA equations

Outlook

Introduction

ODE/IM correspondence

[Dorey-Tateo 9812211, Bazhanov-Lukyanov-Zamolodchikov 9812247, ...]

- a relation between spectral analysis approach of ordinary differential equation (ODE), and the "functional relations" approach to 2d quantum integrable model (IM).
- non-trivial correspondence between classical and quantum integrable models.

ODE/IM (2)

the Schrödinger equation with centrifugal potential term

$$\left[-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + x^{2M} - E\right] y(x, E, \ell) = 0$$

ODE	XXZ
order of the potential $2M$	anisotropy parameter η
angular momentum ℓ	twist parameter ϕ
energy E	spectral parameter θ

$$\eta = \frac{\pi M}{2M+2}, \quad \phi = \frac{\pi(2\ell+1)}{2M+2}, \quad E = e^{\frac{2M\theta}{M+1}}$$

ODE/IM (3)

$$\left[-\frac{d^2}{dx^2}+\frac{\ell(\ell+1)}{x^2}+x^{2M}-E\right]y(x,E,\ell)=0$$

$x=0$: regular singularity

$x=\infty$: irregular singularity

ODE	IM
Connection coefficients between 0 and ∞	Q-function (T-Q relation)
Stokes coefficients $=$Wronskians	T-functions
Plücker relation	T-system
Voros symbols (exact WKB periods)	Y-functions
DDP discontinuity formula	Y-system, TBA equation

exact WKB method

the Schrödinger equation ($m = 1/2$)

$$-\hbar^2\psi''(q) + (V(q) - E)\psi(q) = 0.$$

WKB solution: $\psi(q) = \exp\left[\frac{i}{\hbar}\int^q Q(q')dq'\right].$

Riccati equation:

$$Q^2(q) - i\hbar\frac{dQ(q)}{dq} = p^2(q), \quad p(q) = (E - V(q))^{1/2},$$

$$Q(q) = \sum_{k=0}^{\infty} Q_k(q)\hbar^k = Q_{\text{even}} + Q_{\text{odd}}$$

$$P(q) = Q_{\text{even}} = \sum_{n\geq 0} p_n(q)\hbar^{2n}, \quad Q_{\text{odd}} = \frac{i\hbar}{2}\frac{d}{dq}\log P(q)$$

$p_0(q) = p(q)$ and $p_n(q)$ are determined recursively.

WKB periods and Voros symbols

potential: polynomial in q

$$V(q) = q^{r+1} + u_1 q^r + \cdots + u_r q$$

the WKB curve: $\Sigma_{\mathrm{WKB}} : y^2 = E - V(q)$.

WKB periods (*quantum periods*)

$$\Pi_\gamma(\hbar) = \oint_\gamma P(q)dq = \sum_{n=0}^{\infty} \hbar^{2n} \Pi_\gamma^{(n)}, \quad \Pi_\gamma^{(n)} = \oint_\gamma p_n(q)dq \qquad \gamma \in H_1(\Sigma_{\mathrm{WKB}}).$$

- $\Pi_\gamma^{(n)} \sim (2n)!$, $\Pi_\gamma(\hbar)$: asymptotic series in $\hbar$.
- exact WKB periods: $s(\Pi_\gamma)$: Borel resummation of Π_γ
- *Voros symbol* $\mathcal{V}_\gamma = \exp\left(\frac{i}{\hbar} s(\Pi_\gamma)\right)$

Borel resummation

[Ecalle, Voros, Delabaere-Pham, Dillinger-Delabaere-Pham, Sato-Kawai-Aoki-Takei, ...]

asymptotic series | analytic function

$$\boxed{\phi(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}} \quad \xrightarrow{\text{Borel transf.}} \quad \boxed{\tilde{\phi}(\xi) = \sum_{n=0}^{\infty} c_n \frac{\xi^n}{n!}}$$

Asym. exp. $\nwarrow$ | $\swarrow$ Laplace transf.

$$\boxed{s[\phi](z) = \int_0^{\infty} d\xi e^{-\xi z} \tilde{\phi}(\xi)}$$

Borel resummation

Voros symbols and Y-functions

$$\mathcal{V}_\gamma = \exp\left(\frac{i}{\hbar} s(\Pi_\gamma)\right) \Leftrightarrow Y_\gamma(\theta) = \exp(\epsilon_\gamma(\theta))$$

- the WKB periods $s(\Pi_\gamma)(\theta)$ show the discontinuity in $\theta(= -\log \hbar)$-plane [Delabaere-Dillinger-Pham, Iwaki-Nakanishi]
- In the limit $\theta \to \infty$, $s(\Pi_\gamma)(\theta) \to \Pi_\gamma^0$: classical WKB period
- Introducing pseudo energies $\epsilon_\gamma(\theta) = \frac{i}{\hbar} s(\Pi_\gamma)(\theta)$, $\epsilon_\gamma(\theta)$ is shown to obey the Non-linear Integral equation (TBA equation).[I-Mariño-Shu]
- T/Y-functions are generating function of the quantum integrals of motion.

TBA and 4d SUSY gauge theory

- ODE also appears in the context of 4-dim $\mathcal{N} = 2$ SUSY gauge theories in the $\frac{1}{2}\Omega$-background [Nekrasov-Shatashivili] as the quantum SW curve [Mironov-Morozov]. The WKB periods satisfies the TBA equations [Gaiotto-Moore-Neitzke, Gaiotto].
- In particular, the linear system for MATFT is regarded as the quantum SW curve for the Argyres-Douglas theories. [I-Shu].
- A class of AD theories is classified by (G, G') [Cecotti-Neitzke-Vafa,Xie,Wang-Xie], where linear system associated with G and degree $N+1$ polynomial p corresponds to (G, A_N)-type AD theory.
- Y-functions Y_γ are associated with the basis of the BPS states, or the set of cycles on the SW curve.
- Wall-crossing phenomena, which changes the basis of the BPS states, introduces the change of TBA system.

exact WKB for higher order ODE

exact WKB periods and IM for higher order ODE?

- Higher order ODE appears naturally in the context of SW theory.
- The exact WKB analysis for higher order ODE has been started by Berk-Nevins-Roberts and Aoki-Kawai-Takei, where the turning points are simple (first order zero).
- In the context of the ODE/IM correspondence, there are many examples of higher order ODE [Dorey-Dunning-Tateo, Suzuki, Dorey-Dunning-Masoero-Suzuki-Tateo] and the 1st order linear differential systems [Sun, I-Locke,Masoero et al.] associated with affine Lie algebras. These turning points are non-simple.

affine Toda field equation and linear problem

affine Toda field equation: $\varphi(z,\bar{z})$: r-component scalar fields

$$\partial_a \partial_{\bar{z}} \varphi(z,\bar{z}) + \frac{m^2}{\beta^2} \sum_{i=0}^{r} \exp(\beta \alpha_i \cdot \varphi(z,\bar{z})) = 0$$

$\Longleftrightarrow$Lax form $[\partial_z + A_z, \partial_{\bar{z}} + A_{\bar{z}}] = 0$

$$A_z = \frac{\beta}{2} \partial \varphi \cdot H + m e^{\lambda} \sum_{i=0}^{r} E_{\alpha_i} e^{\frac{\beta}{2} \alpha_i \cdot \varphi}$$

$$A_{\bar{z}} = -\frac{\beta}{2} \partial_{\bar{z}} \varphi \cdot H - m e^{-\lambda} \sum_{i=0}^{r} E_{-\alpha_i} e^{\frac{\beta}{2} \alpha_i \cdot \varphi}$$

λ: spectral parameter

$\Longleftrightarrow$ compatibility condition of the linear problem

$(\partial_z + A_z)\psi = 0$, $(\partial_{\bar{z}} + A_{\bar{z}})\psi = 0$

modified affine Toda field equation

conformal transformation $z \to f(z)$, $\varphi \to \tilde{\varphi} = \varphi - \frac{1}{\beta}\rho^\vee \log\bigl(\partial_z f \partial_{\bar{z}} \bar{f}\bigr)$

modified affine Toda field equation

$$\partial_a \partial_{\bar{z}} \varphi + \frac{m^2}{\beta^2}\left\{\sum_{i=1}^{r} \exp(\beta\alpha_i \cdot \varphi) + p(z)\bar{p}(\bar{z})\alpha_0 e^{\beta\alpha_0\cdot\varphi}\right\} = 0$$

linear problem $(\partial_z + A_z)\psi = 0$, $(\partial_{\bar{z}} + A_{\bar{z}})\psi = 0$

$$A = \frac{\beta}{2}\partial\phi \cdot H + m e^{\lambda}\left\{\sum_{i=1}^{r} E_{\alpha_i} e^{\frac{\beta}{2}\alpha_i\phi} + p(z) E_{\alpha_0} e^{\frac{\beta}{2}\alpha_0\phi}\right\},$$

$$\bar{A} = -\frac{\beta}{2}\bar{\partial}\phi \cdot H - m e^{-\lambda}\left\{\sum_{i=1}^{r} E_{-\alpha_i} e^{\frac{\beta}{2}\alpha_i\phi} + \bar{p}(\bar{z}) E_{-\alpha_0} e^{\frac{\beta}{2}\alpha_0\phi}\right\}.$$

- $p(z) = (\partial f)^h$: a polynomial in z $p(z) = z^{hM} - s^{hM}$
- boundary condition of φ
  $\varphi \sim M\rho^\vee \log|z|^2 + \cdots$ for large $|z|$
  $\varphi \sim g \log|z|^2 + \cdots$ for small $|z|$

Linear differential equation for affine Lie algebra

conformal limit: $z, \bar{z} \to 0$, $x = (me^{\lambda})^{\frac{1}{M+1}} z$,$E = s^{hM}(me^{\lambda})^{\frac{hM}{M+1}}$ fixed

linear problem $\mathcal{L}\Psi = 0$

$$\mathcal{L} = \frac{d}{dx} - \frac{1}{x} g \cdot H + \sum_{a=1}^{r} \sqrt{n_a^{\vee}} E_{\alpha_a} + \sqrt{n_0^{\vee}} p(x,E) E_{\alpha_0}$$

- $p(x,E) = x^{hM} - E$ (h: Coxeter number)
- Ψ takes values in a representation V of $\mathfrak{g}$.
- $x = 0$ (regular sing.) , $x = \infty$ (irregular sing.)
- invariant under the Sibuya rotation $(x, E) \to (\omega^k x, \Omega^k E)$
  If $\Psi(x, E, g)$ is a solution,

$$\Psi_{[k]}(x, E, g) = \omega^{-k\rho^{\vee} \cdot H} \Psi(\omega^k x, \Omega^k E, g)$$

  is also a solution. $\omega = \exp\left(\frac{2\pi i}{h(M+1)}\right)$, $\Omega = \exp\left(\frac{2\pi i M}{M+1}\right)$

Example

- $A_1^{(1)}$

$$\left[\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} - \frac{l_1}{x} & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}x} + \frac{l_1}{x} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ p(x,E) & 0 \end{pmatrix}\right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0\,.$$

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{l_1(l_1-1)}{x^2} + x^{2M} - E\right]\psi_1 = 0.$$

Asymptotic solution around the singularities

$x = \infty$, gauge transformation $\Psi'(x) = U(x)\Psi(x)$, $\mathcal{L}' = U\mathcal{L}U^{-1}$

$$\mathcal{L}' = \frac{d}{dx} + p(x)^{\frac{1}{h}}\Lambda_+ + \cdots, \quad \Lambda_+ = \sum_{a=1}^{r} E_{\alpha_a} + \zeta E_{\alpha_0}$$

$$\Psi^{WKB}(x) = \sum_{i=1}^{n=\dim V} C_i \exp\left(-\nu_i \int^x p^{\frac{1}{h}}(x')dx' - \frac{1}{h}\log p(x)\rho^\vee \cdot H\right)\boldsymbol{\nu}_i$$

Subdominant solution

$$\Psi = \sim C \exp\left(-\mu \int^x p^{\frac{1}{h}}(x')dx' - \frac{1}{h}\log p(x)\rho^\vee \cdot H\right)\boldsymbol{\nu}$$

$\Lambda_+\boldsymbol{\nu}_i = \nu_i\boldsymbol{\nu}_i$, μ: eigenvalue with the largest real part

Asymptotic solution around the singularities (2)

$x = 0$, regular singularity
$\mathcal{L} = \frac{d}{dx} - \frac{g \cdot H}{x} + \cdots$

$$\mathcal{X}_i = x^{-h_i \cdot g} \mathbf{e}_i + \cdots$$

$\mathbf{e}_i$: weight vector of V with weight h_i
($\mathbf{e}_1$: highest weight vect., $\mathbf{e}_n$: lowest weight vect.)

Q-function

Fix a representation $V = V^{(a)}$: a-th fundamental representation with highest weight ω_a:

$\{\Psi_{[k]}\}$: a basis of the solutions around $x = \infty$

$\{\mathcal{X}_i\}$: a basis of the solutions around $x = 0$

$$\Psi(x, E, g) = \sum_{i=1}^{\mathrm{dim} V} \mathcal{Q}_i^{(a)}(E, g)\mathcal{X}_i(x, E, g)$$

$Q^{(a)}(E, g) = \mathcal{Q}_1^{(a)}(E, g)$: Q-function (associated with the representation $V^{(a)}$)

ψ-system and BAE

A_r: $V^{(a)} = V^{(1)} \wedge \cdots \wedge V^{(1)}$

$V^{(a)} \wedge V^{(a)} \hookrightarrow V^{(a-1)} \otimes V^{(a+1)} = \bigotimes_{b=1}^{r} (V^{(b)})^{2\delta_{ab} - C_{ab}}$

- ψ-system[Dorey-Dunning-Masoero-Suzuki-Tateo, Sun,. . .]

$$\Psi^{(a)}_{[-\frac{1}{2}]} \wedge \Psi^{(a)}_{[-\frac{1}{2}]} = \bigotimes_{b=1}^{r} (\Psi^{(b)})^{2\delta_{ab} - C_{ab}}$$

- quantum Wronskian relations

$$\omega^{-\frac{1}{2}\left(\lambda_1^{(a)} - \lambda_2^{(a)}\right)} \mathcal{Q}^{(a)}_{1[-1/2]} \mathcal{Q}^{(a)}_{2[1/2]} - \omega^{\frac{1}{2}\left(\lambda_1^{(a)} - \lambda_2^{(a)}\right)} \mathcal{Q}^{(a)}_{1[1/2]} \mathcal{Q}^{(a)}_{2[-1/2]} = \prod_{b=1}^{r} \left[\mathcal{Q}^{(b)}_1\right]^{2\delta_{ab} - C_{ab}}.$$

- Bethe Ansatz equation for the Langlands dual $\hat{\mathfrak{g}}^{\vee}$

$$\prod_{b=1}^{r} \Omega^{C_{ab}\gamma_b/2} \left. \frac{Q^{(b)}_{[C_{ab}/2]}}{Q^{(b)}_{[-C_{ab}/2]}} \right|_{E_i^{(a)}} = -1, \qquad i = 0, 1, \ldots,$$

$\gamma_a = \frac{2}{hM} \omega_a (g - \rho^{\vee})$

dual representation and the Q-function

$$\Psi(x,E,g)=\sum_{i=1}^{n}\mathcal{Q}_i(E,g)\mathcal{X}_i(x,E,g)$$

- Wronskian type formula $V^{(a)}=V^{(1)}\wedge\cdots\wedge V^{(1)}$

$$\mathcal{Q}_i(E,g)=\frac{\det(\mathcal{X}_1,\ldots,\Psi,\ldots,\mathcal{X}_n)}{\det(\mathcal{X}_1,\ldots,\mathcal{X}_n)}$$

- inner product formula [I-Kondo-Kuroda-Shu]
  V^*: dual representation, $\mathcal{L}^{dual}\bar{\Psi}=0$, $\mathcal{L}^{dual}=\partial+\bar{A}$
  $\langle\bar{\Psi},\Psi\rangle$ is x-independent
  $\bar{\mathcal{X}}_i$: the basis of the solutions of the dual problem around $x=0$ such that $\langle\bar{\mathcal{X}}_i,\mathcal{X}_j\rangle=\delta_{ij}$.

$$\mathcal{Q}_i(E,g)=\langle\bar{\mathcal{X}}_i,\Psi\rangle$$

NLIE and ODE

Q-function from ODE

$\hat{\mathfrak{g}}$ and the representation $V^{(a)}$

- $Q^{(a)}(E,g) = \langle \bar{\mathcal{X}}_1, \Psi^{(a)} \rangle$
- Solve the dual problem $\mathcal{X}_i^{(a)}$ around the origin (Cheng's algorithm)
- $\Psi^{(a)}_{WKB} = (\psi_1, \ldots, \psi_n)$: most divergent component is the lowest weight component ψ_n
- Use truncated solution and evaluate the Q-function at $x = x_{fixed} \gg 1$

  $$\mathcal{Q}_i(E,l) \ \sim \ \bar{\chi}_{i,n}(x_{\text{fixed}}, E, l) \ \psi_n(x_{\text{fixed}}) \qquad \text{for } x_{\text{fixed}} \gg 1.$$

  the lowest weight component of $(\mathcal{X}_1)^{(a)}_n(x_{fixed}, E) = 0 \Longrightarrow$ solution to $Q^{(a)}(E,g) = 0$, $E_i^{(a)}$

NLIE [Zinn-Justin]

counting function

$$\mathfrak{a}^{(a)}(E) \coloneqq \prod_{b=1}^{r} \Omega^{-C_{ab}\gamma_b/2} \frac{Q^{(b)}_{[-C_{ab}/2]}(E)}{Q^{(b)}_{[C_{ab}/2]}(E)}, \qquad a = 1, \ldots, r.$$

BAE eq. $\mathfrak{a}^{(a)}(E^{(a)}_k) = -1.$

$$\ln \mathfrak{a}^{(a)}(\theta) = i\pi\hat{\alpha}_a - ib_0 M_a e^{\theta} + \sum_{b=1}^{r} \int_{\mathcal{C}_1} \mathrm{d}\theta' \, \varphi_{ab}(\theta - \theta') \ln\Big(1 + \mathfrak{a}^{(b)}(\theta')\Big)$$

$$- \sum_{b=1}^{r} \int_{\mathcal{C}_2} \mathrm{d}\theta' \, \varphi_{ab}(\theta - \theta') \ln\left(1 + \frac{1}{\mathfrak{a}^{(b)}(\theta')}\right),$$

$$\varphi_{ab}(\theta) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} e^{ik\theta} \left(\delta_{ab} - \frac{\sinh(\mu\pi k)}{\sinh((h\mu - 1)\pi k/h)\cosh(\pi k/h)}\right) C^{-1}_{ab}(k)),$$

$$C_{ab}(k) \coloneqq \begin{cases} 2, & a = b, \\ \dfrac{C_{ab}}{\cosh(\pi k/h)}, & a \neq b, \end{cases}$$

Numerical Test

Bethe roots for $E_6^{(1)}$ $P = x^2 - E$,
$g = (5/12, 1/3, 0, -1/3, -5/12, 1/10)$.

	ODE					
i	$E_i^{(1)}$	$E_i^{(2)}$	$E_i^{(3)}$	$E_i^{(4)}$	$E_i^{(5)}$	$E_i^{(6)}$
0	26.16492	19.04286	16.95324	19.98072	29.04567	21.54848
1	76.14709	37.06396	28.48668	38.50821	80.49209	52.00437
2	146.8766	64.52501	44.61201	66.15524	152.5322	93.90413
3	236.0037	95.98720	63.83908	97.97408	242.8648	145.7213

	IM					
i	$E_i^{(1)}$	$E_i^{(2)}$	$E_i^{(3)}$	$E_i^{(4)}$	$E_i^{(5)}$	$E_i^{(6)}$
0	26.16452	19.04232	16.95299	19.98020	29.04519	21.54807
1	76.14715	37.06297	28.48688	38.50782	80.49126	52.00351
2	146.8773	64.52390	44.61186	66.15433	152.5313	93.90137
3	236.0021	95.98496	63.83727	97.97372	242.8597	145.7216

Effective central charge and 2d/4d correspondence

- effective central charge

$$c_{eff} = 2 \times \frac{3}{\pi^2} \sum_{a}^{r} i b_0 M_a \left[\int_{\mathcal{C}_1} \mathrm{d}\theta \, e^{\theta} \log\Big(1 + \mathfrak{a}^{(a)}(\theta)\Big) - \int_{\mathcal{C}_2} \mathrm{d}\theta \, e^{\theta} \log\Big(1 + \frac{1}{\mathfrak{a}^{(a)}(\theta)}\Big) \right]$$

$$= r - \frac{12}{(M+1)h^2}(-g + \rho^{\vee})^2$$

- $g = 0$, $p = x^2 - E$, $c_{eff} = r - \frac{\dim \mathfrak{g}}{h+2}$
  - (A_{2n}, A_1) $c_{eff} = \frac{2n}{2n+3}$
  - (E_6, A_1) $c_{eff} = \frac{6}{7}$
  - (E_8, A_1) $c_{eff} = \frac{1}{2}$

  c_{eff} agrees with that from the Schur index of (G, A_1)-type AD theory [Cordova-Shao] through $c_{4d} = -\frac{1}{12} c_{2d}$. [Beem et al.]

WKB analysis of the linear problem for the modfied ATFT equations

$A_r^{(1)}$-type linear problem and its abelianization

$\mathcal{L}_m \psi = 0$

$$\left\{ \partial_z + \begin{pmatrix} \partial_z\phi_1 & \lambda & & & \\ & (\partial_z\phi_2 - \partial_z\phi_1) & \lambda & & \\ & & \ddots & & \\ & & & (\partial_z\phi_r - \partial_z\phi_{r-1}) & \lambda \\ \lambda p(z) & & & & -\partial_z\phi_r \end{pmatrix} \right\} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_r \\ \psi_{r+1} \end{pmatrix} = 0.$$

ODE for ψ_1:

$$(-\lambda)^{-h}(\partial_z - \partial_z\phi_r)(\partial_z + \partial_z\phi_r - \partial_z\phi_{r-1}) \dots (\partial_z + \partial_z\phi_2 - \partial_z\phi_1)(\partial_z + \partial_z\phi_1)\psi_1 = p(z)\psi_1$$

with $h = r + 1$.

$D_r^{(1)}$-type linear problem

$$\left\{\partial_z + \begin{pmatrix} \partial\boldsymbol{\phi}_1 & \lambda & & & & & & & & & \\ & \partial\boldsymbol{\phi}_1 & \lambda & & & & & & & & \\ & & \ddots & & & & & & & & \\ & & & \partial\boldsymbol{\phi}_{r-1} & \lambda & \lambda & & & & & \\ & & & & \partial\boldsymbol{\phi}_r & & \lambda & & & & \\ & & & & & -\partial\boldsymbol{\phi}_r & \lambda & & & & \\ & & & & & & -\partial\boldsymbol{\phi}_{r-1} & \lambda & & & \\ & & & & & & & \ddots & & & \\ \lambda p(z) & & & & & & & & -\partial\boldsymbol{\phi}_2 & \lambda \\ & \lambda p(z) & & & & & & & & -\partial\boldsymbol{\phi}_1 \end{pmatrix}\right\}\psi = 0$$

$$\lambda^{-h} D(-\boldsymbol{\phi}_1)\cdots D(-\boldsymbol{\phi}_r)\partial_z^{-1} D(\boldsymbol{\phi}_r)\cdots D(\boldsymbol{\phi}_1)\psi_1 - 4\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi_1 = 0,$$

$h = 2r - 2.$

$A_r^{(1)}$	$D(\mathbf{h})\psi = (-me^{\lambda})^h p(z)\psi$
$D_r^{(1)}$	$D(\mathbf{h}^\dagger)\partial^{-1}D(\mathbf{h})\psi = 2^{r-1}(me^{\lambda})^h\sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$B_r^{(1)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 2^{r}(me^{\lambda})^h\sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$A_{2r-1}^{(2)}$	$D(\mathbf{h}^\dagger)D(\mathbf{h})\psi = -2^{r-1}(me^{\lambda})^h\sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$C_r^{(1)}$	$D(\mathbf{h}^\dagger)D(\mathbf{h})\psi = (me^{\lambda})^h p(z)\psi$
$D_{r+1}^{(2)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 2^{r+1}(me^{\lambda})^{2h} p(z)\partial^{-1}p(z)\psi$
$A_{2r}^{(2)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = -2^{r}\sqrt{2}(me^{\lambda})^h p(z)\psi$
$G_2^{(1)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi = 8(me^{\lambda})^h\sqrt{p(z)}\partial\sqrt{p(z)}\psi$
$D_4^{(3)}$	$D(\mathbf{h}^\dagger)\partial D(\mathbf{h})\psi + (\omega+1)2\sqrt{3}(me^{\lambda})^4 D(\mathbf{h}^\dagger)p(z)$ $-(\omega+1)2\sqrt{3}(me^{\lambda})^4 pD(\mathbf{h}) - 8\sqrt{3}\omega(me^{\lambda})^3 D(-h_1)\sqrt{p}\partial\sqrt{p}D(h_1)$ $+(\omega-1)^3 12(me^{\lambda})^8 p\partial^{-1}p\}\psi = 0$

$D(\mathbf{h}) = D(h_r)\cdots D(h_1)$, $D(\mathbf{h}^\dagger) = D(-h_1)\cdots D(-h_r)$ for $\mathbf{h} = (h_r, \cdots, h_1)$
$D(h) := \partial + \frac{h}{x}$.

abelianization vs diagonalization

linear problem in a representation V of $\mathfrak{g}$

- **abelianization** ODE for the the highest state component ψ_1
  It can contain the pseudo-differential operator ∂^{-1}
  For exceptional Lie algebra, the abelianized ODE is highly complicated.
- **diagonalization** Diagonalization of the linear problem in the eigenspace of Λ. [I-Zhu]

The linear problem can be diagonalized and the diagonal elements are classical conserved currents [Zakharov-Shabat 1974,Drinfeld-Sokolov (1984)].

$$\mathcal{L} = \partial_z + q(z,\lambda) + \lambda\Lambda,$$

$$\mathcal{L}_0 = T\mathcal{L}T^{-1} = \partial_z + \lambda\Lambda + \sum_{i=0}^{\infty} \lambda^{-i} I_i(z) \Lambda^{-i}$$

Diagonalization of the Lax operator

$$\epsilon\mathcal{L}_m = \epsilon\partial_z + A(z) = \epsilon\partial_z + \epsilon\sum_{i=1}^{r}\partial_z\phi_i(z)H_i + \sum_{i=1}^{r}E_{\alpha_i} + p(z)E_{\alpha_0}.$$

step-by-step gauge transformation: [Babelon-Bernard-Talon]

$$\mathbf{Gau}_T[A(z)] = T^{-1}(z)A(z)T(z) + \epsilon T^{-1}(z)\partial_z T(z).$$

$$T(z) = T_d T_{d-1}\ldots T_3 T_2 T_1,$$

$$T_i(z) = \begin{pmatrix} 1 & & & & & & \\ & & \ddots & & & & \\ g_{i,1} & g_{i,2} & \ldots & 1 & g_{i,i+1} & \ldots & g_{i,d} \\ & & & & \ddots & & \\ 0 & & & & & & 1 \end{pmatrix}$$

Diaginalization (2)

$$\mathbf{Gau}_{T_d}[A(z)] = \begin{pmatrix} \\ \\ \mathbf{Gau}_{T_d}[A(z)]_{d,1} & \dots & \mathbf{Gau}_{T_d}[A(z)]_{d,d-1} & \mathbf{Gau}_{T_d}[A(z)]_{d,d} \end{pmatrix}$$

Eliminate $g_{d,1}, \dots, g_{d,d-1}$ by solving
$\mathbf{Gau}_{T_d}[A(z)]_{d,1} = \cdots = \mathbf{Gau}_{T_d}[A(z)]_{d,d-1} = 0$

$$A_{\mathsf{diag}}(z) = \mathbf{Gau}_{T_1} \circ \mathbf{Gau}_{T_2} \dots \mathbf{Gau}_{T_{d-2}} \circ \mathbf{Gau}_{T_{d-1}} \circ \mathbf{Gau}_{T_d}[A(z)].$$

In the diagonalized form, the linear problem $(\epsilon\partial + A_{\mathsf{diag}}(z))\Psi(z) = 0$ can be solved exactly:

$$\Psi(z) = diag\left[\exp\left(-\frac{1}{\epsilon}\int dz A_{\mathsf{diag}}(z)\right)\right].$$

diagonalization of $A_1^{(1)}$ linear problem

$$\mathcal{L}_m = \epsilon\partial_z + \epsilon\partial_z\phi(z)H + E_\alpha + p(z)E_{-\alpha}$$

with

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Gauge transformation $\mathcal{L} \to \mathcal{L}' \equiv T^{-1}\mathcal{L}T$ by $T(z) = T_1 T_2$:

$$T_2(z) = \begin{pmatrix} 1 & 0 \\ g_{2,1}(z,\epsilon) & 1 \end{pmatrix}, \quad T_1(z) = \begin{pmatrix} 1 & g_{1,2}(z,\epsilon) \\ 0 & 1 \end{pmatrix}.$$

$T_2(z)$ is determined to diagonalize the second row

$$\mathbf{Gau}_{T_2}[A(z)] = \begin{pmatrix} g_{2,1} + \phi' & 1 \\ -2\epsilon g_{2,1}\epsilon\phi' + \epsilon g_{2,1}' - g_{2,1}^2 + p & -g_{2,1} - \epsilon\phi' \end{pmatrix}.$$

$$g_{2,1}^2(z,\epsilon) + 2\epsilon g_{2,1}(z,\epsilon)\phi'(z,\bar{z}) - \epsilon g_{2,1}'(z,\epsilon) - p(z) = 0.$$

The diagonal element $f(z,\epsilon) := -g_{2,1}(z,\epsilon) - \epsilon\phi'(z)$ satisfies

$$f^2(z,\epsilon) + \epsilon f'(z,\epsilon) - \epsilon^2 u_2(z) - p(z) = 0$$

with $u_2(z) = \phi'(z)^2 - \phi''(z)$. After the diagonalization of the second row, the second gauge transformation $T_1(z)$ gives

$$A_{\mathsf{diag}}(z) = \mathbf{Gau}_{T_1} \circ \mathbf{Gau}_{T_2}[A(z)] = \begin{pmatrix} -f(z,\epsilon) & 1 - 2g_{1,2}(z,\epsilon)f(z,\epsilon)(=0) \\ 0 & f(z,\epsilon) \end{pmatrix}.$$

- $f(z,\epsilon) = \sum_{i=0}^{\infty} \epsilon^i f_i(z)$
  $f_{odd}(z)$ total derivative

$$A_{\mathsf{diag}}(z) = \begin{pmatrix} -f(z,-\epsilon) + d(*) & 0 \\ 0 & f(z,\epsilon) \end{pmatrix},$$

- $f(z,\epsilon)$ is the Riccati equation for the ODE:

$$\left(\epsilon^2\partial_z^2 + \epsilon^2\partial_z^2\phi(z) - \epsilon^2(\partial_z\phi)^2 - p(z)\right)\psi(z,\epsilon) = 0.$$

Conserved charges vs WKB periods

conformal transformation:

$$dw = \sqrt{p(z)}dz, \quad \hat{u}_2\big(w(z)\big) = \frac{1}{p(z)}\Big[u_2(z) + \frac{4pp'' - 5p'^2}{16p^2}\Big],$$

$\hat{f}(z,\epsilon)dw = f(z,\epsilon)dz$

$$\hat{f}_2(w) = \frac{\hat{u}_2(w)}{2},$$

$$\hat{f}_4(w) = \frac{\partial_w^2 \hat{u}_2(w) - \hat{u}_2^2(w)}{8}, \dots$$

These are the classical conserved current densities of the sine-Gordon models.

$$\Pi_i \equiv \oint dz\ f_i(z) = \oint dz\ \sqrt{p(z)}\hat{f}_i = \oint dw\ \hat{f}_i \equiv \mathcal{Q}_i.$$

The quantum period is the generating function of the classical/quantum conserved charges. The classical/quantum correspondence

$A_r^{(1)}$

- diagonalization

$$A_{\mathsf{diag}}(z) = \textbf{Diag}\{e^{-\frac{2\pi i}{h}} f(z, e^{\frac{2\pi i}{h}}\epsilon) + d(*), \ldots, e^{-\frac{2\pi i r}{h}} f(z, e^{\frac{2\pi i r}{h}}\epsilon) + d(*), f(z,\epsilon)\}.$$

$h = r + 1$

- The traceless condition

$$\sum_{n=0}^{h-1} e^{-\frac{2\pi i n}{h}} f(x, e^{\frac{2\pi i n}{h}}\epsilon) = d(*),$$

the $(1 + hk)$-th term $(k \in \mathbf{Z})$ is the total derivative.

- $f(z,\epsilon)$ satisfies the Riccati equation for :

$$(-\epsilon)^h(\partial_z - \partial_z\phi_1)(\partial_z - \partial_z\phi_2 + \partial_z\phi_1)\cdots(\partial_z + \partial_z\phi_r)\psi(z,\epsilon) = p(z)\psi(z,\epsilon),$$

which is the adjoint ODE of the abelianization problem:

$$(-\lambda)^{-h}(\partial_z - \partial_z\phi_r)(\partial_z + \partial_z\phi_r - \partial_z\phi_{r-1})\ldots(\partial_z + \partial_z\phi_2 - \partial_z\phi_1)(\partial_z + \partial_z\phi_1)\psi_1 = p(z)\psi_1$$

The diagonalization procedure is also applied to other classical affine Lie algebras $D_r^{(1)}$, $B_r^{(1)}$, $D_{r+1}^{(2)}$, and $A_{2r-1}^{(2)}$.

WKB analysis of 3rd order ODE

$$\left[\epsilon^3\partial_x^3 + p(x)\right]\psi(x) = 0$$

$$\psi(x) = \exp\left(\frac{1}{\epsilon}\int^x P(x';\epsilon)dx'\right), \quad P(x;\epsilon) = \sum_{n=0}^{\infty}\epsilon^n p_n(x).$$

Riccati equation:

$$\epsilon^2\partial^2 P + 3\epsilon P\partial P + P^3 + p(x) = 0$$

$$\begin{aligned}
p_0 &= (-p)^{1/3}, \\
p_1 &= -\frac{p_0'}{p_0}, \\
p_2 &= -\frac{(p_0')^2}{p_0^3} + \frac{2}{3}\frac{p_0''}{p_0^2}, \\
p_3 &= -\frac{2(p_0')^3}{p_0^5} + \frac{2p_0'p_0''}{p_0^4} - \frac{1}{3}\frac{p_0^{(3)}}{p_0^3}, \\
p_4 &= -\frac{4(p_0')^4}{p_0^7} + \frac{16}{3}\frac{(p_0')^2p_0''}{p_0^6} - \frac{2}{3}\frac{(p_0'')^2}{p_0^5} - \frac{10}{9}\frac{p_0'p_0^{(3)}}{p_0^5} + \frac{1}{9}\frac{p_0^{(4)}}{p_0^4}.
\end{aligned}$$

- p_{odd}: total derivative
- $p_{6n+4} = p_4, p_{10}, \ldots$: total derivative

  $p_4 = \partial \left(\frac{2}{3} \frac{(p_0')^3}{p_0^6} - \frac{2}{3} \frac{p_0' p_0^{(2)}}{p_0^5} + \frac{1}{9} \frac{p_0^{(3)}}{p_0^4} \right)$

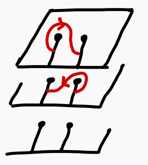
  This is consistent with the TBA equations and also order of the integral of motions in IM.

WKB periods

$$\Pi_\gamma(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n \Pi_\gamma^{(n)}, \quad \Pi_\gamma^{(n)} = \int_\gamma p_n(x) dx$$

WKB curve: super elliptic curve

$$y^3 = -p(x), \quad p(x) = u_0 x^{N+1} + \cdots + u_{N+1}$$



classical WKB period

$$\Pi_\gamma^{(0)} = \int_\gamma y dx$$

The SW differentials $y dx$ and $y^2 dx$ and their periods:

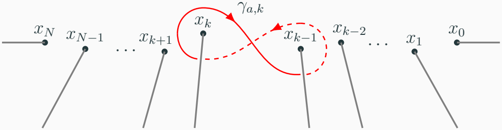
$$(\hat{\Pi}_a)_\gamma = \int_\gamma y^a dx \quad (a = 1, 2).$$

SW differentials are generating functions of the basis of the meromorphic differentials on the WKB curve:

$$\partial_{u_a} y = -\frac{1}{3} \frac{x^{N+1-a}}{y^2}, \partial_{u_a} y^2 = -\frac{2}{3} \frac{x^{N+1-a}}{y},$$

classical SW periods

$\gamma_{a,k}$:1-cycles on the a and $(a+1)$-th sheets of the WKB curve $y^3 = u_0(x-x_0)\dots(x-x_N)$



$$
\begin{aligned}
(\hat{\Pi}_a)_{\gamma_{l,k}} &= \left(e^{\frac{2\pi i}{3}l} - e^{\frac{2\pi i}{3}(l+1)}\right)\oint_{\gamma_{l,k}} p(x)^{\frac{a}{3}}\mathrm{d}x \\
&= -2ie^{\frac{\pi i}{3}(2l+1)a}u_0^{\frac{a}{3}}\sin\frac{\pi a}{3}\int_{x_k}^{x_{k-1}}\prod_{i=0}^{N}(x-x_i)^{\frac{a}{3}}\mathrm{d}x,
\end{aligned}
$$

Quantum corrections

a basis of meromorphic differential

$$\frac{dx}{y}, \frac{xdx}{y}, \ldots, \frac{x^{N-1}dx}{y}, \quad \frac{dx}{y^2}, \frac{xdx}{y^2}, \ldots, \frac{x^{N-1}dx}{y^2}$$

$p_n dx$ defines the meromorphic 1-form on the WKB curve:

$$p_n dx = \sum_{a=1}^{2} \sum_{i=1}^{N-1} B_{ai}^{(n)} \frac{x^{i-1}}{y^a} dx + d(*),$$

For $p(x) = u_0 x^3 + u_1 x^2 + u_2 x + u_3$,

$$B_{21}^{(2)} = \frac{1}{18} \frac{D_0}{\Delta} \frac{u_1}{3}, \qquad B_{22}^{(2)} = \frac{1}{18} \frac{D_0}{\Delta} u_0,$$

$$B_{11}^{(6)} = -\frac{1}{174960} \frac{D_0}{\Delta^4} \left(21983 D_0^4 - 823446 u_0^2 D_0^2 \Delta + 6633171 u_0^4 \Delta^2\right).$$

$$\Delta = -u_1^2 u_2^2 + 4u_3 u_1^3 + 4u_0 u_2^3 - 18 u_0 u_2 u_3 u_1 + 27 u_0^2 u_3^2,$$

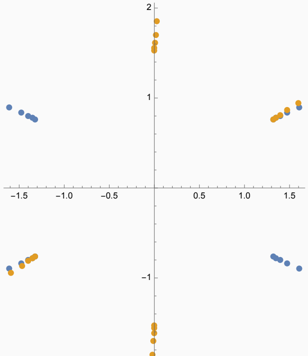
$$D_0 = 2u_1^3 - 9u_0 u_2 u_1 + 27 u_0^2 u_3.$$

Borel resummation

Formal WKB expansions

$$\Pi_\gamma(\epsilon) = \sum_{a=1}^{2} \sum_{i=1}^{N-1} B_{ai}(\epsilon)(\Pi_{ai})_\gamma,$$

$$B_{ai}(\epsilon) = \sum_{n=0}^{\infty} B_{ai}^{(n)} \epsilon^n, \quad (\Pi_{ai})_\gamma = \int_\gamma \frac{x^{i-1}dx}{y^a}.$$



Singularity of $\Pi_{\gamma_{1,1}}$ (blue) and $\Pi_{\gamma_{3,2}}$ (yellow) in the Borel plane for $p(x) = -x^3 + 7x + 6$

Integrals of motion and quantum periods

eigenvalue of the T-operator of the $(W_3)_{p,p'}$-minimal model with $c = 50 - 24(g + g^{-1})$ $(g = p/p')$ [Bazhanov-Hibberd-Khoroshkin]

$$\log \mathbf{T}(t) \sim mt^{\frac{1}{3(1-g)}}\mathbf{I} - 2\sum_{n=1}^{\infty} C_{2n}\cos\left(\frac{\pi n}{3}\right)t^{-\frac{2n}{3(1-g)}}\mathbf{I}_{2n}$$
$$+ 2i\sum_{n=1}^{\infty} C_{2n-1}\sin\left(\frac{\pi(2n-1)}{6}\right)t^{-\frac{2n-1}{3(1-g)}}\mathbf{I}_{2n-1}.$$

On the module V_{Δ_2,Δ_3}

$$I_1^{(\mathrm{vac})} = \Delta_2 - \frac{c}{24},\ I_2^{(\mathrm{vac})} = \Delta_3, \quad I_3^{(\mathrm{vac})} = \Delta_3(\Delta_2 - \frac{c+6}{24}),$$
$$I_4^{(\mathrm{vac})} = \Delta_2^3 + \frac{4\Delta_3^2}{3} - \frac{c+8}{8}\Delta_2^2 + \frac{(c+2)(c+15)}{192}\Delta_2 - \frac{c(c+23)(7c+30)}{96768}.$$

$p(x) = u_0 x^2 + u_2$ corresponds to $g = 3/5$, $\Delta_2 = -1/5$, $\Delta_3 = 0$.

$$\Pi_{\gamma_1} = \Pi_{\gamma_1}^{(0)} e^{\theta} + \Pi_{\gamma_1}^{(2)} e^{-\theta} + \Pi_{\gamma_1}^{(6)} e^{-5\theta} + \cdots,$$

$$\log T(t) = \Pi_{\gamma_1}, \quad t = \frac{(-1)^{\frac{4}{5}} u_2}{5^{\frac{6}{5}}\Gamma\left(\frac{2}{5}\right)^3 u_0^{\frac{3}{5}}} e^{\frac{6\theta}{5}}$$

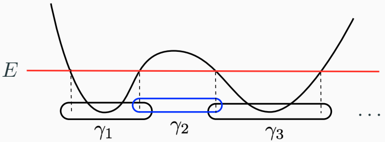
exact WKB periods and TBA equations

minimal chamber

Schrödinger equation with a generic polynomial potential $V(q)$ with order $r+1$.

$\left(-\hbar^2\partial_q^2 + q^{r+1} - \sum_{a=1}^{r} u_a q^{r-a}\right)\hat{y}(q,u_a,\zeta) = 0.$

all the turning points are real and distinct (minimal chamber)



γ_{2i-1}: allowed cycle, $m_{2i-1} = \Pi^{(0)}_{\gamma_{2i-1}} = 2\int_{q_{2i-1}}^{q_{2i}} y dx$
γ_{2i}: forbidden cycle $m_{2i} = i\,\Pi^{(0)}_{\gamma_{2i}} = 2i\int_{q_{2i}}^{q_{2i+1}} y dx,$

m_a are real and positive.

exact WKB periods $s(\Pi_{\gamma_a})(\hbar)$

Exact WKB periods and TBA equations

- the subdominant solution on the positive real axis and their Sibuya rotations y_k
- discontinuity formula
- cross-ratio of the Wronskians $W[y_k, y_{k'}]$
- $\hbar = e^{-\theta}$, asymptotics $\hbar \to 0$ $s(\Pi_{\gamma_a})(\hbar) \to \Pi_{\gamma_a}^{(0)}$
- analyticity

TBA equations in the minimal chamber

$$\epsilon_a(\theta) = m_a e^{\theta} - K * L_{a-1} - K * L_{a+1}, \quad K = \frac{1}{2\pi}\frac{1}{\cosh\theta}$$

$$L_a(\theta) = \log\bigl(1 + e^{-\epsilon_a(\theta)}\bigr)$$

$$\epsilon_{2i-1}\left(\theta + \frac{i\pi}{2} \pm i\delta\right) = \frac{i}{\hbar} s_{\pm}\left(\Pi_{\gamma_{2i-1}}\right)(\hbar), \quad \epsilon_{2i}(\theta) \quad = \frac{i}{\hbar} s\left(\Pi_{\gamma_{2i}}\right)(\hbar),$$

Y-function: $Y_a(\theta) = e^{-\epsilon_a(\theta)}$

Wall-crossing of TBA equations

In general, turning points can be complex:

$$m_a = |m_a| e^{i\phi_a}, \quad \widetilde{\epsilon}_a(\theta) = \epsilon_a(\theta - i\phi_a), \qquad \widetilde{L}_a(\theta) = L_a(\theta - i\phi_a).$$

$$\widetilde{\epsilon}_a = |m_a| e^{\theta} - K_{a,a-1} \star \widetilde{L}_{a-1} - K_{a,a+1} \star \widetilde{L}_{a+1}, \quad K_{r,s} = K(\theta + i(\phi_s - \phi_r)$$

- TBA is valid for ϕ_a satisfying $|\phi_a - \phi_{a\pm1}| < \frac{\pi}{2}$
- $\phi_a - \phi_{a\pm1} = \pm\frac{\pi}{2}$, $\Pi^{(0)}_{\gamma_a} \propto \Pi^{(0)}_{\gamma_{a\pm1}}$ marginal stability condition
- Introducing new Y-functions for the cycle $\gamma_a + \gamma_{a+1}$, the TBA system is modified Wall-crossing of the TBA equations
- TBA kernels correspond to the S-matrices in IM $K_{ab}(\theta) = -i\frac{d}{d\theta}\log S_{ab}(\theta)$.
- For each BPS chamber, we can associate S-matrix theory.
- at special pt in a chamber, some Y-functions are identified. enhanced symmetry

2nd order ODE with $(r+1)$-th order potential

- minimal chamber
  - r-term TBA-system for Homogeneous sine-Gordon model $SU(r+1)_2/U(1)^r$ [Castro-Alvaredo-Fring-Korff-Miramontes]
  - effective central charge

$$c_{\text{eff}} = \frac{r(r+1)}{r+3}.$$

- maximal chamber $r(r+1)/2$-term TBA, at the monomial potential pt. reduced to
  $V(x) = x^{2n+1}$, $A_{2n}^{(2)}$-TBA, $M_{2,2n+3}$
  $V(x) = x^{2n}$, $A_{2n-1}/\mathbb{Z}_2$-TBA, Z_{n+1}-parafermions
  [Al. B. Zamolodchikov, Klassen-Meltzer,Ravanini-Tateo-Valleriani]
- enhanced symmetry occurs when $V(x) - E = (x^{hM/K} - E')^K$
  $SU(2)_K \times SU(2)_L/SU(2)_{K+L}$ [DDMST]

r-th order ODE with $(N+1)$-th order potential

minimal chamber

$$\left[(-1)^r \epsilon^{r+1} \partial_x^{r+1} + \left(u_0 x^{N+1} + \cdots + u_{N+1}\right)\right] \psi(x) = 0.$$

- (A_r, A_N)-type Y-system

$$Y_{a,l}^{[+1]} Y_{a,l}^{[-1]} = \frac{(1+Y_{a+1,l})(1+Y_{a-1,l})}{(1+Y_{a,l+1}^{-1})(1+Y_{a,l-1}^{-1})}.$$

  [F.Ravanini, 1992]

- $c_{eff} = \frac{rN(N+1)}{N+r+1}$, CFT $\frac{SU(N+1)_{r+1}}{U(1)^N}$

Wall-crossing of TBA (arXiv:2111.11047)

maximal chamber

- (A_2, A_2)
  minimal chamber 2-TBA $\Longrightarrow$ maixmal chamber 4-term TBA
  For the monomial potential, D_4-type TBA appears ($c_{eff} = 1$: sine-Gordon model at reflectionless point) [Al. Zamolodchikov, Klassen-Meltzer]

- (A_2, A_3)
  minimal chamber 3-TBA $\Longrightarrow$ maixmal chamber 12-term TBA
  For the monomial potential E_6-type TBA appears ($c_{eff} = \frac{6}{7}$: tri-critical 3-state Potts model) [Al. Zamolodchikov, Klassen-Meltzer]

- Duality of AD theories $(A_r, A_1) \sim (A_1, A_r)$, $(A_2, A_2) \sim (D_4, A_1)$, $(A_2, A_3) \sim (E_6, A_1)$ [Cecotti-Neitzke-Vafa]

Outlook

Outlook

- ODE= the linear problem of the classical integrable equation
  massive ODE/IM, RG flow, $T - \bar{T}$ deformation
  [Lukyanov-Zamolodchikov, Gaiotto-Lee-Vicedo-Wu, Aramini-Brizio-Negro-Tateo,...]
- make a (complete) dictionary
  Construct BAE, T-Q relation, Y-system for an ODE
  Identifying CFT, quantum Integrable model
- Integrable model with SUSY [I-Zhu 2206.08024]
  $osp(2|2)^{(2)}$ linear problem $\leftrightarrow$ $N = 1$ SCFT
- ODE/IM for effective potential $V_0 + \hbar V_1 + \hbar^2 V_2 + \cdots$
  [work in progress w/ H.Shu]
- application to the Stark effect [I-Yang 2307.03504]
- mathematical proof (Riemann-Hilbert problem)