How to get into an excited state

Bologna 2023

(work in progress with Brandon Morrison; and also
earlier work with Roberto Tateo, Lorenz Hilfiker, Istvan Szecsenyi)

1. Basic problem
2. A simple example
3. Yang-Lee & TBA continuation
4. Sinh-Gordon case
5. Results (including movies)

1. Basic problem

Suppose we know the ground-state energy of some system.

ⓐ Does this tell us the energies of its excited states?

ⓑ If so, how are they connected?

Often the answer to ⓐ is yes, if we can do analytic continuation in a parameter: an old idea, see eg Bender & Wu's work for the Q.M. case.

<u>Big goal</u>: get a similar level of understanding for a QFT.

- Back in ancient times, we (PED+RT) used this as a trick to find excited-state TBA equations for some simple models, around the time Francesco and collaborators were developing excited-state equations using NLIE:

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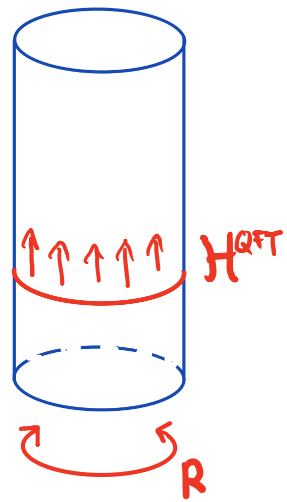
- New angle: look at the sinh-Gordon model, which has an extra parameter - systematic patterns appear as this parameter becomes small.

2. A simple example

Ising field theory on a cylinder

- Off-critical Ising model in thermal direction.
- One mass m, correlation length $\xi = 1/m$.



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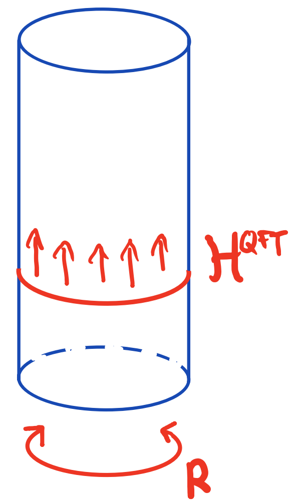

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Ground state energy on a circle is known in closed form:

$$E_0(m,R) = E_{bulk}(m,R) - \frac{\pi}{6R}\, c^{(ising)}(\underbrace{mR}_{=\, r})$$

Sometimes called the "effective central charge".

r (system size in units of correl. length)

with
$$c^{(ising)}(r) = \frac{1}{2} - \frac{3r^2}{2\pi^2}\left[\log\frac{1}{r} + \frac{1}{2} + \ln\pi - \gamma_E\right] + \frac{6}{\pi}\sum_{k=1}^{\infty}\left(\sqrt{r^2+(2k-1)^2\pi^2} - (2k-1)\pi - \frac{r^2}{2(2k-1)\pi}\right)$$

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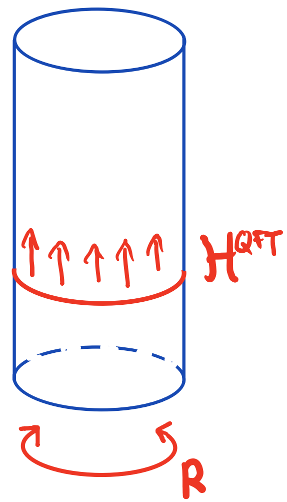

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Correct for the ground state $(d=\bar{d}=0)$ in the $c=1/2$ CFT found as $R \to 0$

Note: $c(0) = \frac{1}{2}$ so $E_0(R\to 0) \sim -\frac{\pi}{6R}\cdot\frac{1}{2} = \frac{2\pi}{R}\underbrace{\left(0+0-\frac{1/2}{12}\right)}_{d+\bar{d}-c/12} = \frac{2\pi}{R}\overbrace{\left(d+\bar{d}-\frac{c}{12}\right)}$

Continuation in r:

We have $E_0(m,R) = E_{bulk}(m,R) - \frac{\pi}{6R} c^{(ising)}(r)$

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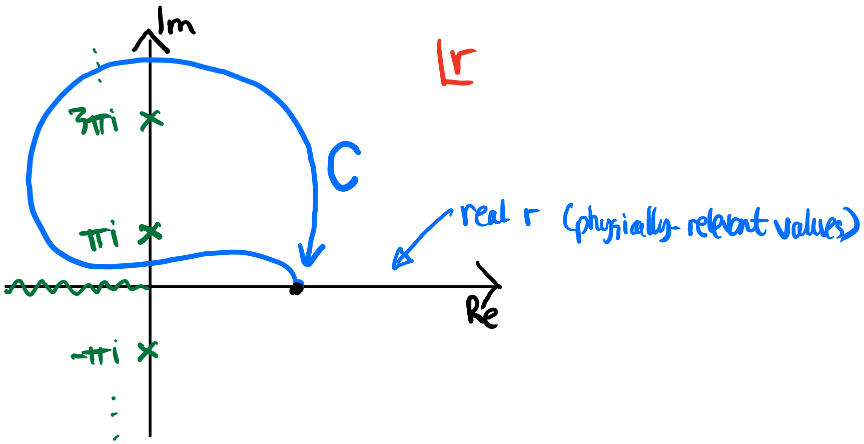
∞ sequence of $\sqrt{\ }$s

Complex r plane:



$\sqrt{\ }$ branch points at $r = (2k-1)\pi i$, $k \in \mathbb{Z}$.

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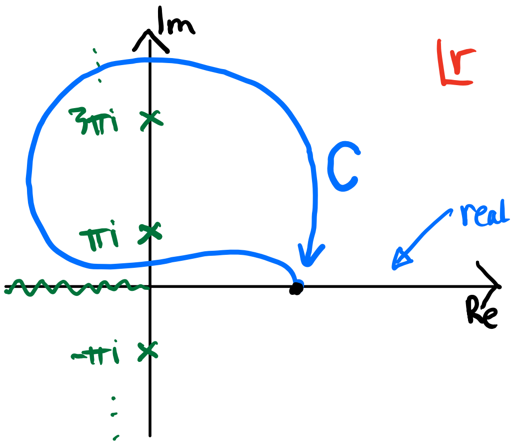
Complex r plane: $\quad \boxed{r}$



real r (physically relevant values)

$\sqrt{\ }$ branch points at $r = (2k-1)\pi i$, $k \in \mathbb{Z}$.

Take C around $k_1, k_2, \ldots k_n$. This flips the signs of the square roots in E_0 from minus to plus.

Return to real axis to find $E_{k_1 \ldots k_n}(m,R) = E_0(m,R) + \frac{2}{R}\sum_{i=1}^{n}\sqrt{r^2+(2k_i-1)^2\pi^2}$ $\leftarrow$ An excited state!

[+ sign since we flipped $-\sqrt{\ }$ to $+\sqrt{\ }$]

[Correct for $2n$ particles on a circle, momenta $\pm(2k_i-1)$, BA states with S matrix $S=-1$.]

Alternative viewpoint, closer to the Thermodynamic Bethe Ansatz:

- Another expression for $c^{(\text{ising})}(r)$:

$$c^{(\text{ising})}(r) = \frac{3}{\pi^2}\int_{-\infty}^{\infty} d\theta \; r\cosh\theta \, \log\left(1+e^{-r\cosh\theta}\right)$$

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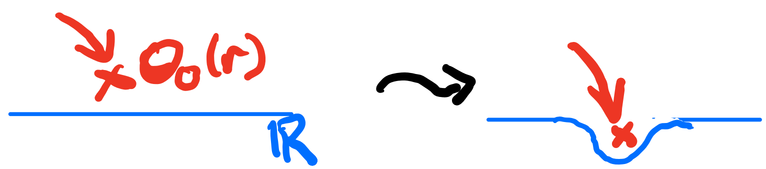

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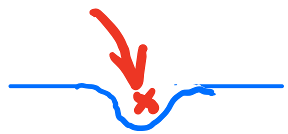
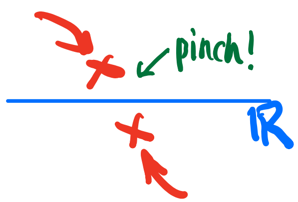


and all is well.

- This fails if two singularities approach the contour from opposite sides: a <u>pinch singularity</u>:



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This generates the branch points!

3. Yang-Lee and TBA continuation

Problem: in general we don't have a closed form for $E_0(R)$. But for integrable QFTs we do know the TBA equation exactly. This is enough to continue it to an "excited TBA" equation, and use this to explore the connectivity of the finite-size energy levels in the complex r plane.

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Simplest example is the Yang-Lee model, a perturbation of the M_{25} minimal CFT:

$$H^{QFT}_{YangLee} = \underbrace{H^{CFT}_{M(2,5)}}_{c=-22/5} + \lambda \int \underbrace{\varphi(x)}_{d=\bar{d}=-1/5} d^2x$$

- The ∞ volume theory has just one particle type and a very simple S-matrix $S(\theta) = \dfrac{\sinh\theta + i\sin\pi/3}{\sinh\theta - i\sin\pi/3}$

  (θ = rapidity; momentum = $(p_0, p_1) = m(\cosh\theta, \sinh\theta)$)
- Its mass is $m(\lambda) = (2.642\ldots)(-\lambda)^{5/12}$
- Since $r = m(\lambda)R$ we can equivalently think of our procedure as analytic continuation in the coupling λ.

  (inf. vol. spectrum is real for λ negative)

TBA equation to find $c(r)$:

Solve $\varepsilon(\theta) = r\cosh\theta - \phi * L(\theta)$

where $L(\theta) = \log(1 + e^{-\varepsilon(\theta)})$

$f * g(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta' f(\theta - \theta') g(\theta')$

$\phi(\theta) = -i \frac{\partial}{\partial\theta} \log S(\theta)$

Then $c(r) = \frac{3}{\pi^2}\int_{-\infty}^{\infty} d\theta \, r\cosh\theta \, L(\theta)$

and $E_0(m,R) = E_{bulk}(m,R) - \frac{\pi}{6R} c(r)$

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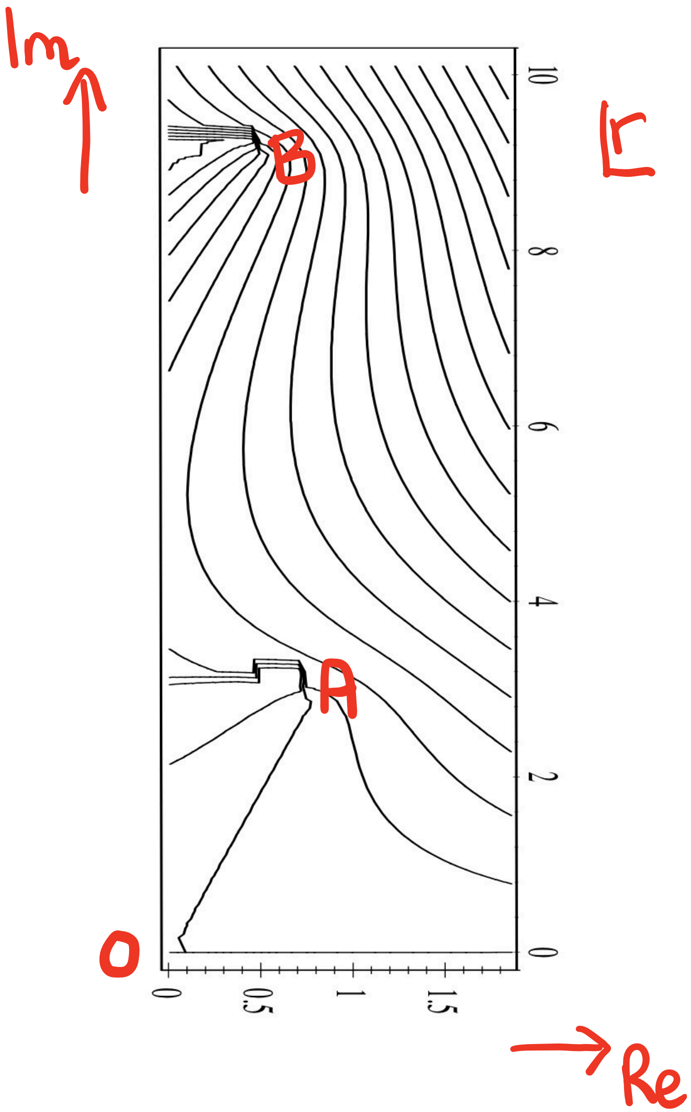
Usually we solve this for real r. But nothing stops us from making r complex, solving on a computer, and plotting the results...

Plot of $\mathrm{Im}(c(r))$ from ground-state YL TBA at complex r



Note A and B look like $\sqrt{\ }$ branch points - can suspect similar causes to the Ising field theory case, but more subtle since the TBA equation, as well as the integral giving $c(r)$, may undergo monodromy.

(NB: $\mathrm{Im}(c(r))=0$ on real axis, but also on the line OA. This line corresponds to λ real but <u>positive</u> rather than <u>negative</u> - so $\arg(r)=\frac{5\pi}{12}$ since $r=mR$ and $m\propto(-\lambda)^{5/12}$.)

Basic mechanism:

$L(\theta) = \log(1 + e^{-\varepsilon(\theta)})$ has singularities in the complex θ plane when $e^{-\varepsilon(\theta)} = -1$ (& also when $e^{\varepsilon(\theta)} = 0$ but these won't be so important)

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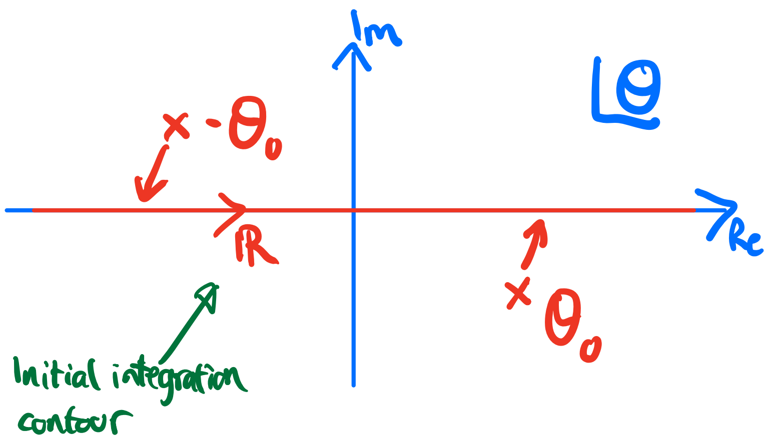
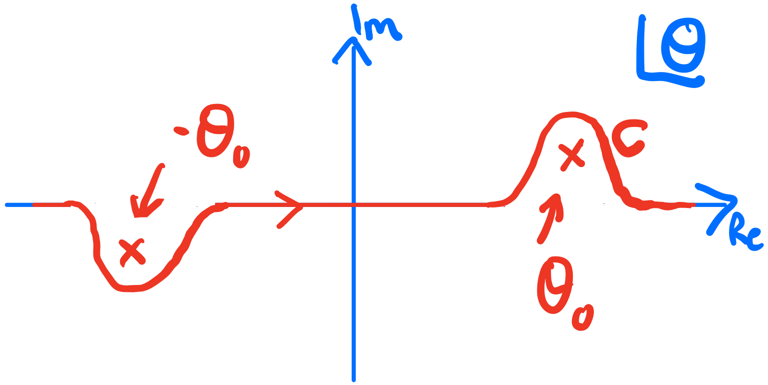


Then the TBA convolution

$$\phi * L(\theta) = \frac{1}{2\pi}\int_{\mathbb{R}} \phi(\theta-\theta')\log(1+e^{-\varepsilon(\theta')})\,d\theta'$$

is in danger...

Step 1: avoid the problem by distorting the contour:



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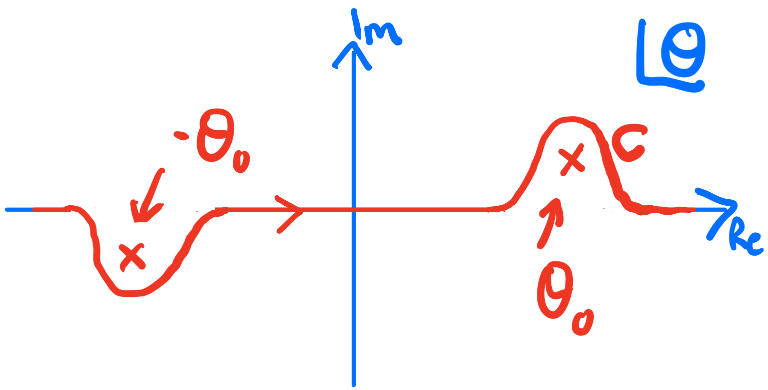

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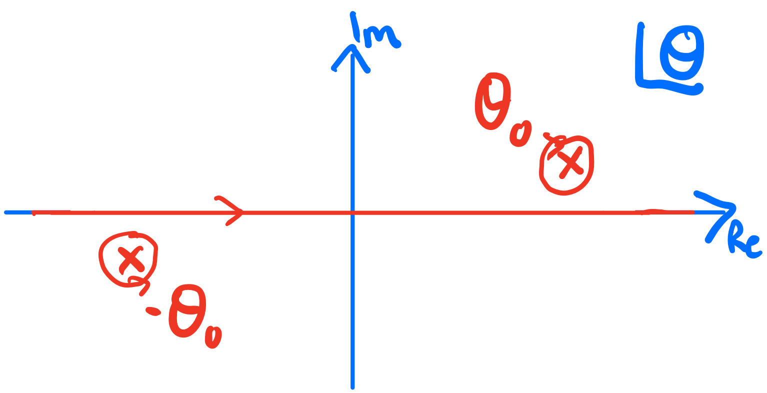
Step 2: return the contour to the real axis:



The relevant residue terms can be found explicitly...

To find the residue, integrate by parts:

$$\phi *_c L(\theta) = \frac{1}{2\pi i}\int_c \log S(\theta-\theta') \frac{\varepsilon'(\theta')}{1+e^{-\varepsilon(\theta')}}\, d\theta'$$

[Recall $\varsigma(\theta) = -i\frac{\partial}{\partial\theta} S(\theta)$]

Useful fact: if $e^{-\varepsilon(\theta')} = -1$, the residue of this at θ_0 is equal to -1.

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Hence $$\phi *_C L(\theta) = \underbrace{\frac{1}{2\pi}\int \phi(\theta-\theta')L(\theta')\,d\theta'}_{\phi * L} - \underbrace{\log \frac{S(\theta-\theta_0)}{S(\theta+\theta_0)}}_{\text{Extra term from the two residues at } \theta=\pm\theta_0.}$$

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Likewise $c(r)$ gets an extra bit:

$$c(r) = \frac{12r}{\pi}\cdot i\sinh\theta_0 + \frac{3}{\pi^2}\int_{\infty}^{\infty} r\cosh\theta\, L(\theta)\,d\theta$$

- The new TBA equation has an extra unknown: θ_0, the location of the singularity which crossed the integration contour.

- It is fixed by imposing $e^{\varepsilon(\theta_0)} = -1 \Rightarrow \varepsilon(\theta_0) = (2N+1)\pi i$

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A natural question: what is the full surface of $c(r)$, and how are levels with diffent Ns connected to each other?

Grand plan:

- Start with ground-state TBA, continue to find connectivity with first-excited-state TBA, continue that to find connectivities with higher-excited-state TBAs, and so on, to find the Riemann surface of the holomorphic function $c(r)$.

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Despite some progress with Istvan & Lorenz, full picture is still missing.

Instead try a different model: sinh-Gordon.

4. Sinh-Gordon case

Again a single type of particle, but now the S-matrix depends on a parameter p:

$$S(\theta) = \frac{\sinh(\theta) - i\sin\pi p}{\sinh(\theta) + i\sin\pi p}$$

This has zeroes (not poles) at $i\pi p$, $i\pi(1-p)$, and maps to itself under $p \to 1-p$.

The TBA is as before

$$\varepsilon(\theta) = r\cosh\theta - \frac{1}{2\pi}\int_{\mathbb{R}} \phi(\theta,\theta')L(\theta')\,d\theta'$$

where $L(\theta) = \log(1+e^{-\varepsilon(\theta)})$, $\phi(\theta) = -i\frac{\partial}{\partial\theta}\log S(\theta)$.

This provides a representation of $\varepsilon(\theta)$ for $-\pi p < \operatorname{Im}\theta < \pi p$.

This was analysed in detail by Al. Zamolodchikov (JPA 2008).

The continuous parameter p compliactes the story!

He introduced

$$Y(\theta) = e^{-\varepsilon(\theta)} = \exp\left(-r\cosh\theta + \frac{1}{2\pi}\int_{\mathbb{R}} \phi(\theta-\theta')L(\theta')d\theta'\right)$$

(holds for $-\pi p < \mathrm{Im}(\theta) < \pi p$)

and

$$X(\theta) = \exp\left(-\frac{r}{2\sin\pi p}\cosh\theta + \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{\cosh(\theta-\theta')} L(\theta')d\theta'\right)$$

(holds for $-\frac{\pi}{2} < \mathrm{Im}(\theta) < \frac{\pi}{2}$)

Note the initial definition of X holds on a wider strip than Y.

X, Y and X·Y systems

Set $a = 1 - 2p$. Then

$$X\left(\theta + \frac{i\pi}{2}\right) X\left(\theta - \frac{i\pi}{2}\right) = 1 + X\left(\theta + \frac{ia\pi}{2}\right) X\left(\theta - \frac{ia\pi}{2}\right)$$

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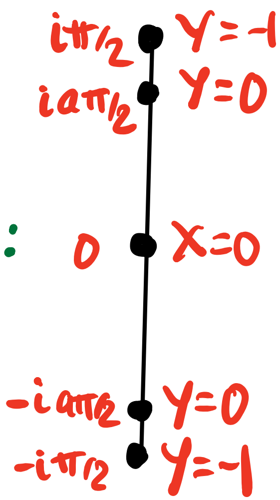
Special values of X and Y form quintets:



Armed with these relations, can extend X and then Y to the whole complex plane, starting from ε on the real axis.

The $p \to 0$ limit

As $p \to 0$ the kernel $\phi(\theta)$ concentrates near $\theta = 0$ and $\phi(\theta - \theta')$ can be replaced in the TBA by $2\pi\delta(\theta-\theta')$. Then the equation becomes

$$\varepsilon(\theta) = r\cosh\theta - \int_{\mathbb{R}} \delta(\theta-\theta') L(\theta') d\theta' = r\cosh\theta - L(\theta)$$

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This resembles $c^{(\text{ising})}(r)$ but with some sign flips – it is $c_0(r)$, the effective central charge of a free boson on a circle, with $c_0(0) = 1$.

Maybe not surprising, but there's a problem here...

The problem:

Just as with $c^{(\text{ising})}(r)$, there's an alternative formula for $c_0(r)$ [see, e.g., Klassen & Melzer 1991]:

$$c_0(r) = 1 - \frac{3r}{\pi} + \frac{3r^2}{2\pi^2}\left[\ln\frac{1}{r} + \frac{1}{2} + \ln 4\pi - \gamma_E\right] - \frac{6}{\pi}\sum_{k=1}^{\infty}\left(\sqrt{(2k\pi)^2 + r^2} - 2k\pi - \frac{r^2}{4k\pi}\right)$$

cf: $$c^{(\text{ising})}(r) = \frac{1}{2} - \frac{3r^2}{2\pi^2}\left[\log\frac{1}{r} + \frac{1}{2} + \ln\pi - \gamma_E\right] + \frac{6}{\pi}\sum_{k=1}^{\infty}\left(\sqrt{r^2 + (2k-1)^2\pi^2} - (2k-1)\pi - \frac{r^2}{2(2k-1)\pi}\right)$$

- This time the branch points are at even multiples of $i\pi$, not odd ones.
- But more crucially, the signs of the square roots are reversed!

Why might this be a problem?

- When continuing $c^{(\mathrm{ising})}(r)$ around branch points from the ground state, the flipped signs of the square roots led to an <u>increase</u> in $E(m,R)$ and excited states with higher energies than the ground state. [good!]

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- But the square roots in $c_0(r)$ start with the opposite sign, so flipping any of them <u>decreases</u> $E(m,R)$, leading to "states" with lower energy than the ground state. [bad!]

Why might this not be a problem?

- Toy example: SHO (free boson in a universe with one point)

$$\left(-\frac{d^2}{dx^2}+v^2x^2\right)\psi = H_v\psi = E\psi \quad (*)$$

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$$\to E = (2n+1)\omega \quad (n = 0, 1, 2, \ldots)$$

- But if we set $\nu = re^{i\phi}$ (r real) and continue ϕ from 0 to π, then the eigenvalues change sign even though $H_\nu \to H_{-\nu} = H_\nu$.

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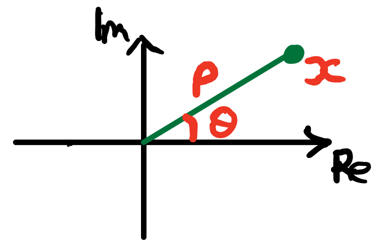
  - while the other shrinks (is subdominant)



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  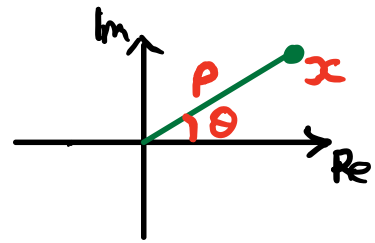

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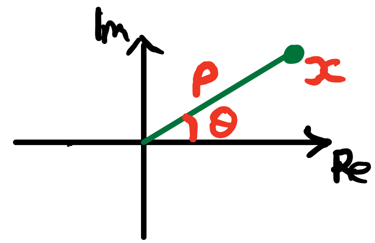

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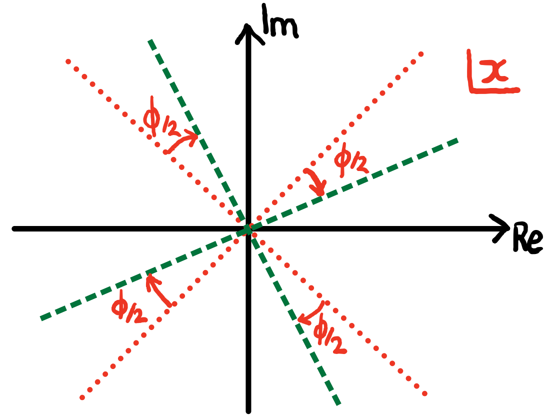
- Such θ define the **anti-Stokes lines**, and split the complex plane into **Stokes sectors**:



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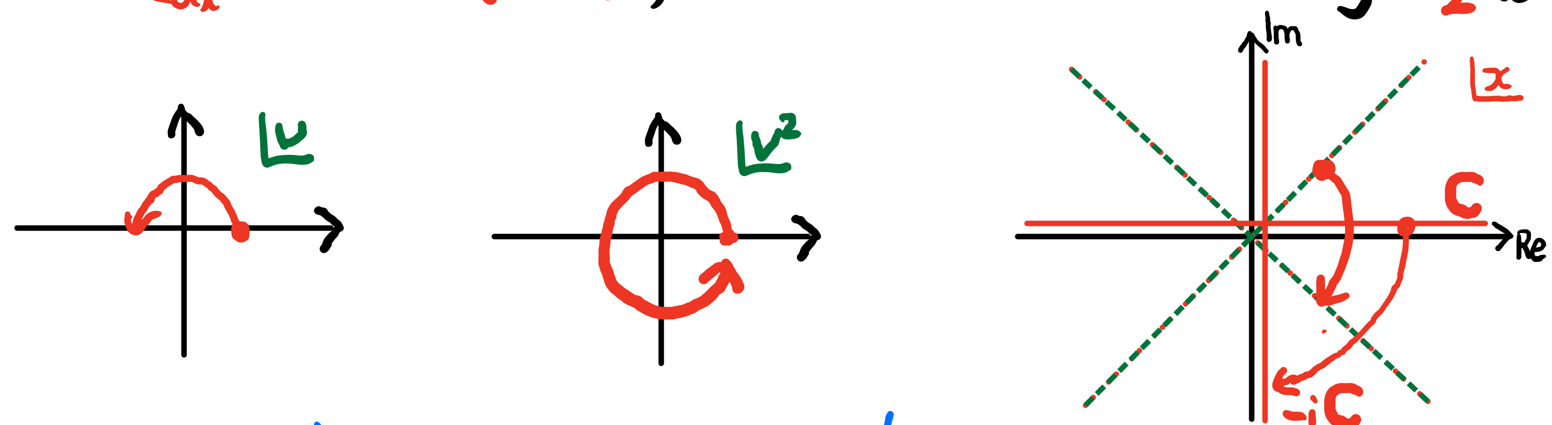
- To keep out of trouble during continuation, C must be distorted so as to track the same pair of Stokes sectors at $\pm\infty$, to avoid anti-Stokes lines being crossed.

- For the case in hand, as ϕ increases from 0 to π, $\nu = re^{i\phi} \to -\nu$,

  $H_\nu = \left(-\frac{d^2}{dx^2} + \nu^2 x^2\right) \to H_{-\nu} = H_\nu$, but the Stokes sectors rotate by $-\pi/2$ so $C \to -iC$



... so the b.c.'s for ψ have changed.

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Claim: this is what is happening for the free boson. By contrast we would not expect to see this in sinh-Gordon, where the growth of the potential $\cosh(\phi(x))$ at each point x is much stronger.

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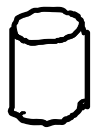
Further reassurance comes from numerical work (next section) which shows that for $p>0$ continuation round branch points & back to the real axis does indeed lead to states of higher energy.

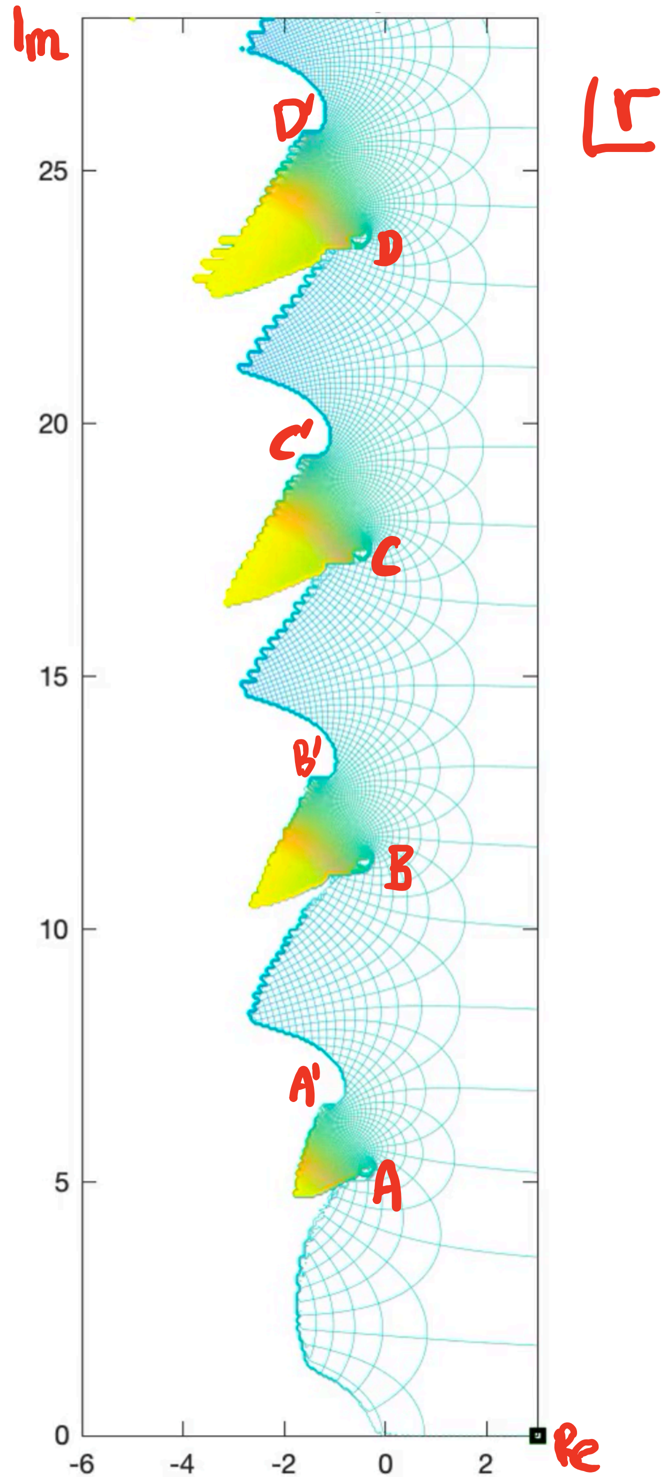
5 Results

Ground state TBA solution for $p=0.04$:

contours of $\mathrm{Re}(c(r))$ and $\mathrm{Im}(c(r))$.



Note paired branch points at $\{A, A'\}$, $\{B, B'\}$, $\{C, C'\}$, $\{D, D'\}$.



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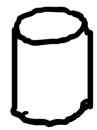
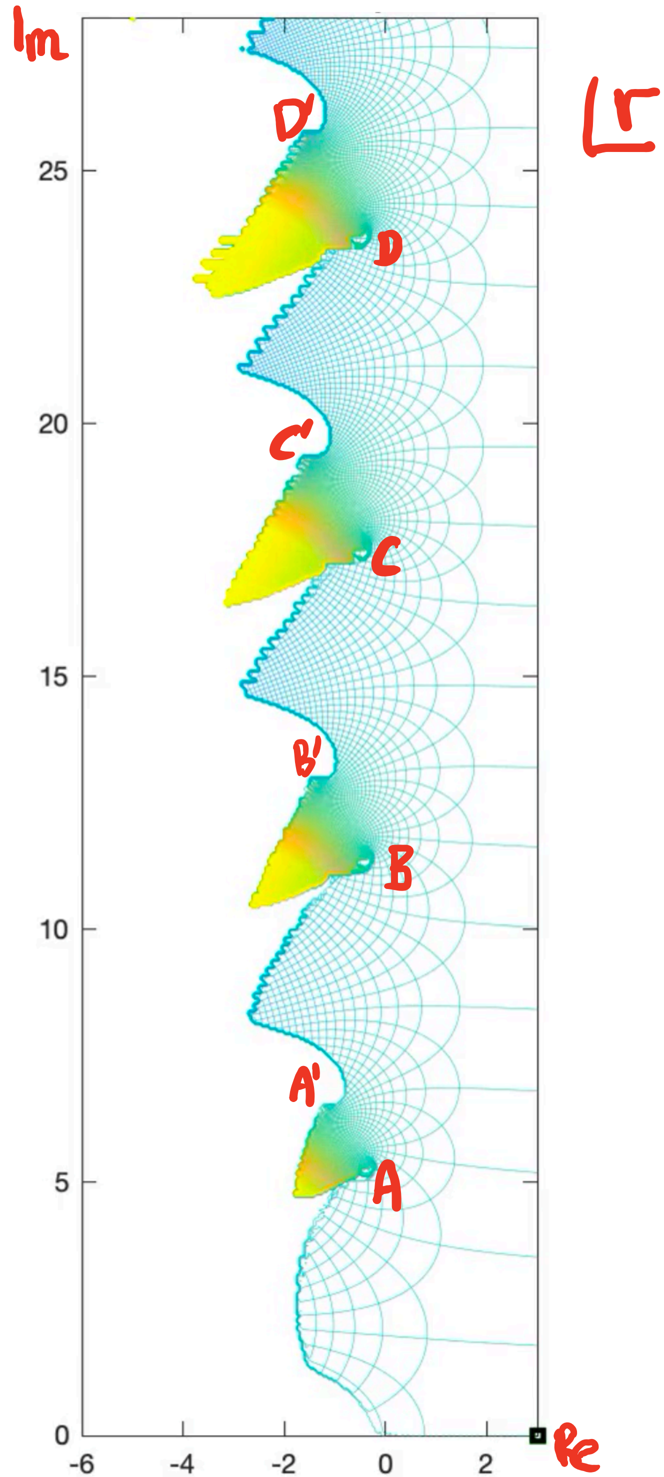


0 particles



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As $p \to 0$,

$A \,\&\, A' \to 2\pi i$ $\qquad$ $B \,\&\, B' \to 4\pi i$

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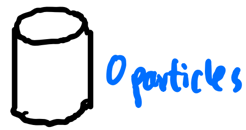

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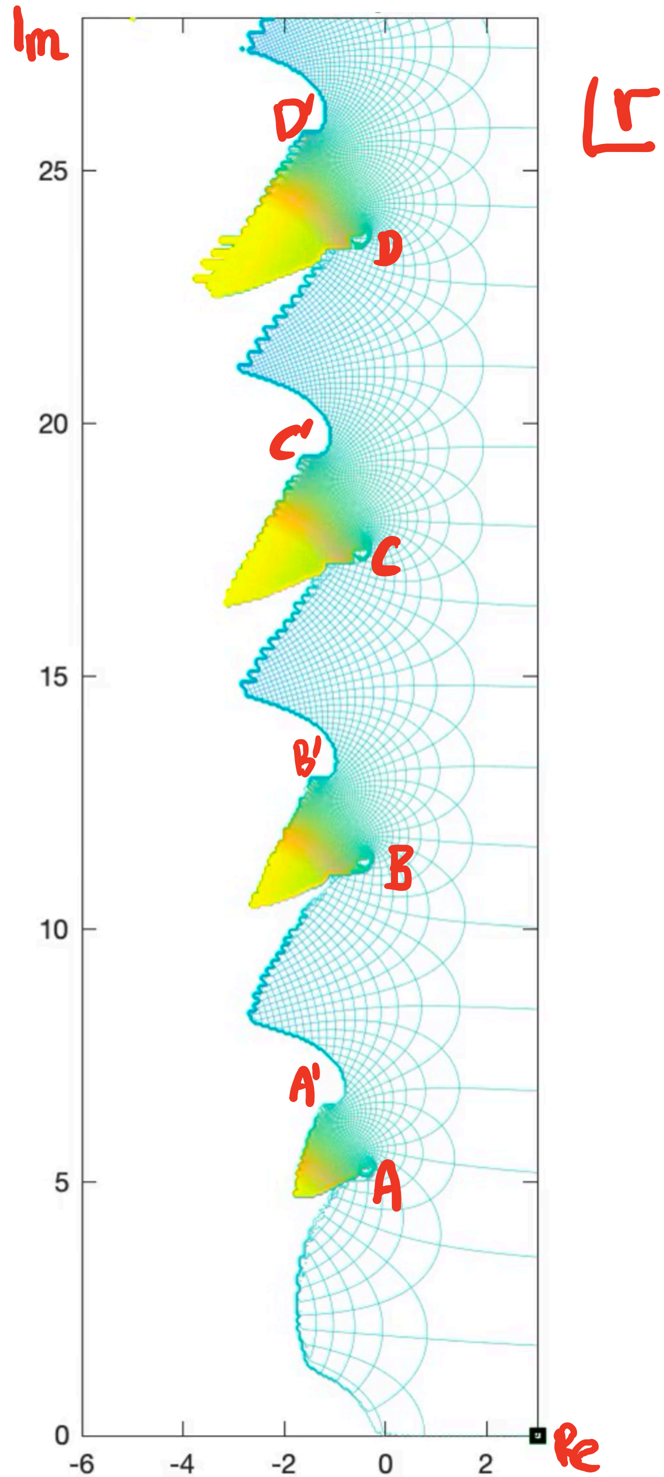
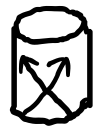
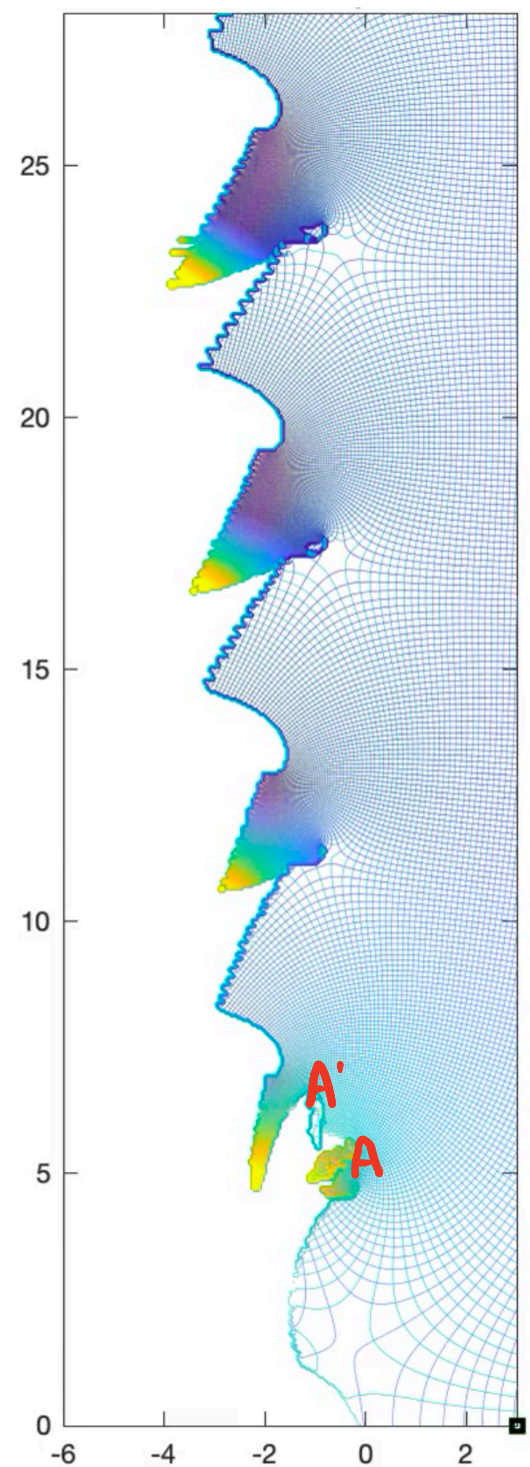
Each connects to a 2-particle TBA solution, with Bethe number $N=1$ ($A \& A'$), $N=2$ ($B \& B'$), $N=3$ ($C \& C'$) & $N=4$ ($D \& D'$) and so on...



Two-particle sinh-Gordon TBA

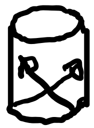
with $N=1$

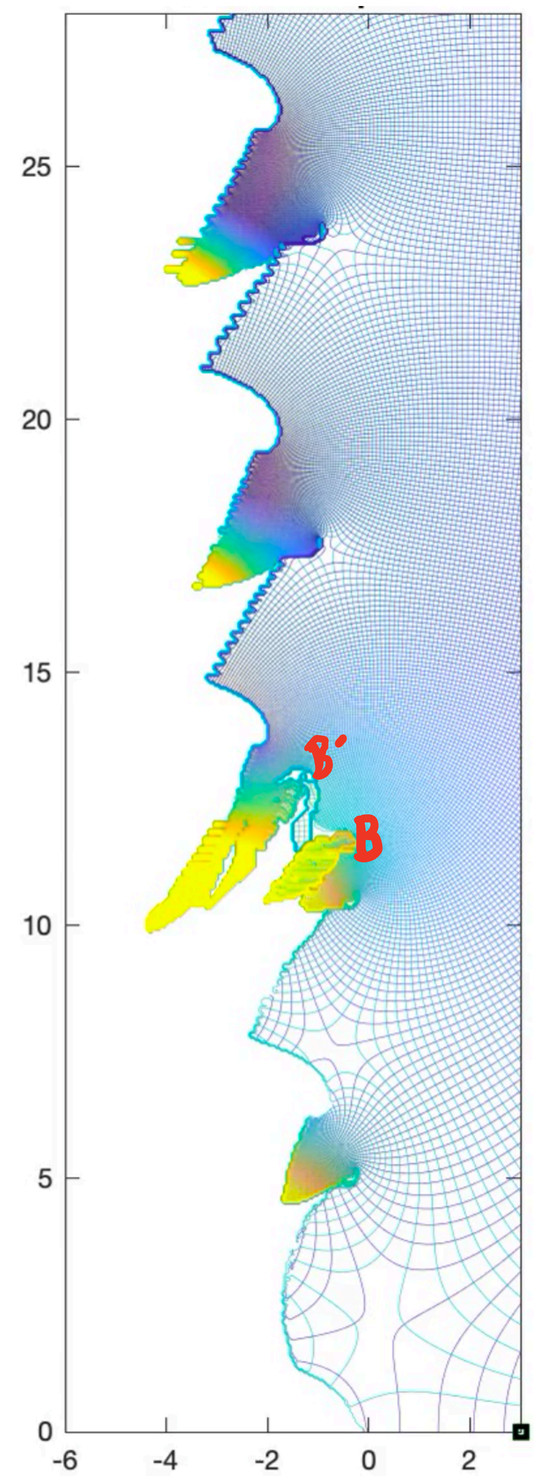
2 particles: 



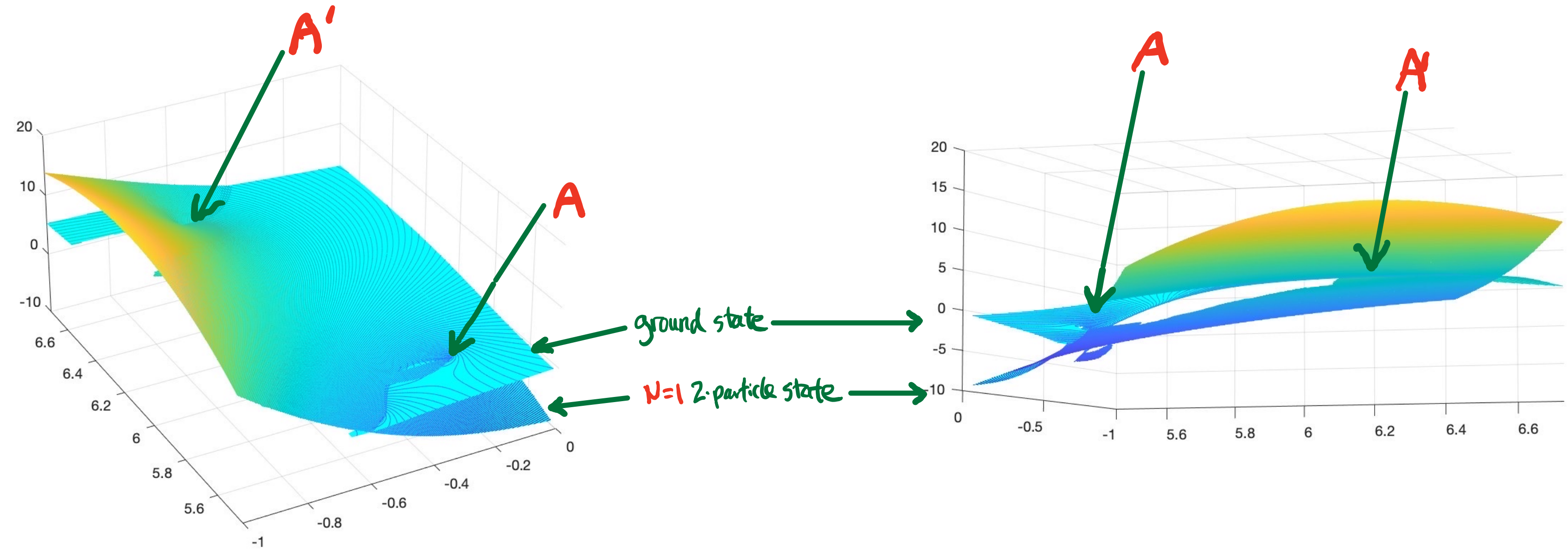
Two-particle sinh-Gordon TBA with $N=2$

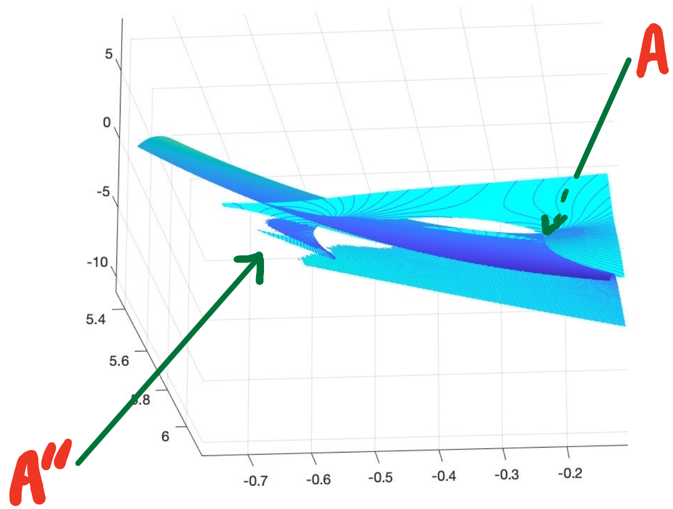
2 particles, going a bit faster:



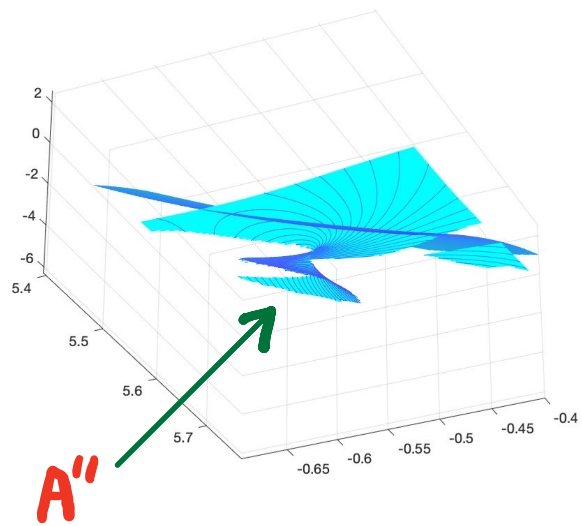


Some 3d plots of Re(c(r)) (p=0.02 this time)



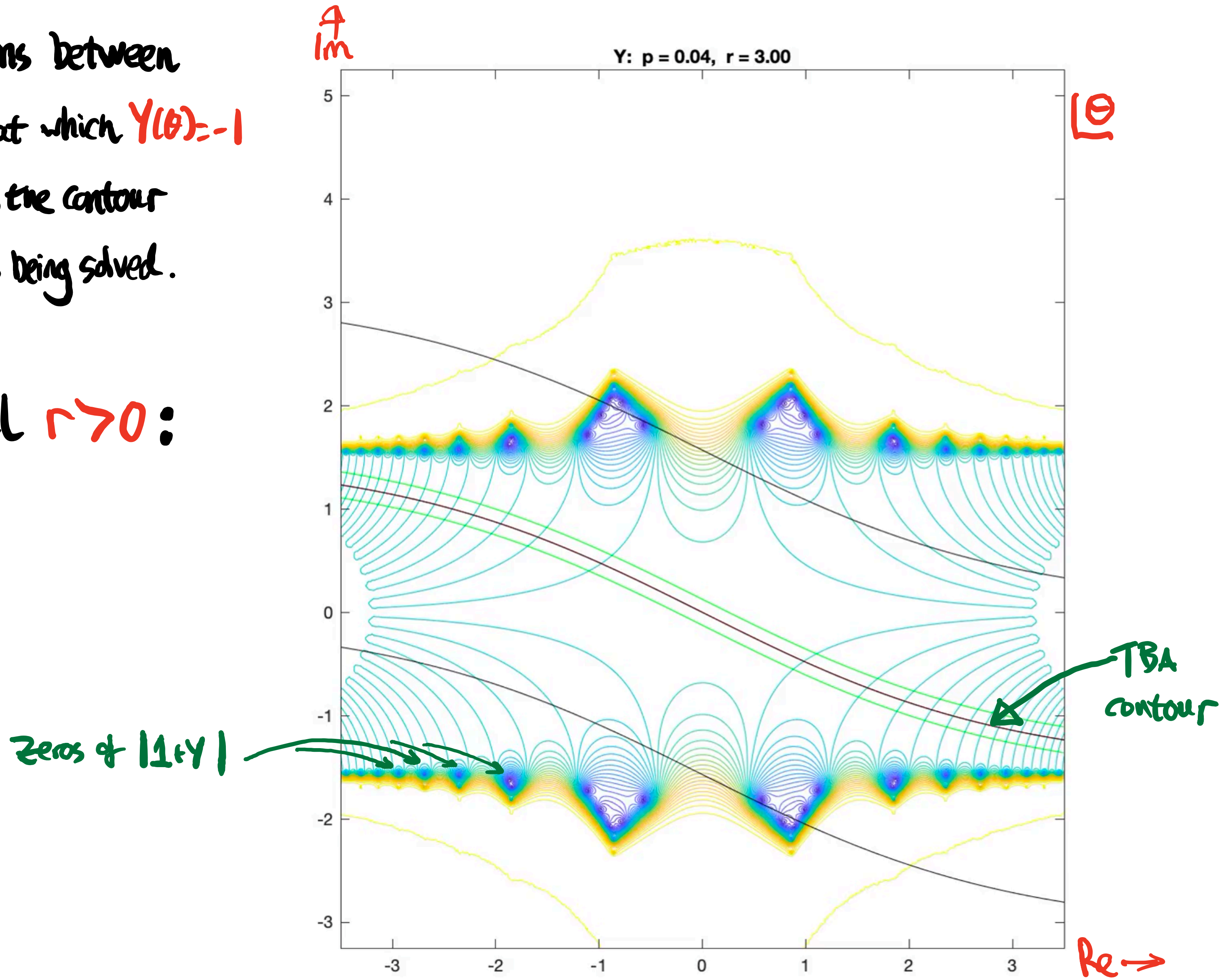


(a further branch point, the first of a sequence, near to A)

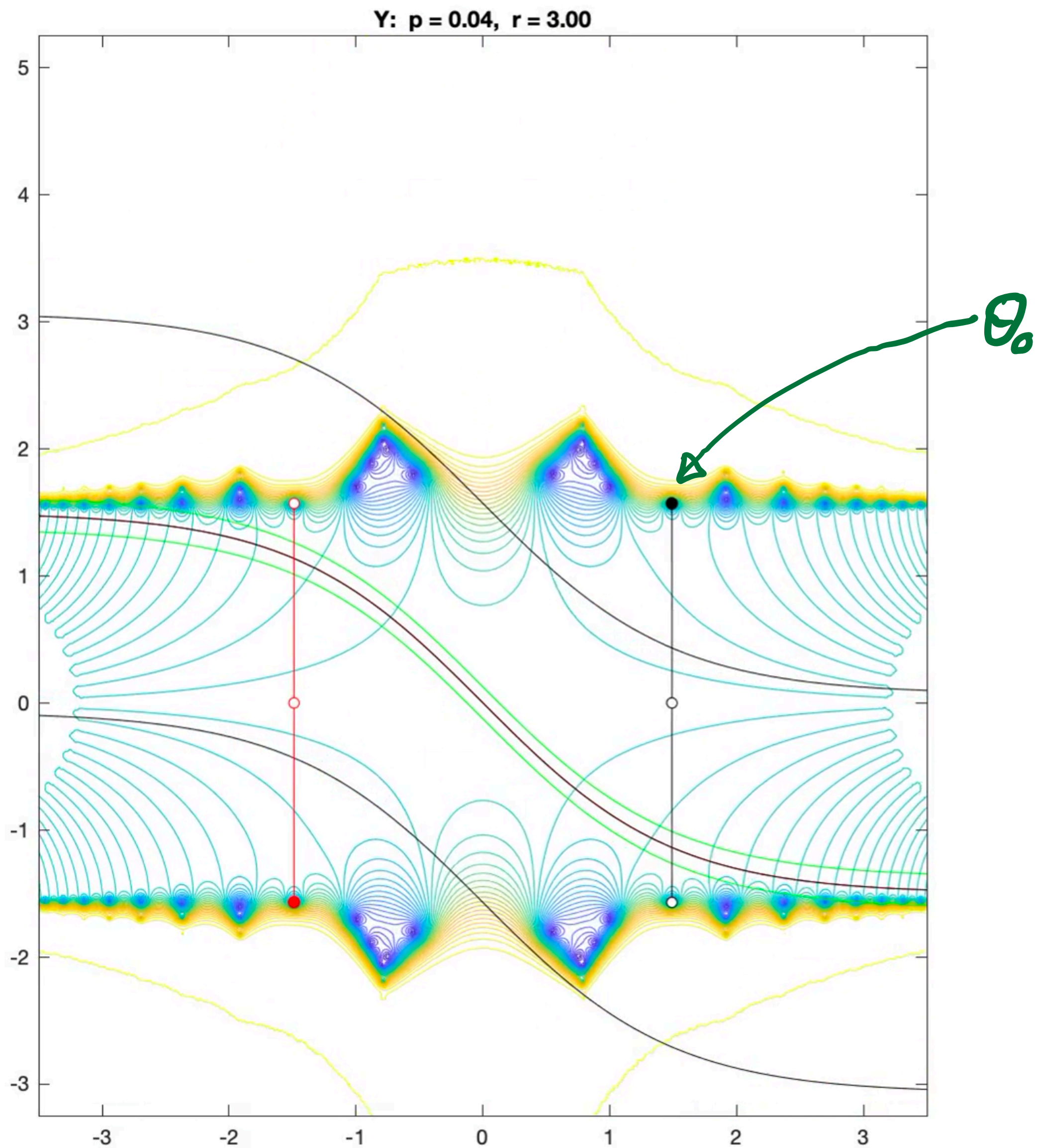


To track transitions between TBAs, monitor points at which $Y(\theta)=-1$ to see when they cross the contour along which the TBA is being solved.

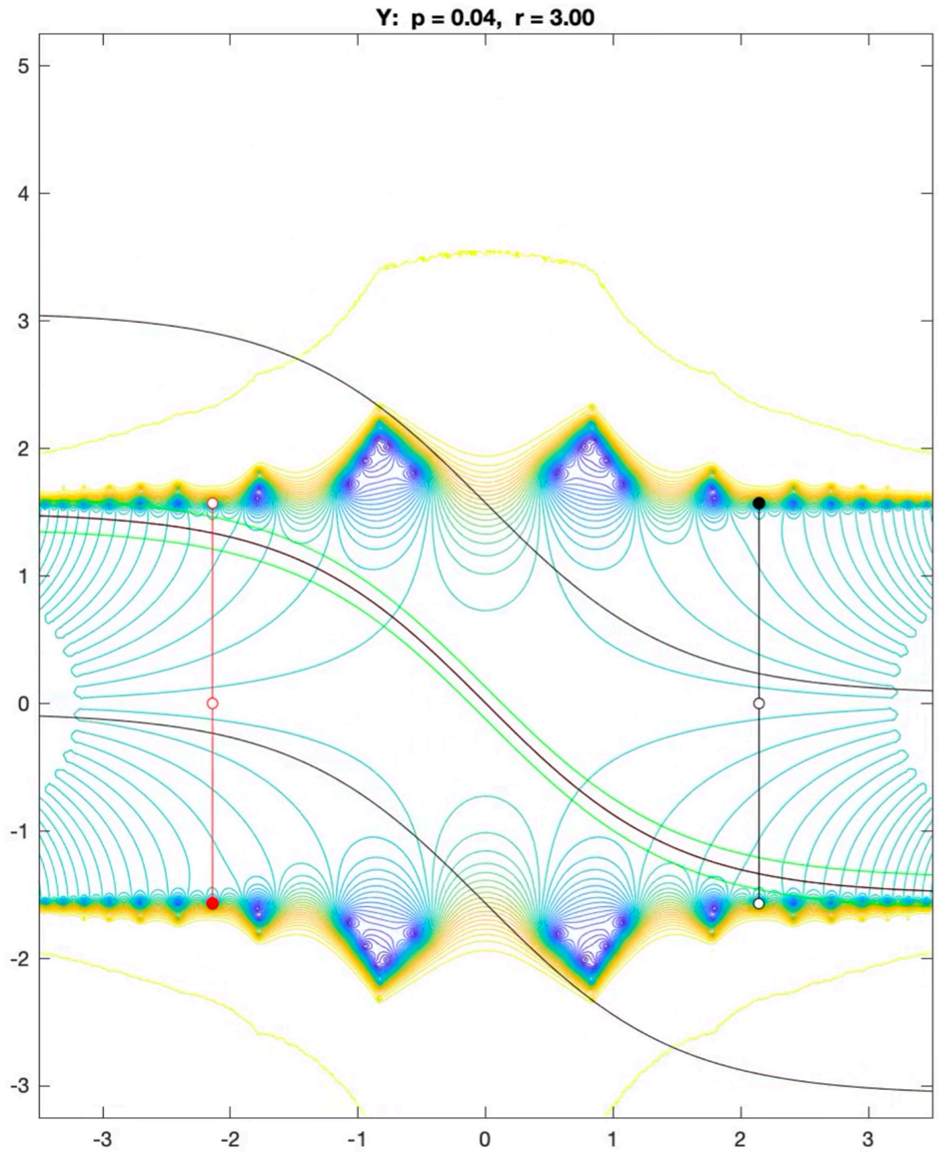
Situation for real $r>0$:



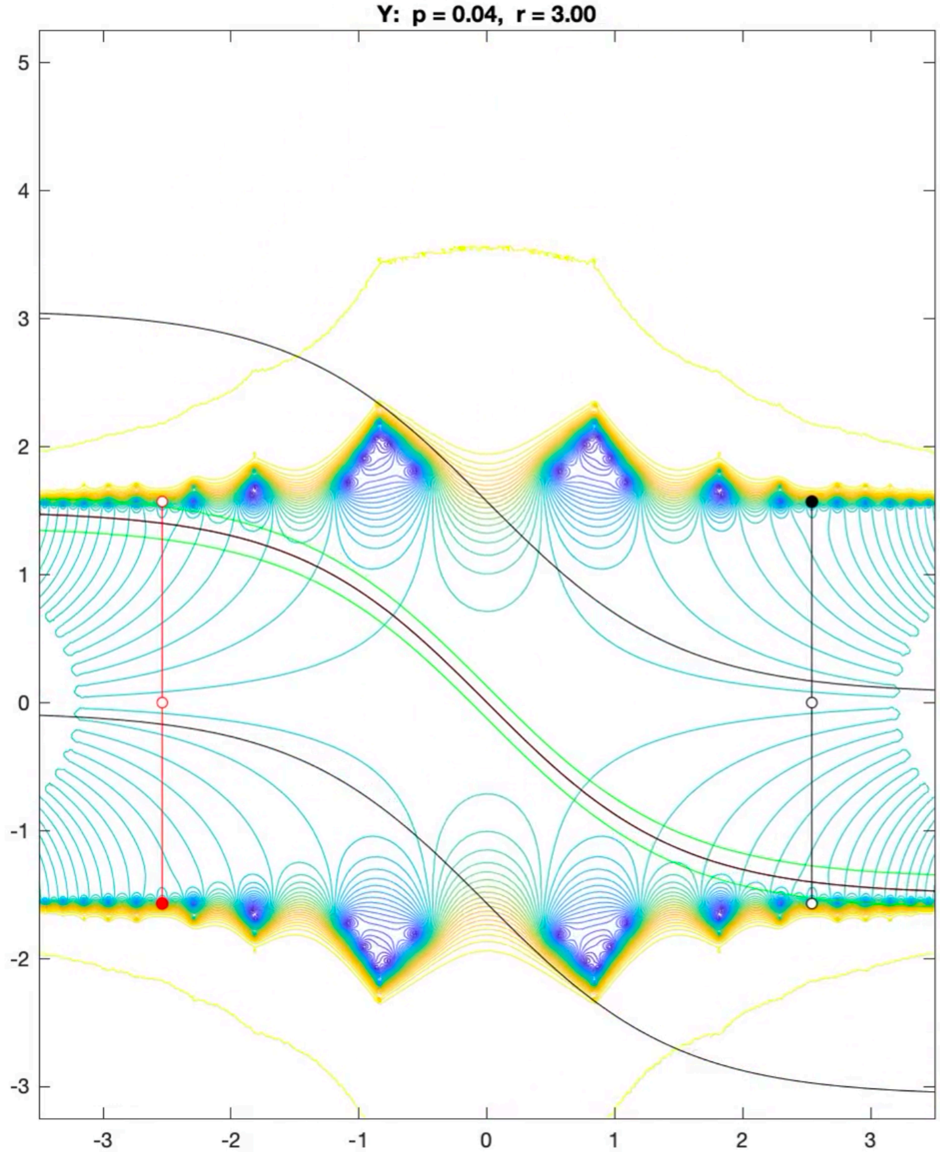
Plot of $|1+Y(\theta)|$ for complex θ : ground state



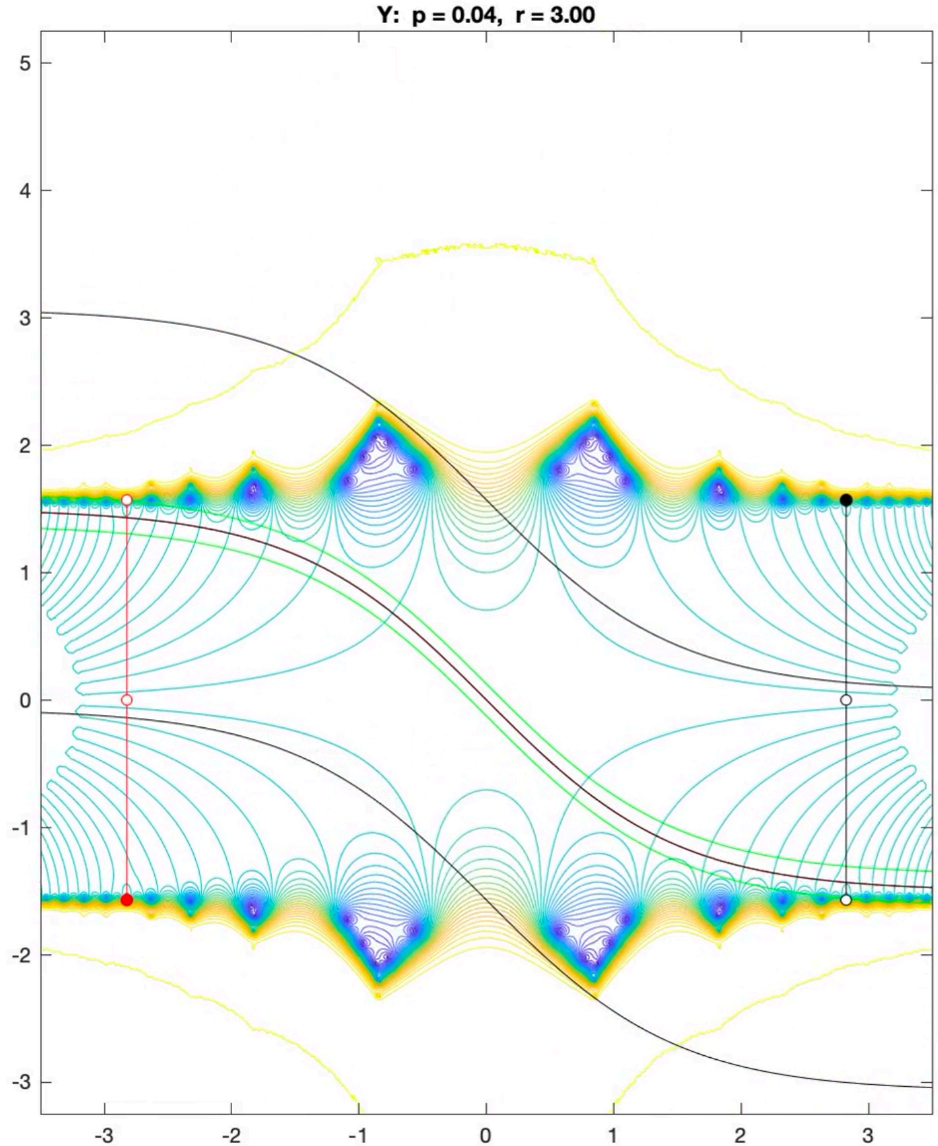
$|1+Y|$: 2 particles, $N=1$



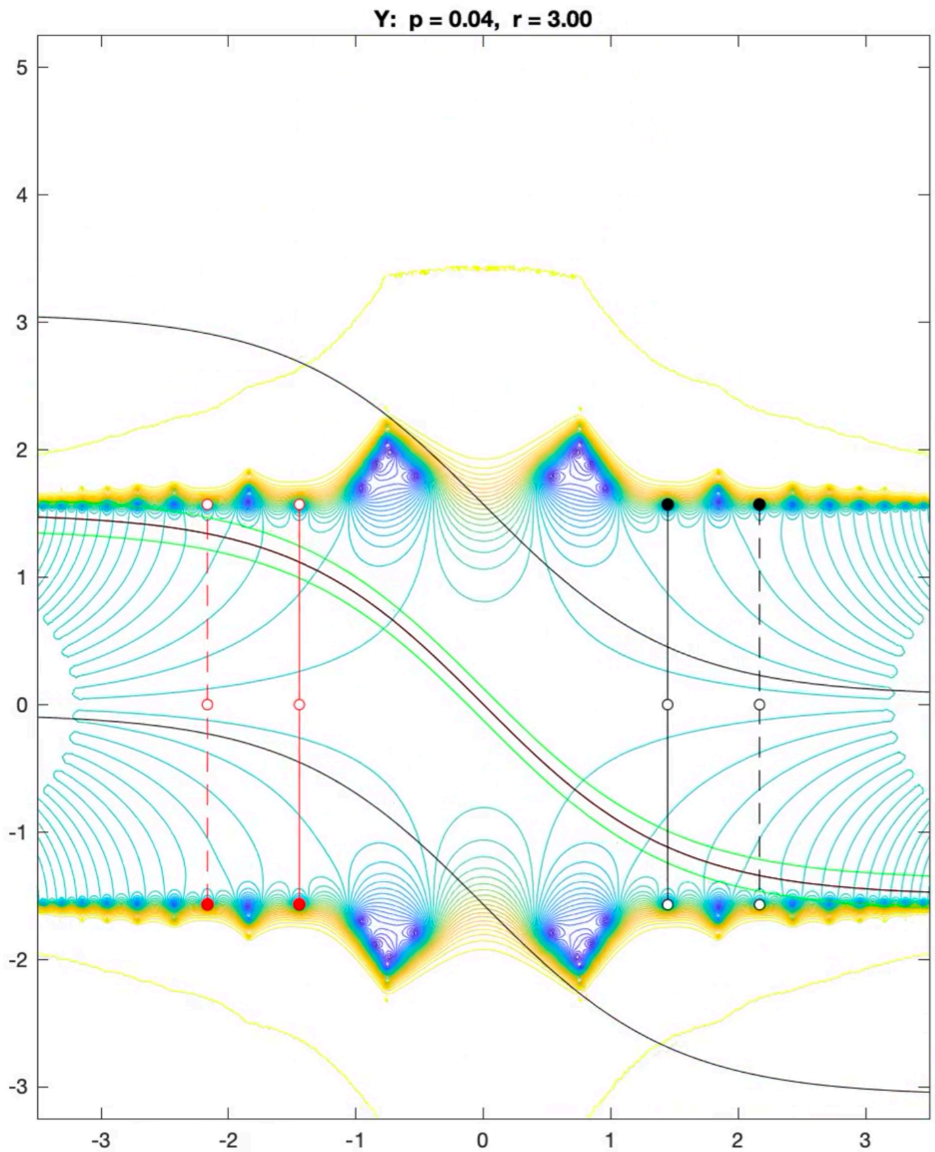
$|1+Y|$: 2 particles, $N=2$



$|1+Y|$: 2 particles, $N=3$



$|1+Y|$: 2 particles, $N = 4$



$|1+Y|$: 4 particles, $N_1, N_2 = 1, 2$

As r leaves the real axis the zeros of $|1+Y|$ wander about and induce transitions between the different TBAs.

<see movies>

Conclusions:

- The $p \to 0$ limit shows regularities of structure which make us expect that a more-complete picture of $c(r)$ in the complex plane will be possible.

- It would be interesting to study other models in a similar way, & also other limits of sinh-Gordon such as the staircase models...

The end...