Sine-Gordon - a beautiful model of quantum fields

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Bologna Workshop on “CFT and Integrable Models”

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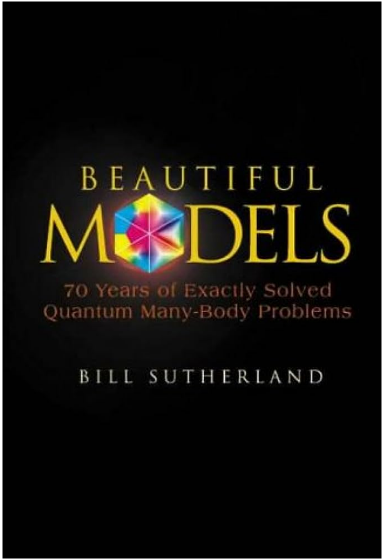
Outline of talk

1. Studying Sine-Gordon with Francesco
2. Sine-Gordon out of equilibrium
3. Truncated Hamiltonian method
4. Generalised hydrodynamics
5. Conclusions



Bologna 1997-99

1997-99: INFN postdoctoral fellow in Bologna

Then recent developments in sine-Gordon:

- New thermodynamic Bethe ansatz equations without strings
  C. Destri and H. J. de Vega, Phys. Rev. Lett. 69: 2313, 1992

- Excited State Destri - De Vega Equation for Sine-Gordon and Restricted Sine-Gordon Models,
  D. Fioravanti, A. Mariottini, E. Quattrini and F. Ravanini, Phys. Lett. B390: 243-251,1997

- Non linear integral equation and excited--states scaling functions in the sine-Gordon model
  C. Destri and H.J. de Vega, Nucl. Phys. B504: 621-664, 1997

However: some issues persisted...

Sine-Gordon model

$$\mathcal{A} = \int d^2x \left[\frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}(\partial_x\varphi)^2 + \frac{m^2}{\beta^2}\cos\beta\varphi\right]$$

Quantum dynamics: governed by Hamiltonian

$$H = \int dx \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x\varphi)^2 + \lambda :\cos\beta\varphi:\right]$$

$$[\lambda] = 2 - 2\Delta \qquad \Delta = \frac{\beta^2}{8\pi}$$

Soliton/antisoliton with mass M

Breathers: $m_n = 2M\sin\frac{n\pi\xi}{2} \qquad n = 1, \cdots < 1/\xi \qquad \xi = \frac{\beta^2}{8\pi - \beta^2}$

Exact S matrix: A.B. Zamolodchikov and Al.B. Zamolodchikov, Annals Phys. 120: 253-291, 1979.

Exact form factors: F.A. Smirnov: Form Factors in Completely Integrable Models of Quantum Field Theory, World Scientific, 1992.

Sine-Gordon in finite volume: the NLIE

$$H = \int_0^L dx \left[\frac{1}{2}\pi^2 + \frac{1}{2} (\partial_x \varphi)^2 + \lambda : \cos \beta\varphi : \right]$$

$$\varphi(x+L) = \varphi(x) + \frac{2\pi}{\beta} Q \qquad Q \in \mathbb{Z}$$

NLIE:

$$Z(\vartheta) = ML \sinh \vartheta + \alpha + g(\vartheta|\vartheta_j) + 2\Im m \int_{-\infty}^{\infty} dx G(\vartheta - x - i\eta) \log\left(1 + (-1)^\delta e^{iZ(x+i\eta)}\right)$$

$$g(\vartheta|\vartheta_j) = \sum_{k=1}^{N_H} \chi(\vartheta - h_k) - 2\sum_{k=1}^{N_S} \chi(\vartheta - y_k) \sum_{k=1}^{N_C} \chi(\vartheta - c_k) - \sum_{k=1}^{N_W} \chi(\vartheta - w_k)_{II}$$

$$E = E_{bulk} + M\sum_{j=1}^{N_H} \cosh h_j - 2M\sum_{j=1}^{N_S} \cosh y_j - M\sum_{j=1}^{N_C} \cosh c_j + M\sum_{j=1}^{N_W} (\cosh w_j)_{II}$$

$$- M \int_{-\infty}^{\infty} \frac{dx}{2\pi} 2\Im m \left[\sinh(x+i\eta) \log(1 + (-1)^\delta e^{iZ(x+i\eta)})\right]$$

Originally derived for Q even, works also for Q odd

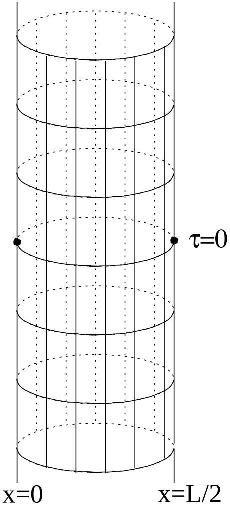
Choice of δ!

Hamiltonian truncation

1. Separate Hamiltonian into exactly solvable part H_0 + deformation H_1

$$
\begin{array}{rcc}
H = & H_0 & + \; H_1 \\
\text{sine-Gordon}: & \int dx \frac{1}{2}\left[(\partial_t\varphi)^2 + (\partial_x\varphi)^2\right] & + \lambda \int dx \cos\beta\varphi
\end{array}
$$

2. Put system in finite volume $x \equiv x + L$ (PBC) - discrete spectrum
3. Use eigenstates of H_0 as computational basis
4. Truncate Hilbert space by restricting to the subspace $H_0 \leq \Lambda$



Method is non-perturbative - equivalent to Rayleigh-Ritz for spectrum

H_0 is conformal: truncated conformal space approach (TCSA)

original idea: V.P. Yurov & A.B. Zamolodchikov, Int.J.Mod.Phys. A5 (1990) 3221-3245.

sine-Gordon: G. Feverati, F. Ravanini & GT, Phys. Lett. B430 (1998) 264-273.

Results from 1997-99

G. Feverati, F. Ravanini and G. Takács:

Truncated conformal space at c=1, nonlinear integral equation and quantization rules for multi-soliton states

Phys. Lett. B430: 264-273, 1998.

Sine-Gordon TCSA + pure even soliton excited states - $\delta = +1$

Non-linear integral equation and finite volume spectrum of sine-Gordon theory, Nucl. Phys. B540: 543-586, 1999.

Correct derivation of NLIE with complex roots, states with even # of solitons/anti-solitons

Scaling functions in the odd charge sector of sine-Gordon/massive Thirring theory

Phys. Lett. B430: 264-273, 1998.

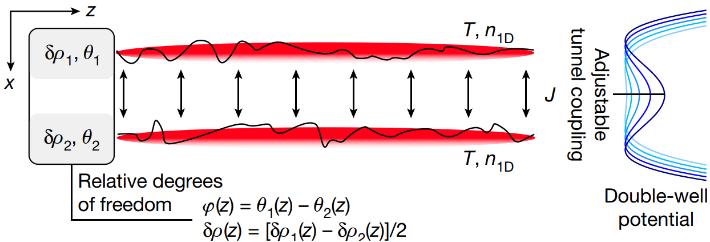
States with odd topological charge, $\delta = \pm 1$ *fermionic (mTH) / bosonic (sG)*

Non-linear integral equation and finite volume spectrum of minimal models perturbed by $\Phi_{1,3}$.

Nucl. Phys. B570: 615-643, 1999.

Breather states, perturbed minimal models

Experimental realisation of Sine-Gordon model



T. Schweigler et al., Nature 545: 323–326, 2017

1D quasi-condensates: Luttinger liquid

[V. Gritsev, A. Polkovnikov and E. Demler, Phys. Rev. B75: 174511, 2007]

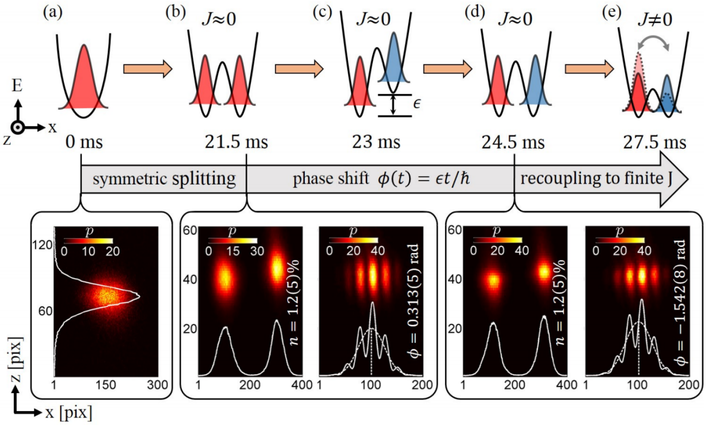
$$H = \sum_{i=1}^{2} \frac{\hbar c}{2} \int dx \left[\frac{\pi}{K} \delta\rho_i^2 + \frac{K}{\pi} (\partial_x \theta_i)^2 \right] - 2\hbar J \rho_0 \int dx \cos\theta_r$$

$$\delta\rho_c = \delta\rho_1 + \delta\rho_2 \qquad \theta_c = \frac{\theta_1 + \theta_2}{2} \quad K_c = 2K \qquad \delta\rho_r = \frac{\delta\rho_1 - \delta\rho_2}{2} \qquad \theta_c = \theta_1 - \theta_2 \quad K_r = K/2$$

Deviations from sG!

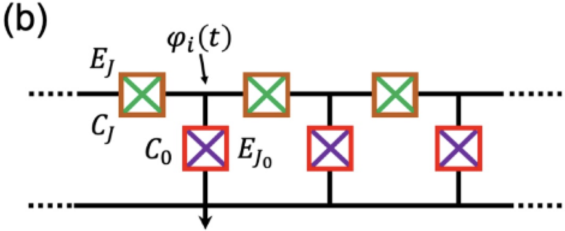
$\xi = 1/(8K-1) \approx 1/215$ very small coupling - 215 breathers

Quantum quenches with coupled condensates



M.Pigneur et al.: Phys. Rev. Lett. ***120*** *(2018) 173601*

Another proposal: Josephson circuits



A. Roy, D. Schuricht, J. Hauschild, F. Pollmann and H. Saleur, Nucl. Phys. B968: 115445, 2021.

$$H = E_{C_0} \sum_{i=1}^{L} n_i^2 + \delta E_{C_0} \sum_{i=1}^{L-1} n_i n_{i+1} - E_g \sum_{i=1}^{L} n_i - E_J \sum_{i=1}^{L-1} \cos(\varphi_i - \varphi_{i+1}) - E_{J_0} \sum_{i=1}^{L} \cos\varphi_i \qquad E_J$$

$$S = \frac{1}{2\pi K} \int d^2x \left[\frac{1}{u} (\partial_t \varphi)^2 + u (\partial_x \varphi)^2 + M_0 \cos\varphi \right]$$

$$u \simeq a\sqrt{2E_{C_0} E_J} \qquad K \simeq \sqrt{2E_{C_0}/E_J} \qquad M_0 = E_{J_0} a^{-(1-K/4)}$$

Quantum quenches in QFT

Quantum quench: a protocol for non-equilibrium dynamics

$$\text{Pre-quench:} \quad H_{pre}|\Omega_{pre}\rangle = E_0|\Omega_{pre}\rangle$$

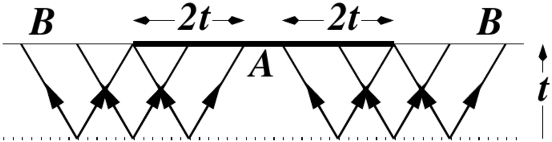
$$\text{Post-quench:} \quad i\frac{\partial}{\partial t}|\Psi(t)\rangle = H_{post}|\Psi(t)\rangle$$

$$|\Psi(0)\rangle = |\Omega_{pre}\rangle$$

Global quantum quench: both H_{pre} and H_{post} are translationally invariant.
Global QQ injects finite energy density $\mathcal{E}$ into the system:

$$\langle\Psi(0)|H_{post}|\Psi(0)\rangle - \langle\Omega_{post}|H_{post}|\Omega_{post}\rangle = \mathcal{E}\cdot\text{volume}$$

Semiclassical picture for time evolution (Calabrese & Cardy, 2006)



Computing the time evolution

Use eigenbasis of postquench Hamiltonian

$$|\Psi(0)\rangle = \sum_r C_r |r\rangle \qquad H_{post}|r\rangle = E_r|r\rangle$$

$$\langle\Psi(t)|\mathcal{O}|\Psi(t)\rangle = \sum_{r,s} C_r^* C_s e^{-it(E_s - E_r)} \langle r|\mathcal{O}|s\rangle$$

Many ingredients needed + difficult to compute:

- C_r: initial state overlaps
- E_r: post-quench energy levels
- $\langle r|\mathcal{O}|s\rangle$: form factors of observable
- (partially) performing double sum, dominated by finite density states

D. Schuricht and F.H.L. Essler, J. Stat. Mech. 1204 (2012) P04017
B. Bertini, D. Schuricht and F.H.L. Essler, J. Stat. Mech. 1410 (2014) P10035

But in many interesting cases (e.g. attractive sine-Gordon), it cannot be used :(

A.C. Cubero and D. Schuricht, J. Stat. Mech. 1710 (2017) 103106 **???**
D.X. Horváth, M. Kormos and G. Takács, JHEP 2018 (2018) 170 **!!!**

FF expansion

$$|\Psi(0)\rangle = \sum_r C_r |r\rangle \qquad H_{post}|r\rangle = E_r|r\rangle$$

$$\langle\Psi(t)|\mathcal{O}|\Psi(t)\rangle = \sum_{r,s} C_r^* C_s e^{-it(E_s-E_r)}\langle r|\mathcal{O}|s\rangle \quad \text{for} \quad \mathcal{O} = e^{i\beta\varphi/2}$$

$$|\Psi_0\rangle = \exp\left(\frac{g_B}{2}A^\dagger(0) + \int \frac{d\vartheta}{2\pi} K_B(\vartheta) A_B^\dagger(\vartheta) A_B^\dagger(-\vartheta) + \text{solitons} + \dots\right)|0\rangle$$

Linked cluster expansion $\Rightarrow$

$$\langle\Psi(t)|\mathcal{O}|\Psi(t)\rangle = \langle 0|\mathcal{O}|0\rangle\left[1 - \frac{t}{\tau} + \frac{t^2}{2\tau^2} + \dots\right] = \langle 0|\mathcal{O}|0\rangle e^{-t/\tau}$$

$$\tau^{-1} = \tau_S^{-1} + \tau_B^{-1} + \tau_{BB}^{-1} + \dots$$

B. Bertini, D. Schuricht and F.H.L. Essler, J. Stat. Mech. 1410 (2014) P10035

$$\tau_S^{-1} = \frac{2M}{\pi}\int_0^\infty d\theta\, |K_{s\bar{s}}(\theta)|^2 \sinh\theta + O\left(K^4\right)$$

Overlap singularity and other problems

Breather contribution A.C. Cubero and D. Schuricht, J. Stat. Mech. 1710 (2017) 10310

$$\tau_n^{-1} = \frac{M}{\pi}\int_0^\infty d\theta \left|K_{s\bar{s}}(\theta)\right|^2 \left(1 + S_{SB_n}(\theta)\right)\sinh\theta + O\left(K^4\right)$$

$$\tau_{nm}^{-1} = \frac{m_n}{\pi}\int_0^\infty d\theta \left|K_n(\theta)\right|^2 \left(1 - S_{B_nB_m}(\theta)\right)\sinh\theta + O\left(K^4\right)$$

$g \neq 0$ for breathers $\Rightarrow$ overlap singularity:

$$g \neq 0 \Rightarrow K(\vartheta \sim 0) \sim -i\frac{g^2}{2}\frac{1}{\vartheta}$$

M. Kormos and B. Pozsgay, JHEP 1004:112, 2010

FF expansion: can be performed using finite volume regularisation

B. Pozsgay and G. Takács, J. Stat. Mech. 1011: 110120, 2010

The singularity cancels! But: we do not know how to perform resummation of the series

D.X. Horváth, M. Kormos and G. Takács, JHEP 2018 (2018) 170

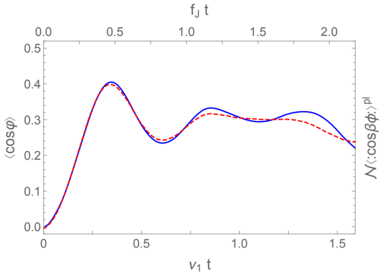
+: overlaps are hard to know - only partial information for some simple quenches

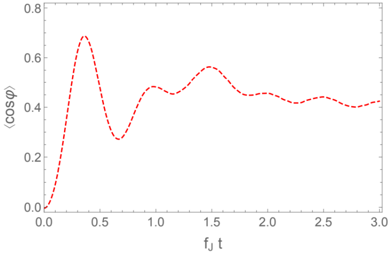
D.X. Horváth and G. Takács, Phys. Lett. B771: 539–545, 2017

Hamiltonian truncation for time evolution

Motivation:

- obtain full quantum evolution
- cross-check other (semiclassical) methods e.g. truncated Wigner approximation (TWA)





TCSA vs. TWA $K = 1.56$ TWA $K = 27$

Initial state: free boson ground state (conformal vacuum)

Nonequilibrium time evolution and rephasing in the quantum sine-Gordon model

D. X. Horváth, I. Lovas, M. Kormos, G. Takács and G. Zaránd

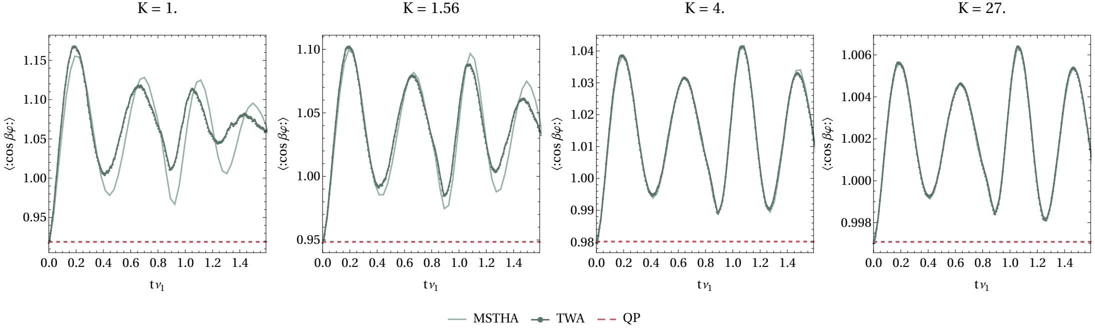
Phys. Rev. A 100: 013613, 2019.

Minisuperspace improvement: MSTHA

Idea: use quantum pendulum eigenbasis for zero mode

$$H_{QP} = \frac{1}{2}\pi_0^2 + \lambda L \left(\frac{L}{2\pi}\right)^{2\Delta} \cos\beta\varphi_0$$

Small quenches (starting from pendulum ground state)

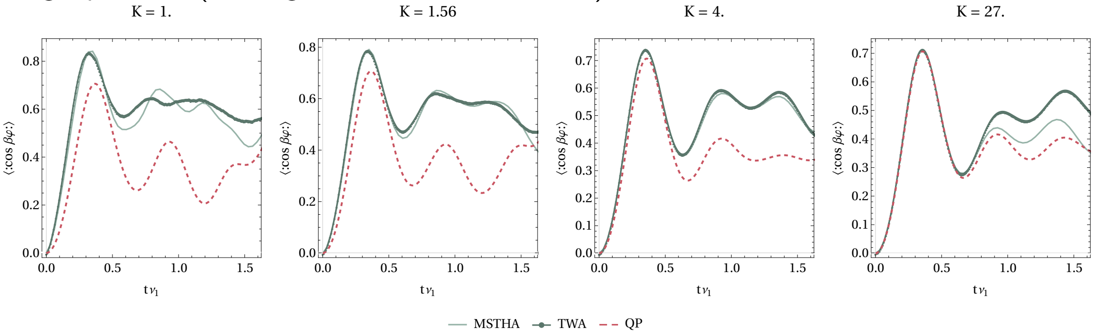


Minisuperspace improvement: MSTHA

Idea: use quantum pendulum eigenbasis for zero mode

$$H_{QP} = \frac{1}{2}\pi_0^2 + \lambda L \left(\frac{L}{2\pi}\right)^{2\Delta} \cos \beta\varphi_0$$

Large quenches (starting from conformal vacuum)



$K = 27$: MSTHA has not converged, TWA is expected to be accurate

D. Szász-Schagrin, I. Lovas and G. Takács, to appear - see poster at workshop!

Thermodynamics and GHD in sine-Gordon

Alternative for time evolution: generalised hydrodynamics.
However: TBA not known explicitely for generic coupling!
For N_s kinks/antikinks and N_{B_i} breathers of type B_i :

$$e^{iM_{B_i}L\sinh\theta^{(j)}_{B_i}}\prod_{\substack{k=1\\(k,l)\neq(i,j)}}^{n_B}\prod_{l=1}^{N_{B_k}}S_{B_i,B_k}\left(\theta^{(j)}_{B_i}-\theta^{(l)}_{B_k}\right)\prod_{k=1}^{N_s}S_{\pm,B_i}\left(\theta^{(j)}_{B_i}-\theta_k\right)=\quad 1\,,\qquad j=1,...,N_{B_i},\quad i=$$

$$e^{iML\sinh\theta_k}\Lambda(\theta_k|\{\mu_m\},\{\theta_l\})\prod_{i=1}^{n_B}\prod_{j=1}^{N_{B_i}}S_{\pm,B_i}\left(\theta_k-\theta^{(j)}_{B_i}\right)=-1\,,\qquad k=1,...,N_s$$

$$\Lambda\left(\theta_k|\{\mu_m\},\{\theta_l\}\right)=\prod_{m=1}^{N_m}\frac{1}{S_T(\mu_m-\theta_k)}\prod_{l=1}^{N_s}S_0(\theta_k-\theta_l)$$

$$\text{Magnons}:\prod_{l=1}^{N_s}\frac{1}{S_T(\mu_r-\theta_l)}=\prod_{\substack{q=1\\q\neq r}}^{N_m}\frac{S_T(\mu_r-\mu_q)}{S_T(\mu_q-\mu_r)}\,,\qquad r=1,\ldots,N_m$$

TBA for sine-Gordon

String hypothesis (magnons: same as Takahashi's for XXZ)

$$\xi = \cfrac{1}{n_B + \cfrac{1}{\nu_1 + \cfrac{1}{\nu_2 + \dots}}} = \cfrac{1}{n_B + \cfrac{1}{\alpha}}$$

Fully coupled TBA equations

$$\epsilon_a = \frac{M_a}{T}\cosh\theta - \frac{\mu}{T}q_a - \sum_b \eta_b \Phi_{ab} * \log(1 + e^{-\epsilon_b})$$

$$\frac{f}{T} = -\sum_a \int \frac{\mathrm{d}\theta}{2\pi} \eta_a M_a \cosh\theta \Big[\log(1 + \mathrm{e}^{-\epsilon_a})\Big]$$

Partial decoupling:

M. Takahashi and M. Suzuki, Prog. Theor. Phys. 48: 2187-2209, 1972.

Al.B.Zamolodchikov, Phys. Lett. B253: 391-394, 1991.

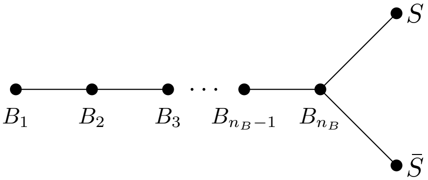
F.Ravanini, R.Tateo and A.Valleriani, Int.J.Mod.Phys. A8: 1707-1728, 1993.

E. Boulat, arXiv:1912.03872.

TBA for sine-Gordon: reflectionless

$$\epsilon_a = w_a + \sum_b K_{ab} * \left(\sigma_b^{(1)} \epsilon_b - \sigma_b^{(2)} w_b + L_b\right)$$

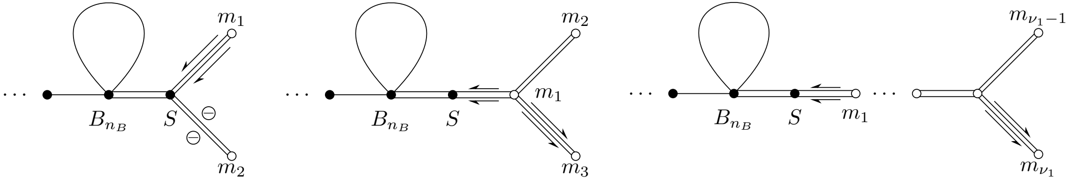
Reflectionless points: $\xi = \dfrac{1}{n_B - 1}$



Excitations	Labels	w	q	η	$\sigma^{(1)}$	$\sigma^{(2)}$
Breathers	$B_i,\ i = 1, \ldots, n_B$	$M_{B_i} \cosh(\theta)/T$	0	$+1$	$+1$	$+1$
Soliton	S	$M \cosh(\theta)/T - \mu/T$	$+1$	$+1$	$+1$	$+1$
Antisoliton	$\bar{S}$	$M \cosh(\theta)/T + \mu/T$	-1	$+1$	$+1$	$+1$

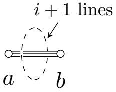
TBA for sine-Gordon: level 1

Level 1: $\xi = \dfrac{1}{n_B + \dfrac{1}{\nu_1}}$

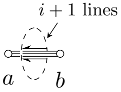


Excitations	Labels	w	q	η	v	$\sigma^{(1)}$	$\sigma^{(2)}$
Breathers	$B_i,\ i = 1, ..., n_B$	$M_{B_i} \cosh(\theta)/T$	0	+1	0	+1	+1
Soliton	S	$M \cosh(\theta)/T$	+1	+1	0	0	0
Intermediate magnons	$m_j,\ i = j, ...\nu_1 - 2$	0	$-2 \cdot j$	−1	+1	+1	0
Next-to-last magnon	$m_{\nu_1 - 1},\ (j = \nu_1 - 1)$	$\nu_1 \cdot \mu/T$	$-2 \cdot j$	−1	+1	+1	0
Last magnon	m_{ν_1}	$\nu_1 \cdot \mu/T$	−2	+1	−1	0	0

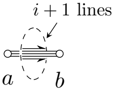
TBA for sine-Gordon: kernels



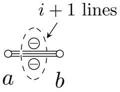
$$\tilde{K}_{ab} = \tilde{\Phi}_{p_i}(t)$$



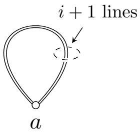
$$-\tilde{K}_{ab} = \tilde{K}_{ba} = \tilde{\Phi}_{p_i}(t)$$



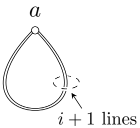
$$\tilde{K}_{ab} = -\tilde{K}_{ba} = \tilde{\Phi}_{p_i}(t)$$



$$\tilde{K}_{ab} = -\tilde{\Phi}_{p_i}(t)$$



$$\tilde{K}_{aa} = \tilde{\Phi}^{(i)}_{\text{self}}(t)$$



$$\tilde{K}_{aa} = -\tilde{\Phi}^{(i)}_{\text{self}}(t)$$

$$\tilde{\Phi}_{p_i}(t) = \frac{1}{2\cosh\left(p_i \frac{\pi}{2} \frac{\xi}{\alpha} t\right)}$$

$$\tilde{\Phi}^{(i)}_{\text{self}}(t) = \frac{\cosh\left((p_i - p_{i+1}) \frac{\pi}{2} \frac{\xi}{\alpha} t\right)}{2\cosh\left(p_i \frac{\pi}{2} \frac{\xi}{\alpha} t\right)\cosh\left(p_{i+1} \frac{\pi}{2} \frac{\xi}{\alpha} t\right)}$$

TBA for sine-Gordon: densities and cross-check

$$\eta_a\,\rho_a^{\rm tot} = \frac{\partial_\theta p_a}{2\pi} + \sum_b K_{ab} * \left[\left(\sigma_b^{(1)} - \vartheta_b\right)\eta_b\,\rho_b^{\rm tot} - \sigma_b^{(2)}\frac{\partial_\theta p_b}{2\pi}\right]$$

with $p_a = M_a \sinh\theta$ and the filling fractions

$$\vartheta_a(\theta) = \frac{\rho_a^{\rm r}(\theta)}{\rho_a^{\rm tot}(\theta)} = \frac{1}{1+e^{\epsilon_a}}$$

T/M	$\xi = 1/(1+\frac{1}{3})$		$\xi = 3$	
	DdV	TBA	DdV	TBA
20	-10.4512155761	-10.4512155756	-10.3511969646	-10.3511979842
10	-5.19859206357	-5.19859206333	-5.10437699605	-5.10437749589
5	-2.55324704632	-2.55324704620	-2.47030920896	-2.47030944780
2	-0.92684324811	-0.92684324807	-0.87159594008	-0.87159602002
1	-0.35869323482	-0.35869323480	-0.33218987940	-0.33218990576
0.5	-0.08847607829	-0.08847607828	-0.08339026939	-0.08339027349
0.2	-0.00258436645	-0.00258436645	-0.00256472781	-0.00256472782

TBA for sine-Gordon

Also checked against
R. Tateo, Int. J. Mod. Phys. A10: 1357-1376, 1995.

! Talk ! Wednesday 14:30-15:20
M. Kormos
Thermodynamics, transport, and fluctuations in the sine-Gordon model

B.C. Nagy, M. Kormos, and G. Takács, arXiv:2305.15474

Also related:

A. Bastianello (Wednesday 15:20-15:50)
Sine-Gordon: from coupled condensates to hydrodynamics

Conclusions

1. Sine-Gordon is beautiful!

   Introduced by Edmond Bour (1862) and still of great interest

2. Recent interest fueled by non-equilibrium dynamics

   Existing and proposed experimental realisations

3. Several approaches:
   3.1 Spectral expansion (exact form factors!)
   3.2 Semiclassical (truncated Wigner approximaton, mean field)
   3.3 Truncated Hamiltonian
   3.4 GHD

4. Numerous interesting challenges remain!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