The Frustration of being Odd

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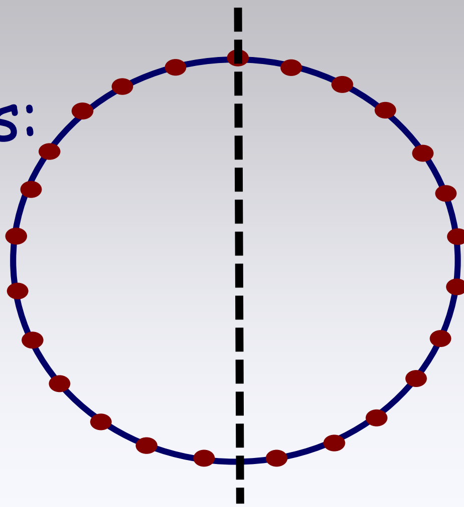
and others...

Based on:

- J. Phys. Commun. 3, 081001 (2019);
- New J. Phys. 22 083024 (2020);
- Nature's Comm. Phys. 3, 220 (2020);
- J. Phys. A 54 025201 (2020);
- Sci Rep 11, 6508 (2021);
- Phys. Rev. B 103, 014429 (2021);
- Phys. Rev. B 105, 064408 (2022);
- SciPost Phys. 12, 075 (2022);
- Phys. Rev. B 105, 184424 (2022);
- Phys. Rev. B 106, 125145 (2022);
- arXiv:2209.10541; arXiv:2210.13495; arXiv:2307.02529 & work in progress

Support: **Croatian Science Fundation** Project No. IP-2019-04-3321;
European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (KK.01.1.1.01.0004)
Horizon 2020 Marie Skłodowska-Curie MoQS ITN (GA #955479).

Frustrated Boundary Conditions



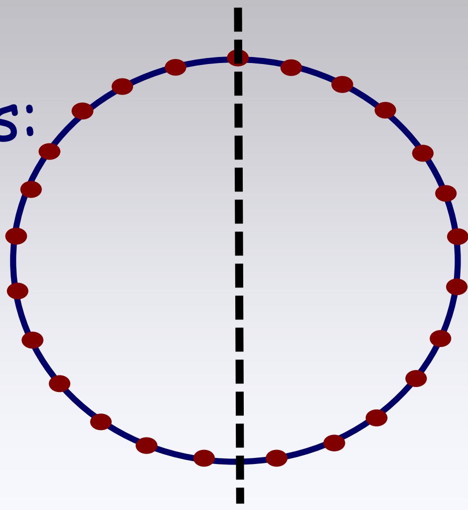
- We revisited an old problem for spin chains:

  the effect of periodic b.c. with

  an ODD number of sites

- Referred as: Frustrated Boundary Conditions (FBC)

Frustrated Boundary Conditions



- We revisited an old problem for spin chains:

  the effect of periodic b.c. with an ODD number of sites

- Referred as: Frustrated Boundary Conditions (FBC)
- Why such interest?
- From one side: b.c. can only matter in finite systems

  (are we sure?)

- From another side: FBC are special

1D Classical Ising

$$H_{\text{Ising}} = J \sum_{l=1}^{2M+1} \sigma_l^x \sigma_{l+1}^x$$

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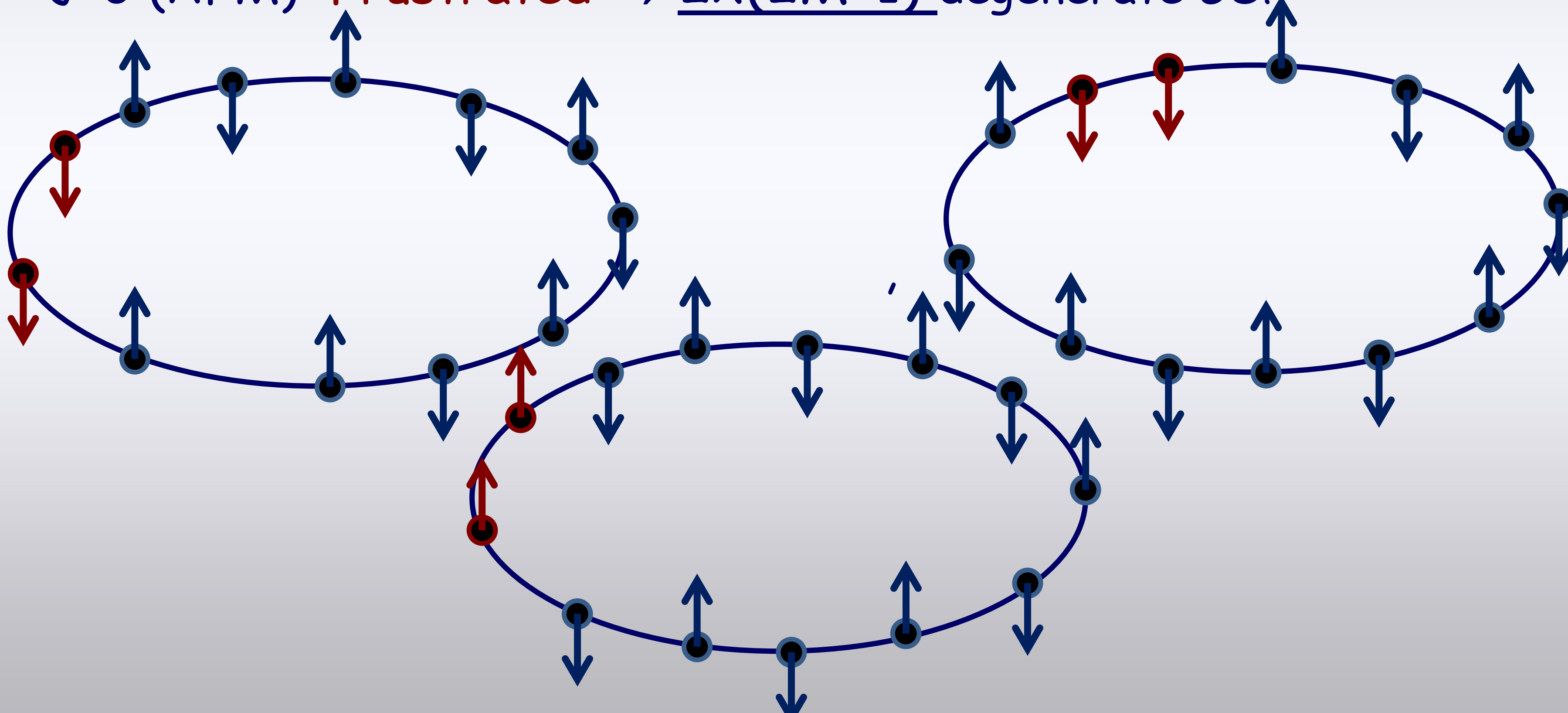
- J<0 (Ferromagnetic): 2 deg ground states
- J>0 (AFM): Frustrated → 2x(2M+1) degenerate GS!



Perturbative picture

$$H = \frac{1}{2} \sum_{l=1}^{2M+1} \left(\sigma_l^x \sigma_{l+1}^x + \lambda h_l \right)$$

- At λ=0: 2N-degenerate GS (2 x Neel with 1 domain wall) (compare to 2-degenerate for N even, i.e not frustrated)
- Turn on $\lambda \neq 0$: degeneracy lifted, GS part of a band
- Perturbative picture: low-energy eigenstates as a traveling domain wall with different momenta

Laumann et al, PRL (2012)

Order parameter

$$H_{\text{Ising}} = \frac{1}{2} \sum_{l=1}^{2M+1} \left(\sigma_l^x \sigma_{l+1}^x + \lambda \sigma_l^y \sigma_{l+1}^y \right)$$

- Without frustration, Z_2 broken phase: $\langle \sigma_j^x \rangle = \pm(-1)^j \ m_x$
- Staggered order not compatible with pbc and odd # sites

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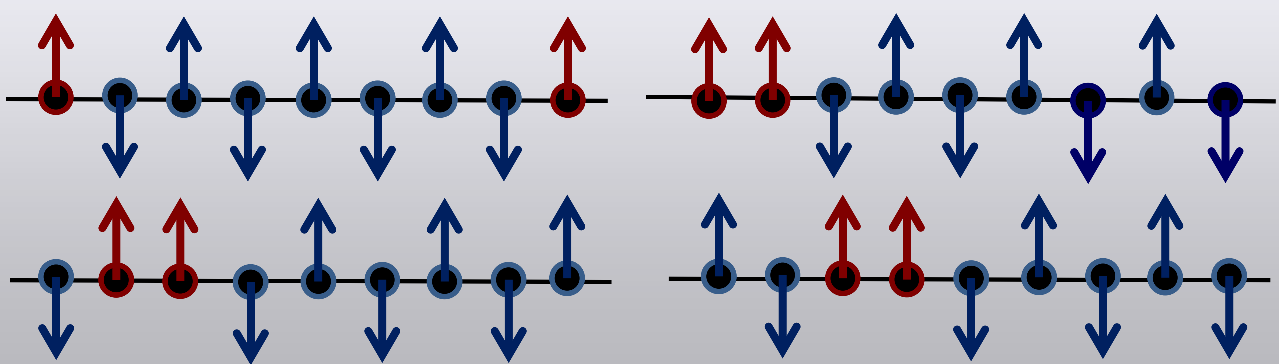

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- Alternatively: 2 counterpropagating waves

  ⇒ non perfect staggerization (& beats)

$$\langle \sigma_j^x \rangle = (-1)^j \cos\left(\pi \frac{j}{N} + \theta \right) m_x$$

Paradoxical conclusion?

$$H_{\text{Ising}} = \frac{1}{2} \sum_{l=1}^{2M+1} \left(\sigma_l^x \sigma_{l+1}^x + \lambda \sigma_l^y \sigma_{l+1}^y \right)$$

- XY chain posterchild for SSB $\Rightarrow$ order parameter
- Perturbative calculation show fragility against FBC

Can boundary conditions affect bulk order?

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Can boundary conditions affect bulk order?

We perform non-perturbative analytically exact and numerical analysis to analyze the subtle effects of FBC

- Non perturbative calculations confirm perturbative picture!

Frustrated Systems

- Certain degree of frustration is very common
- In any dimension, due to closed AFM loops
- Typically: extensive frustration
  (# loops scale with system size)
  - Ordered (ANNNI model, spin-ice…)
  - Disordered (Sherrington-Kirkpatrick model, spin glasses…)
- Peculiar physics: residual entropy, local zero-modes, algebraic decay, artificial EM, monopoles, Dirac strings…
- Hard problem

Frustrated Boundary Conditions

- We pursue a bottom-up approach
- We consider a simple setting:

  spin chains with frustration only from b.c. (FBC)

- We find emergent phenomenologies reminiscent of extensive frustration case!

Chain in zero field

$$H = \sum_{j=1}^{2M+1} \left[\cos\delta \left(\cos\phi\, \sigma_j^x \sigma_{j+1}^x + \sin\phi\, \sigma_j^y \sigma_{j+1}^y \right) - \sin\delta\, \sigma_j^z \sigma_{j+1}^z \right]$$

- In absence of external fields, H is T-Symmetric

  $\Rightarrow$ Kramer's degeneracy

- GS is at least 2-fold degenerate even at finite size

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- GS is at least 2-fold degenerate even at finite size

- Any GS choice breaks a parity symmetry: $\Pi^\alpha \equiv \prod_{j=1}^{N} \sigma_j^\alpha, \;\; \alpha = x, y, z$

  $\Rightarrow$ we can SSB at finite N!

- Avoid usual tricky order of limits with external field

XYZ Chain in zero field

$$H = \sum_{j=1}^{2M+1} \left[\cos\delta \left(\cos\phi \, \sigma_j^x \sigma_{j+1}^x + \sin\phi \, \sigma_j^y \sigma_{j+1}^y \right) - \sin\delta \, \sigma_j^z \sigma_{j+1}^z \right]$$

- Exact finite size degeneracy allow "SSB at finite N"
- We also found a way to compute 1-point fct. directly

  ⇒ Exact finite size expressions for observables (magnetization) and follow them to the thermodynamic limit

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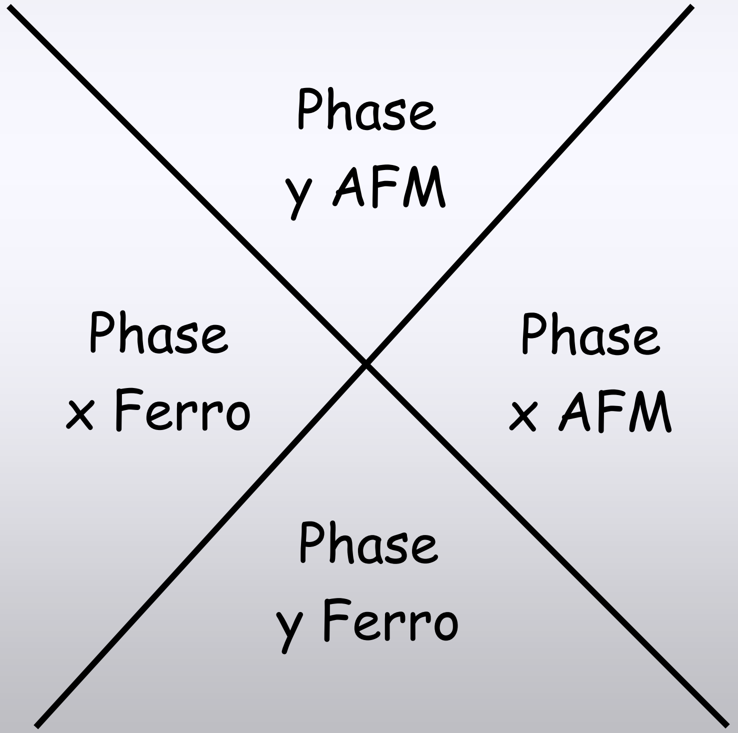
Duality
$\sigma_j^x \leftrightarrow \sigma_j^y$

N=2M: not frustrated



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N=2M+1

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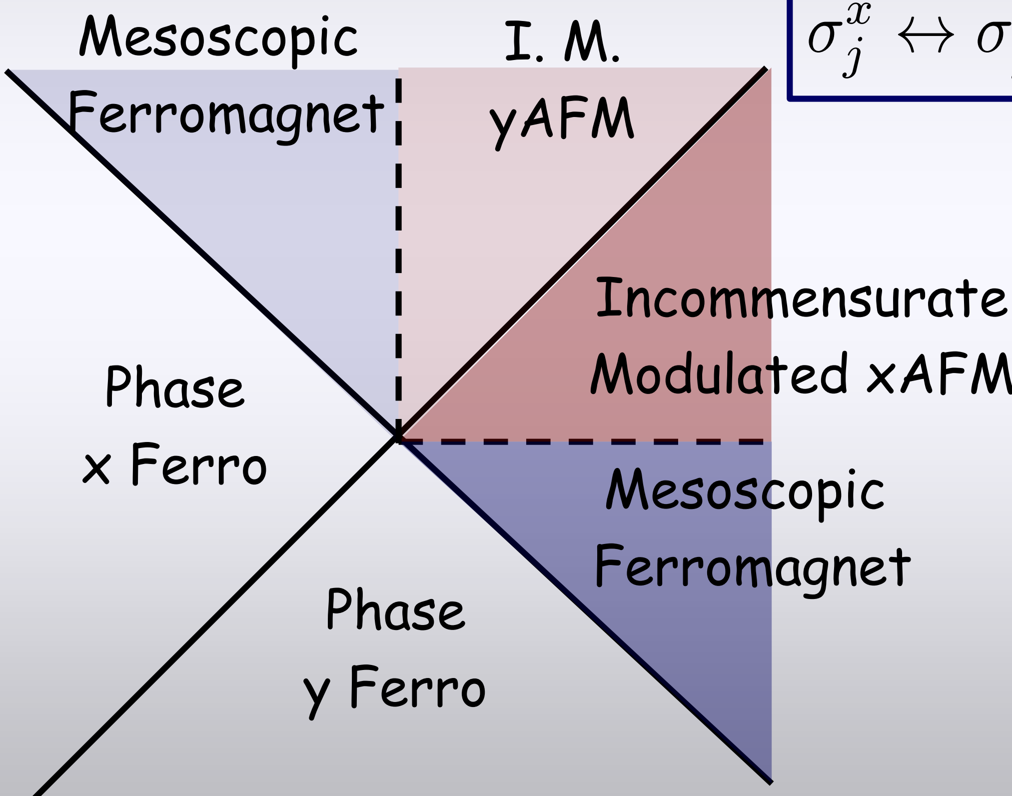

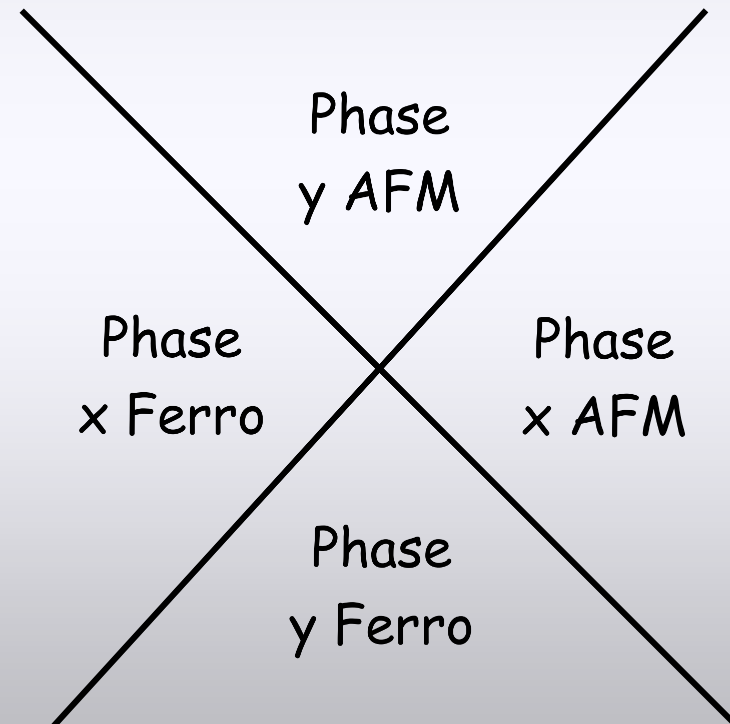

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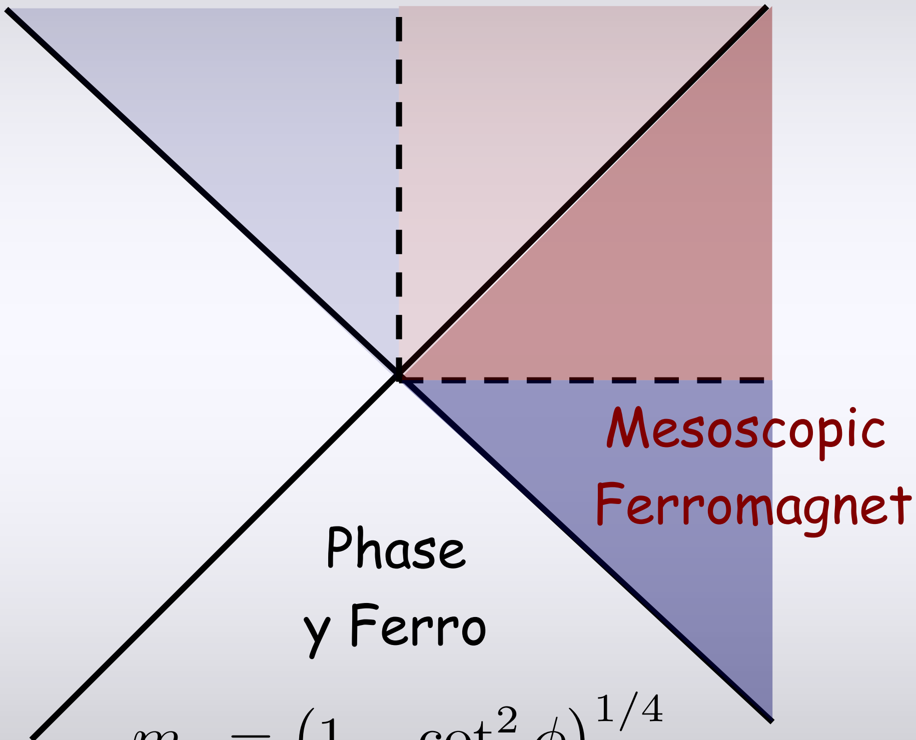


$$m_y = \left(1 - \cot^2\phi\right)^{1/4}$$

Consistency check on methodology!

$$m_\alpha \simeq \frac{\tilde{m}_\alpha}{N^\gamma} \overset{N\to\infty}{\longrightarrow} 0$$

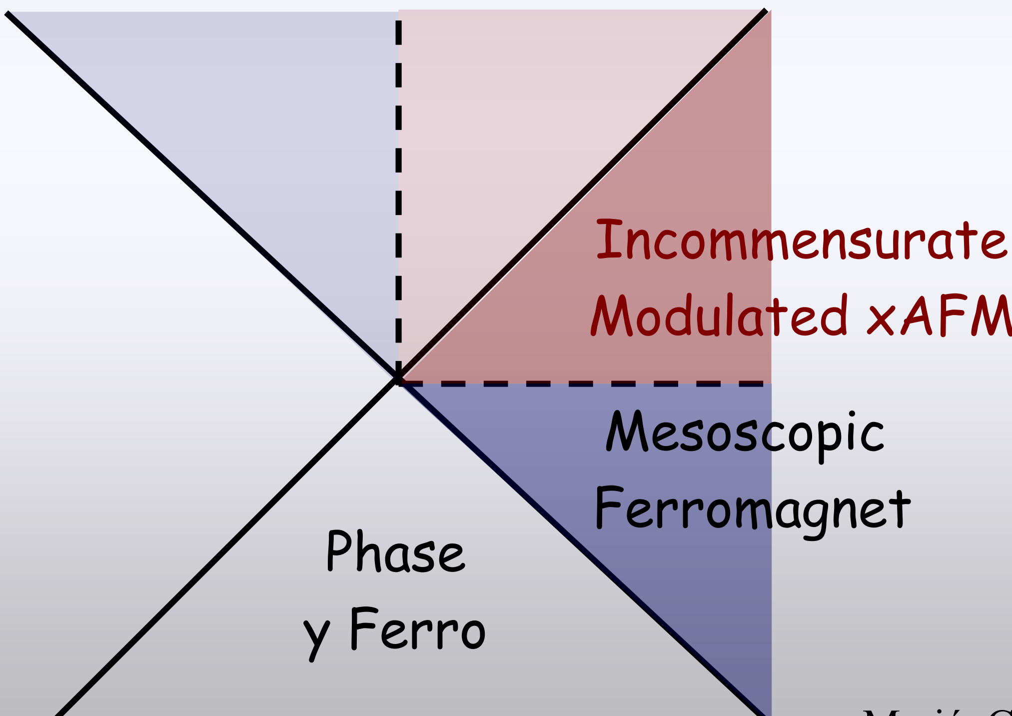
All magnetizations decay algebraically to zero and are not staggered!

Marić, Giampaolo, Kuić, Franchini, New J. Phys. 22 083024 (2020)

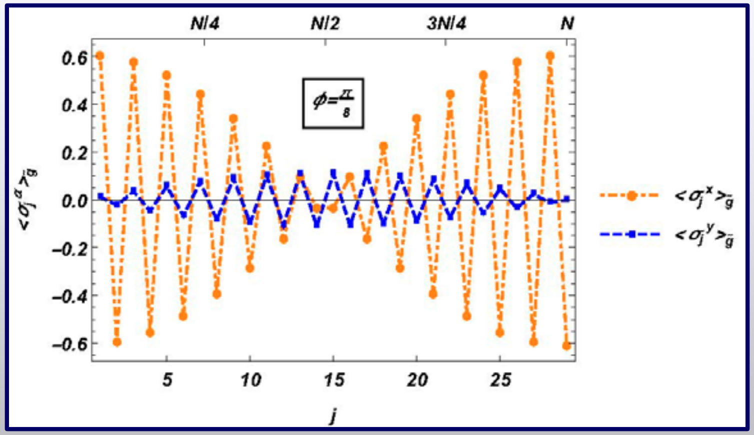
Incommensurate Modulated AFM

$$H = \sum_{j=1}^{2M+1} \left[\cos\phi \, \sigma_j^x \sigma_{j+1}^x + \sin\phi \, \sigma_j^y \sigma_{j+1}^y \right]$$

- IMAFM: $\phi \in (0, \pi/4)$



- 2 frustrated) AFM int.
- Lowest energy states have finite momentum ±π/2
- GS can break transl. Inv.



Marić, Giampaolo, Franchini, Comm. Phys. 3, 220 (2020)

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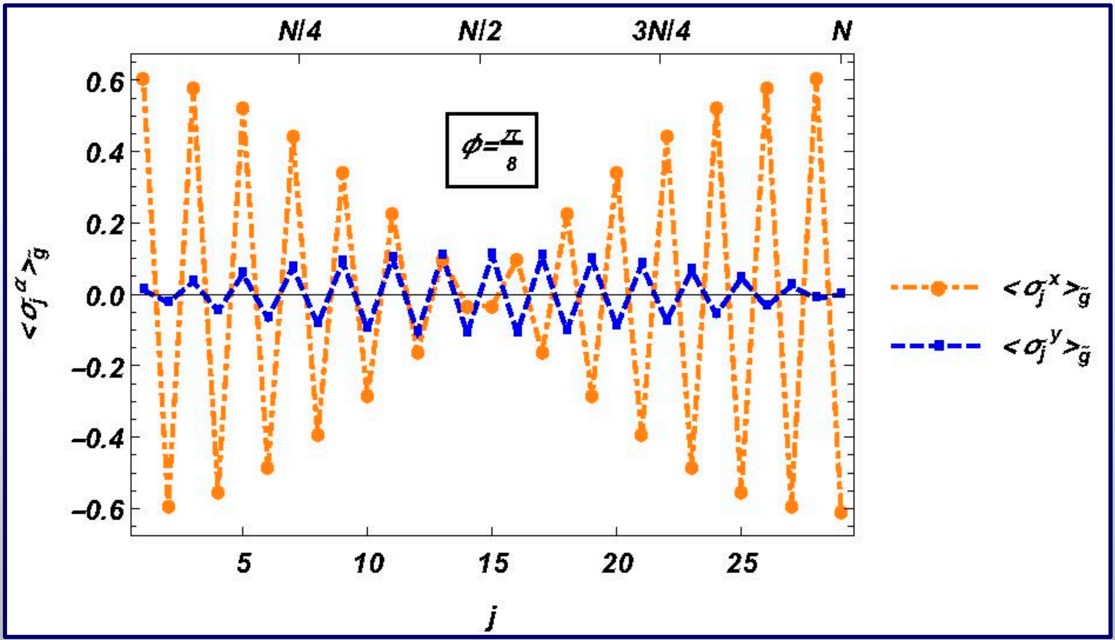
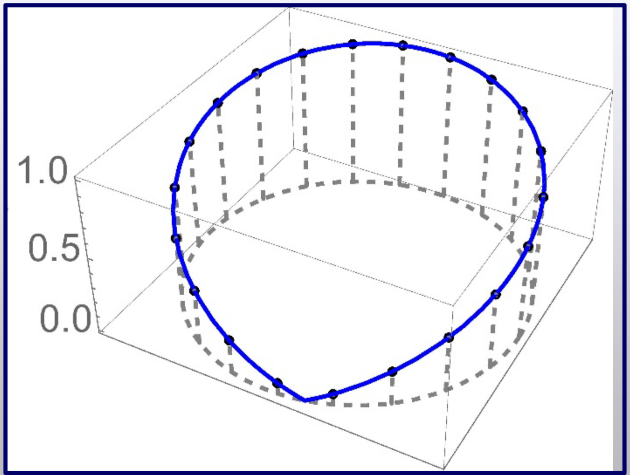
$$\langle \tilde{g} | \sigma_j^\alpha | \tilde{g} \rangle = (-1)^j \cos\left(\pi \frac{j}{N} + \tilde{\theta}_\alpha\right) f_\alpha$$





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$$p \equiv \frac{\pi}{2}\left(1 + \frac{1}{N}\right)$$

$$|\tilde{g}\rangle \equiv \frac{1}{\sqrt{2}}\left[|\pm p, +\rangle + e^{i\theta}|\mp p, -\rangle\right]$$

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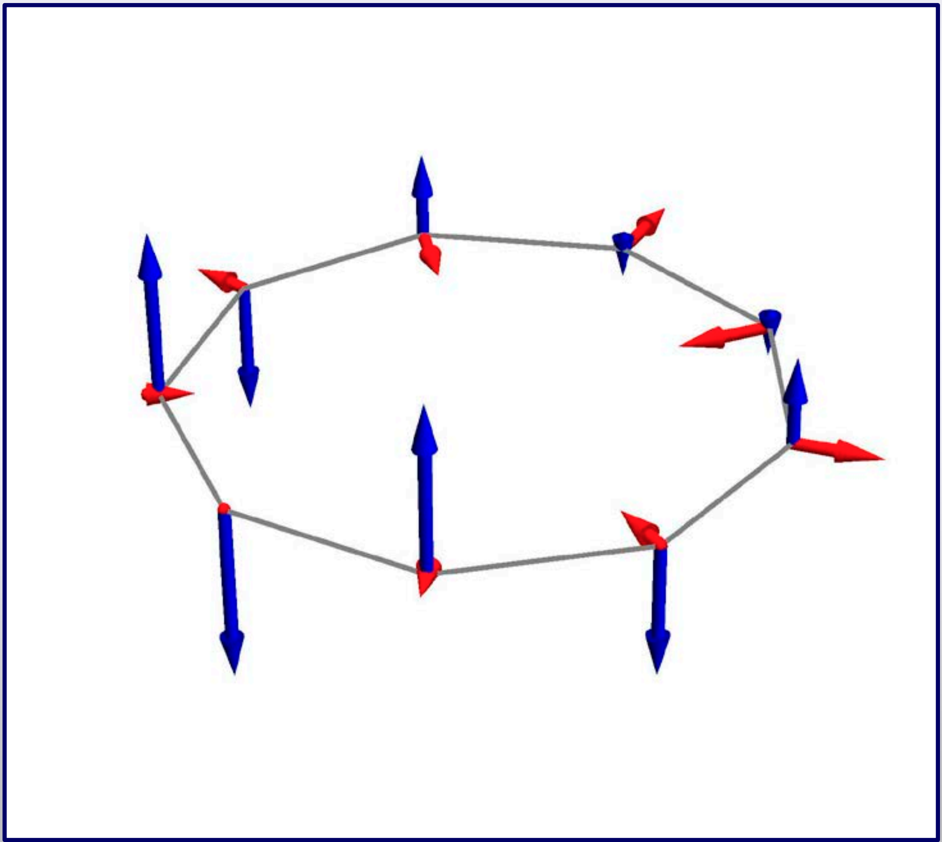
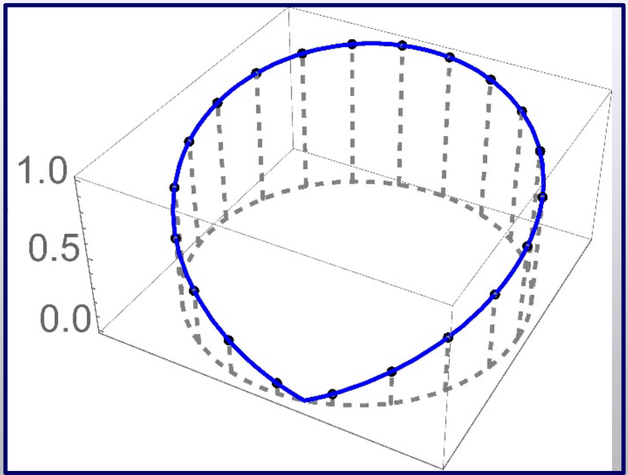
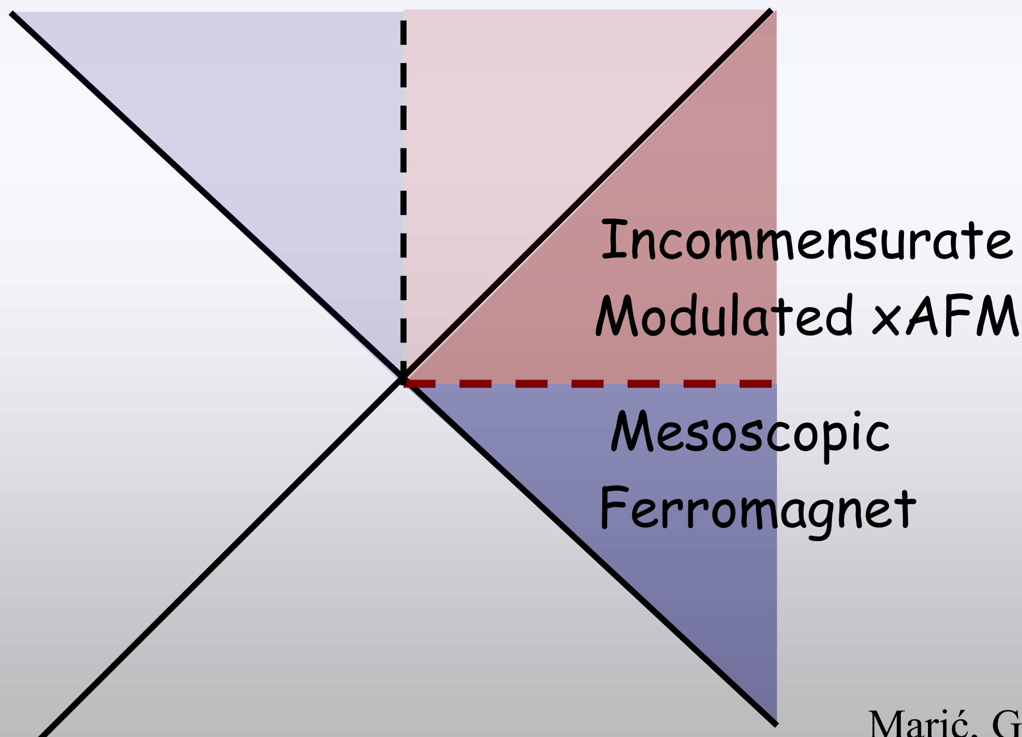




Quantum phase transition?

$$H = \sum_{j=1}^{2M+1} \left[\cos\phi \, \sigma_j^x \sigma_{j+1}^x + \sin\phi \, \sigma_j^y \sigma_{j+1}^y \right]$$

$$\left.\frac{dE_g}{d\phi}\right|_{\phi\to 0^-} - \left.\frac{dE_g}{d\phi}\right|_{\phi\to 0^+} = 2\left(1 + \cos\frac{\pi}{N}\right)$$



- ϕ=0 (classical Ising)
- Level crossing (change in GS degeneracy: 2 $\leftrightarrow$ 4)
- Finite discontinuity in 1° derivative of GS energy
- Akin to a 1° order b-QPT

$\Rightarrow$ Boundary-less b-QPT

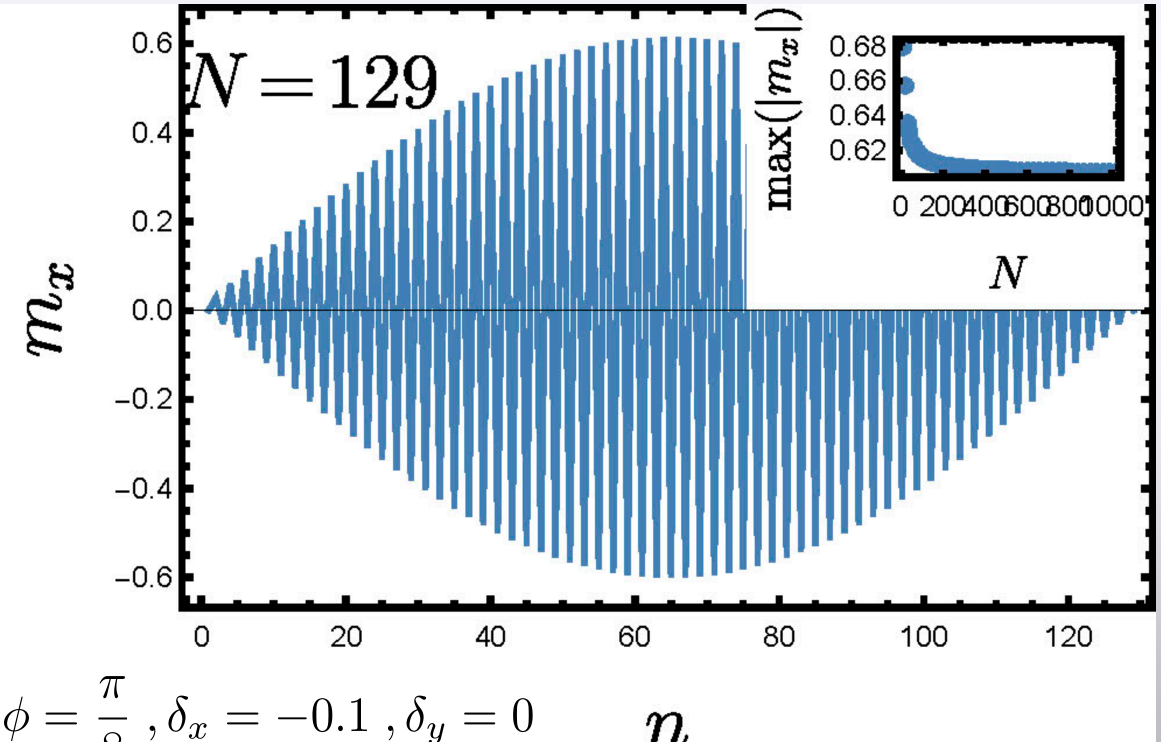
Marić, Giampaolo, Franchini, Comm. Phys. 3, 220 (2020)

Defects

$$H = \sum_{j=1}^{2M} \left[\cos\phi \; \sigma^x_j \sigma^x_{j+1} + \sin\phi \; \sigma^y_j \sigma^y_{j+1}\right] + \cos(\phi+\delta_x)\sigma^x_{2M+1}\sigma^x_1 + \sin(\phi+\delta_y)\sigma^y_{2M+1}\sigma^y_1$$

- Physics discussed so far often dismissed as fragile
- A single AFM defect stabilizes the incommensurate AFM order!
- No defect as QPT between Ising and kink phases

Campostrini et al, PRE 91 (2015)



$$\phi = \frac{\pi}{8} \;, \delta_x = -0.1 \;, \delta_y = 0$$

Torre, Marić, FF, Giampaolo, PRB 103 (2021)

Loschmidt Echo

- FBC: massive to gapless, but deemed unobservable
- Detect difference in spectrum through a Quantum Quench

$$H_0 = \sum_{l=1}^{N} \left(\sigma_l^x \sigma_{l+1}^x + h \sigma_l^z \right) \longrightarrow H_1 = H_0 + \lambda \sigma_N^z$$

- Loschmidt Echo: $\mathcal{L}(t) = |\langle 0 | e^{-\imath H_1 t} | 0 \rangle|^2, \; |0\rangle$ GS of H_0

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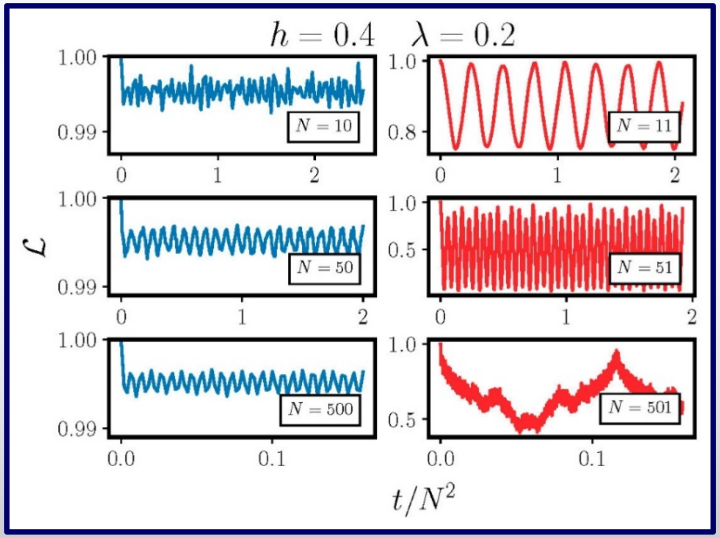

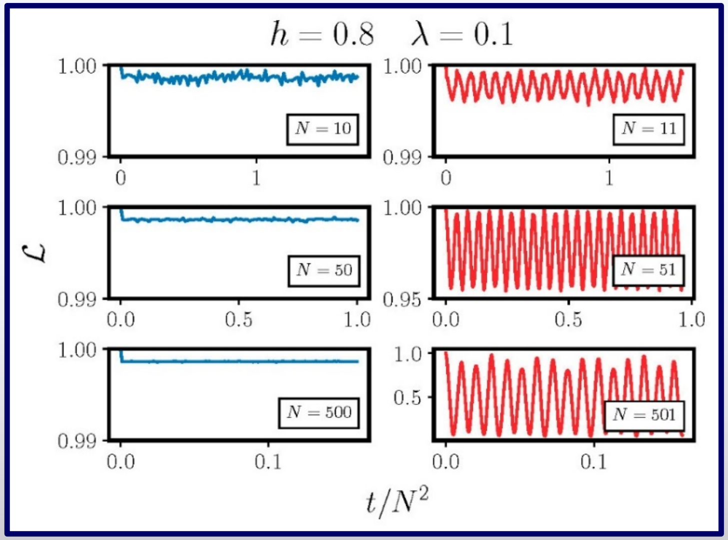

LE distinguishes between the parity of the chain length!

Torre, Marić, Kuić, F. F. S.M. Giampaolo, PRB (2022)

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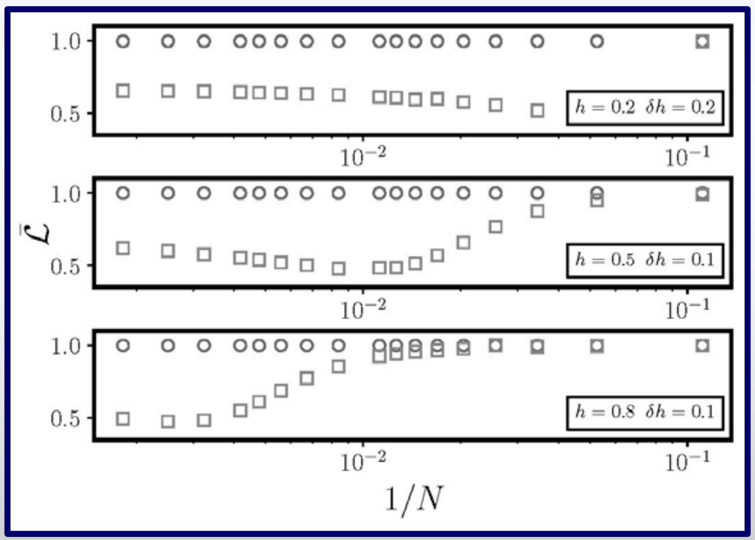

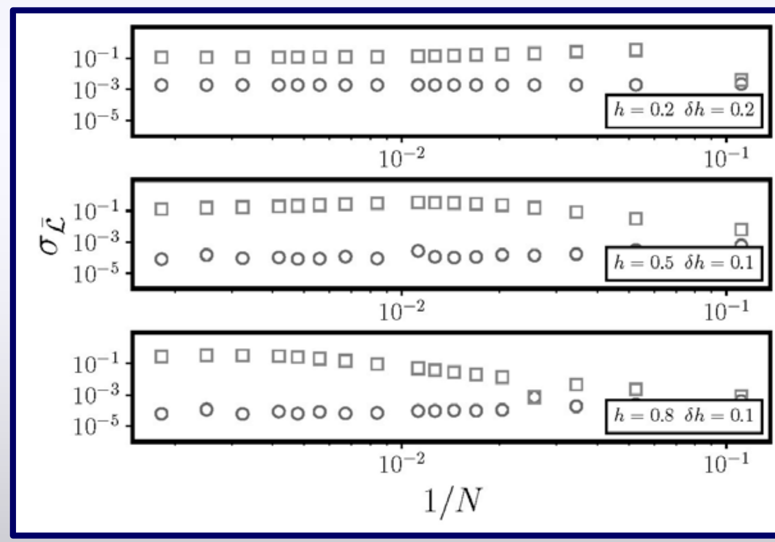

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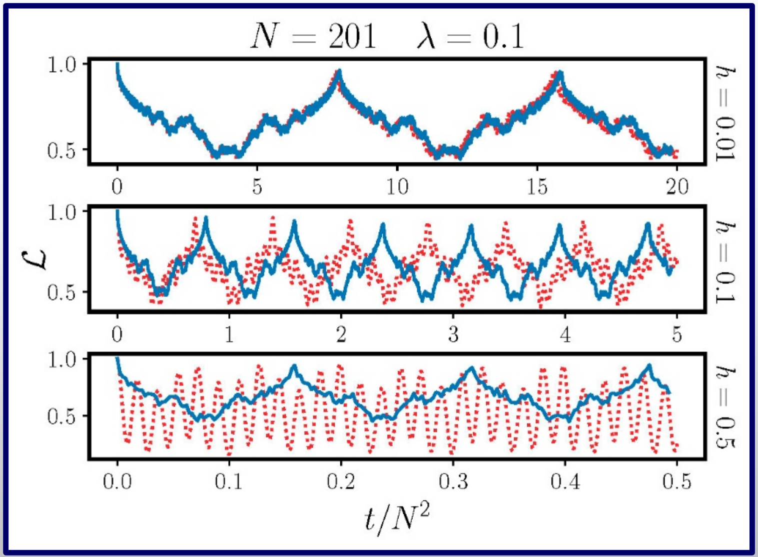
- Perturbative calculation (domain wall basis):

$$\mathcal{L}(t) \simeq \mathcal{F}\left(\frac{2ht}{N^2}\right)$$

$$\mathcal{F}(x) = \lim_{M\to\infty} \left| \frac{1}{2M^2} \sum_{k=1}^{M} \tan^2\left[\frac{(2k-1)\pi}{4M}\right] \times \right.$$
$$\left. \times \exp\left\{ -\imath x (2M+1)^2 \cos\left[\frac{(2k-1)\pi}{2M}\right]\right\} \right|^2$$



- LE continuous, but nowhere differentiable

Torre, Marić, Kuić, F. F. S.M. Giampaolo, PRB (2022)

Complexity

- Loschmidt echo's features point for greater complexity
- We measure it as Stabilizer Renyi Entropy (SRE or magic)

$$\mathcal{M}_2(|\psi\rangle) = -\log_2\left(\frac{1}{2^L}\sum_P \langle\psi|P|\psi\rangle^4\right)$$

$$P = \bigotimes_{j=1}^{L} P_j, \;\; P_j \in \{\sigma_j^0, \sigma_k^x, \sigma_j^y, \sigma_j^z\}$$

- Distance from Clifford states that can be efficiently classically simulated

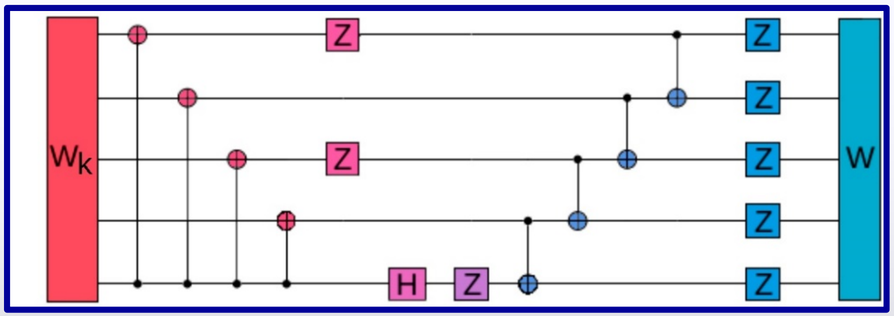
Odavić, Torre, Mijić, Davidović, F. F., Giampaolo, arXiv:2210.13495

W-States

- W-states: entangled & not separable after measurement:

$$|W\rangle = -\frac{1}{\sqrt{N}} \sum_{j=1}^{N} \sigma_j^+ |0\rangle^{\otimes N}$$

- Near classical point, topologically frustrated gs as superposition of kink states realizes a W-state



(exact, SRE preserving, map between kink and W-state)

$$\mathcal{M}_2^W(L) = 3\log_2(L) - \log_2(7L-6)$$

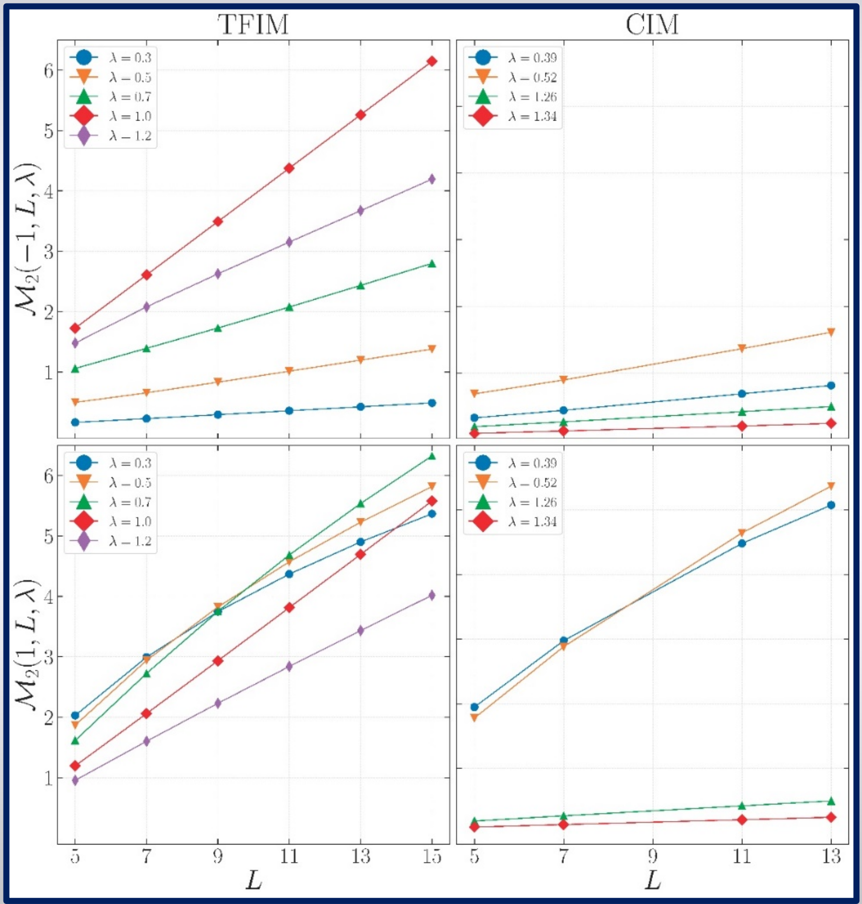
- SRE of W-state growth logarithmically!

Odavić, Torre, Mijić, Davidović, F. F., Giampaolo, arXiv:2210.13495

SRE in topologically frustrated chains

- SRE can be decomposed as non-frustrated volume contribution and W-state logarithmic one

$$\mathcal{M}_2(J=1,L,\lambda) = \mathcal{M}_2(J=-1,L,\lambda) + \mathcal{M}_2^W(L)$$



- FBC add non-local complexity of W-state nature

Odavić, Torre, Mijić, Davidović, F. F., Giampaolo, arXiv:2210.13495

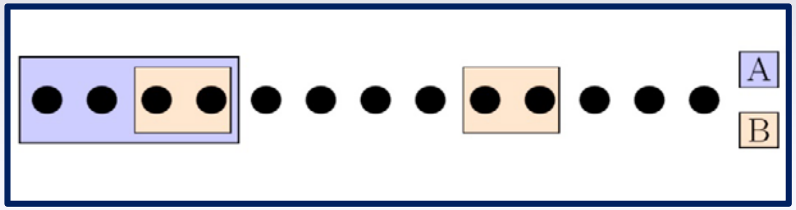
Disconnected Topological Entropy

- Consider tripartition of chain: A, B and their complement $\overline{A \cup B}$:

$$S_{\alpha}^{\mathrm{D}} = S_{\mathrm{A},\alpha} + S_{\mathrm{B},\alpha} - S_{\mathrm{A}\cup\mathrm{B},\alpha} - S_{\mathrm{A}\cap\mathrm{B},\alpha}$$

$$S_{A,\alpha} = \frac{1}{1-\alpha} \log \mathrm{Tr}_{A} \rho_{\mathrm{A}}^{\alpha}$$

$$\rho_{\mathrm{A}} = \mathrm{Tr}_{\overline{A}} |0\rangle\langle 0$$



- It is called "disconnected entropy" and was shown to be non-zero for Symmetry Protected Topological phases.

Micallo et Al. SciPost Phys. Core 3, 012 (2020)

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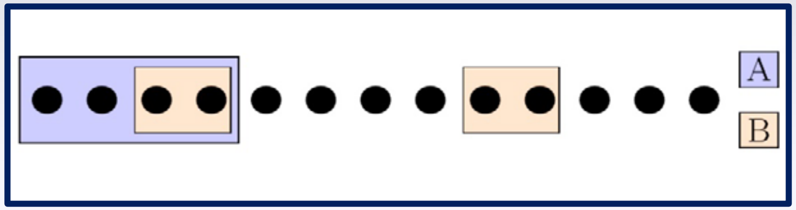

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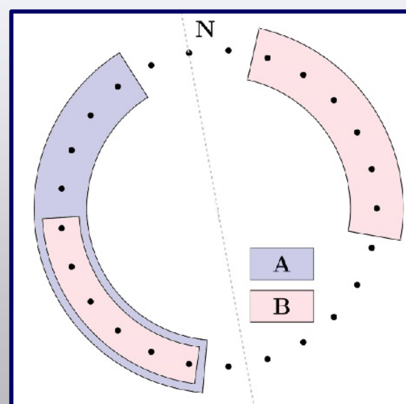
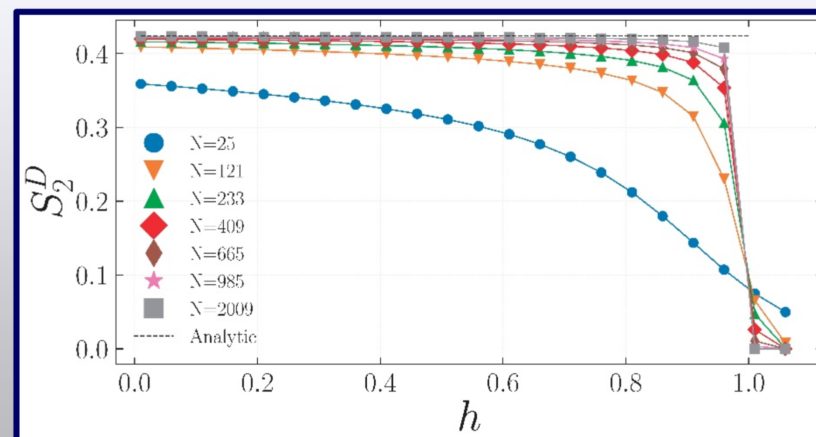
Micallo et Al. SciPost Phys. Core 3, 012 (2020)

- Topologically frustrated Ising chain:

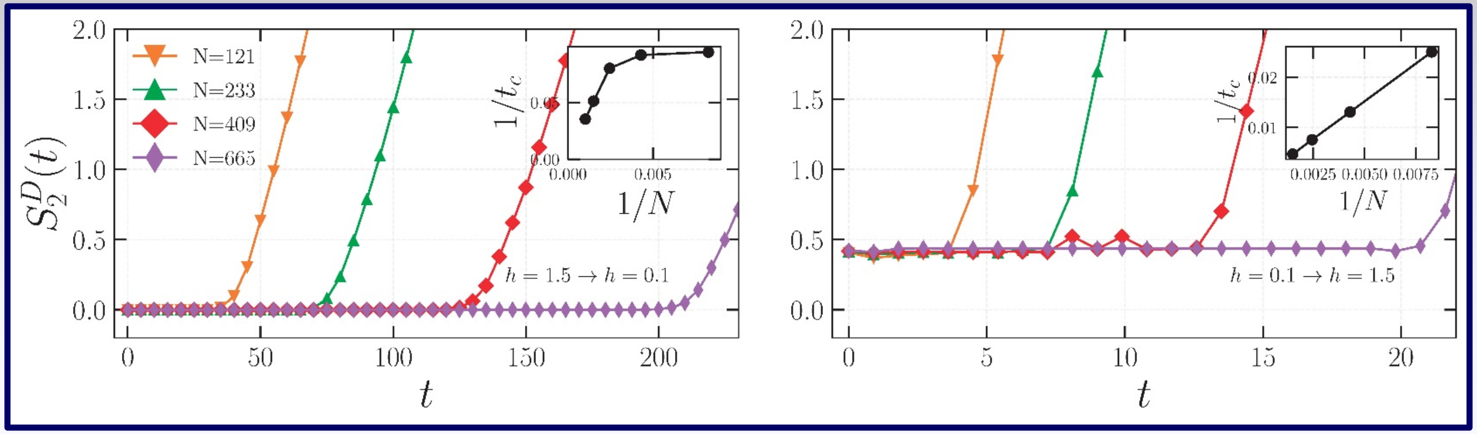
$$H = \sum_{j=1}^{2M+1} \sigma_j^x \sigma_{j+1}^x - h\sigma_j^z$$



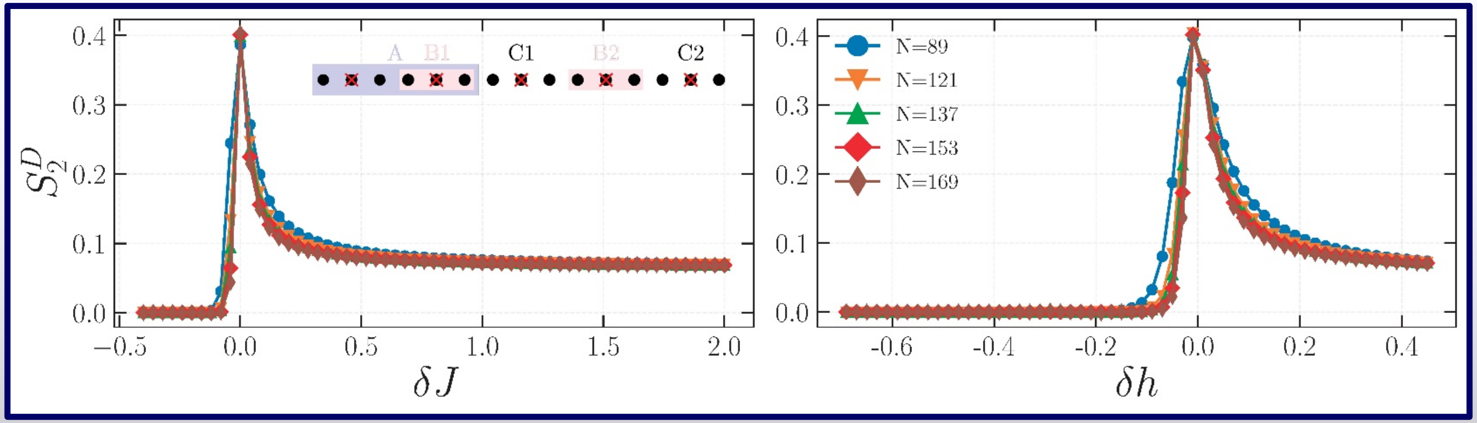


Disconnected Topological Entropy

- It is finite and resilient:



with global quench



with AFM defects

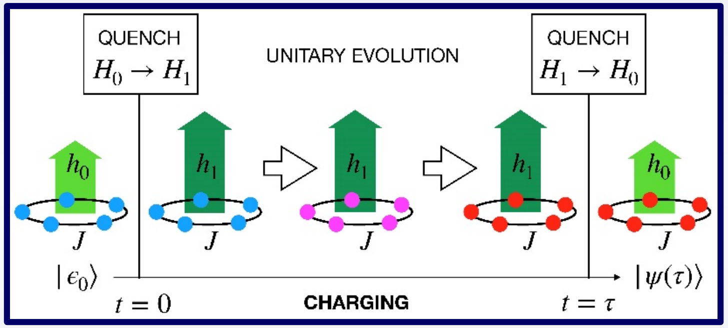
S^D detects long-range entanglement due to fractional nature of single excitation in ground state

Technological applications

- Quantum Battery: quantum mechanical system that can store and transfer energy coherently

- Our charging protocol: global quantum quench on Ising chain

$$H_0 = \sum_{l=1}^{N} \left(\sigma_l^x \sigma_{l+1}^x + h \sigma_l^z \right)$$



Catalano, Giampaolo, Morsch, Giovannetti, F. F., arXiv:2307.02529

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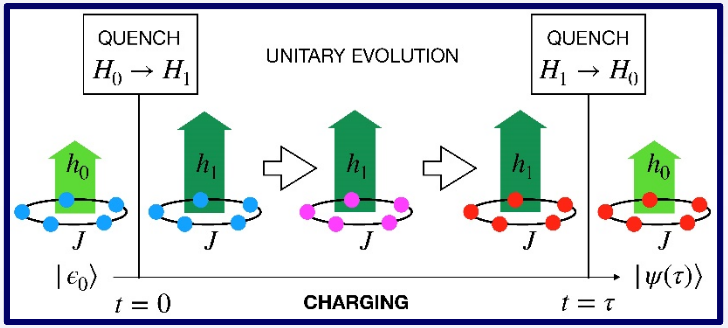
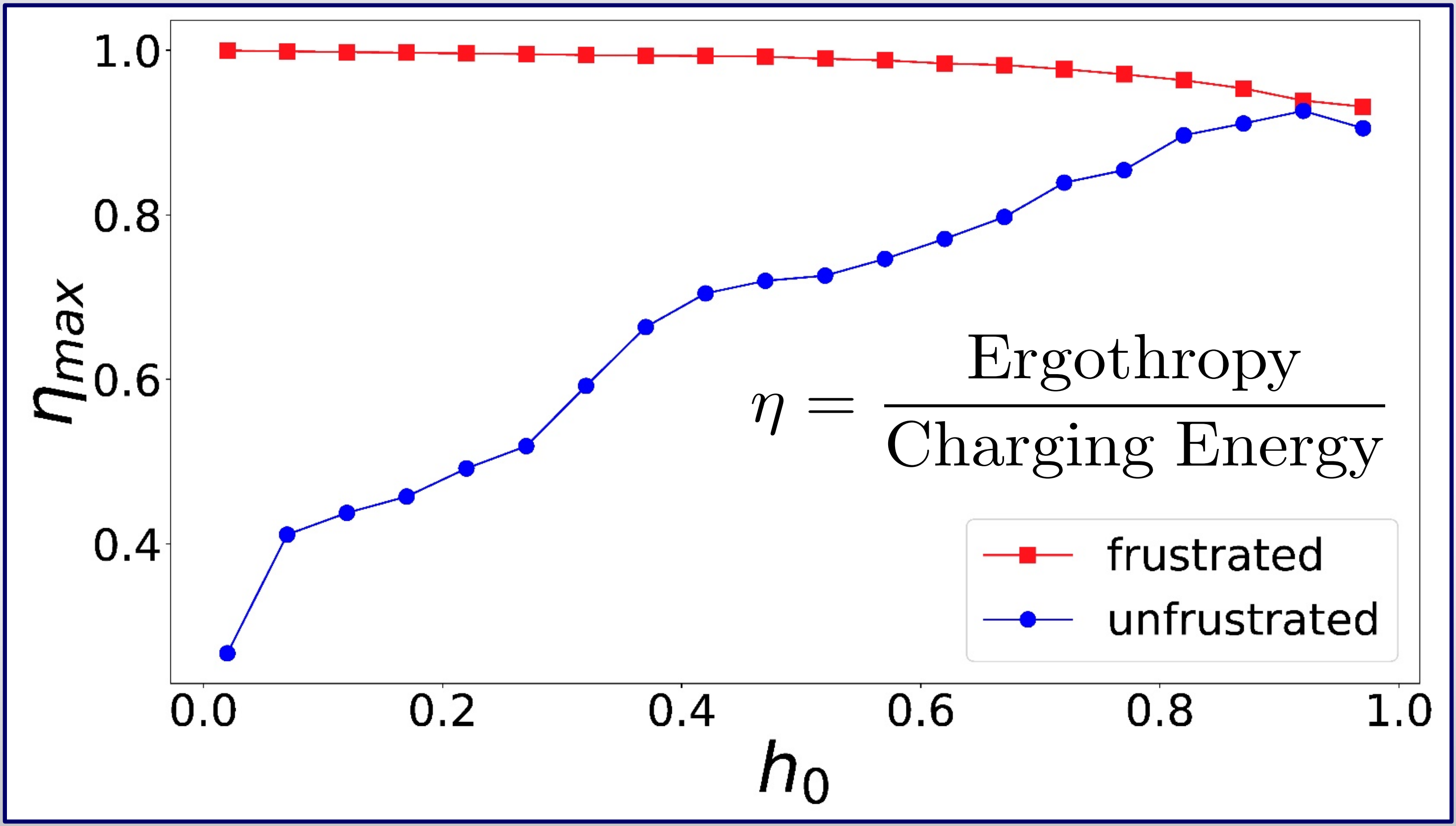


- We model intrinsic decoherence after quench as emergence of diagonal ensamble (pure → mixed state)
- We use ergotropy as figure of merit: energy extractable through unitary transformation

Catalano, Giampaolo, Morsch, Giovannetti, F. F., arXiv:2307.02529

Robustness against decoherence

- Frustrated chain more resilient against decoherence than non-frustrated countepart



$N = 25$

$\Delta h = 0.01$

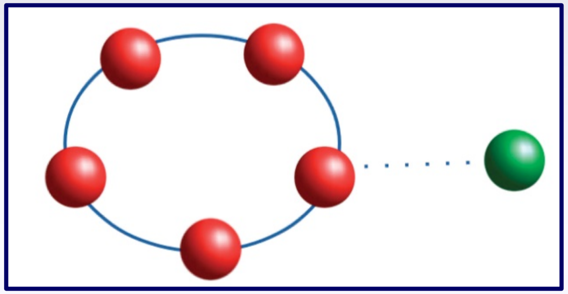
Catalano, Giampaolo, Morsch, Giovannetti, F. F., arXiv:2307.02529

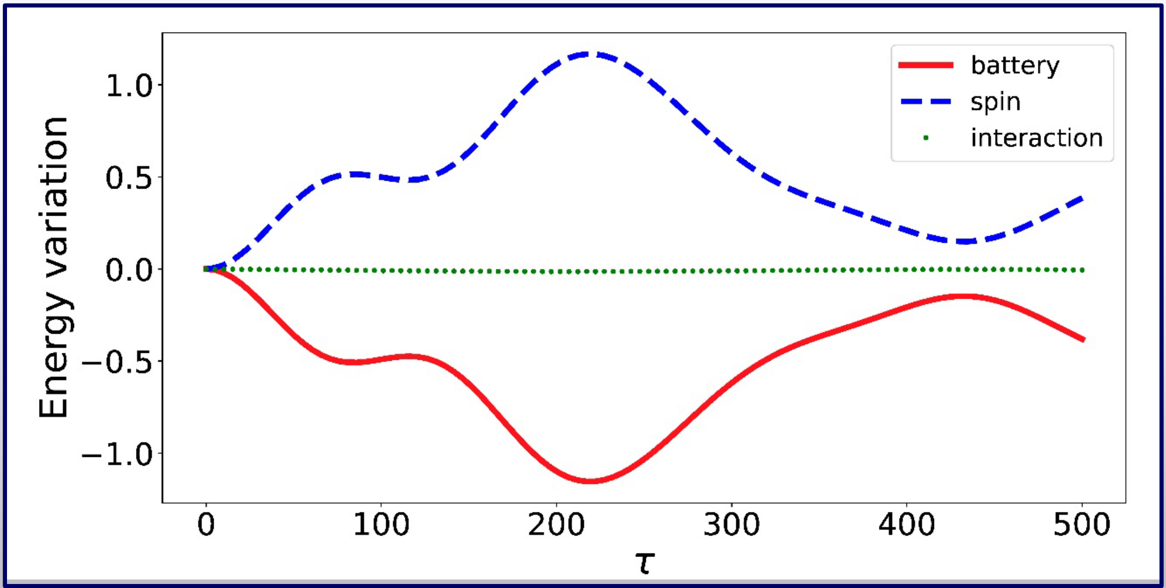
Discharging

- Test: energy extraction to additional, isolated spin

$$H = \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x - h \sum_{j=1}^{N} \sigma_j^z + \lambda H_{int} + \omega \sigma_S^z$$

$$H_{int} = \sigma_1^+ \sigma_S^- + \sigma_1^- \sigma_S^+$$





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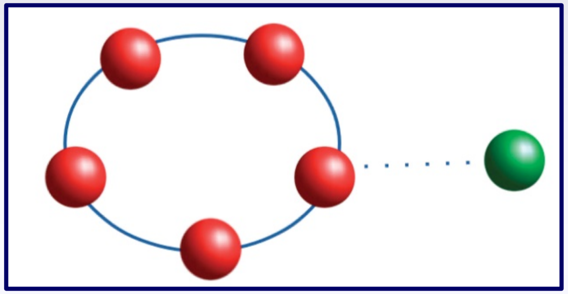

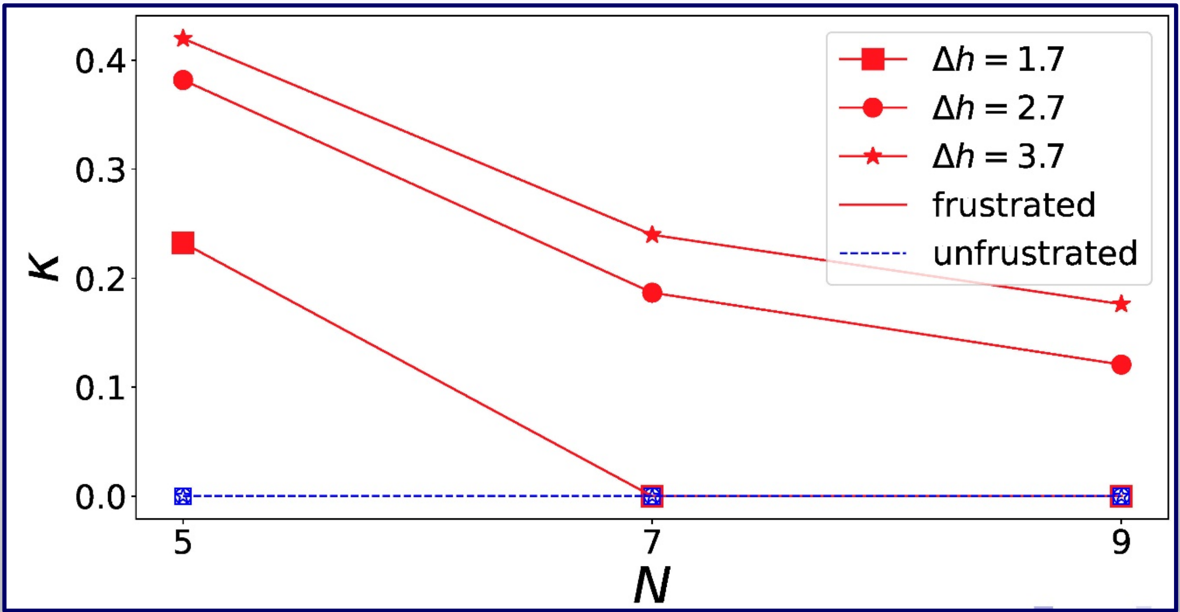

Frustrated chain transfers work, not just heat as non-frustrated chain

Catalano, Giampaolo, Morsch, Giovannetti, F. F., arXiv:2307.02529

FBC and beyond

- Frustrated boundary conditions can change the local, bulk order (mesoscopic, modulated) and the quantum phase transitions (new b-QPT and replacing 2° order QPT)
- More complex ground state, non-local correlations
- Frustration known to give new physics in quantum systems

<u>Outlook:</u>

- Add more frustration and follow its effects
- Add disorder (spin glass, many body localization)
- Technological exploitation:
  - Quantum batteries
  - New platform for quantum computation (new q-bit encoding)

Thank you!