Gauge-String Duality and Arithmetic?

Rajesh Gopakumar, ICTS-TIFR, Bengaluru,
"Connections between Number Theory and Physics",
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Based on works with:
M. Gaberdiel, E. Mazenc, L. Eberhardt,
P. Maity, B. Knighton,

In Outline

- What is gauge-string duality?
  Why deriving it is important (even for physicists!).

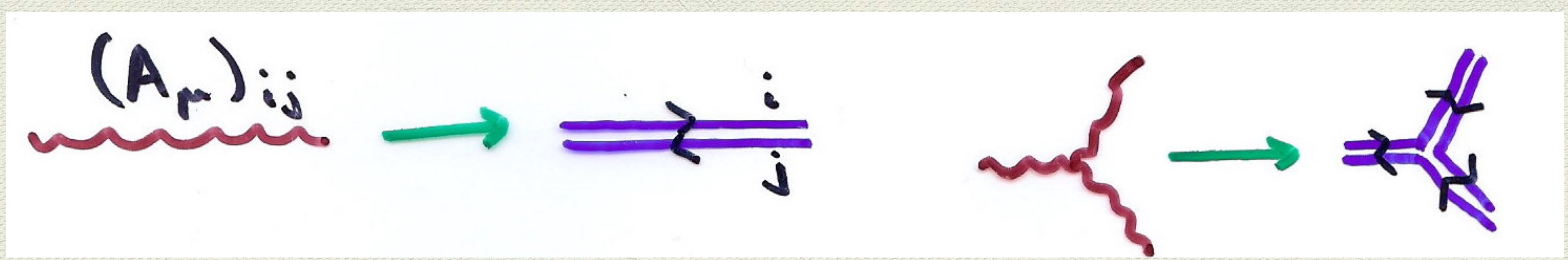
- Sketch of a Program (``*l'Esquisse d'un Programme*"):
  From worldlines to worldsheets.

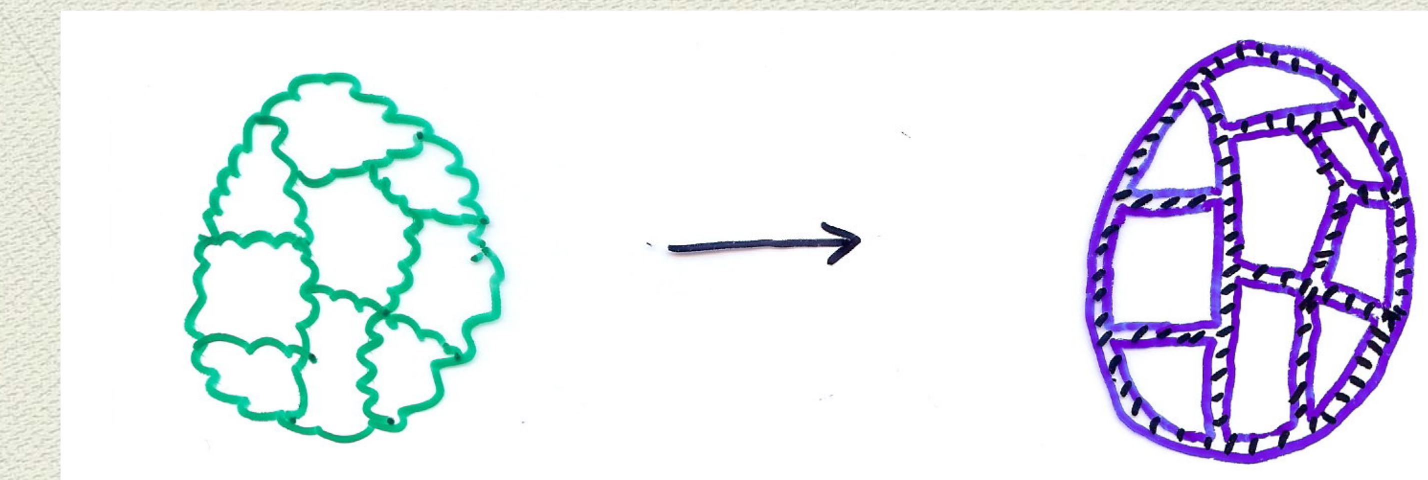
- Illustrating the general idea:
  1) The Simplest Gauge-String Duality (large N Matrix Integrals and Top. Strings).
  2) AdS_3/CFT_2 Duality [cf. Matthias' talk]

- Why Arithmetic Riemann Surfaces might have a special role to play.

Gauge-String Duality

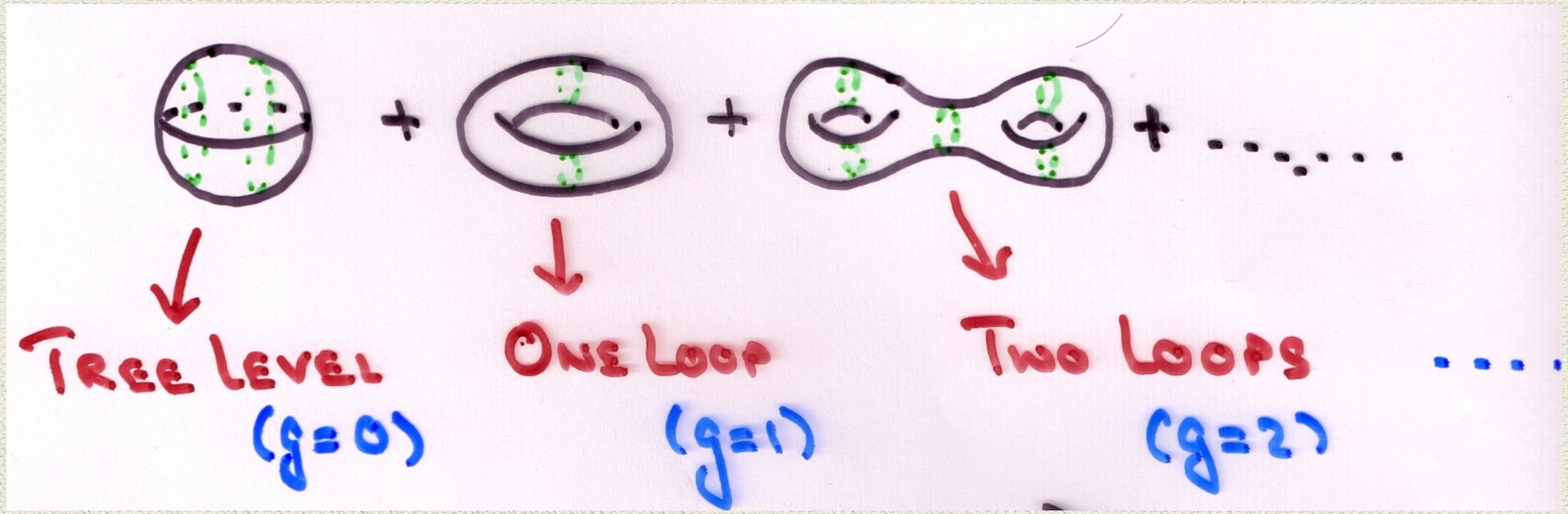
- A broader term than the AdS/CFT Correspondence. Originates in 't Hooft's large N limit of U(N) gauge theories ('74)





- Feynman diagrams →Ribbon Graphs. Associate a topology to the graph, weight N^{2-2g}.

- Reminiscent of a perturbative string expansion (with $g_s^2 \sim \frac{1}{N^2}$).

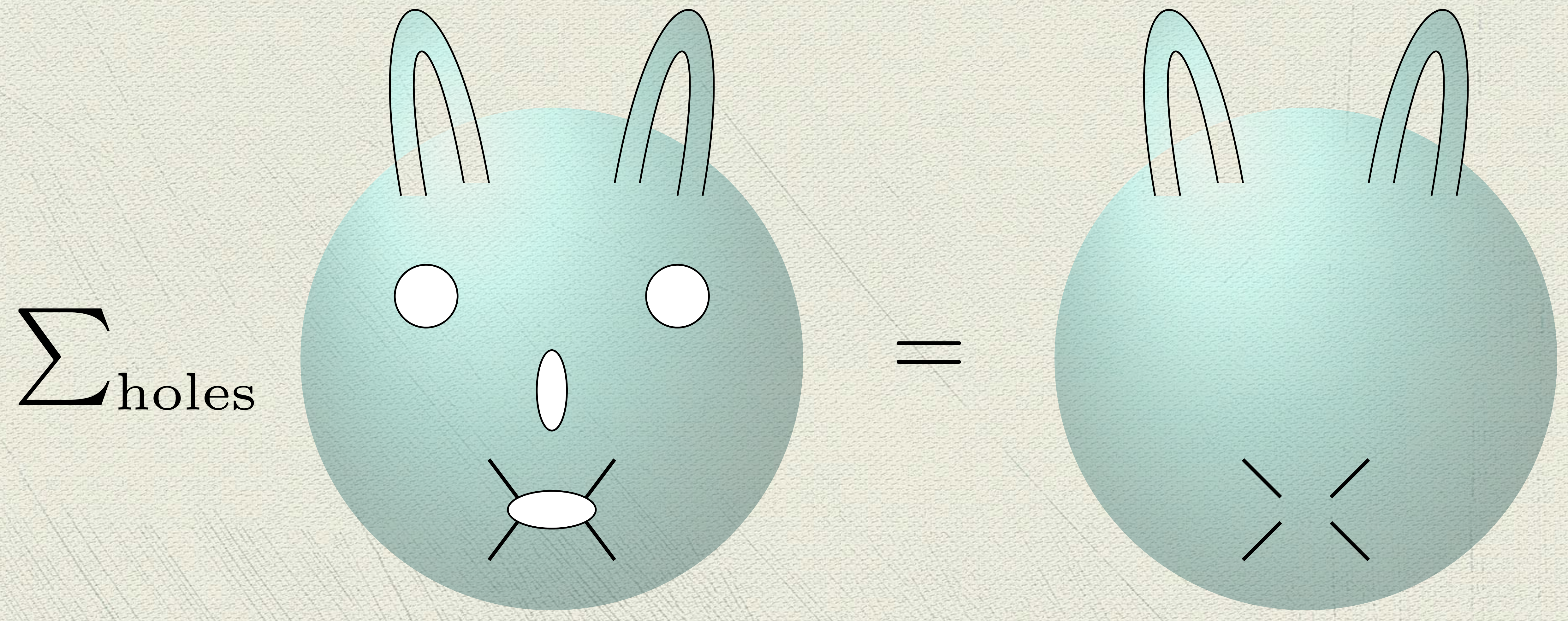


- Relates Strings to Fields (cf. Langlands).

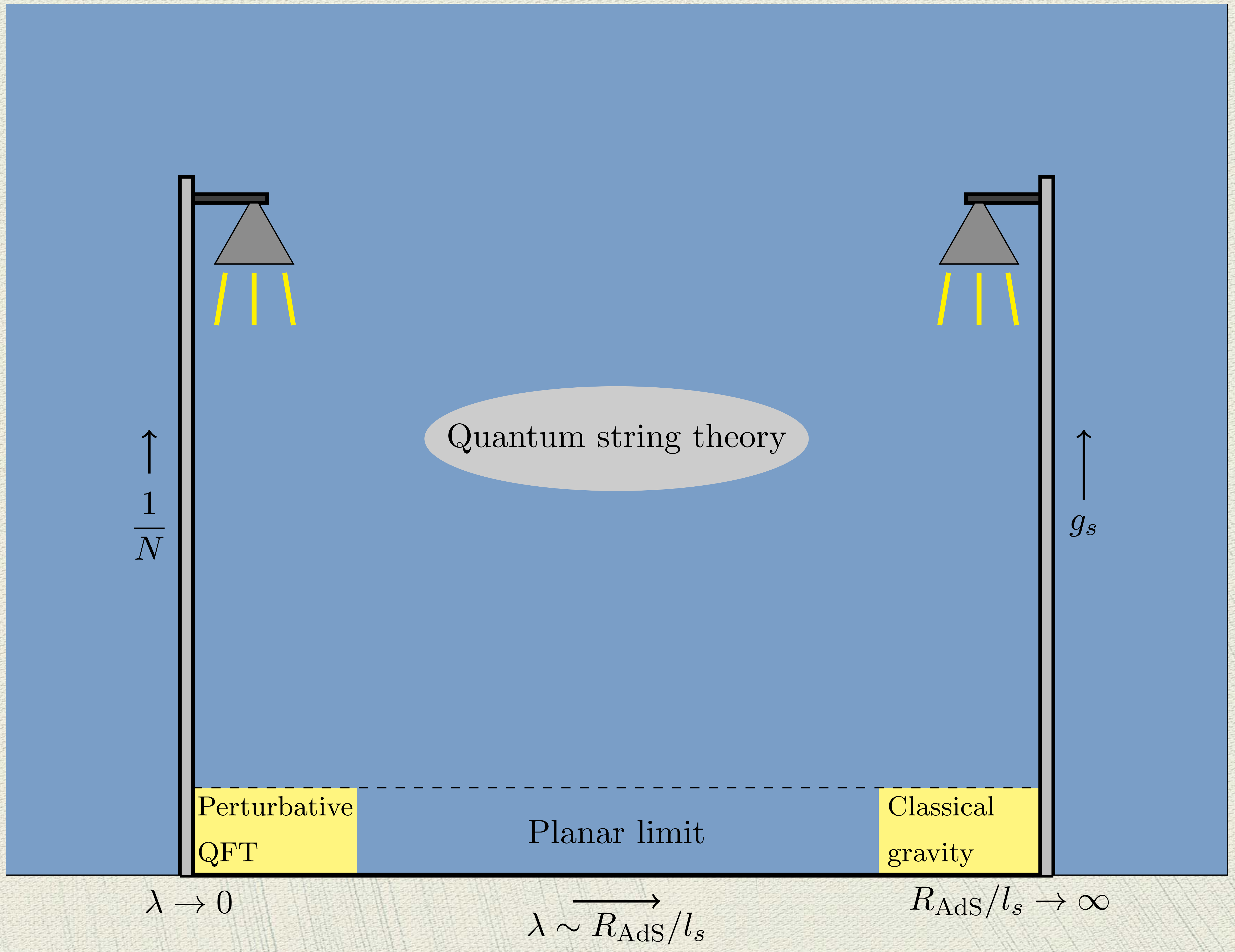
- Remained a suggestive picture till the AdS/CFT Duality [Maldacena'97].

Deriving Gauge-String Duality

- How exactly do large N QFTs reorganise themselves into theories of strings?
- D-brane physics indicates open-closed string duality as an underlying reason.
- Can we turn this into a precise recipe which enables one to derive the string dual to QCD?
- Large N Gauge theory Feynman diagrams are open string diagrams.
- Holes close up (and deform the background).
- But difficult to see this explicitly happen at large $g_s N = \lambda$.



The Other Lamp Post

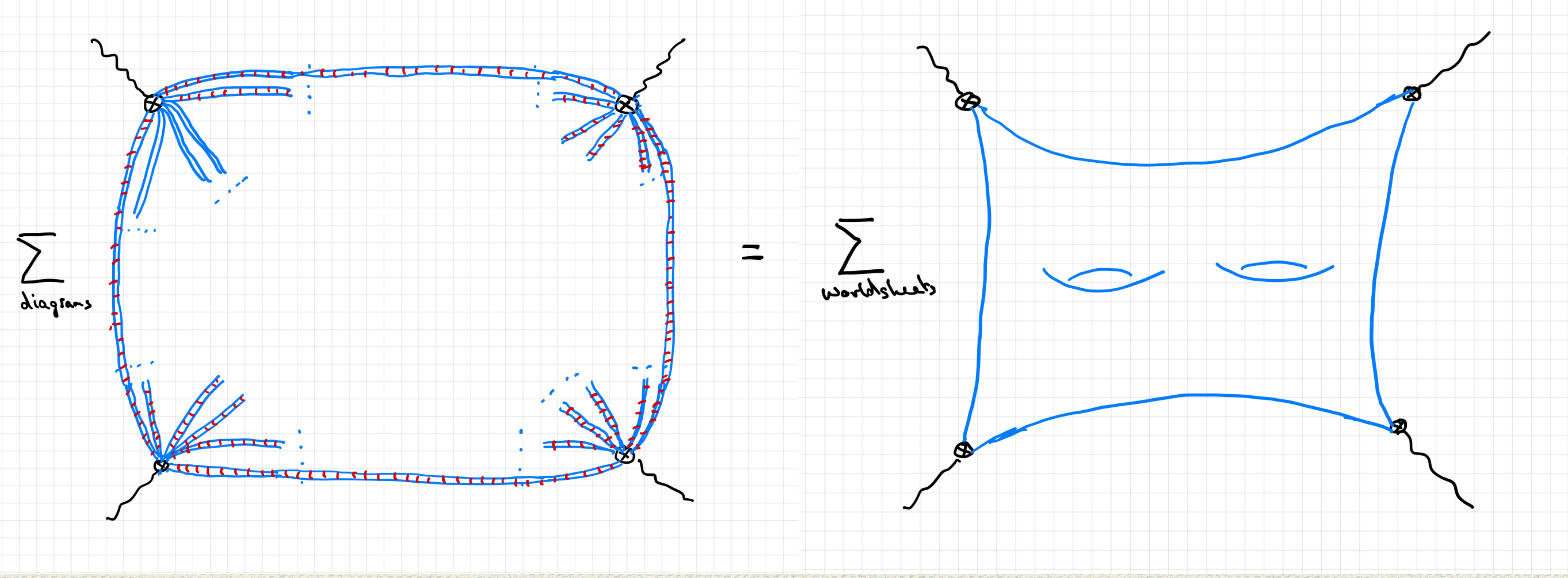


- Focus on the corner where we understand the QFT but not necessarily the bulk.

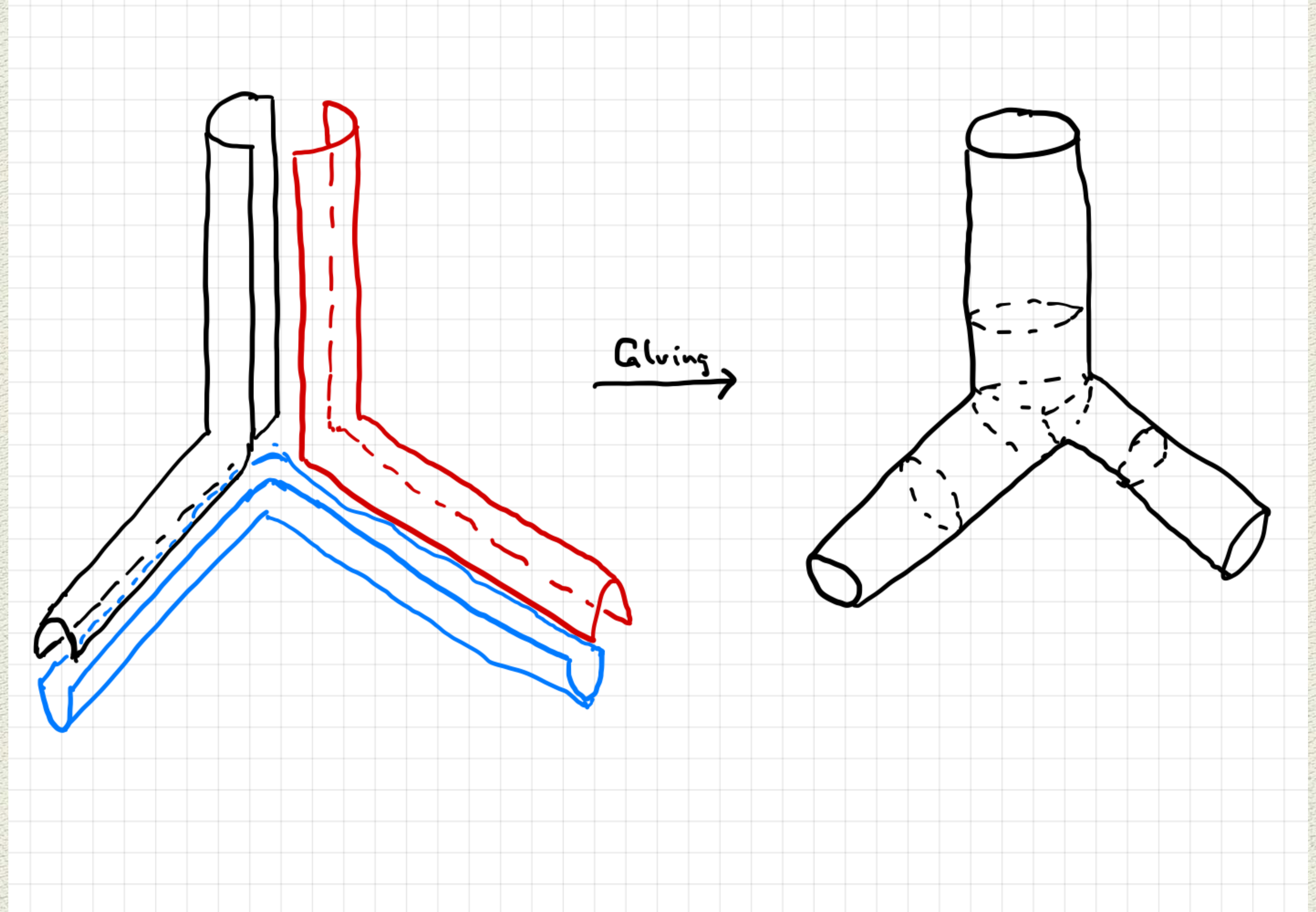
- Free (perturbative) QFT as $\lambda \to 0$ $\leftrightarrow$ `Tensionless' limit of dual string theory.
  [Sundborg, Sezgin-Sundell, Witten…]
- Finite # of Feynman diagrams, of any genus. Power series in λ given by Wick contractions.

- Test Cases:
  a) Matrix Model $\leftrightarrow$ Top. A/B-Model String.
  b) $(T^4)^N/S_N \leftrightarrow AdS_3 \times S^3 \times T^4$.
  c) Pert. $\mathcal{N} = 4$ SYM $\leftrightarrow AdS_5 \times S^5$.

From Worldlines to Worldsheets



Implementation of
open-closed string duality.



SLOGAN: EACH FEYNMAN GRAPH
↔ A CLOSED WORLDSHEET.

Moduli space

Exploits the Strebel parametrisation of $\mathscr{M}_{g,n}$ [R.G.'05].

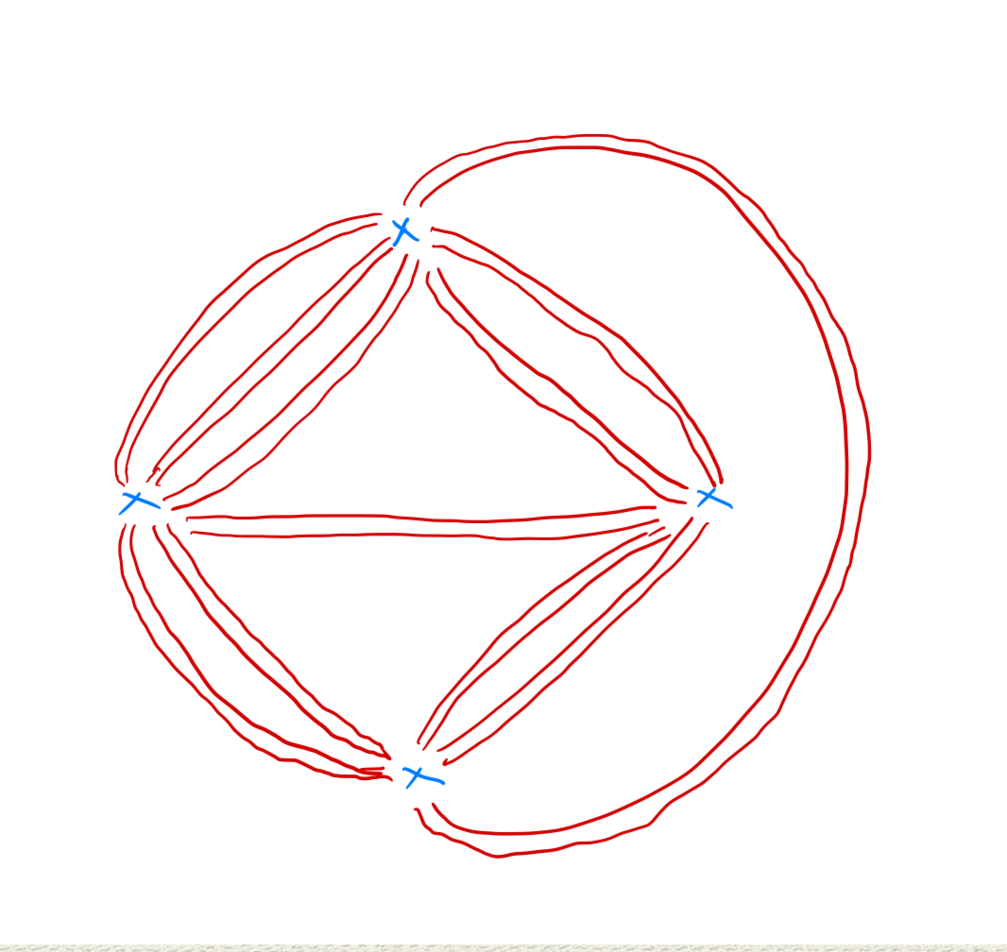
A refinement of 't Hooft's idea of associating a genus to double line Feynman graphs [R.G. '04].

Bridge from Fields to Strings

- Explain how individual Feynman diagrams translate to specific world sheets - points on $\mathscr{M}_{g,n}$.
- Pattern seen in our test cases: Matrix Models, Free Symmetric Product Orbifold CFT (and very likely, Perturbative Yang-Mills).
- Feyn. Diags. $\leftrightarrow$ Permutations $\leftrightarrow$ Branched Coverings $\leftrightarrow$ Special points on $\mathscr{M}_{g,n}$ (often arithmetic).
- Matches with worldsheet amplitude - delta functions on $\mathscr{M}_{g,n}$. Weight also seems to be the natural geometric one (Nambu-Goto).

Feynman diagrams $\leftrightarrow$ Permutations

- Feynman diagrams for a correlator enumerate Wick contractions. View in terms of counting permutations. Equivalent to Grothendieck's *Dessins*.



- Matrix Model: $N^n \langle \prod_{i=1}^{n} \mathrm{Tr} M^{k_i} \rangle_{conn}^{(g)} = N^{2-2g} \sum_{\alpha \in S_k} \delta(1, \alpha \cdot \beta \cdot \gamma) \mathbf{1}.$

  α, β, γ are conjugacy classes in S_k, $(k = \sum k_i)$

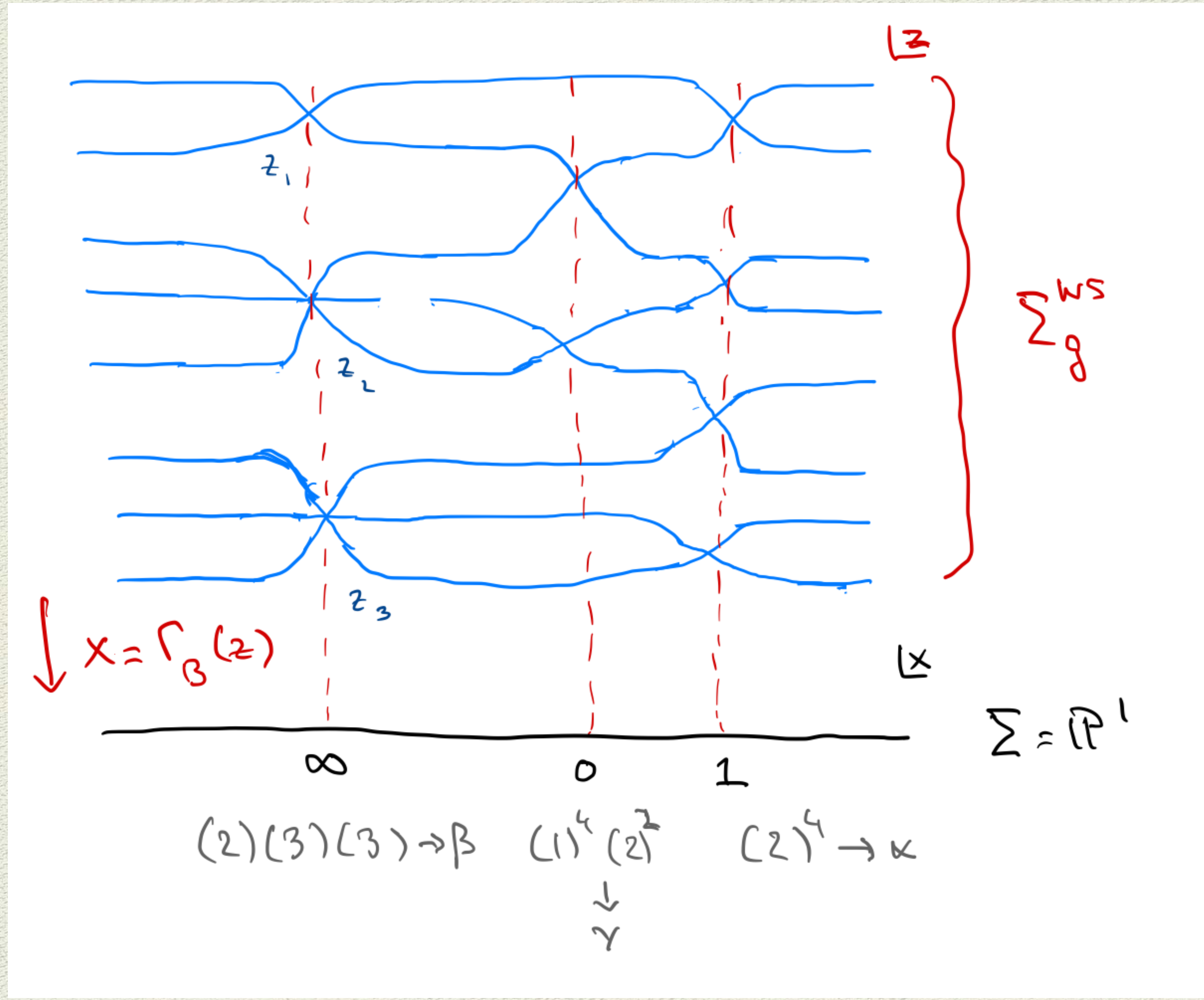
  - β indicates the cyclic structure of edges around each vertex $(k_1) \ldots (k_n)$;
  - $\alpha \in (2)^{k/2}$ labels different Wick contractions of edges;
  - $\gamma = \beta^{-1} \alpha^{-1}$ labels edges around a face.

[Itzykson et.al., de Mello Koch-Ramgoolam]

- Symm. Orbfld. CFT: $\langle \prod_{i=1}^{n} \sigma_{k_i}(x_i) \rangle_{conn}^{(g)}$ labelled by n distinct single cycles (k_i). Correlator is a sum over permutations that sew together these cycles.

Permutations ↔ Branched Coverings

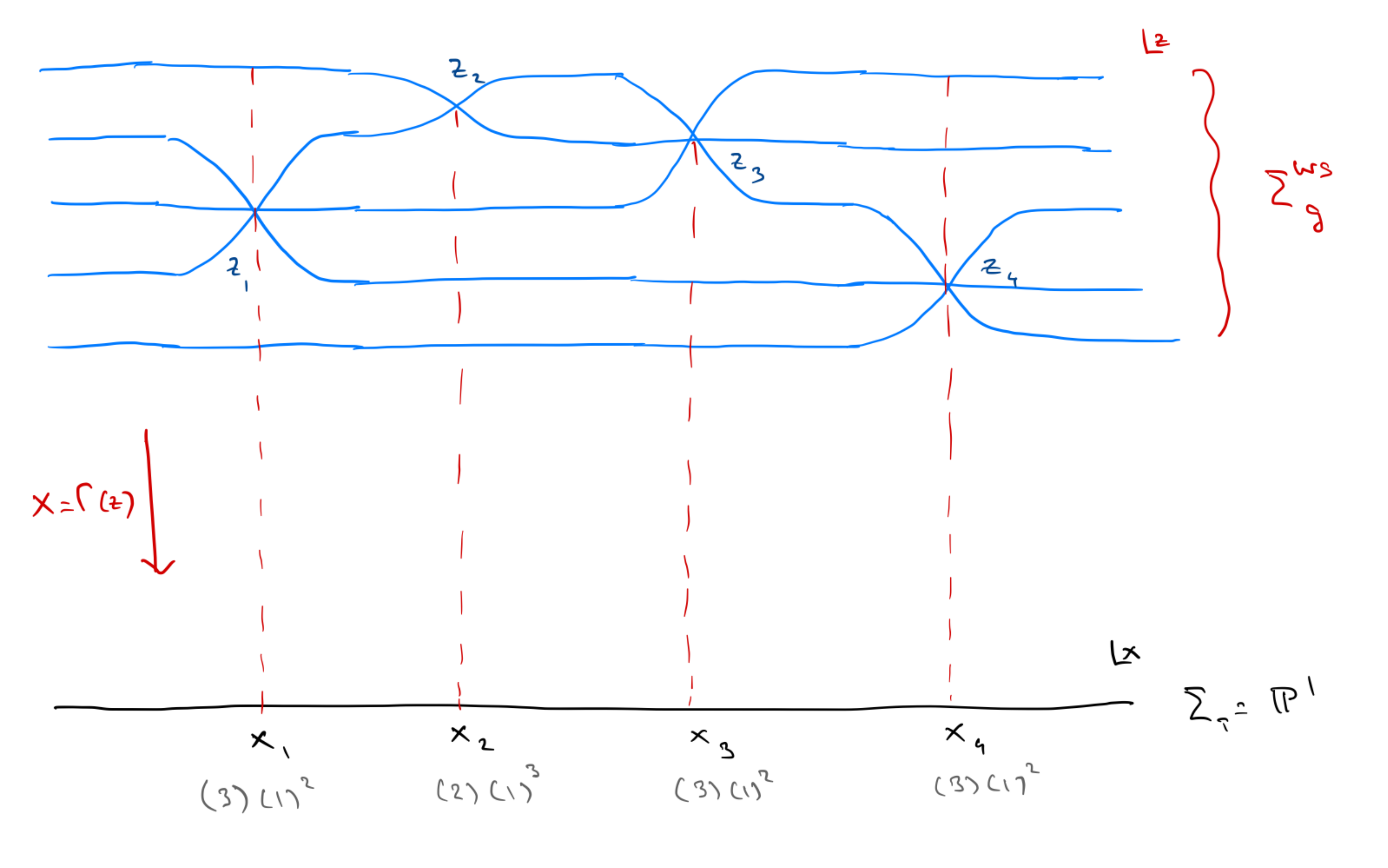
- Enumerating such permutations is the same as counting branched covers.
- Each permutation in a conjugacy class associated to a branch point with definite ramification (cycle structure).
- For Matrix correlators, (α, β, γ) associated with three branch points of Σ_g over $\mathbb{P}^1$ $(\alpha \cdot \beta \cdot \gamma = 1)$.
- `Belyi maps' - only admissible at arithmetic points on $\mathscr{M}_{g,n}$.



[A contribution to $\langle \mathrm{Tr}M^2 \, (\mathrm{Tr}M^3)^2 \rangle$]

Permutations ↔ Branched Coverings

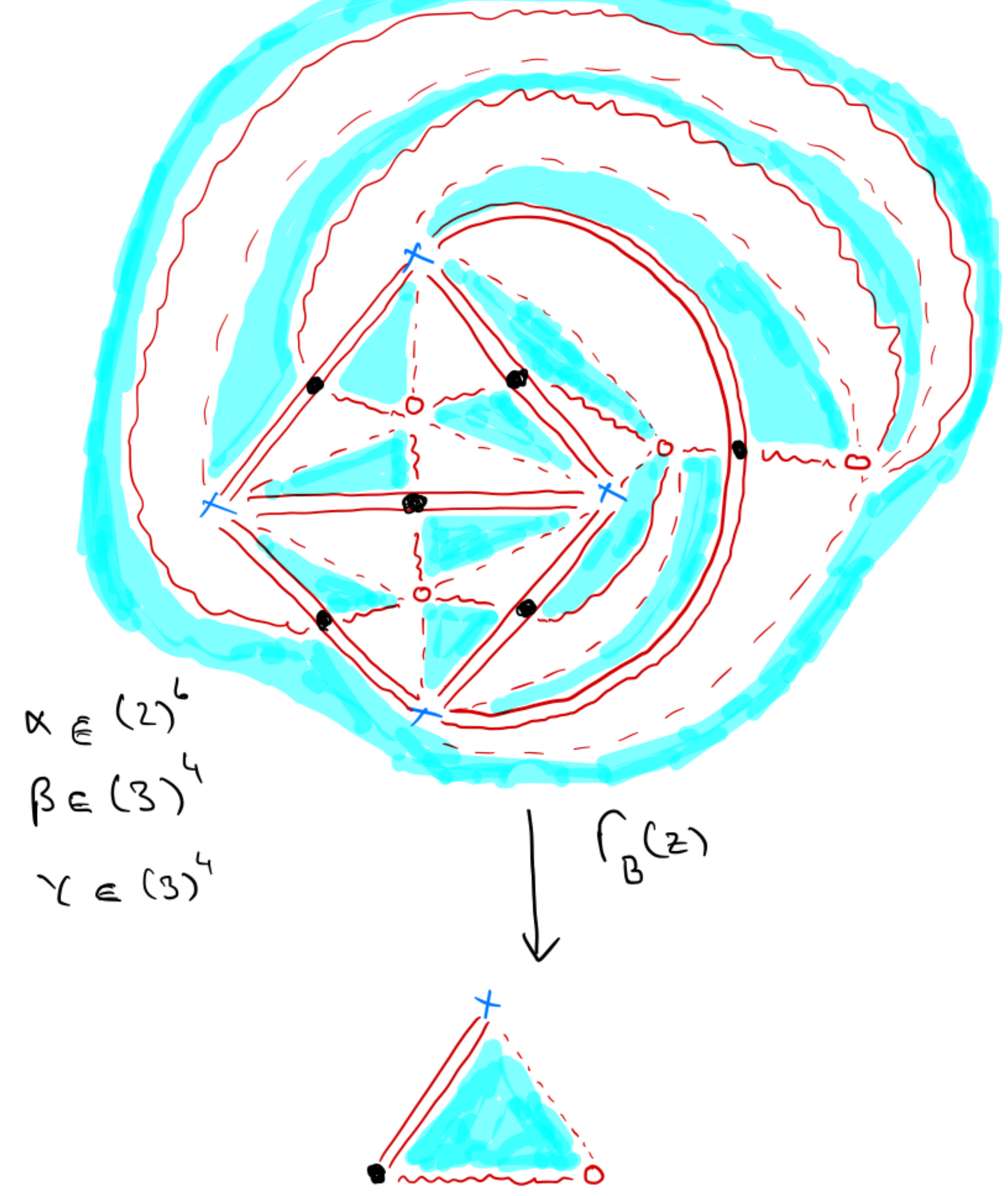
- This connection also underlies the Lunin-Mathur computation of symm. orbfld. CFT correlators.
- Lift to covering space geometrises permutations.
- x_i-dependence of correlators now means n points w/ branching k_i.
- Admissible only at discrete points on $\mathscr{M}_{g,n}$ (depend on x_i).

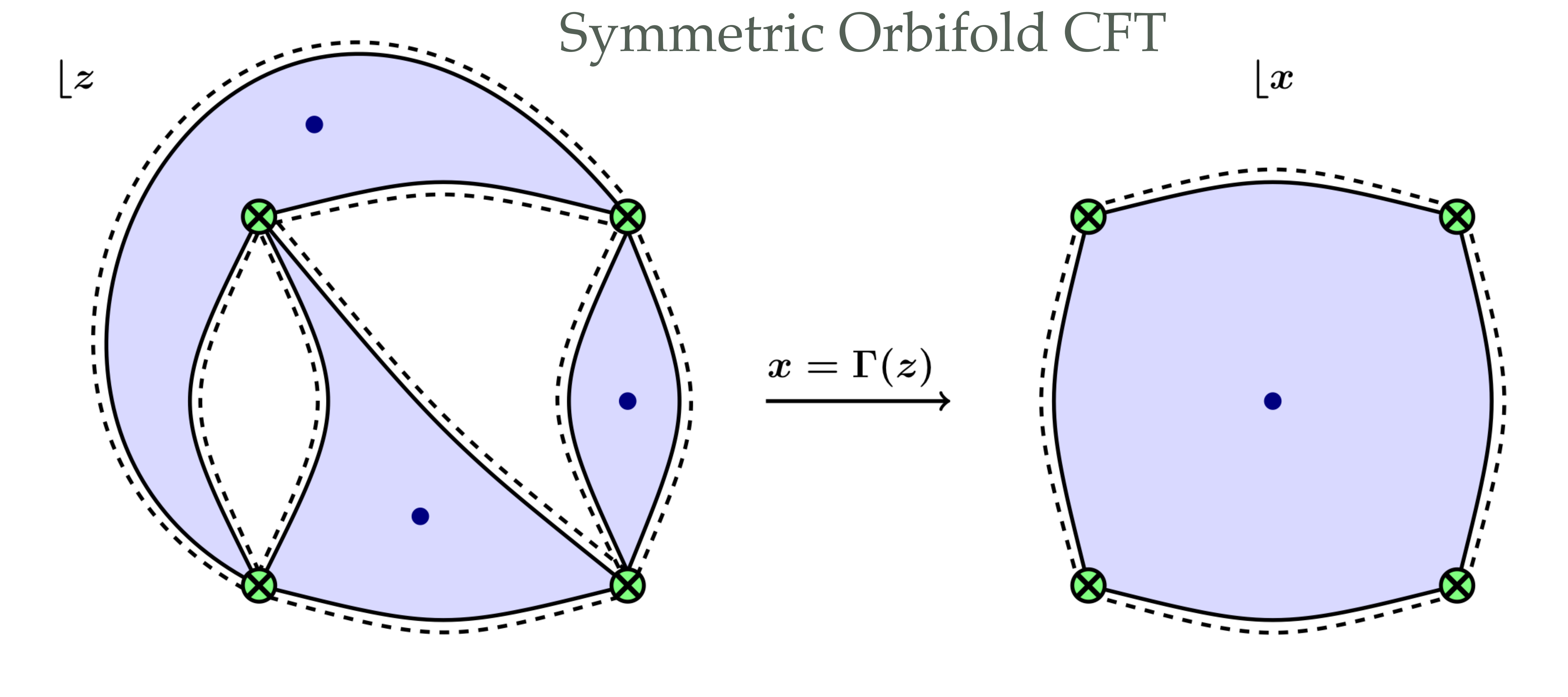


[A contribution to $\langle \sigma_3(x_1)\sigma_2(x_2)\sigma_3(x_3)\sigma_3(x_4)\ldots\rangle$]

Feynman diagrams ↔ Branched Covers

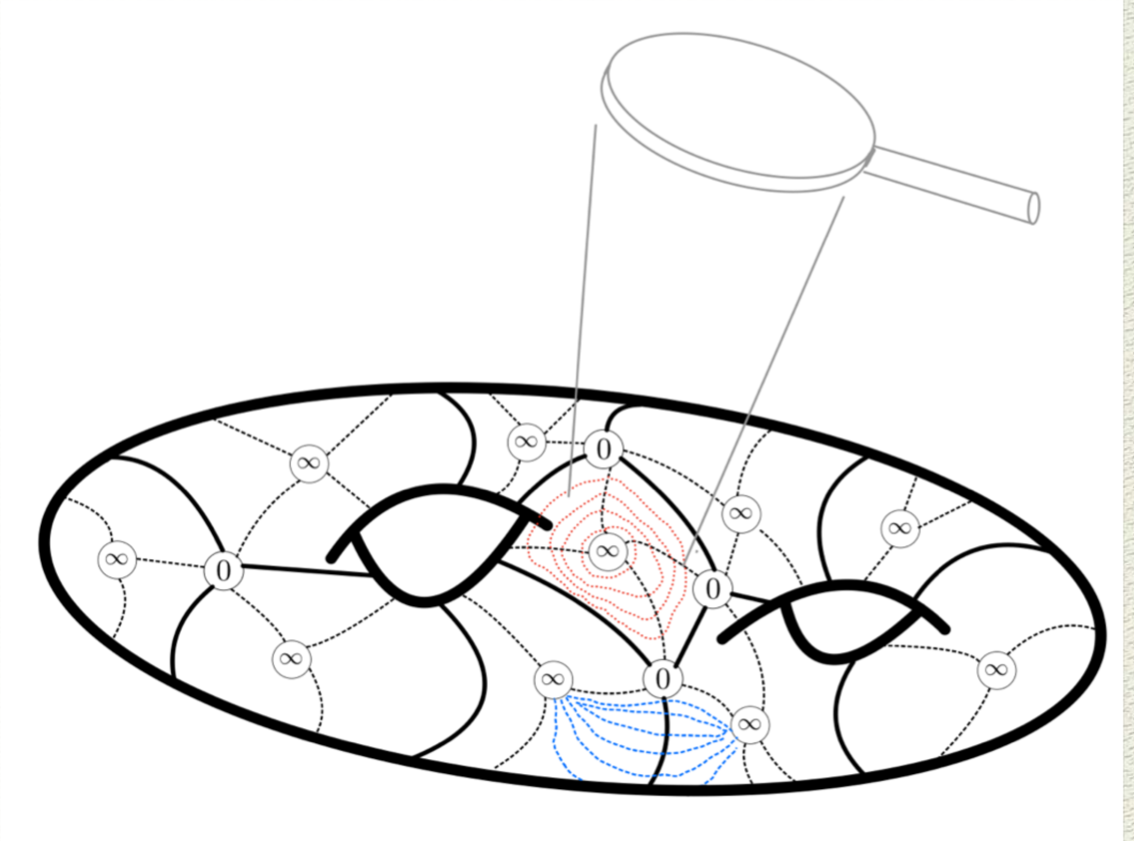
Matrix Model

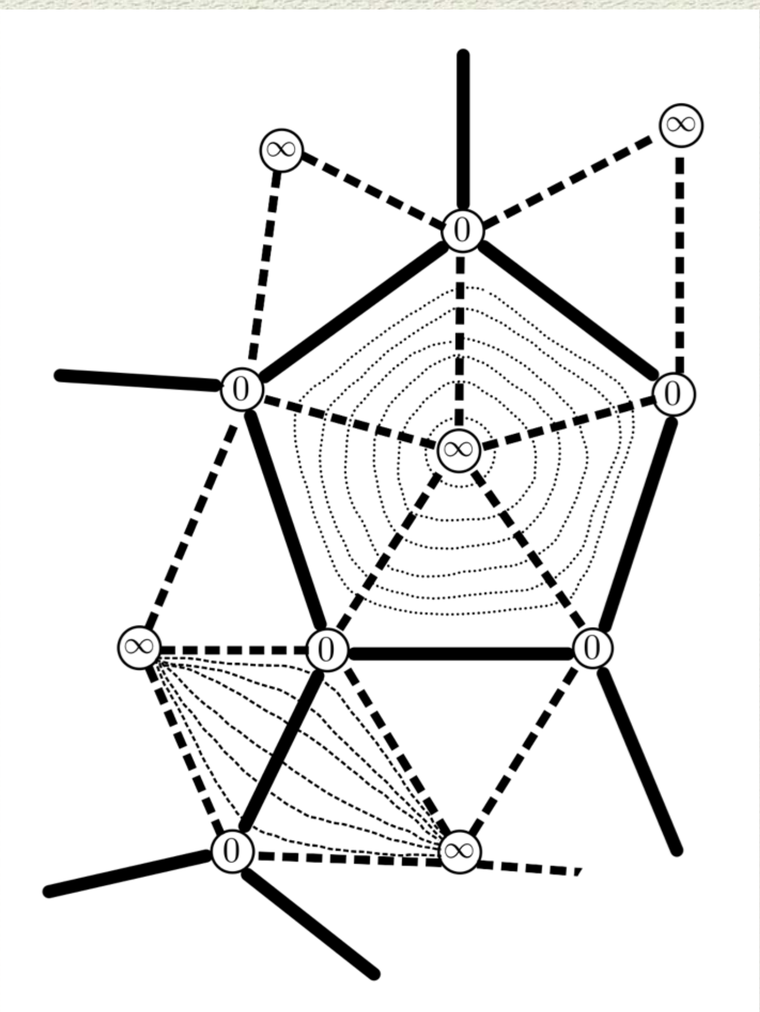




- Feynman diagrams directly pictured as covering space worldsheets pulled back from target space.
  [cf. Pakman, Rastelli, Razamat]
- 1-1 correspondence between branched covers and distinct Feynman diagrams.

The Strebel Construction

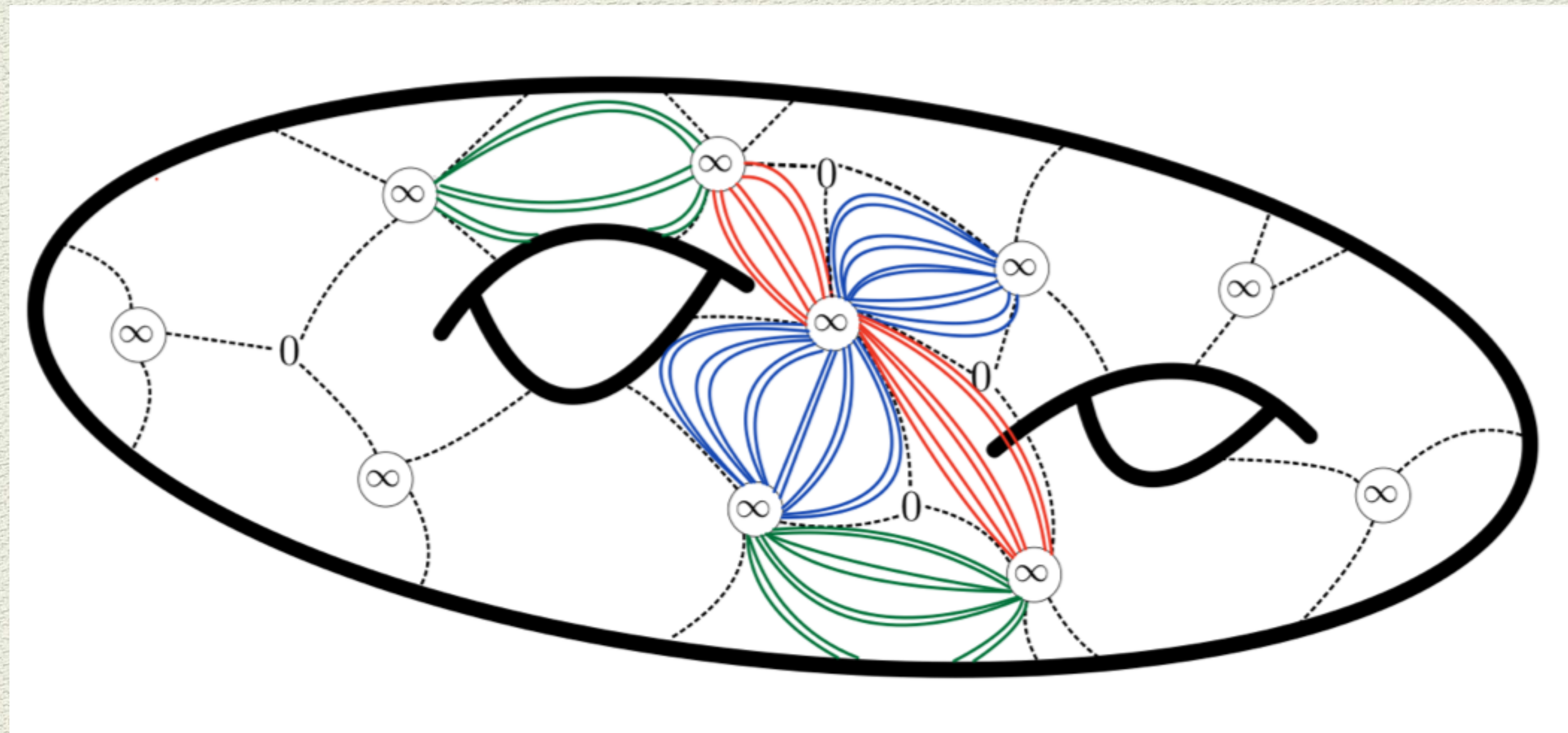


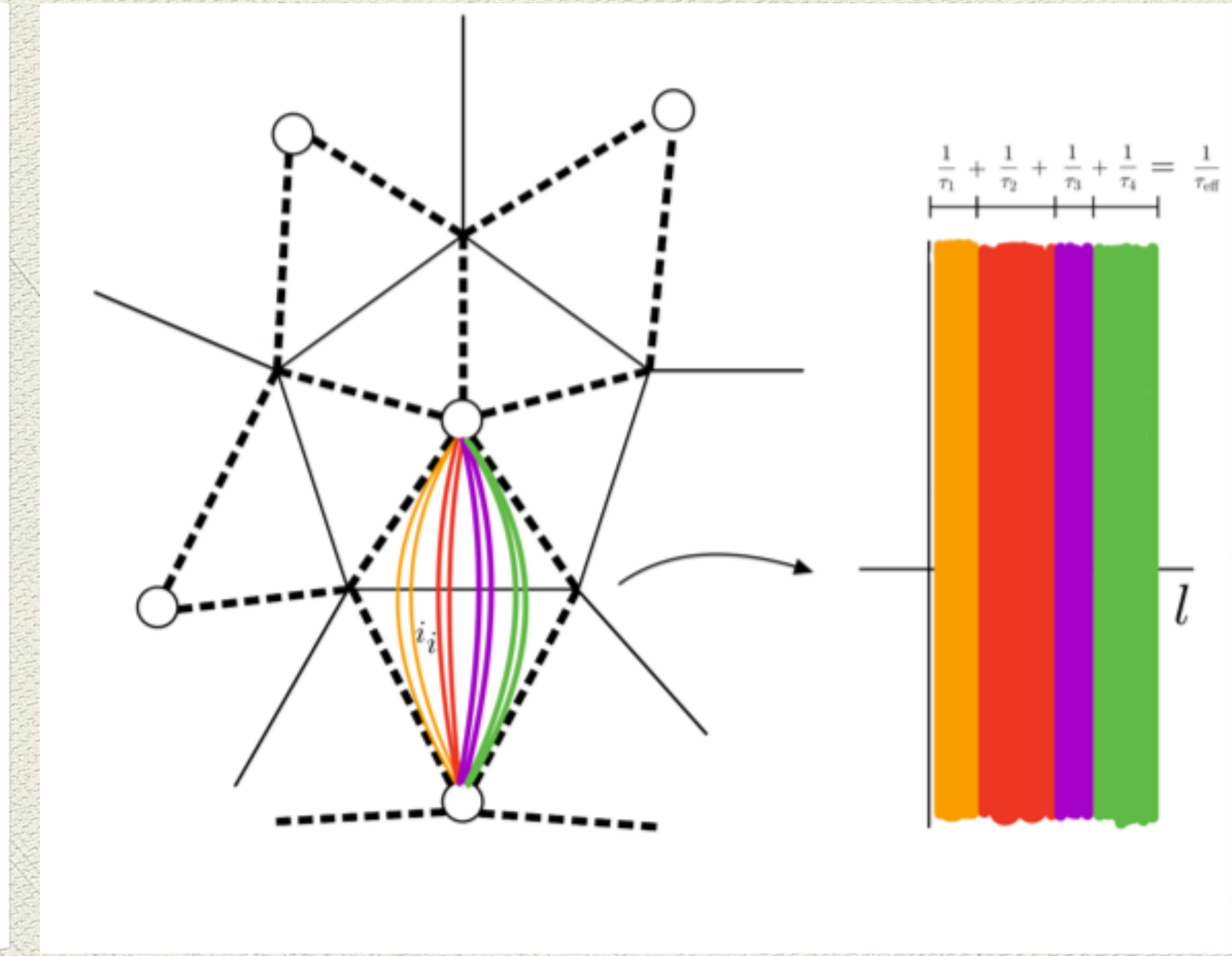


- The point on $\mathscr{M}_{g,n}$ associated to a Feynman diagram given by an explicit gluing construction of strips - string bit worldsheets.
- Relies on the parametrisation of $\mathscr{M}_{g,n}$ by a unique quadratic (Strebel) differential $\phi_S(z)dz^2$. Has n double poles at marked points.
- It's \`horizontal trajectories' - $\phi_S(z(t))\left(\frac{dz(t)}{dt}\right)^2 > 0$ - foliate $\Sigma_{g,n}$ into n disk faces each with pole at z_i, separated by a critical graph. Vertices of this Strebel graph are zeroes a_m of ϕ_S.
- Real Strebel lengths $l_{km} = \int_{a_k}^{a_m} \sqrt{\phi_S(z)}dz$ and critical graph topology parametrises $\mathscr{M}_{g,n}$.

Strebel Construction (Contd.)

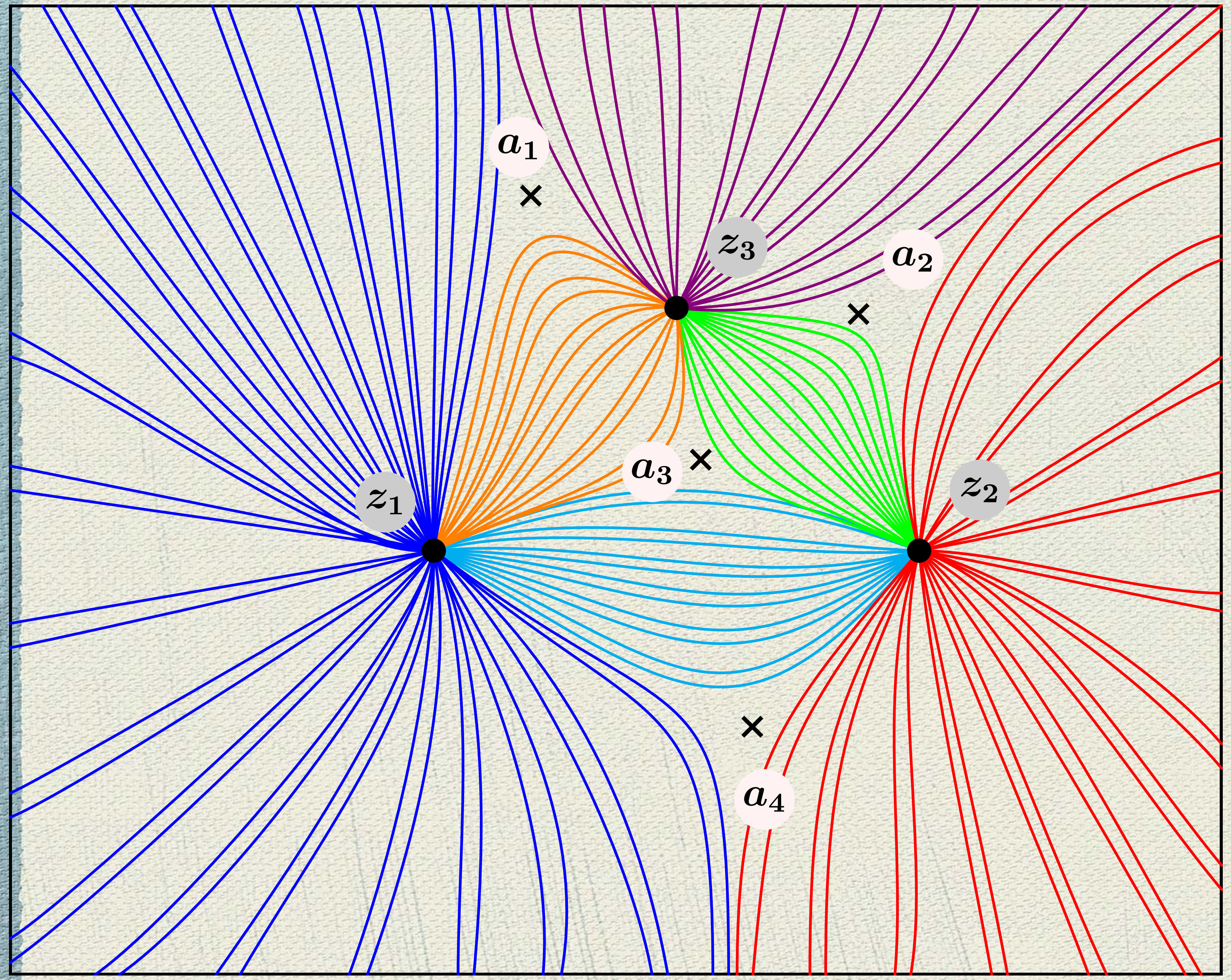
- `Vertical trajectories' - $\phi_S(z(t))\left(\frac{dz(t)}{dt}\right)^2 < 0$, begin and end on poles z_i. Generates strips, glued at centres of faces (poles) and vertices of Strebel graph (zeroes) correspond to Feynman propagators.
- Feynman graph of corresponding worldsheet is dual to the Strebel graph. [Strebel graph another open string description - Triality!]
- Integer Strebel lengths (# of wick contractions in matrix model) $\leftrightarrow$ arithmetic points on $\mathscr{M}_{g,n}$. [Mulase-Penkava, Razamat, R.G.]
- Gross-Mende like limit ($k_i \to \infty$) in symm. orbfld. Also picks out the same arithmetic points. [Gaberdiel-R.G.-Knighton-Maity]
- Nambu-Goto weight in `Strebel gauge' $e^{-\int_\Sigma d^2z|\phi_S(z)|}$ gives right Liouville weight in this limit. $\phi_S(z) \approx S[\Gamma(z)]$ - Schwarzian.



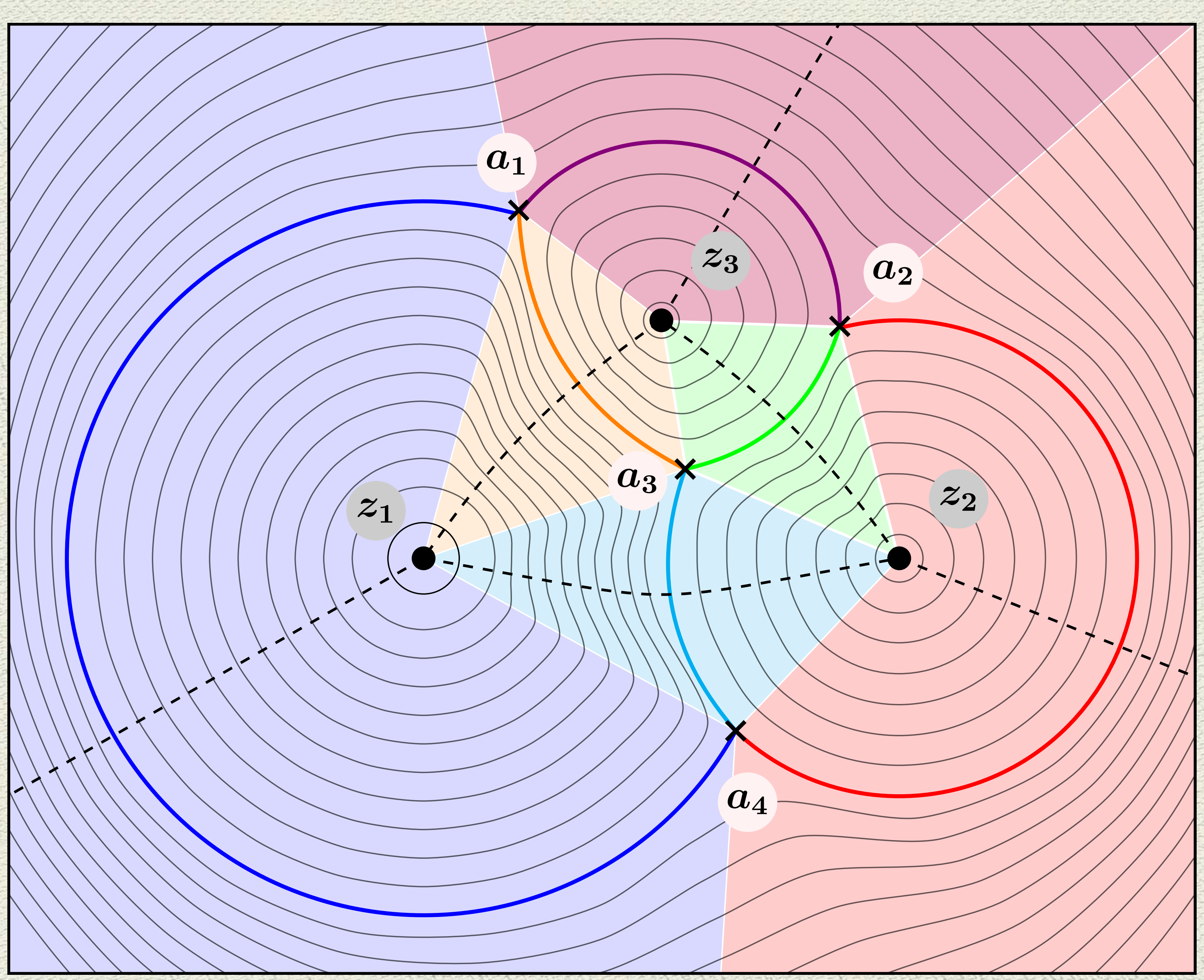


Open-Closed Triptych

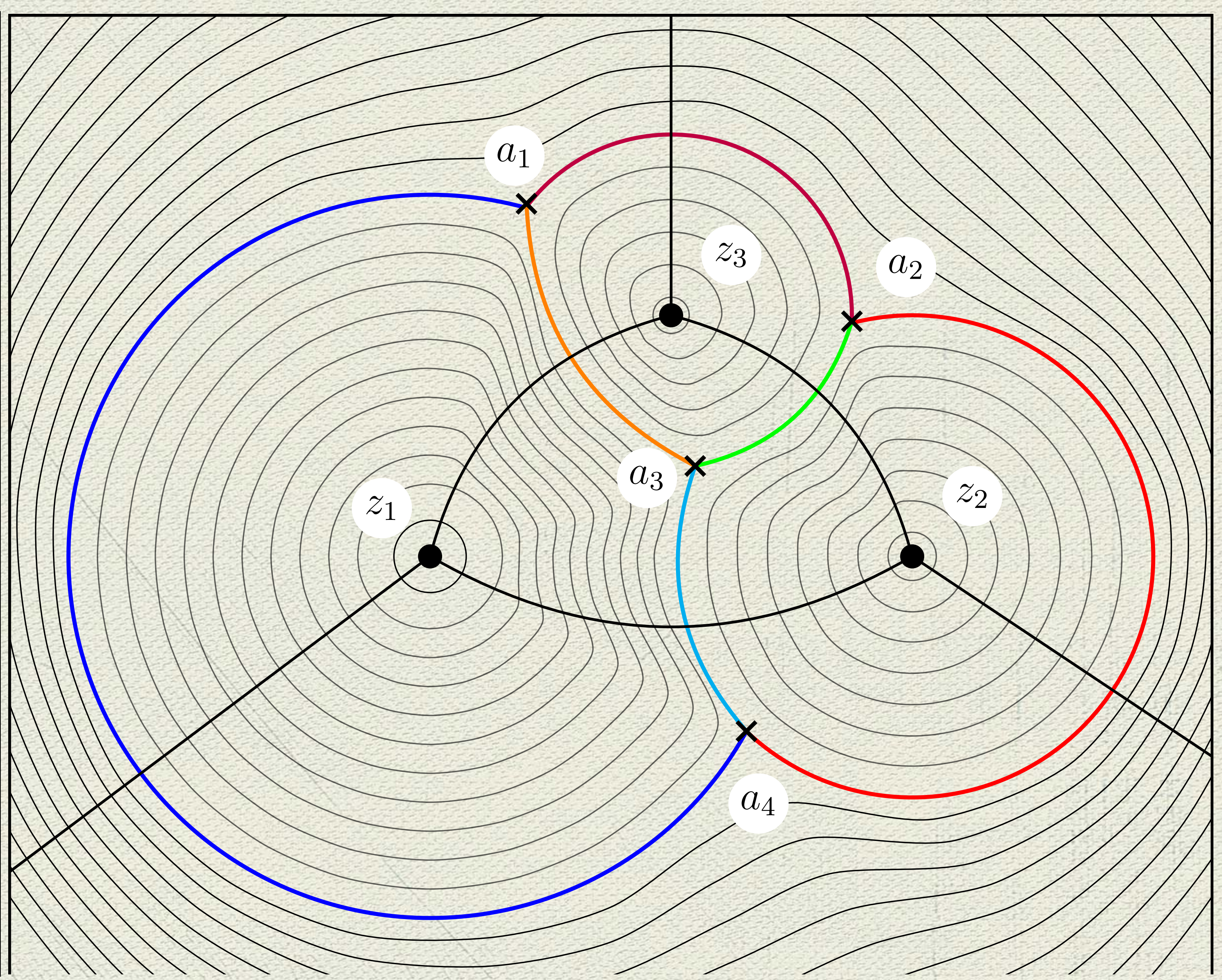
1) Ribbon Graphs



2) Glued up Strips



3) Strebel Surface



Bridge from Strings to Fields

- This picture of large N Feynman diagrams strikingly predicts that the dual worldsheet amplitudes should have (a presentation with) δ-function support on points of $\mathscr{M}_{g,n}$.
- Realised precisely in tensionless worldsheet theory of $AdS_3 \times S^3 \times T^4$. [Eberhardt-Gaberdiel-R.G.]
- Correlators of $\mathfrak{sl}(2,\mathbb{R})_1$ spectrally flowed representations $\mathscr{V}^{k_i}_{j=1/2}(x_i, z_i)$ are zero unless there exists $x = \Gamma(z)$ s.t. $x \approx x_i + (z - z_i)^k + \ldots$ in the vicinity of each z_i.
- The δ-fn. behaviour transparent in a twistorial free field description of worldsheet.

[Dei-Gaberdiel-R.G.-Knighton]

- Liouville weight associated with the non-zero points also matches with Lunin-Mathur computation.

Belyi Maps from Strings

- Dual to the Hermitian Matrix model proposed [R.G.-Mazenc '22]: A-model Topological String theory on Kazama-Suzuki coset $\mathfrak{sl}(2,\mathbb{R})_1/\mathfrak{u}(1)$ (with momentum deformation).
- Physical cohomology given by operators $\mathscr{Y}^{(k_i)}_{j=1/2}(z_i)$ in $\mathfrak{sl}(2,\mathbb{R})_1$ spectrally flowed representations - exactly like in the AdS_3 case. [Mukhi-Vafa-Frenkel; Ashok-Murthy-Troost]
- Ward Identity arguments again imply δ-fn. support on covering maps. But now no x_i dependence. All $\mathscr{Y}^{(k_i)}_{j=1/2}(z_i)$ mapped to ∞ on target space cigar. Branching $(k_1)\ldots(k_n) \rightarrow \beta$.
- Need $(k = \sum_i k_i)/2$ momentum (2) insertions for mom. conservation. Branching $(2)^{k/2} \rightarrow \alpha$.
- Together with permutation γ, structure exactly that of Belyi maps.

Questions, Questions

- Are Arithmetic Riemann Surfaces universal saddle points (a la Gross-Mende) for tensionless strings (e.g. on AdS)?
- Is there a mathematically natural sense in which one might localise (in the spirit of Atiyah-Bott) to arithmetic Riemann surfaces? "*Arithmetic Localisation*"
- Can such a localisation be realised physically by a special class of topological strings?
- Would the topological string description help you to better understand $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ and its representations? The String dual to *Dessins d'Enfants*.

Thanks for your attention