

SCALE-INVARIANT GRAVITY IN COSMIC INFLATION

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OUTLINE

- Quick facts about quadratic gravity theories
- Spontaneous scale-invariance breaking
- Inflation
- Conclusions and further developments

MAIN GOALS

- Consider an **exactly scale-invariant** $f(R, \phi)$ theory
- Show that a **mass scale emerges dynamically** (no need for quantum correction)
- Study an **inflationary scenario** in this theory and show that it fits data
- **terminology:** here “quadratic” means “exactly scale-invariant”

Quick facts about quadratic gravity

In the literature one often refers to the action

$$S = \int d^4x \sqrt{g} \left(\frac{M^2}{2} (R - 2\Lambda) + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} \right) + S_m, \quad (R^\alpha_{\mu\lambda\sigma} = \partial_\lambda \Gamma^\alpha_{\mu\sigma} - \dots)$$

as to “quantum gravity”.

This theory is renormalisable (Stelle '77). DOF are (flat space)

- massless spin-2 field
- ghost massive spin-2 field
- massive scalar field

At large curvature the theory becomes scale-invariant.

Invariance is typically broken by quantum corrections or by the linear term in R .

...But it can also be broken dynamically!

Simplest case of scale-invariant theory:

$$\mathcal{L}_J = \frac{\alpha}{36} \sqrt{|g|} R^2$$

$$S_J = \int \mathcal{L}_J d^4x \quad \text{invariant under} \quad \bar{g}_{\mu\nu}(x) = g_{\mu\nu}(\ell x)$$

Weyl - rescaling from Jordan frame to Einstein frame

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \Omega = \frac{\sqrt{\alpha R}}{3M}$$

Note that the rescaling can be done only for $R \neq 0$

Weyl - rescaling from Jordan frame to Einstein frame:

$$\mathcal{L}_E = \sqrt{|\tilde{g}|} \left[\frac{M^2}{2} (\tilde{R} - 2\Lambda) - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\partial}_\mu \psi \tilde{\partial}_\nu \psi \right], \quad \Lambda = \frac{9M^2}{4\alpha}$$

$$\psi = \sqrt{6}M \ln \Omega = \sqrt{\frac{3}{2}}M \ln \left(\frac{\alpha R}{9M^2} \right)$$

M is arbitrary
and "redundant"

E-frame Lagrangian is still scale-invariant!

E-frame equations of motion:

$$\tilde{G}_{\mu\nu} + \Lambda \tilde{g}_{\mu\nu} = M^2 \left(\tilde{\partial}_\mu \psi \tilde{\partial}_\nu \psi - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{\partial}_\mu \psi \tilde{\partial}^\mu \psi \right)$$

By comparing eom:

$$\text{JF: } RR_{\mu\nu} - \frac{1}{4}R^2 g_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu \nabla_\nu)R = 0$$

$$\text{EF: } \tilde{G}_{\mu\nu} + \Lambda \tilde{g}_{\mu\nu} = M^2 \left(\tilde{\partial}_\mu \psi \tilde{\partial}_\nu \psi - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{\partial}_\mu \psi \tilde{\partial}^\mu \psi \right)$$

The JF admits Ricci-flat solutions, the EF does not!

Minkowski space is a solution in JF but it is not in EF.

Same for any other Ricci-flat solution.

...careful when quantising around flat space in EF!

Emergence of a scale

SCALE-INVARIANT INFLATIONARY SCENARIO

$$\mathcal{L}_J = \frac{\alpha}{36} \sqrt{|g|} R^2, \quad ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j$$

EOM: $2HH'' + H'^2 + 6HH' = 0 \quad N = \ln a, \quad H = a'/a$

One class of solutions is (in terms of e-foldings N)

$$H = (c_1 + c_2 e^{-3N})^{2/3}, \quad N = \ln a$$

The solution interpolates between a radiation-dominated Universe ($H = \exp(-2N)$) and a de Sitter Universe ($H = \text{const}$).

Note that there is no EM source!

However, the solution is no good for inflation. We need another DOF.

Scalar-tensor theory (higher-derivative "induced gravity"):

$$\mathcal{L}_{\text{inv}} = \sqrt{g} \left[\frac{\alpha}{36} R^2 + \frac{\xi}{6} \phi^2 R - \frac{1}{2} (\partial\phi)^2 - \frac{\lambda}{4} \phi^4 \right]$$

Effective **scalar potential** for fixed R:

$$V_{\text{eff}} = -\frac{\xi}{6} \phi^2 R + \frac{\lambda}{4} \phi^4$$

Max and min correspond to **fixed points**:

$$\phi = 0, \quad \phi_0^2 = \frac{\xi R}{3\lambda}$$

EOM for **flat** RW metric. $ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j$:

$$H^2 \phi'' + (HH' + 3H^2) \phi' - 2\xi \phi H H' - \phi(4\xi H^2 - \lambda \phi^2) = 0,$$

$$\alpha H^2 (2HH'' + H'^2 + 6HH') + 2\xi H^2 \phi \phi' - \frac{1}{2} \phi'^2 H^2 + \frac{\phi^2}{4} (4\xi H^2 - \lambda \phi^2) = 0.$$

NB if we add $\beta R_{\mu\nu} R^{\mu\nu}$ then $\alpha \rightarrow \alpha + 12\beta$ in the above eq.

There are only **two fixed points**

$$(H, \phi) = (H, 0)$$

H arbitrary, **saddle** point

$$(H, \phi) = (H, 2\sqrt{\xi/\lambda}H)$$

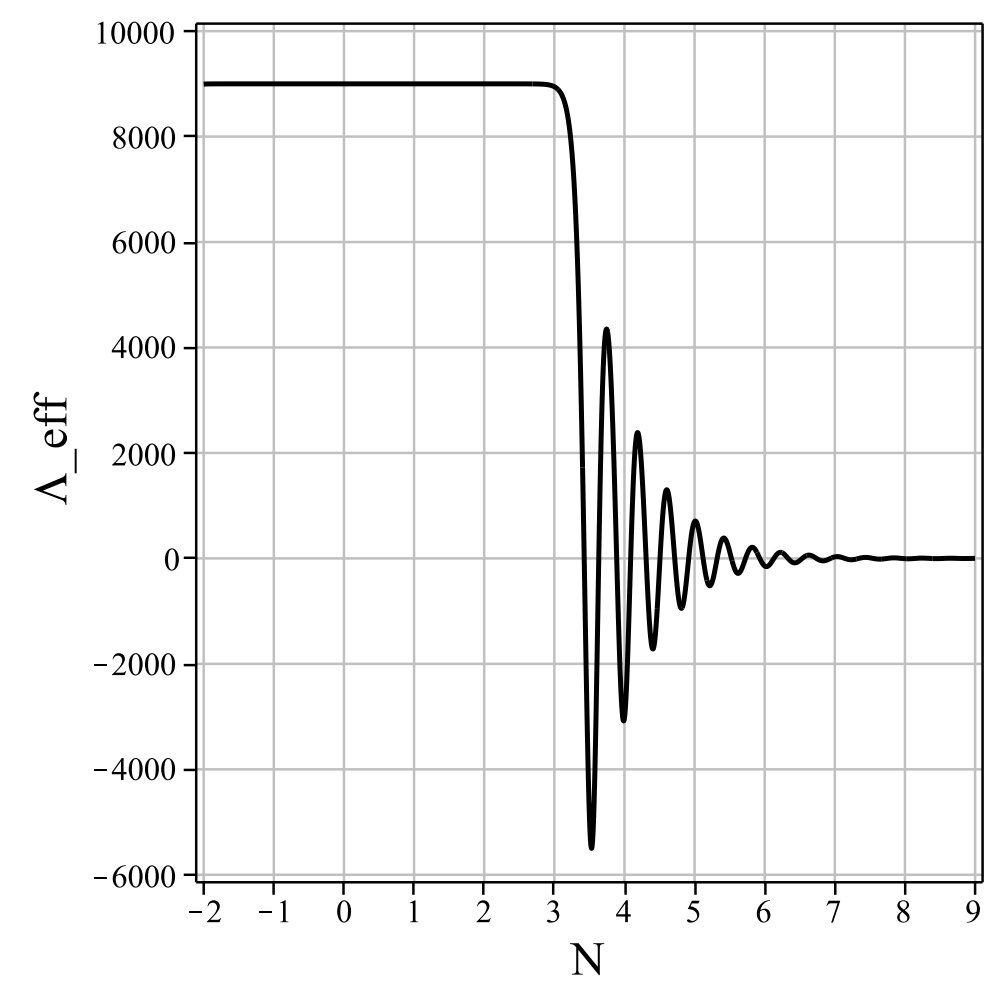
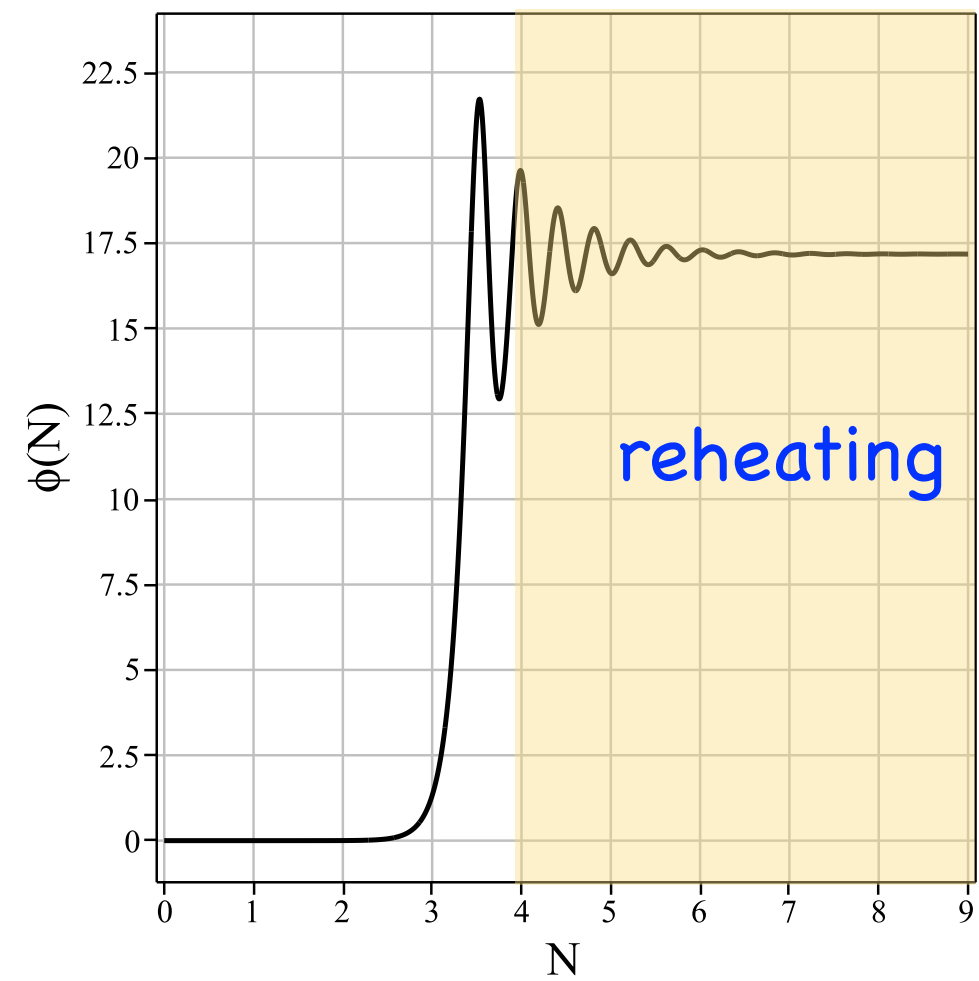
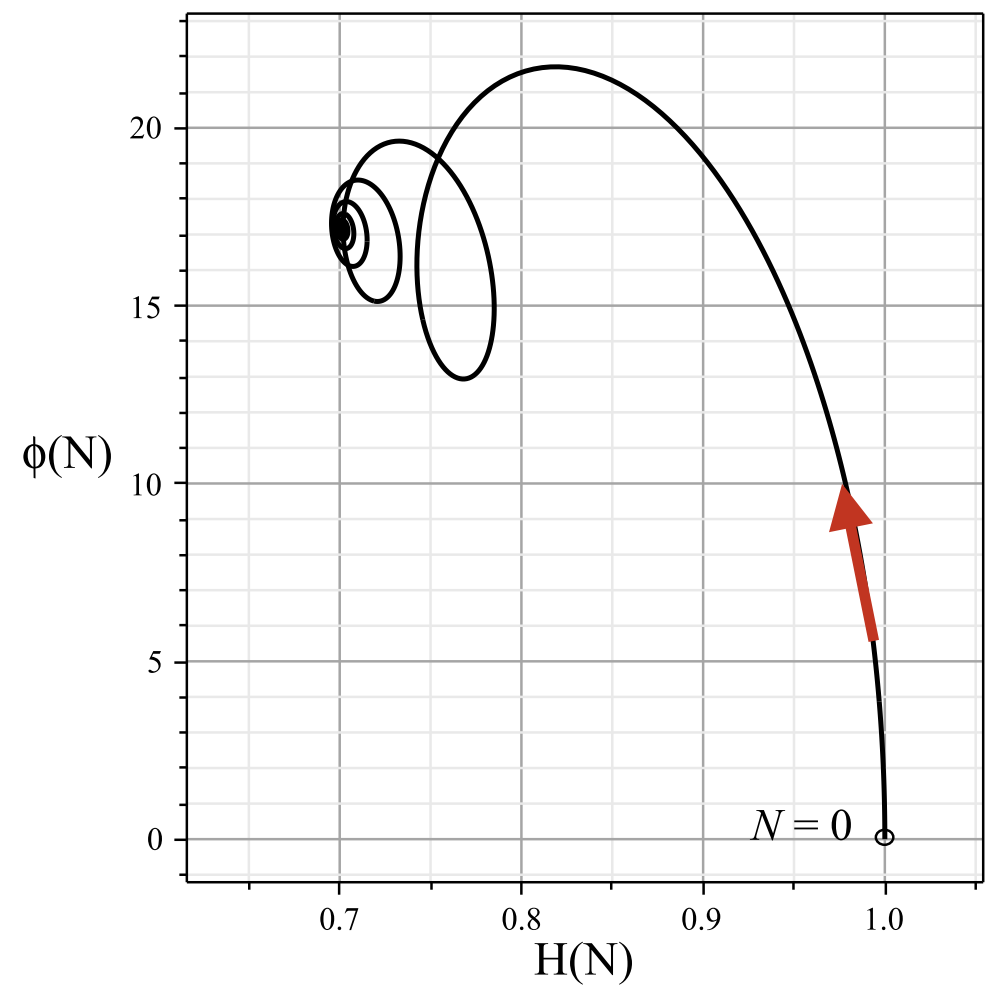
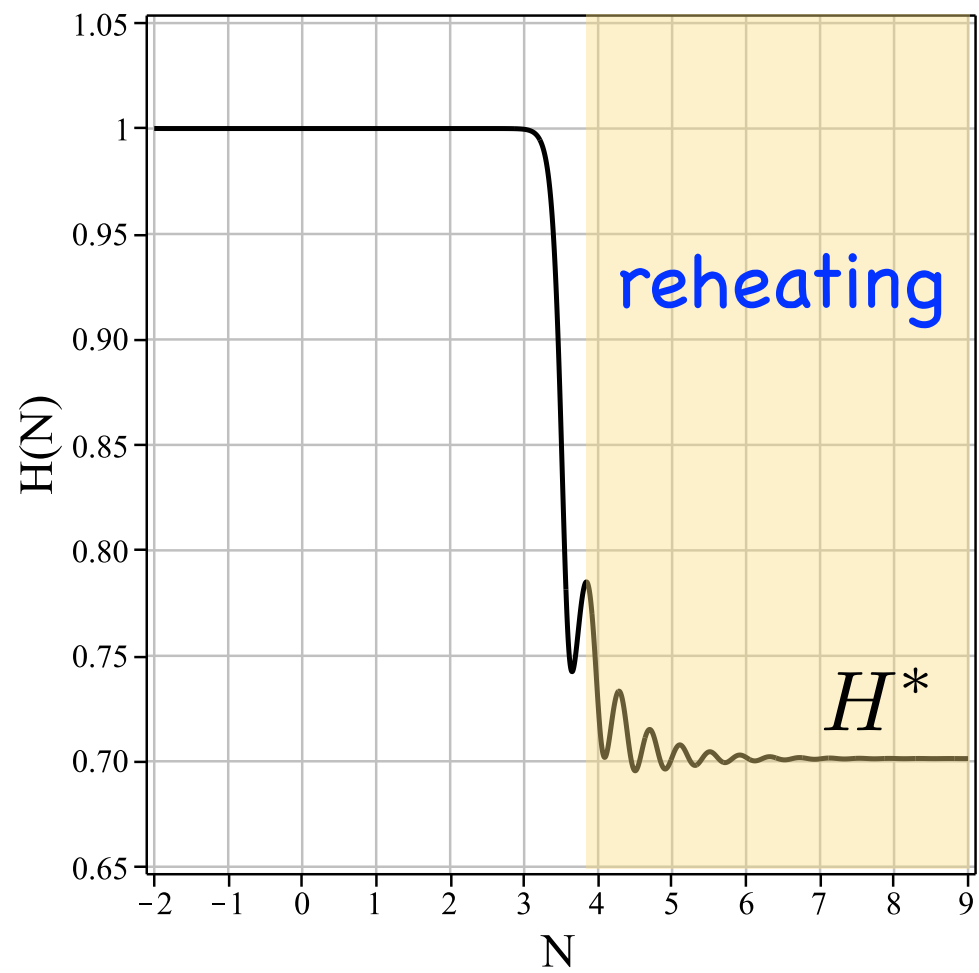
H arbitrary, **stable** point

At the stable fixed point: $\Lambda_{\text{eff}} = \frac{\xi^2 R^2}{36\lambda} - \frac{\lambda\phi^4}{4} \rightarrow 0$

The scalar field is stabilized and a mass scale emerges:

$$M = \sqrt{\xi/3} \phi_0 \Leftrightarrow \alpha = \xi^2/\lambda$$

$$\mathcal{L}_{\text{inv}} = \sqrt{|\det g|} \left[\cancel{\frac{\alpha}{36} R^2} + \boxed{\frac{\xi}{6} \phi^2} R - \frac{1}{2} (\cancel{\partial\phi})^2 - \cancel{\frac{\lambda}{4} \phi^4} \right]$$



What happens in the Einstein frame?

$$\mathcal{L}_{JF} = \sqrt{g} \left[\frac{\alpha}{36} R^2 + \frac{\xi}{6} \phi^2 R - \frac{1}{2} (\partial\phi)^2 - \frac{\lambda}{4} \phi^4 \right] \quad \text{Fields redefinitions} \rightarrow$$

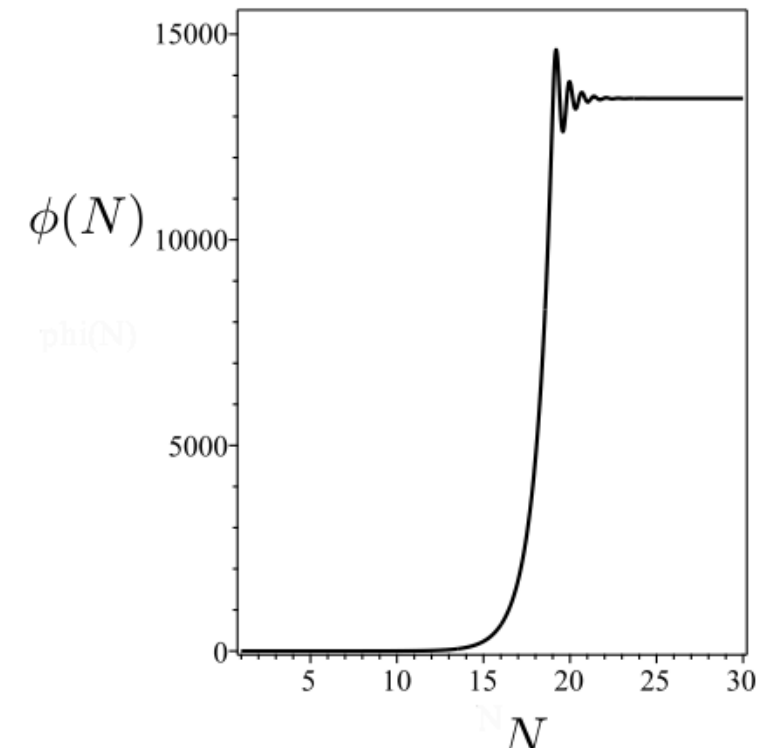
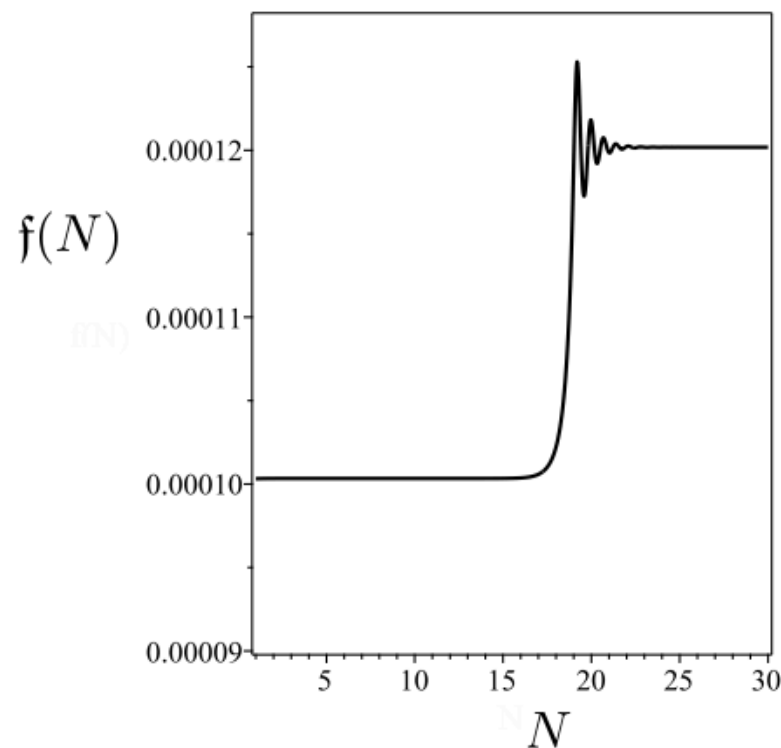
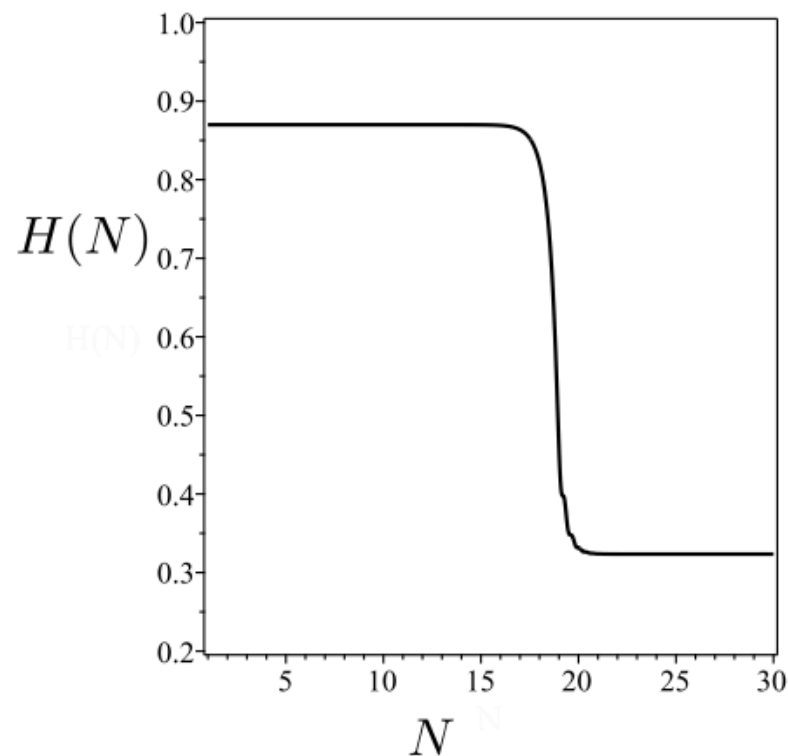
$$\mathcal{L}_{EF} = \sqrt{g} \left[\underbrace{\frac{M^2}{2} R - \frac{9M^4}{4\alpha}}_{\text{EH term}} - \underbrace{\frac{3M^2}{f^2} (\partial f)^2 - \frac{f^2}{2M^2} (\partial\phi)^2}_{\text{non-canonical kinetic terms}} + \underbrace{\frac{3\xi\phi^2 f^2}{2\alpha} - \frac{\Omega\phi^4 f^4}{4\alpha M^4}}_{\text{quartic potential}} \right]$$

EH term

non-canonical
kinetic terms

quartic potential

$$\Omega = \alpha\lambda + \xi^2$$



Inflationary dynamics

Further simplification – due to scale-invariance

Perform a new field redefinition:

$$\zeta = \sqrt{6}M \operatorname{arcsinh} \left(\frac{f\phi}{\sqrt{6}M^2} \right), \quad \rho = \frac{M}{2} \ln \left(\frac{\phi^2}{2M^2} + \frac{3M^2}{f^2} \right)$$

New lagrangian:

$$\mathcal{L} = \sqrt{g} \left[\frac{M^2}{2} R - \frac{1}{2} (\partial\zeta)^2 - 3 \cosh^2 \left(\frac{\zeta}{\sqrt{6}M} \right) (\partial\rho)^2 - U(\zeta) \right]$$

One-field potential only:

$$U(\zeta) = -\frac{9\xi M^4}{\alpha} \sinh^2 \left(\frac{\zeta}{\sqrt{6}M} \right) + \frac{9\Omega M^4}{\alpha} \sinh^4 \left(\frac{\zeta}{\sqrt{6}M} \right) + \frac{9M^4}{4\alpha}$$

Just recall where we started off:

$$\mathcal{L}_{\text{inv}} = \sqrt{|\det g|} \left[\frac{\alpha}{36} R^2 + \frac{\xi}{6} \phi^2 R - \frac{1}{2} (\partial\phi)^2 - \frac{\lambda}{4} \phi^4 \right]$$

We now search for an inflationary solution (1 field inflation):

Small field limit $\zeta \ll M$ leads to "hilltop inflation"? No!

$$U(\zeta) = -\frac{9\xi M^4}{\alpha} \sinh^2\left(\frac{\zeta}{\sqrt{6}M}\right) + \frac{9\Omega M^4}{\alpha} \sinh^4\left(\frac{\zeta}{\sqrt{6}M}\right) + \frac{9M^4}{4\alpha}$$

$$U(\zeta) \approx \frac{9M^4}{4\alpha} \left[1 - \frac{2\xi}{3} \frac{\zeta^2}{M^2} - \frac{1}{9} \left(\frac{\xi}{3} - \Omega \right) \frac{\zeta^4}{M^4} + \mathcal{O}\left(\frac{\zeta^6}{M^6}\right) \right]$$

should vanish (Planck)

then it has the wrong sign

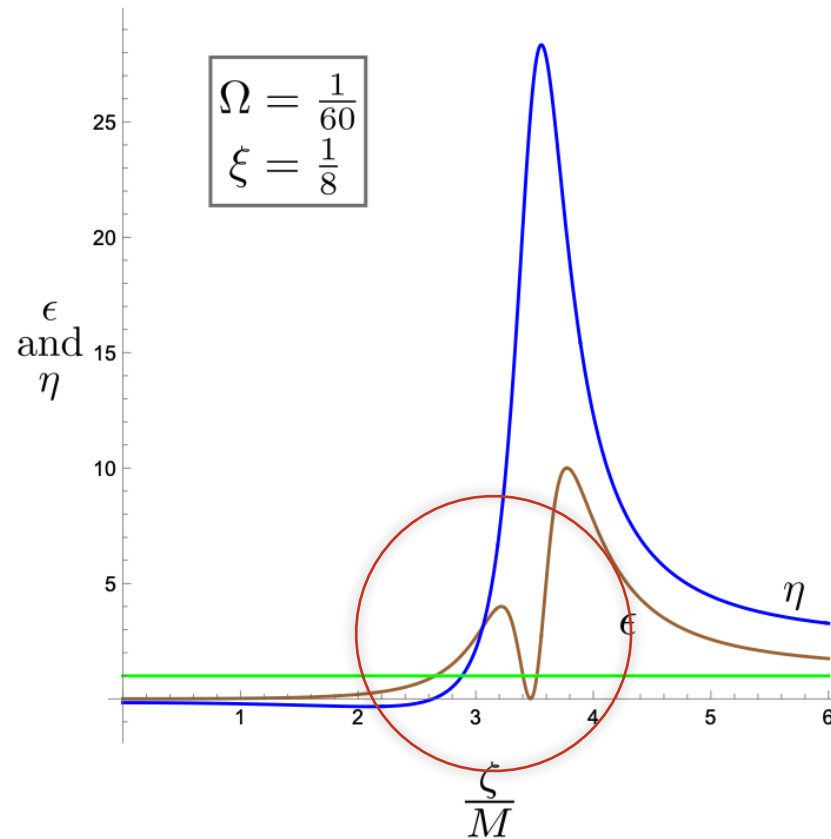
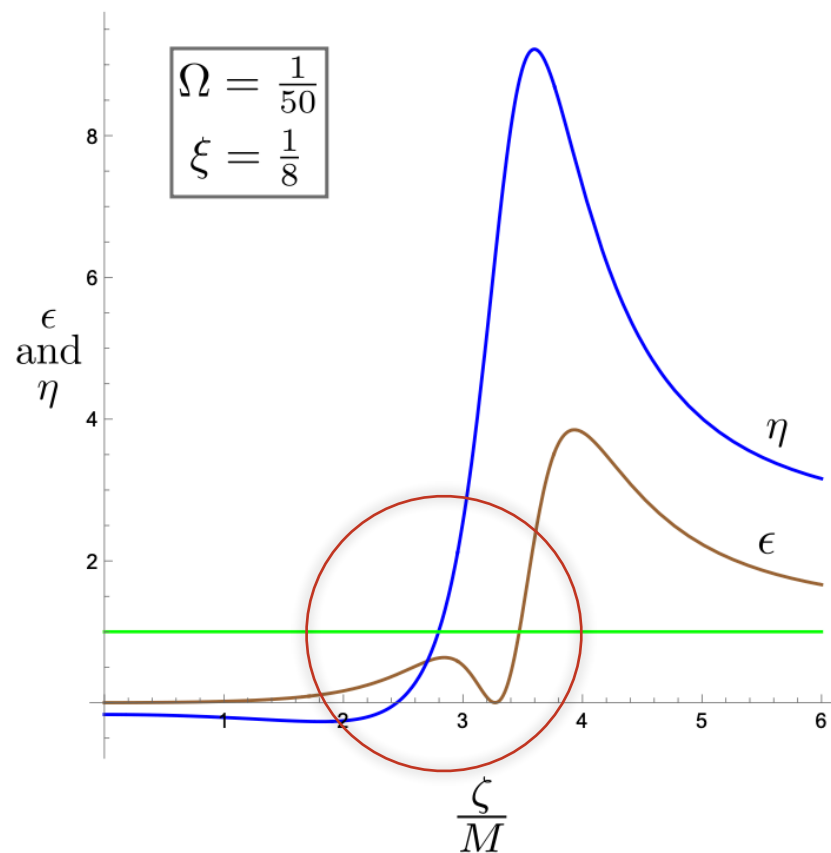
One can also show that $\Omega > \frac{4}{3}$, $n_s \simeq 1 + \frac{8}{3}\Omega \frac{\zeta^2}{M^2} > 1$

Consider $\zeta \geq M$:

Slow-roll parameters

$$\epsilon = \frac{M^2}{2} \left(\frac{1}{U} \frac{dU}{d\zeta} \right)^2, \quad \eta = \frac{M^2}{U} \frac{d^2U}{d\zeta^2}$$

$$\Delta N = -\frac{1}{M} \int_{\zeta_*}^{\zeta_{\text{end}}} \frac{d\zeta}{\sqrt{2\epsilon}} < \infty \Leftrightarrow \Omega < \frac{2\sqrt{3}}{3} \xi^2 \simeq 1.1547 \xi^2$$



Also, slow-roll parameters are well-defined for $\Omega < \frac{1}{64}$

All this implies $\xi < 0.11$

Analytic solutions if approximate (large-ish field approx)

$$U(\zeta) = -\frac{9\xi M^4}{\alpha} \sinh^2\left(\frac{\zeta}{\sqrt{6}M}\right) + \frac{9\Omega M^4}{\alpha} \sinh^4\left(\frac{\zeta}{\sqrt{6}M}\right) + \frac{9M^4}{4\alpha}$$

$$U \rightarrow \tilde{U} = \frac{9M^4}{4\alpha} \left[1 - \xi \exp\left(\frac{\sqrt{6}\zeta}{3M}\right) + \frac{\Omega}{4} \exp\left(\frac{2\sqrt{6}\zeta}{3M}\right) \right]$$

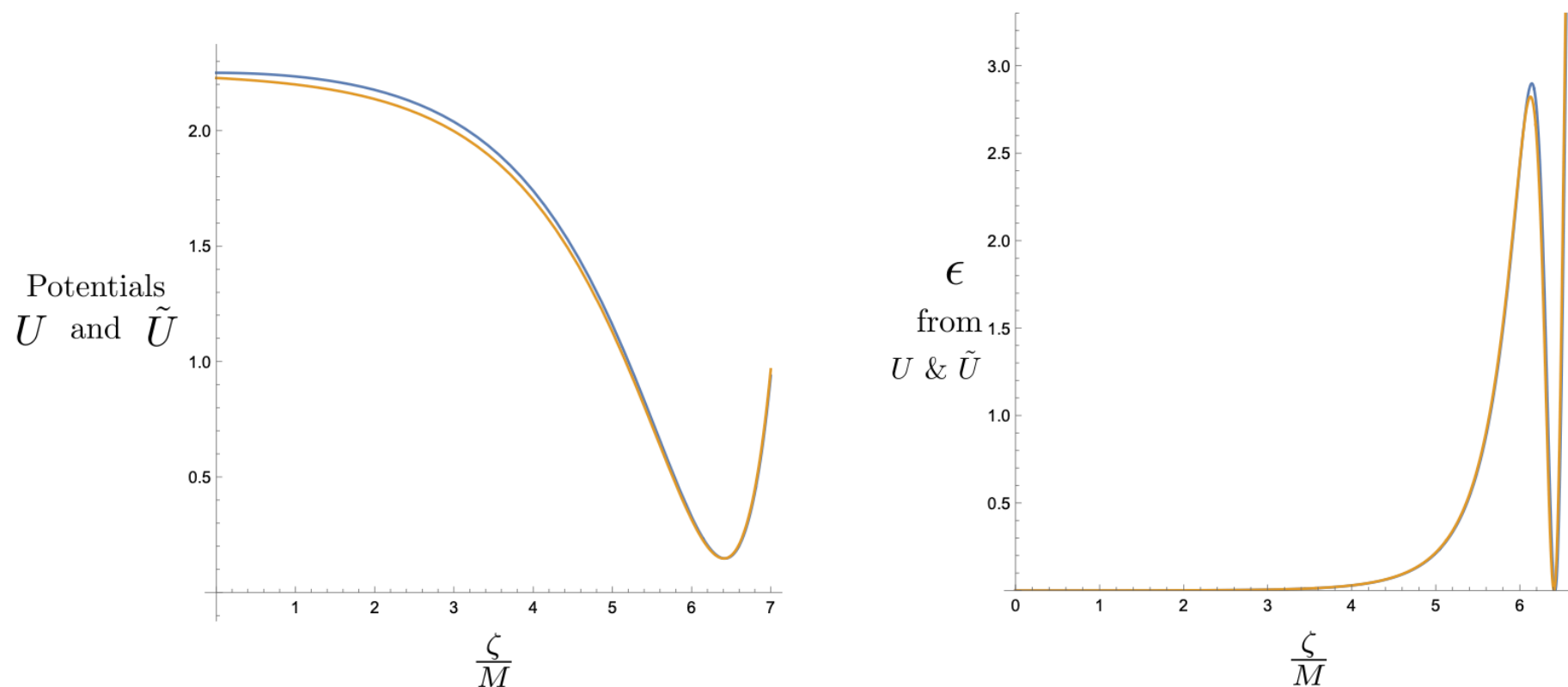


Figure 5. Plot of U (blue curve) and $\tilde{U}$ (yellow curve) on the left and of ϵ constructed with U (blue curve) and $\tilde{U}$ (yellow curve) with $\Omega = 1.07\xi^2$, $\xi = 10^{-2}$, $\alpha = 1$ and $M = 1$.

Example:

$$\Omega = 1.07 \xi^2, \quad \xi = 10^{-2}, \quad \Delta N = 60$$

$$\zeta_i = 1.04 M \rightarrow \zeta_f = 5.637 M \quad \text{Large field}$$



$$n_s = 0.9679, \quad r = 0.003, \quad \xi_v^2 = 0.00024.$$

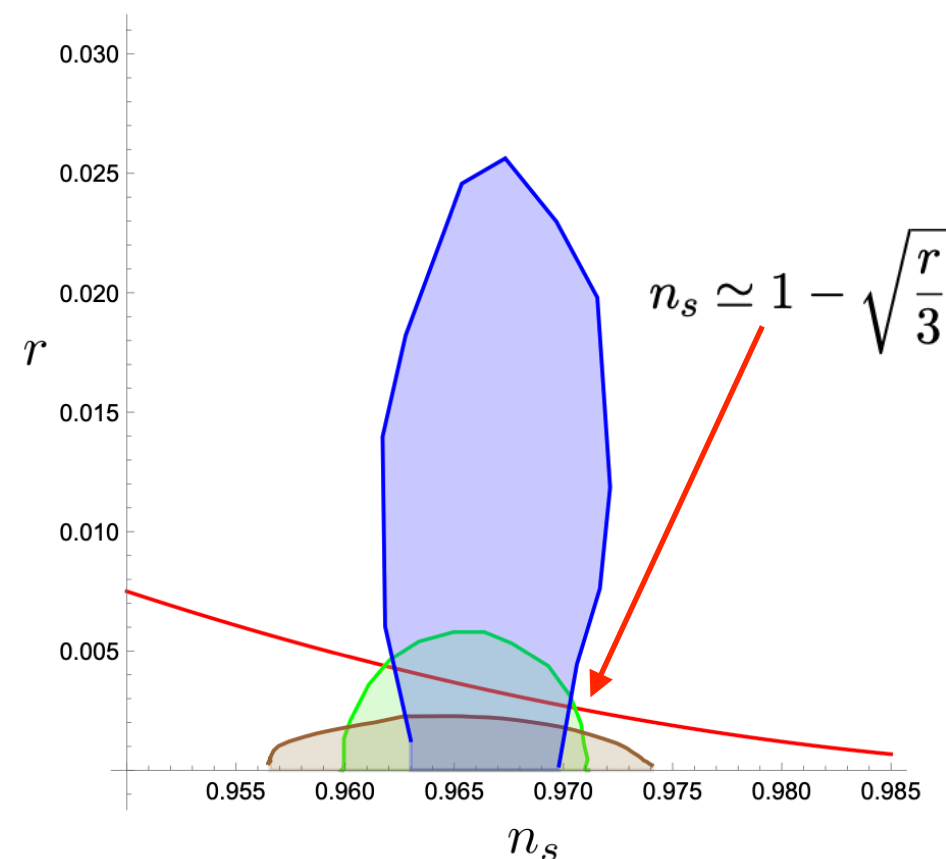


Figure 4. Plot of r vs n_s following from (5.26). The blue region represents the current PLANCK + BICEP constraints [66–69], the green region represents the future reach of Simon's Observatory [70] while the brown region depicts the detection range of LiteBIRD [71]

Summary: starting from the **scale-invariant theory**

$$\mathcal{L}_{JF} = \sqrt{g} \left[\frac{\alpha}{36} R^2 + \frac{\xi}{6} \phi^2 R - \frac{1}{2} (\partial\phi)^2 - \frac{\lambda}{4} \phi^4 \right]$$

we show that a **mass scale emerges dynamically** on a flat RW metric.

In the process, **inflation occurs**.

The three-parameter space is mostly restricted by the inflationary mechanism (not by observations – the only input is the number of e-foldings).

Compared to the **Starobinski model** $R + R^2/m^2$: one extra DOF but no energy scale m to be determined.

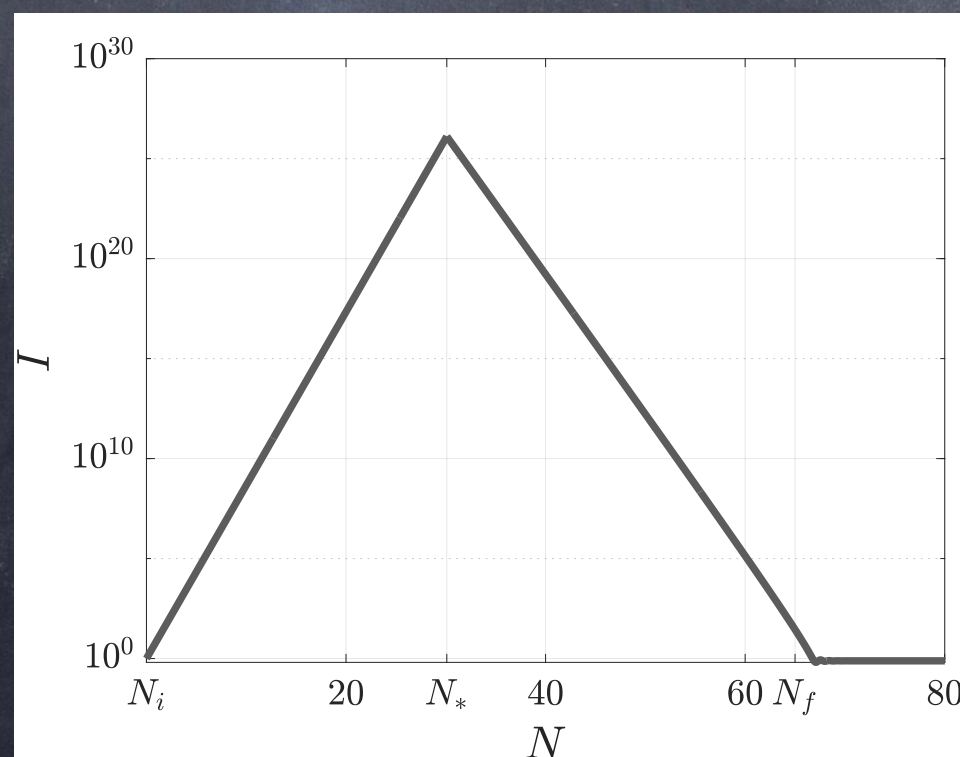
Compared to **Higgs inflation**: ξ must be small.

Other directions

- add spatial (positive) curvature: quasi-De Sitter bounce
- non-symmetric solution (hierarchy problem?)

Primordial Magnetogenesis (Chiara)

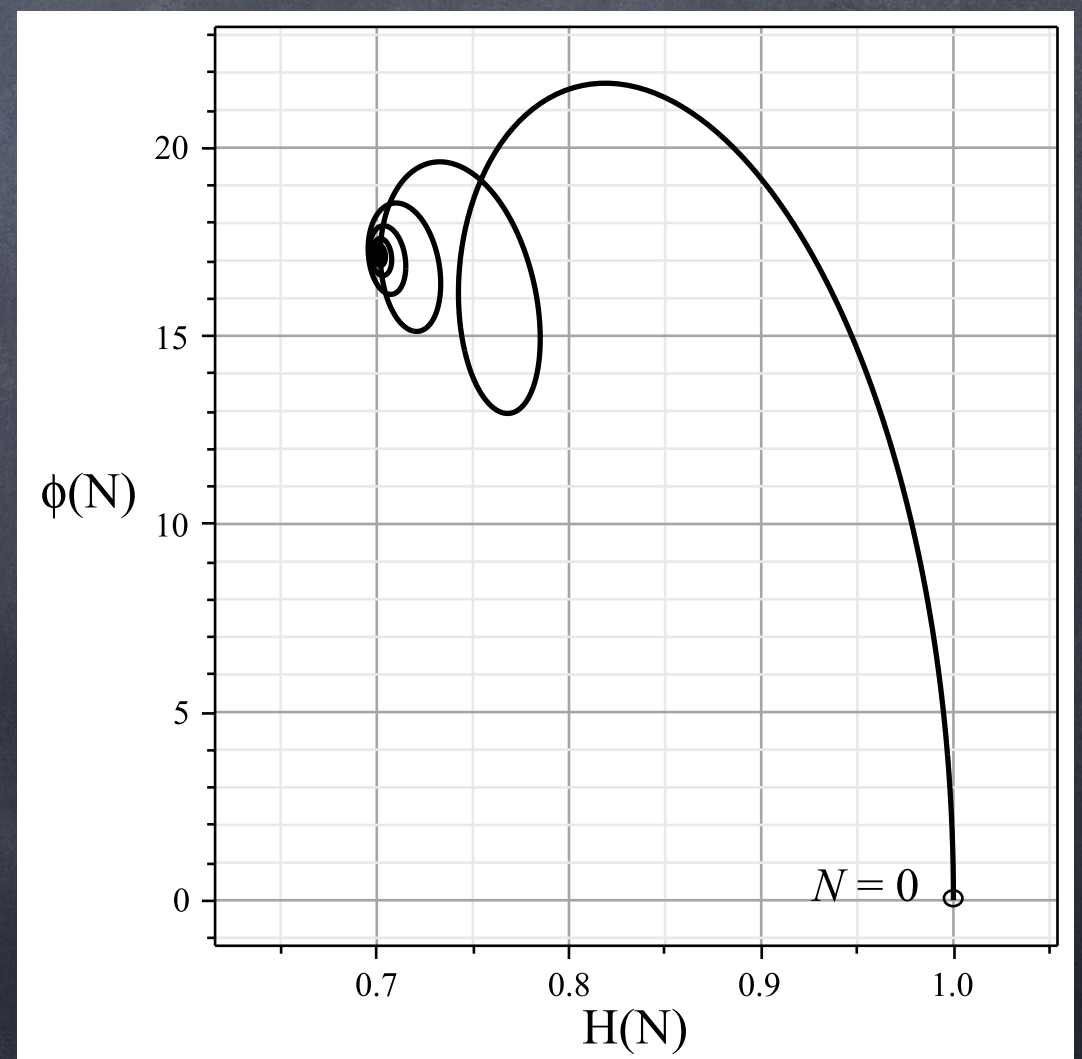
$$S = -\frac{1}{16\pi} \int d^4x \sqrt{-g} I^2(\phi) \left(F_{\mu\nu} F^{\mu\nu} + \gamma F_{\mu\nu} \tilde{F}^{\mu\nu} \right) + \int d^4x \sqrt{-g} \mathcal{L}_\phi$$



Other directions

Black holes in the full $f(R, \phi)$ theory

- two dS black hole solutions
- stability/instability transition (Simon & Samuele)



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Inflation and reheating in theories with spontaneous scale invariance symmetry breaking

#21

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Nov 11, 2022

Inflation in scale-invariant theories of gravity

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