Renormalization of composite operators in $2+\varepsilon$ dimensional quantum gravity

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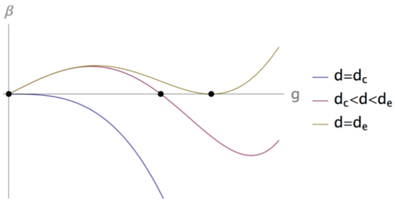
Motivation I

- Original formulation of asymptotic safety [**Weinberg, 1979**] based on results in gravity close to $d = 2$



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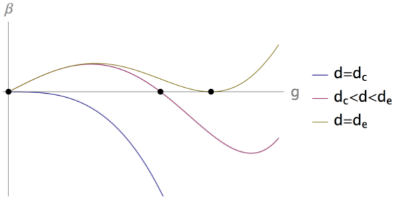

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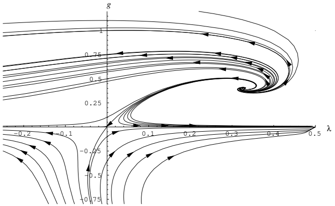

Reuter & Saueressig, 2001

- First application of non-perturbative RG flow to test the conjecture in $d = 4$ [**Reuter, 1998**]
- Non-perturmative computations based on Functionale renormalization group [**Wetterich, 1993**]

Motivations: II

Functional Renormalization Group cons:

- Theory space is infinitely dimensional $\rightarrow$ need of truncation
- Strong regulator and scheme dependence
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Physics lies in on-shell observables

imposing equations of motion along the RG flow •

Essential Renormalization Group •

[Weinberg, 1979; Anselmi, 2013; Baldazzi, Zinati, Falls; 2021]

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Revise perturbative methods

- Revise perturbative computations in $d = 2 + \varepsilon$
- Does the analytic continuation $d = 2 \rightarrow 4$ interpolate higher derivative gravity? [**Stelle, 1976;...**]

Wave function renormalization

Given an action $S[g]$ for a field g_x over a background $\bar{g}_x$ we have the functionals

$$Z[j;\bar{g}] = e^{W[j;\bar{g}]} = \int \mathcal{D}h \ \exp\left\{-S[\bar{g}+h] + j_x h_x\right\} ,$$

$$\Gamma[H;\bar{g}] = j_x[H]H_x - W[j[H]] \quad \text{with} \quad H_x \equiv \frac{\delta W[j]}{\delta J_x} = \langle h_x \rangle_c$$

One loop approximation close to some critical dimension $d = d_c - \varepsilon$

$$\frac{1}{2}\mathrm{Tr}\log \bar{S}'' = -\frac{1}{\varepsilon}\mathcal{J}_x \bar{S}_x + \frac{1}{\varepsilon}\Delta\Gamma_\infty + \Delta\Gamma_f$$

$$\Gamma[H;\bar{g}] = \bar{S} + \bar{S}'_x\left(H_x - \frac{1}{\varepsilon}\mathcal{J}_x\right) + \frac{1}{2}H_x\bar{S}''_{xy}H_y + \frac{1}{\varepsilon}\Delta\Gamma_\infty + \Delta\Gamma_f$$

However $j_x[H] = 0 \Leftrightarrow H_x = -G_{xy}\bar{S}'_y \overset{\text{o.s.}}{=} 0$

$$\Gamma[0;\bar{g}] = \bar{S} - \bar{S}'_x\frac{1}{\varepsilon}\mathcal{J}_x + \frac{1}{\varepsilon}\Delta\Gamma_\infty + \Delta\Gamma_f$$

Field redefinition

Change variable $\tilde{h}_x = h_x - \frac{1}{\varepsilon}\mathcal{J}_x$ and add the counterterm

$$Z_R[j;\bar{g}] = e^{\frac{1}{\varepsilon}\Delta\Gamma_\infty} \int \mathcal{D}\tilde{h}\ \exp\left\{-S\left[\bar{g} + \tilde{h} + \frac{1}{\varepsilon}\mathcal{J}\right] + j_x \tilde{h}_x\right\}$$

We can read the renormalized equations of motion from the relation between j_x and $\tilde{H}_x$

$$j_x[\tilde{H}] = \bar{S}''_{xy}\tilde{H}_y + \bar{S}'_x + \frac{1}{\varepsilon}\bar{S}''_{xy}\mathcal{J}_x \tag{1}$$

leading to

$$\Gamma_R[\tilde{H};\bar{g}] = \bar{S} + \Delta\Gamma_f + \tilde{H}_x\left(\bar{S}'_x + \frac{1}{\varepsilon}\bar{S}''_{xy}\mathcal{J}_y\right) + \frac{1}{2}\tilde{H}_x\bar{S}''_{xy}\tilde{H}_y + \frac{1}{2\varepsilon^2}\mathcal{J}_x\bar{S}''_{xy}\mathcal{J}_y$$

$$\Downarrow$$

$$\Gamma_R[0;\bar{g}] = \bar{S} + \Delta\Gamma_f + \frac{1}{2\varepsilon^2}\mathcal{J}_x\bar{S}''_{xy}\mathcal{J}_y$$

Composite operators

We are interested in computing the expectation value of an operator $O[g]$

$$\langle O[g]\rangle \equiv \int \mathcal{D}h\; O[\bar{g}+h]\exp\left\{-S[\bar{g}+h]+j_x h_x\right\}$$
$$= \frac{\partial}{\partial\alpha}\bigg|_{\alpha=0}\int \mathcal{D}h\;\; \exp\left\{-S[\bar{g}+h]+\alpha O[\bar{g}+h]+j_x h_x\right\} \equiv \frac{\partial}{\partial\alpha}\bigg|_{\alpha=0}\exp\{W[j,\alpha;\bar{g}]\}$$

define

$$\Gamma[H,\alpha;\bar{g}] = H_x j_x - W[j,\alpha;\bar{g}] \quad \Rightarrow \quad \langle O[\bar{g}+h]\rangle = -\frac{\partial}{\partial\alpha}\bigg|_{\alpha=0}\Gamma[j^{-1}[0],\alpha,\bar{g}]$$

Effectively the action is shifted $A[g] = S[g] - \alpha O[g]$

$$\frac{1}{2}\mathrm{Tr}\log A'' = \frac{1}{2}\mathrm{Tr}\log\left(\bar{S}'' - \alpha\bar{O}''\right) = -\frac{1}{\varepsilon}\mathcal{J}_x\bar{S}'_x + \frac{1}{\varepsilon}\Delta\Gamma^O_\infty + \Delta\Gamma^O_f\,,$$

Composite operators and wave function renormalization

$$j_x[H] = \left(\bar{S}''_{xy} - \alpha\bar{O}''_{xy}\right) H_y + \bar{S}'_x - \alpha\bar{O}'_x \,,$$

leading to

$$\begin{aligned}\Gamma[H,\alpha;\bar{g}] =& \bar{S} - \alpha\bar{O} + (H_x - \frac{1}{\varepsilon}\mathcal{J}_x)\bar{S}'_x - \alpha H_x \bar{O}'_x + \frac{1}{2} H_x \left(\bar{S}''_{xy} - \alpha\bar{O}''_{xy}\right) H_y \\ &+ \frac{1}{\varepsilon}\Delta\Gamma^O_\infty + \Delta\Gamma_f\end{aligned}$$

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- $H[0] = j^{-1}[0] = 0 \quad \Leftrightarrow \quad \bar{S}'_x = \alpha \bar{O}'_x$

  Then $-\frac{1}{\varepsilon}\mathcal{J}_x \bar{S}' \to -\frac{\alpha}{\varepsilon}\mathcal{J}_x \bar{O}'$ new counterterm to include

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- Shift H_x as before to take care of $\mathcal{J}_x$

  Then $-\alpha H_x\bar{O}'_x \to -\alpha\tilde{H}_x\bar{O}'_x - \frac{\alpha}{\varepsilon}\mathcal{J}_x\bar{O}'$ new counterterm

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$$\frac{\langle O\rangle}{\langle 1\rangle} = -\frac{\partial}{\partial\alpha}\Gamma_R[0,\alpha;\bar{g}]\Big|_{\alpha=0} = \bar{O} + \frac{1}{2\varepsilon^2}\mathcal{J}_x\bar{O}''_{xy}\mathcal{J}_y - \frac{\partial}{\partial\alpha}\Delta\Gamma^O_f\Big|_{\alpha=0}$$

$$\bar{S}'_x + \frac{1}{\varepsilon}\bar{S}''_{xy}\mathcal{J}_x = 0$$

Einstein-Hilbert gravity

Consider the Einstein-Hilbert action

$$S_{EH}[g] = \int d^d \sqrt{g}\ (g_0 - g_1 R)$$

Diffeomorphism-invariance plays the role of a gauge symmetry: $\delta_\zeta g_{\mu\nu} = \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu$
We look at the parametrization dependence

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} + \frac{\lambda}{2} h_{\mu\rho} \bar{g}^{\rho\theta} h_{\theta\nu} + \mathcal{O}(h^3)$$

and gauge dependence of the 1-loop divergences: $\delta_\zeta h_{\mu\nu} = \bar{g}_{\mu\rho} \bar{\nabla}_\nu \zeta^\rho + \bar{g}_{\nu\rho} \bar{\nabla}_\mu \zeta^\rho + o(h)$

$$\begin{aligned} S_{gf}[h;\bar{g}] =& \frac{1}{2} \int d^d x \sqrt{\bar{g}}\ \bar{g}^{\mu\nu} F_\mu F \nu\,, \\ F_\mu =& \bar{\nabla}_\rho h^\rho_\mu - \frac{1+\delta\xi}{2} \bar{\nabla}_\mu h^\rho_\rho\,, \\ S_{gh}[h, c, \bar{c}; \bar{g}] =& \int d^d x \sqrt{\bar{g}}\ \ \bar{c}^\mu \delta_\zeta F_\mu|_{\zeta \to c} \end{aligned}$$

On-shell divergences

The 1-loop divergences in $d = 2 + \varepsilon$ can be arranged in the following way

$$\Gamma^{1-loop}_{\rm div} = -\frac{\mu^\varepsilon}{\varepsilon}\int \mathrm{d}^d x\sqrt{\overline{g}}\left\{A\overline{R} + J_{\mu\nu}\left(\overline{G}^{\mu\nu} + \frac{g_0}{2g_1}\overline{g}^{\mu\nu}\right)\right\},$$

$$A = \frac{36+3d-d^2}{48\pi}, \qquad J_{\mu\nu} = \frac{\overline{g}_{\mu\nu}}{4\pi}\left\{\frac{d^2-d-4}{2(d-2)}\lambda - \delta\xi\left(2+\frac{2\lambda}{d-2}\right) - d - 1\right\}$$

And the consequent β-function for $G_N = \frac{1}{g_1}$ is
[Falls, 2015; R.M. et al., 2021; Bastianelli et al., 2022]

$$\beta_G = \varepsilon G - \frac{36+3d-d^2}{48\pi}G^2 \quad \rightarrow \quad \beta_G = -\frac{19}{24\pi}G^2$$

$$G^* = -\frac{48\pi(d-2)}{d^2-3d-36} \quad \text{for} \quad d \lesssim 7.6$$

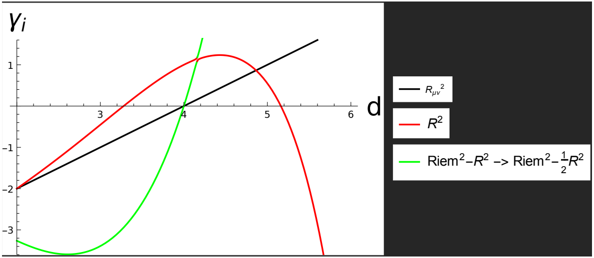
Deformation of Einstein-Hilbert theory

Include higher derivative operators as composite:

$$S_{HD}[g] = -\int d^dx\sqrt{g}\,\{\alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\} \equiv -\alpha\cdot\mathcal{R}_2$$

Divergences of effective action for the composite operator

$$\Gamma_\infty[0,\alpha;\bar g] = -\alpha\left[\frac{1}{2}\mathrm{Tr}\frac{\delta^2\mathcal{R}_2}{\delta h_{\mu\nu}\delta h_{\rho\sigma}}G_{\rho\sigma\mu\nu} - \frac{g_1}{\varepsilon}J_{\mu\nu}\frac{\delta\mathcal{R}_2}{\delta h_{\mu\nu}}\right]_{h=0}$$



Conclusions and Outlook

Conclusions:

- Essential scheme tells us we need to include one-point functions of composite operators

- Non-trivial flow of equations of motion due to the wave function renormalization

Outlook

- Test gauge dependence of composite operators

- Two-loop computations of Einstein-Hilbert gravity in $d = 2 + \varepsilon$

Thank you!