Weyl and projective invariance in metric-affine theories

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Notation

General Relativity's only field variable is the metric tensor $g_{\mu\nu}$

The holonomic connection is the unique torsion-free (symmetric) and metric-compatible, i.e. the Levi-Civita connection

$$\mathring{\Gamma}^{\rho}{}_{\nu\mu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right)$$

In a more general geometric setting the covariant derivative of a vector perceives the non-Riemannian structures

$$\nabla_\mu v^\nu = \mathring{\nabla}_\mu v^\nu + \Phi^{\nu}{}_{\rho\mu} v^\rho$$

Weyl's group

In 1919 Weyl considered the local gauge group $D(1)$ acting as:

$$g_{\mu\nu} \to \mathrm{e}^{2\sigma} g_{\mu\nu} \qquad S_\mu \to S_\mu - \partial_\mu \sigma$$

Gives rise to Weyl non-metricity:

$$\nabla_\mu g_{\nu\rho} = 2 S_\mu g_{\nu\rho}$$

Weyl weights:

$$w(g_{\mu\nu}) = 2, \quad w(\phi) = -\frac{d-2}{2}, \quad w(\psi) = -\frac{d-1}{2}$$

$$w(A_\mu) = -\frac{d-4}{2}, \quad w(\Gamma^\rho{}_{\nu\mu}) = 0, \quad w(\partial_\mu) = 0$$

Metric-affine theories of Gravity (MAGs)

Local gauge group $GL(4,\mathbb{R})$

- Holonomic formulation: field variables $g_{\mu\nu}$, $\Gamma^{\rho}{}_{\nu\mu}$;
- Field strengths:

$$
\begin{aligned}
R^{\rho}{}_{\sigma\mu\nu} &= \partial_{\mu}\Gamma^{\rho}{}_{\sigma\nu} + \Gamma^{\rho}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\sigma\nu} - (\nu \leftrightarrow \mu) \\
T^{\rho}{}_{\nu\mu} &= \Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu} \\
Q_{\mu\nu\rho} &= -\nabla_{\rho} g_{\mu\nu}
\end{aligned}
$$

- Splitting: $\Gamma^{\rho}{}_{\nu\mu} = \mathring{\Gamma}^{\rho}{}_{\nu\mu} + \Phi^{\rho}{}_{\nu\mu} = \mathring{\Gamma}^{\rho}{}_{\nu\mu} + K^{\rho}{}_{\nu\mu} + N^{\rho}{}_{\nu\mu}$;
- Contortion and distortion:

$$
\begin{aligned}
K^{\rho}{}_{\nu\mu} &= \frac{1}{2}(T_{\mu}{}^{\rho}{}_{\nu} + T_{\nu}{}^{\rho}{}_{\mu} - T^{\rho}{}_{\nu\mu}) \\
N^{\rho}{}_{\nu\mu} &= \frac{1}{2}(Q_{\mu}{}^{\rho}{}_{\nu} + Q_{\nu}{}^{\rho}{}_{\mu} - Q_{\mu\nu}{}^{\rho})
\end{aligned}
$$

D.o.f. of the general theory

See Baldazzi, Percacci & Melichev 2112.10193

	Non-metricity		Torsion	
	ts	*hs*	*ha*	*ta*
ttt	3^-, 1^-_1	2^-_1, 1^-_2	2^-_2, 1^-_3	0^-
$ttl + tlt + ltt$	2^+_1, 0^+_1	-	-	1^+_3
$\frac{3}{2} ltt$	-	2^+_2, 0^+_2	1^+_2,	-
$ttl + tlt - \frac{1}{2} ltt$	-	1^+_1	2^+_3, 0^+_3	-
$tll + ltl + llt$	1^-_4	1^-_5	1^-_6	-
lll	0^+_4	-	-	-

$$T^\rho{}_{\mu\nu} = \frac{1}{d-1}(\delta^\rho{}_\nu \tau_\mu - \delta^\rho{}_\mu \tau_\nu) - \frac{1}{(d-1)!}\varepsilon^{\sigma\rho}{}_{\mu\nu}\theta_\sigma + \kappa^\rho{}_{\mu\nu}$$

Weyl's group in MAGs

Assumption: Weyl-invariant holonomic connection

$$\delta^W_\sigma(\Phi^\lambda{}_{\nu\mu} \pm g_{\nu\rho} g^{\lambda\kappa} \Phi^\rho{}_{\kappa\mu}) = -\delta^\lambda{}_\nu (1 \pm 1)\partial_\mu \sigma \\ - \delta^\lambda{}_\mu (1 \mp 1)\partial_\nu \sigma + g_{\mu\nu}(1 \mp 1)\partial^\lambda \sigma$$

- Only vector irreps are affected: τ_μ, $Q_\mu{}^\nu{}_\nu$, $Q_\nu{}^\nu{}_\mu$
- For $Q_{\mu\nu\rho} = 0$ the Weyl-invariant contortion is:

$$\hat{K}^\rho{}_{\nu\mu} = K^\rho{}_{\nu\mu} - \delta^\rho{}_\nu S_\mu + g_{\mu\nu} S^\rho$$

- Weyl distortion:

$$L^\rho{}_{\nu\mu} = \delta^\rho{}_\nu S_\mu + \delta^\rho{}_\mu S_\nu - g_{\mu\nu} S^\rho$$

- $\hat{\Gamma}_\mu = \mathring{\Gamma}_\mu + L_\mu$ is torsion-less and can be integrated by parts

Riemann-Cartan-Weyl, $G = SO(3,1) \times D(1)$

Fundamental field variables: e^a, $\omega^a{}_b$, $S = S_\mu dx^\mu$

- One more field strength

$$\mathcal{W} = dS$$

- Splitting:

$$\Gamma^\rho{}_{\nu\mu} = \mathring{\Gamma}^\rho{}_{\nu\mu} + L^\rho{}_{\nu\mu} + \hat{K}^\rho{}_{\nu\mu}$$

- Covariant derivative:

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\lambda\mu} v^\lambda + w_{v^\nu} S_\mu v^\nu$$

- Commutator:

$$[\nabla_\mu, \nabla_\nu] v^\rho = R^\rho{}_{\sigma\mu\nu} v^\sigma + T^\sigma{}_{\nu\mu} \nabla_\sigma v^\rho + w_{v^\rho} W_{\mu\nu} v^\rho$$

Nöther identities in Riemann-Cartan-Weyl (part 1)

Matter action $S_m = S_m[\phi, e^a, \omega^a{}_b, S]$

- Energy-momentum, spin and dilation currents:

$$T^\mu{}_a \equiv \frac{1}{\underline{e}} \frac{\delta S_m}{\delta e^a{}_\mu} , \qquad \Sigma^\mu{}_{ab} \equiv \frac{1}{\underline{e}} \frac{\delta S_m}{\delta \omega^{ab}{}_\mu} , \qquad \Delta^\mu \equiv \frac{1}{\underline{e}} \frac{\delta S_m}{\delta S_\mu}$$

- Matter field ϕ on-shell, then:

$$\delta S_m = \int \underline{e} \left(T^\mu{}_a \delta e^a{}_\mu + \Sigma^\mu{}_{ab} \delta \omega^{ab}{}_\mu + \Delta^\mu \delta S_\mu \right)$$

- Transformations: $\delta^L_\alpha e^a{}_\mu = \alpha^a{}_b e^b{}_\mu$, $\delta^L_\alpha \omega^a{}_{b\mu} = -\nabla_\mu \alpha^a{}_b$, $\delta^W_\sigma e^a{}_\mu = \sigma e^a{}_\mu$, $\delta^W_\sigma S_\mu = -\partial_\mu \sigma$
- Nöther identities:

$$T_{[ab]} = (\nabla + \tau)_\mu \Sigma^\mu{}_{ab} \qquad T^\mu{}_\mu = -\hat{\nabla}_\mu \Delta^\mu$$

Nöther identities in Riemann-Cartan-Weyl (part 2)

Einstein variation δ^E_ξ is a Lie derivative, it's not gauge covariant

- Define covariant Lie derivatives:

$$\tilde{\delta}^E_\xi = \delta^E_\xi + \delta^L_{\xi\cdot\omega} + \delta^W_{\xi\cdot S}$$

- Actual variations:

$$\tilde{\delta}^E_\xi e^a{}_\mu = \xi^\nu T^a{}_{\nu\mu} + e^a{}_\nu \nabla_\nu \xi^\mu$$

$$\tilde{\delta}^E_\xi \omega^a{}_{b\mu} = \xi^\nu R^a{}_{b\nu\mu} \qquad \tilde{\delta}^E_\xi S_\mu = \xi^\nu W_{\nu\mu}$$

- Nöther identity:

$$(\nabla + \tau)_\mu T^\mu{}_\nu = T^\mu{}_a T^a{}_{\nu\mu} + \Sigma^\mu{}_{ab} R^{ab}{}_{\mu\nu} + \Delta^\mu W_{\mu\nu}$$

- For $\Sigma^\mu{}_{ab} = 0$ and $\Delta^\mu = 0$ we obtain the Riemannian result $\mathring{\nabla}_\mu T^\mu{}_\nu = 0$

Projective invariance in GR

Auto-parallel equation $u^\nu \nabla_\nu u^\mu = 0$

- Whenever $\delta\Gamma^\rho{}_{\nu\mu} \propto \delta^\rho{}_\nu V^1_\mu$ or $\delta\Gamma^\rho{}_{\nu\mu} \propto \delta^\rho{}_\mu V^2_\nu$ the solutions are the same
- The projective transformation $\delta\Gamma^\rho{}_{\nu\mu} = \delta^\rho{}_\nu V_\mu$ is a symmetry of EH
- In a coordinate chart it can always by written as

$$\delta\Gamma^\rho{}_{\nu\mu} = \delta^\rho{}_\nu V_\mu + \delta^\rho{}_\mu V_\nu$$

- Taking into account u^μ light-like we also have

$$\delta\Gamma^\rho{}_{\nu\mu} = g_{\mu\nu} X^\rho$$

Projective invariance in MAGs

Define the hypermomentum $\Xi_{\rho}{}^{\nu\mu} \equiv \frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta \Gamma^{\rho}{}_{\nu\mu}}$

- In general projective invariance gets rid of ghosts and tachyons and reduces number of independent couplings
- Light-cone proj. transf.'s in MAGs

$$\Gamma^{\rho}{}_{\nu\mu} \to \delta^{\rho}{}_{\nu} p_{\mu} + \delta^{\rho}{}_{\mu} q_{\nu} + g_{\mu\nu} r^{\rho} + \varepsilon^{\sigma\rho}{}_{\nu\mu} s_{\sigma}$$

- Only (pseudo-)vectors get shifted: $\delta\tau_{\mu} = 3(p-q)_{\mu}$, $\delta\theta_{\sigma} = 12 s_{\sigma}$, $\delta Q_{\nu}{}^{\nu}{}_{\mu} = (8p + 2q + 2r)_{\mu}$, $\delta Q_{\mu}{}^{\nu}{}_{\nu} = (2p + 5q + 5r)_{\mu}$.
- Nöther identities: $\Xi_{\nu}{}^{\nu\mu} = \Xi_{\nu}{}^{\mu\nu} = 0$, $\Xi_{\mu}{}^{\rho\sigma} g_{\rho\sigma} = 0$, $\varepsilon^{\sigma\rho}{}_{\nu\mu} \Xi_{\rho}{}^{\nu\mu} = 0$

Projective invariance in RCW

In RCW the we have three (light-cone) projective-modes

$$\delta^{p_1} S_\mu = \xi_\mu \qquad \delta^{p_2} \omega^a{}_{b\mu} = (e^a{}_\mu E^\nu{}_b - e_{b\mu} E^{\nu a}) v_\nu$$

$$\delta^{p_3} \omega^a{}_{b\mu} = E^\nu{}_d e^c{}_\mu \varepsilon^{da}{}_{bc} w_\nu$$

Whence

$$\sum_{i=1}^{3} a_i \delta^{p_i} \Gamma^\nu{}_{\rho\mu} = a_1 \delta^\nu{}_\rho \xi_\mu + a_2 \left(\delta^\nu{}_\mu v_\rho - g_{\mu\rho} v^\nu \right) + a_3 \varepsilon^{\sigma\nu}{}_{\rho\mu} w_\sigma$$

Nöther identities:

$$\Delta^\mu = 0 \qquad \Sigma_\nu{}^{\mu\nu} = 0$$

$$\varepsilon^{\sigma\rho}{}_{\nu\mu} \Sigma_\rho{}^{\nu\mu} = 0$$

Torsion gauging and Conformal coupling

Note that $\delta^{p_1}(S_\mu - \frac{1}{d-1}\tau_\mu) = 0$, thus only one vector is physical

We define $\delta^{p_0}_\xi \equiv \delta^{p_1}_\xi + \delta^{p_2}_\xi$, then:

- $\delta^{p_0} S_\mu = \xi_\mu$, whereas $\delta^{p_0}\tau_\mu = 0$
- Fixing δ^{p_0} we obtain torsion-gauged theories from Riemann-Cartan-Weyl
- Now the torsion vector plays the role of the Weyl potential, $\delta^W_\sigma \tau_\mu = -(d-1)\partial_\mu\sigma$

By further fixing δ^{p_2} we obtain the conformal coupling

Application to a scalar toy-model

Denote by $\kappa^{\rho}{}_{\mu\nu}$ the tensor irrep of the torsion ($\kappa_{[\mu\nu\rho]} = 0, \kappa^{\rho}{}_{\mu\rho} = 0$)

- Exploiting the Nöther identities stemming from projective invariance

$$S_\mu = 0 \quad \tau_\mu = 0 \quad \theta_\mu = 0$$

- The full Lagrangian for the scalar is

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{12}\mathring{R}\phi^2 - (\zeta + \frac{1}{2})\kappa^{\rho}{}_{\mu\nu}\kappa_{\rho}{}^{\mu\nu}\phi^2 - \frac{\lambda}{4!}\phi^4$$

- Thus, the Weyl anomaly will depend on $\kappa^{\rho}{}_{\mu\nu}$

Future prospects

What we are doing: study of a Lagrangian quadratic in $\kappa^{\rho}{}_{\mu\nu}$

- There are 8 free parameters
- We are looking for a conformal coupling
- Study of Curtright-like transformations

What we plan to do:

- Relax some of the hypotheses, e.g. $\delta^{P_2}\Gamma^{\nu}{}_{\rho\mu} = \delta^{\nu}{}_{\mu}\partial_{\rho}\varphi - g_{\mu\rho}\partial^{\nu}\varphi$ and $\delta^{P_3}\Gamma^{\nu}{}_{\rho\mu} = \varepsilon^{\sigma\nu}{}_{\rho\mu}\partial_{\sigma}\chi$
- Search for symmetries of the Raychaudhuri equations

The End

Thank you for your attention,
questions are welcomed!