Non-Abelian gauge fields in Graphene

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Outline

1. Motivation: what is graphene and why graphene?
2. Electronic Structure
3. Deformations of the lattice

☞ Based on: **1012.5354**

Graphene and its history

- Graphene is a carbon material with two-dimensional hexagonal lattice
- has been intensively studied since 1940' as a purely theoretical toy model since it captures most of the properties of graphite (nuclear applications)
- "predicted" by Wallace in 1946, who wrote the band structure of the graphene and showed that is possesses unusual properties
- in 2004 when it was for the first time isolated from the graphite by exfoliation [Novoselov et al.'04–Nobel Prize'10]
- Strange properties attracted occasionally interest of "high energy physicists": chirality [Semenoff'84], Klein paradox and Zitterbewegung [Itzykson& Zuber'06] etc.
- has "compactified" allotropes.
  - One-dimensional: *carbon nanotubes*,
  - zero-domensional: *fullerenes*

2D hexagonal lattice

The graphene is a 2D hexagonal/honeycomb lattice made of carbon atoms

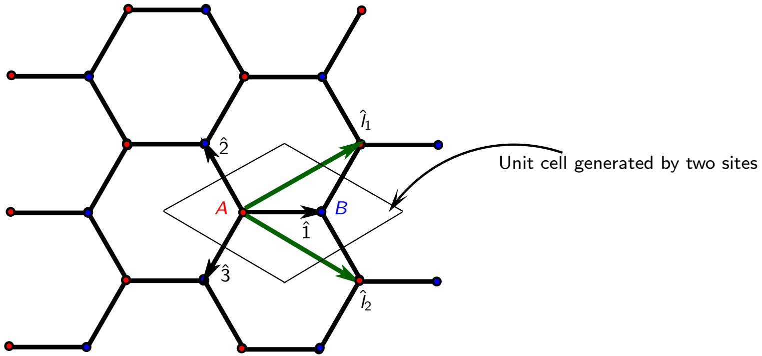


Figure: The hexagonal lattice

Electronic structure of graphene

- The structure of graphene: Three out of four valence electrons of C create hybridized states with three neighbor atoms to form a honeycomb lattice, while the remaining electron can hope between the atoms
- In the undoped case the free electrons fill only $1/2$ of available positions.
- The free electron's wave function can be considered as localized at the atom's site: The tight binding approximation

The tight binding model

The tight binding model describes the dynamics of free electron in terms of Hubbard model on the hexagonal lattice:

$$H = -t \sum_{\mathbf{n},a,\sigma} \left(a^{\dagger}_{\mathbf{n},\sigma} b_{\mathbf{n}+\hat{a},\sigma} + b^{\dagger}_{\mathbf{n}+\hat{a},\sigma} a_{\mathbf{n},\sigma} \right)$$

where $a^{\dagger}_{\mathbf{n},\sigma}$ and $a_{\mathbf{n},\sigma}$ are creation and destruction operators for the electron of spin σ at the A-site $\mathbf{n}$; $b^{\dagger}_{\mathbf{n}+\hat{a},\sigma}$ and $b_{\mathbf{n}+\hat{a},\sigma}$ are the creation/destruction operators for the B-site $\mathbf{n} + \hat{a}$

An alternative form of the Hamiltonian

Another form of the Hamiltonian,

$$H = -t \sum_{\mathbf{n}} \left(\Psi^\dagger_{A,\mathbf{n}} \cdot \Psi_{B,\mathbf{n}} + \Psi^\dagger_{A,\mathbf{n}} \cdot \Psi_{B,\mathbf{n}-\hat{l}_1} + \Psi^\dagger_{A,\mathbf{n}} \cdot \Psi_{B,\mathbf{n}-\hat{l}_2} + \text{h.c.} \right)$$
$$\equiv -t \sum_{\mathbf{n}} \Psi^\dagger_{\mathbf{n}} \cdot D \cdot \Psi_{\mathbf{n}}$$

$$D = \begin{pmatrix} 0 & 1 + T^\dagger_{\hat{l}_1} + T^\dagger_{\hat{l}_2} \\ 1 + T_{\hat{l}_1} + T_{\hat{l}_2} & 0 \end{pmatrix}$$

$T_{\hat{l}_i}$, $i = 1, 2$: elementary translations on the Bravais lattice.

$$\Psi_{\mathbf{n}} = \begin{pmatrix} \Psi_{A,\mathbf{n}} \\ \Psi_{B,\mathbf{n}} \end{pmatrix} \equiv \begin{pmatrix} a_{\mathbf{n}} \\ b_{\mathbf{n}+\hat{1}} \end{pmatrix}$$

just one possible choice...

'Momentum space' action: dispersion relations

Performing a Fourier transform, the action takes the form,

$$S = \frac{1}{A_{\rm FD}} \int_{\rm FD} {\rm d}t{\rm d}^2 k \left[{\rm i}\Psi^\dagger(k)\dot{\Psi}(k) + t\Psi^\dagger(k)\cdot D(k)\cdot\Psi(k)\right].$$

the fermionic operator

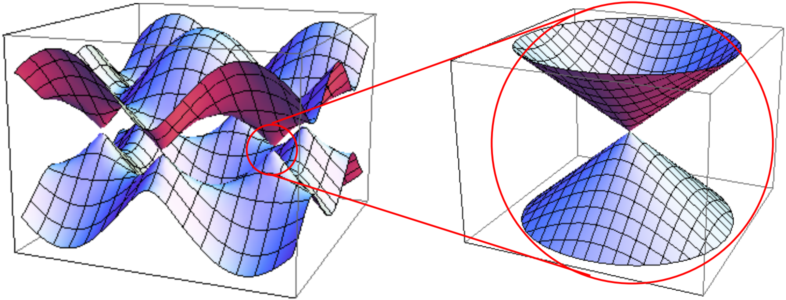
$$D(k) = \begin{pmatrix} 0 & d(k) \\ d^*(k) & 0 \end{pmatrix},$$

$$d(k) = 1 + {\rm e}^{{\rm i}\mathbf{k}\cdot\hat{l}_1} + {\rm e}^{{\rm i}\mathbf{k}\cdot\hat{l}_2}$$

FD=Fundamental domain

Dispersion function

$$\omega(k) = \pm|d(k)|$$



Reciprocal lattice & Fundamental domain

Fundamental domain is defined by $\mathbf{k} = k_1 \hat{k}_1 + k_2 \hat{k}_2$, with $-1/2 \leq k_{1,2} < 1/2$, coordinates of momentum in the dual basis $\{\hat{k}_i\}$:

$$\hat{k}_i \cdot \hat{l}_j = 2\pi\delta_{ij}.$$

$$\hat{k}_1 = \frac{2\pi}{3a}(1, -\sqrt{3}), \quad \hat{k}_2 = \frac{2\pi}{3a}(1, \sqrt{3}),$$

$|\hat{k}_1| = |\hat{k}_2| = 4\pi/3a$.

Return

Fundamental Domain & Brillouin zone

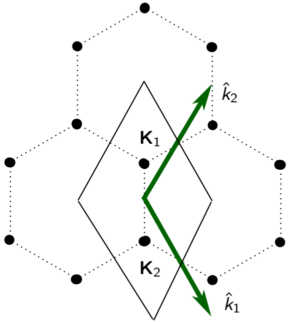


Figure: Momentum Space

Fundamental Domain & Brillouin zone

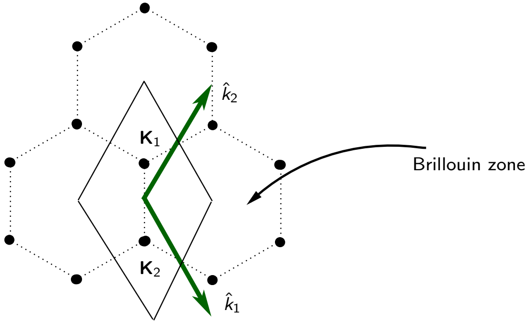


Figure: Momentum Space

Fundamental Domain & Brillouin zone

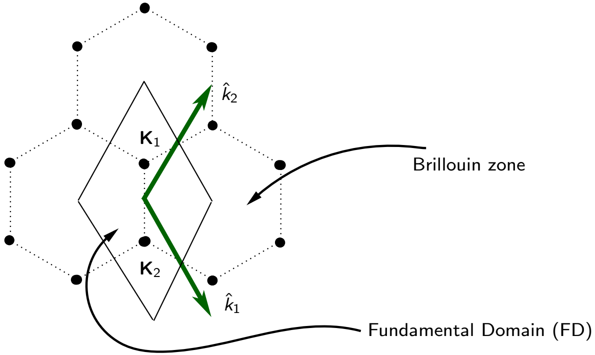


Figure: Momentum Space

Fundamental Domain & Brillouin zone

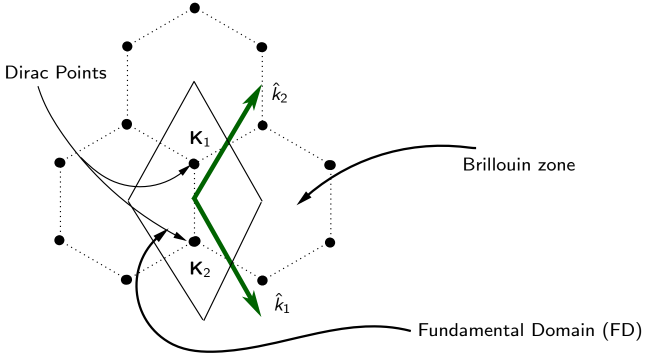


Figure: Momentum Space

Dirac points

The low energy is controlled by the regions near the zeroes of $d(k)$ Dirac points:

$$\mathbf{K}_1 \equiv -\mathbf{K} = (0, -\tfrac{4\pi\sqrt{3}}{9a}) \quad \text{and} \quad \mathbf{K}_2 \equiv \mathbf{K} = (0, \tfrac{4\pi\sqrt{3}}{9a})$$

The fermionic field near each Dirac point

$$\Psi(\pm K + k) = \psi_\pm(k)$$

combine into a Dirac spinor

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

In the low energy limit we have

$$tD(k) = \tfrac{3at}{2}\left[(-\mathrm{i}k_x)\beta + (-\mathrm{i}k_y)\rho\right]$$

$$\beta = \sigma_2 \otimes \mathbb{I}, \quad \rho = \sigma_1 \otimes \sigma_3.$$

Low Energy Action

If we introduce

$$\gamma^0 = -\mathrm{i}\sigma_3 \otimes \sigma_3$$

and

$$\gamma^1 = -\gamma^0\beta = \sigma_1 \otimes \sigma_3, \quad \gamma^2 = -\gamma^0\rho = -\sigma_2 \otimes \mathbb{I}$$

The action in the low energy limit becomes

$$S \sim \int_{\sim \mathrm{Dpts.}} \mathrm{d}k_x \mathrm{d}k_y \left[\mathrm{i}\bar{\Psi}(k)\gamma_0\dot{\Psi}(k) + \tfrac{3ta}{2}\mathrm{i}\bar{\Psi}(k)(-\mathrm{i}k_m)\gamma_m\Psi(k)\right]$$

inverse Fourier transform

$$S = \int \mathrm{d}t\mathrm{d}^2x\,\mathrm{i}\bar{\Psi}\gamma^\mu\partial_\mu\Psi$$

with the speed of light replaced by $v_{\mathrm{F}} = \frac{3at}{2}$

Global nonabelian symmetry of graphene

- The 2+1 dimensional Clifford algebra representation is two-dimensional
- New: 'Lorentz boosts' with v_{F} as the speed of light
- The low energy spinor field is four-dimensional $\Rightarrow$ there is an SU(2) of internal symmetry
- Discrete model is 2D chiral: A and B sites are inequivalent
- The continuum limit restored chiral symmetry (compare: fermionic doubling in lattice QFT)
- We neglect the spin index, otherwise the global internal symmetry is SU(4)

Lattice defects

- The physical lattice is not a perfect hexagon: strain $\rightarrow$ phonons, topological defects $\rightarrow$ disclinations, as well as impurities
- Topological defects lead to deficit/excess angle $\rightarrow$ wave function phase $\rightarrow$ can be absorbed into an effective gauge field through Aharonov-Bohm effect or ‘conical singularity’
- ‘Intrinsic’ curvature is given by the density and character of topological defects
- Bending leads to ‘extrinsic’ curvature, but also some types of bending can be described as coupling to external gauge field (see Kleirnet’s book)
- I will show: U(2) gauge and Yukawa couplings give an universal description of all above types of defects

Lattice deformations

We consider three types of deformations of the Hubbard model

- Local deformation: fluctuations of electron density

$$\Delta H_{\rm n} = -\sum_{\mathbf{n}} (a^\dagger_{\mathbf{n}} z_{A\mathbf{n}} a_{\mathbf{n}} + b^\dagger_{\mathbf{n}} z_{B\mathbf{n}} b_{\mathbf{n}})$$

- Nearest neighbor: fluctuation of transition amplitudes to the nearest atoms (A↔B)

$$\Delta H_{\rm nn} = -\sum_{\mathbf{n},\hat{a}} \left(a^\dagger_{\mathbf{n}} z_{\mathbf{n},\hat{a}} b_{\mathbf{n}+\hat{a}} + b^\dagger_{\mathbf{n}+\hat{a}} \bar{z}_{\mathbf{n},\hat{a}} a_{\mathbf{n}} \right)$$

- Next-to-nearest neighbor: fluctuation of transition amplitudes (A↔A and B↔B)

$$\Delta H_{\rm nnn} = -\sum_{\mathbf{n},\hat{b}\neq\hat{a}} \left(a^\dagger_{\mathbf{n}} z_{A\mathbf{n},\hat{a}\hat{b}} a_{\mathbf{n}+\hat{a}-\hat{b}} + b^\dagger_{\mathbf{n}-\hat{a}} z_{B\mathbf{n},\hat{a}\hat{b}} b_{\mathbf{n}-\hat{b}} \right)$$

Hubbard model deformations map

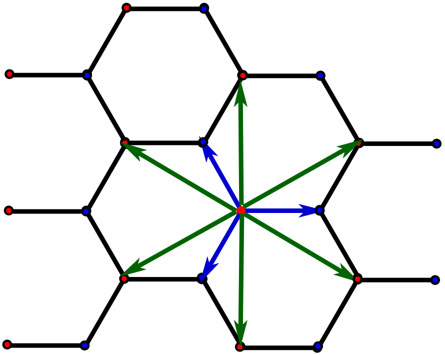


Figure: The nature of deformations

Hubbard model deformations map

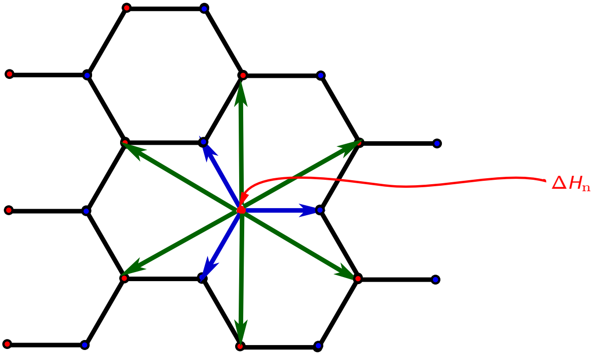
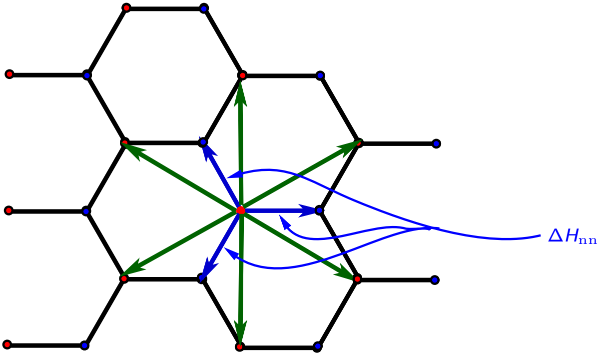


Figure: The nature of deformations

Hubbard model deformations map



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Hubbard model deformations map

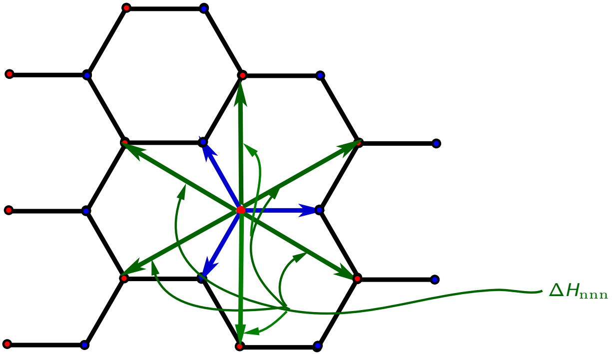


Figure: The nature of deformations

Compact form of deformed Lagrangian

The compact form of the Lagrangian:

$$L = L_0 + \sum_{\mathbf{n}} \Psi^\dagger_{\mathbf{n}} \cdot Z_{\mathbf{n}} \cdot \Psi_{\mathbf{n}} + \sum_{\mathbf{n}, i=1,2} (\Psi^\dagger_{\mathbf{n}} \cdot Z_{\mathbf{n}, \hat{l}_i} \cdot \Psi_{\mathbf{n}-\hat{l}_i} + \Psi^\dagger_{\mathbf{n}-\hat{l}_i} \cdot Z^*_{\mathbf{n}, \hat{l}_i} \cdot \Psi_{\mathbf{n}})$$
$$+ \sum_{\mathbf{n}} (\Psi^\dagger_{\mathbf{n}-\hat{l}_1} \cdot Z_{\mathbf{n}, \hat{l}_1 \hat{l}_2} \cdot \Psi_{\mathbf{n}-\hat{l}_2} + \Psi^\dagger_{\mathbf{n}-\hat{l}_2} \cdot Z^*_{\mathbf{n}, \hat{l}_1 \hat{l}_2} \cdot \Psi_{\mathbf{n}-\hat{l}_1}),$$

should be restricted to low energy modes:

$$\Psi_{\mathbf{n}} \to \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

Fields in the compact form

Fields appearing on the previous slides: matrices act on sublattice space

$$Z_{\mathbf{n}} = \frac{1}{2}(z_{A,\mathbf{n}} + z_{B,\mathbf{n}})\mathbb{I} + \frac{1}{2}(z_{A,\mathbf{n}} - z_{B,\mathbf{n}})\sigma_3 + z_{\mathbf{n},\hat{1}}\frac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2) + \bar{z}_{\mathbf{n},\hat{1}}\frac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2)$$

$$\equiv z^0_{\mathbf{n}}\mathbb{I} + z^i_{\mathbf{n}}\sigma_i, \quad \bar{z}^0_{\mathbf{n}} = z^0_{\mathbf{n}}, \quad \bar{z}^i_{\mathbf{n}} = z^i_{\mathbf{n}}$$

$$Z_{\mathbf{n},\hat{l}_i} = z_{\mathbf{n},\hat{l}_i}\sigma_+ + z^0_{\mathbf{n},\hat{l}_i}\mathbb{I} + z^3_{\mathbf{n},\hat{l}_i}\sigma_3, \quad Z^*_{\mathbf{n},\hat{l}_i} = \bar{z}_{\mathbf{n},\hat{l}_i}\sigma_- + \bar{z}^0_{\mathbf{n},\hat{l}_i}\mathbb{I} + \bar{z}^3_{\mathbf{n},\hat{l}_i}\sigma_3,$$

$$Z_{\mathbf{n},\hat{l}_1\hat{l}_2} = z^0_{\mathbf{n},\hat{l}_1\hat{l}_2}\mathbb{I} + z^3_{\mathbf{n},\hat{l}_1\hat{l}_2}\sigma_3, \quad Z^*_{\mathbf{n},\hat{l}_1\hat{l}_2} = \bar{z}^0_{\mathbf{n},\hat{l}_1\hat{l}_2}\mathbb{I} + \bar{z}^3_{\mathbf{n},\hat{l}_1\hat{l}_2}\sigma_3,$$

$$z_{\mathbf{n},\hat{l}_1} = z_{\mathbf{n},\hat{2}}, \quad z_{\mathbf{n},\hat{l}_2} = z_{\mathbf{n},\hat{3}}, \quad \bar{z}_{\mathbf{n},\hat{l}_1} = \bar{z}_{\mathbf{n},\hat{2}}, \quad \bar{z}_{\mathbf{n},\hat{l}_2} = \bar{z}_{\mathbf{n},\hat{3}},$$

$$z^0_{\mathbf{n},\hat{l}_1} = \frac{1}{2}(z_{A\mathbf{n},\hat{1}\hat{2}} + z_{B\mathbf{n},\hat{1}\hat{2}}), \quad z^0_{\mathbf{n},\hat{l}_2} = \frac{1}{2}(z_{A\mathbf{n},\hat{1}\hat{3}} + z_{B\mathbf{n},\hat{1}\hat{3}}),$$

$$z^3_{\mathbf{n},\hat{l}_1} = \frac{1}{2}(z_{A\mathbf{n},\hat{1}\hat{2}} - z_{B\mathbf{n},\hat{1}\hat{2}}), \quad z^3_{\mathbf{n},\hat{l}_2} = \frac{1}{2}(z_{A\mathbf{n},\hat{1}\hat{3}} - z_{B\mathbf{n},\hat{1}\hat{3}}),$$

$$z^0_{\mathbf{n}\hat{l}_1\hat{l}_2} = \frac{1}{2}(z_{A\mathbf{n},\hat{2}\hat{3}} + z_{B\mathbf{n}\hat{2}\hat{3}}), \quad z^3_{\mathbf{n},\hat{l}_1\hat{l}_2} = \frac{1}{2}(z_{A\mathbf{n},\hat{2}\hat{3}} - z_{B\mathbf{n},\hat{2}\hat{3}})$$

☞ Do a Fourier transform ☞ Keep the low energy modes for the fermion

The low energy (continuum) limit

The action will look like:

$$
\begin{aligned}
S = S_0 + \int \mathrm{d}^2k\mathrm{d}^2q\big[&\Psi_+^\dagger(k)Z(k-q)\Psi_+(q) + \Psi_-^\dagger(k)Z(k-q)\Psi_-(q) \\
&+ \Psi_+^\dagger(k)Z_-(k-q)\Psi_-(q) + \Psi_-^\dagger(k)Z_+(k-q)\Psi_+(q)\big] \\
&- \mathrm{i}\big[\Psi_+^\dagger(k)\big\{\mathbf{Z}(k-q)\cdot\nabla_+ + \mathbf{Z}^*(k-q)\cdot\nabla_-\big\}\Psi_+(q) \\
&+ \Psi_-^\dagger(k)\big\{\mathbf{Z}(k-q)\cdot\nabla_- + \mathbf{Z}^*(k-q)\cdot\nabla_+\big\}\Psi_-(q) \\
&+ \Psi_+^\dagger(k)\big\{\mathbf{Z}_-(k-q)\cdot\nabla_- + \mathbf{Z}_-^*(k-q)\cdot\nabla_-\big\}\Psi_-(q) \\
&+ \Psi_-^\dagger(k)\big\{\mathbf{Z}_+(k-q)\cdot\nabla_+ + \mathbf{Z}_+^*(k-q)\cdot\nabla_+\big\}\Psi_+(q)\big] \\
&+ \Psi_+^\dagger(k)\big\{\mathrm{e}^{2\pi\mathrm{i}/3}Z_{\hat{l}_1\hat{l}_2}(k-q) + \mathrm{e}^{-2\pi\mathrm{i}/3}Z^*_{\hat{l}_1\hat{l}_2}(k-q)\big\}\Psi_+(q) \\
&+ \Psi_-^\dagger(k)\big\{\mathrm{e}^{-2\pi\mathrm{i}/3}Z_{\hat{l}_1\hat{l}_2}(k-q) + \mathrm{e}^{2\pi\mathrm{i}/3}Z^*_{\hat{l}_1\hat{l}_2}(k-q)\big\}\Psi_-(q) \\
&+ \Psi_+^\dagger(k)\big\{Z_{\hat{l}_1\hat{l}_2-}(k-q) + Z^*_{\hat{l}_1\hat{l}_2-}(k-q)\big\}\Psi_-(q) \\
&+ \Psi_-^\dagger(k)\big\{Z_{\hat{l}_1\hat{l}_2+}(k-q) + Z^*_{\hat{l}_1\hat{l}_2+}(k-q)\big\}\Psi_+(q)
\end{aligned}
$$

Low energy coupled modes

New index to parameterize the Dirac space: $\pm$

$$\mathbf{Z}_\pm(\mathbf{k}) \equiv \mathbf{Z}(\pm\mathbf{K} + \mathbf{k})$$

In matrix notations the general form of the low energy Lagrangian:

$$L = L_0 + Z^{IJ}\Psi^\dagger \cdot \sigma_I \otimes \sigma_J \cdot \Psi = L_0 - Z^{IJ}\bar{\Psi} \cdot \gamma^0 \cdot \sigma_I \otimes \sigma_J \cdot \Psi$$

$I, J = 0, 1, 2, 3$: $\sigma_0 = \mathbb{I}$ and Pauli matrices σ_i
We can split the contribution according to the origin of deformation:

$$Z^{IJ} = Z^{IJ}_{\mathrm{n}} + Z^{IJ}_{\mathrm{nn}} + Z^{IJ}_{\mathrm{nnn}}$$

$Z^{IJ}_{\rm n}$, $Z^{IJ}_{\rm nn}$ and $Z^{IJ}_{\rm nnn}$:

$\otimes$	$\mathbb{I}$	σ_1	σ_2	σ_3
$\mathbb{I}$	$\frac{1}{2}(z_A + z_B)$	$\frac{1}{4}(z_{A+} + z_{A-} + z_{B+} + z_{B-})$	$\frac{\rm i}{4}(z_{A-} - z_{A+} + z_{B-} - z_{B+})$	0
σ_1	$\frac{1}{2}(z_{\hat 1} + \bar z_{\hat 1})$	$\frac{1}{4}(z_{\hat 1-} + \bar z_{\hat 1-} + z_{\hat 1+} + \bar z_{\hat 1+})$	$\frac{\rm i}{4}(z_{\hat 1-} + \bar z_{\hat 1-} - z_{\hat 1+} - \bar z_{\hat 1+})$	0
σ_2	$\frac{\rm i}{2}(z_{\hat 1} - \bar z_{\hat 1})$	$\frac{\rm i}{4}(z_{\hat 1-} - \bar z_{\hat 1-} + z_{\hat 1+} - \bar z_{\hat 1+})$	$-\frac{1}{4}(z_{\hat 1-} - \bar z_{\hat 1-} - z_{\hat 1+} + \bar z_{\hat 1+})$	0
σ_3	$\frac{1}{2}(z_A - z_B)$	$\frac{1}{4}(z_{A-} - z_{B-} + z_{A+} - z_{B+})$	$\frac{\rm i}{4}(z_{A-} - z_{B-} - z_{A+} + z_{B+})$	0

$\otimes$	$\mathbb{I}$	σ_1	σ_2	σ_3
$\mathbb{I}$	0	0	0	0
σ_1	$-\frac{3}{8}(z_x + \bar z_x)$	$-\frac{3}{8}[z_{x-} + \bar z_{x-} + z_{x+} + \bar z_{x+}$ $+{\rm i}(z_{y+} + \bar z_{y+} - z_{y-} - \bar z_{y-})]$	$-\frac{3{\rm i}}{8}[z_{x-} + \bar z_{x-} - z_{x+} - \bar z_{x+}$ $-{\rm i}(z_{y+} + \bar z_{y+} + z_{y-} + \bar z_{y-})]$	$-\frac{3{\rm i}}{8}(z_y - \bar z_y)$
σ_2	$-\frac{3{\rm i}}{8}(z_x - \bar z_x)$	$-\frac{3{\rm i}}{8}[z_{x-} - \bar z_{x-} + z_{x+} - \bar z_{x+}$ $+{\rm i}(z_{y+} - \bar z_{y+} - z_{y-} + \bar z_{y-})]$	$\frac{3}{8}[z_{x-} - \bar z_{x-} - z_{x+} + \bar z_{x+}$ $-{\rm i}(z_{y+} - \bar z_{y+} + z_{y-} - \bar z_{y-})]$	$\frac{3}{8}(z_y + \bar z_y)$
σ_3	0	0	0	0

$\otimes$	$\mathbb{I}$	σ_1	σ_2	σ_3
$\mathbb{I}$	$-\frac{3}{4}(z^0_x + \bar z^0_x)$ $-\frac{1}{4}(z'^0 + \bar z'^0)$	$-\frac{3}{4}[z^0_{x-} + \bar z^0_{x-} + z^0_{x+} + \bar z^0_{x+}$ $+{\rm i}(z^0_{y+} + \bar z^0_{y+} - z^0_{y-} - \bar z^0_{y-})]$ $+\frac{1}{4}(z'^0_- + z'^0_+ + \bar z'^0_- + \bar z'^0_+)$	$-\frac{3{\rm i}}{4}[z^0_{x-} + \bar z^0_{x-} - z^0_{x+} - \bar z^0_{x+}$ $-{\rm i}(z^0_{y+} + \bar z^0_{y+} + z^0_{y-} + \bar z^0_{y-})]$ $+\frac{\rm i}{4}(z'^0_- + z'^0_+ - \bar z'^0_- - \bar z'^0_+)$	$-\frac{3{\rm i}}{4}(z^0_y - \bar z^0_y)$ $+\frac{{\rm i}\sqrt{3}}{4}(z'^0 - \bar z'^0)$
σ_1	0	0	0	0
σ_2	0	0	0	0
σ_3	$-\frac{3}{4}(z^3_x + \bar z^3_x)$ $-\frac{1}{4}(z'^3 + \bar z'^3)$	$-\frac{3}{4}[z^3_{x-} + \bar z^3_{x-} + z^3_{x+} + \bar z^3_{x+}$ $+{\rm i}(z^3_{y+} + \bar z^3_{y+} - z^3_{y-} - \bar z^3_{y-})]$ $+\frac{1}{4}(z'^3_- + z'^3_+ + \bar z'^3_- + \bar z'^3_+)$	$-\frac{3{\rm i}}{4}[z^3_{x-} + \bar z^3_{x-} - z^3_{x+} - \bar z^3_{x+}$ $-{\rm i}(z^3_{y+} + \bar z^3_{y+} + z^3_{y-} + \bar z^3_{y-})]$ $+\frac{\rm i}{4}(z'^3_- + z'^3_+ - \bar z'^3_- - \bar z'^3_+)$	$-\frac{3{\rm i}}{4}(z^3_y - \bar z^3_y)$ $+\frac{{\rm i}\sqrt{3}}{4}(z'^3 - \bar z'^3)$

Z^{IJ}_{n}, Z^{IJ}_{nn} and Z^{IJ}_{nnn}:

$\otimes$	$\mathbb{I}$	σ_1	σ_2	σ_3
$\mathbb{I}$	$\frac{1}{2}(z_A+z_B)$	$\frac{1}{4}(z_{A+}+z_{A-}+z_{B+}+z_{B-})$	$\frac{i}{4}(z_{A-}-z_{A+}+z_{B-}-z_{B+})$	0
σ_1	$\frac{1}{2}(z_{\hat{1}}+\bar{z}_{\hat{1}})$	$\frac{1}{4}(z_{\hat{1}-}+\bar{z}_{\hat{1}-}+z_{\hat{1}+}+\bar{z}_{\hat{1}+})$	$\frac{i}{4}(z_{\hat{1}-}+\bar{z}_{\hat{1}-}-z_{\hat{1}+}-\bar{z}_{\hat{1}+})$	0
σ_2	$\frac{i}{2}(z_{\hat{1}}-\bar{z}_{\hat{1}})$	$\frac{i}{4}(z_{\hat{1}-}-\bar{z}_{\hat{1}-}+z_{\hat{1}+}-\bar{z}_{\hat{1}+})$	$-\frac{1}{4}(z_{\hat{1}-}-\bar{z}_{\hat{1}-}-z_{\hat{1}+}+\bar{z}_{\hat{1}+})$	0
σ_3	$\frac{1}{2}(z_A-z_B)$	$\frac{1}{4}(z_{A-}-z_{B-}+z_{A+}-z_{B+})$	$\frac{i}{4}(z_{A-}-z_{B-}-z_{A+}+z_{B+})$	0

$\otimes$	$\mathbb{I}$	σ_1	σ_2	σ_3
$\mathbb{I}$	0	0	0	0
σ_1	$-\frac{3}{8}(z_x+\bar{z}_x)$	$-\frac{3}{8}[z_{x-}+\bar{z}_{x-}+z_{x+}+\bar{z}_{x+}$ $+i(z_{y+}+\bar{z}_{y+}-z_{y-}-\bar{z}_{y-})]$	$-\frac{3i}{8}[z_{x-}+\bar{z}_{x-}-z_{x+}-\bar{z}_{x+}$ $-i(z_{y+}+\bar{z}_{y+}+z_{y-}+\bar{z}_{y-})]$	$-\frac{3i}{8}(z_y-\bar{z}_y)$
σ_2	$-\frac{3i}{8}(z_x-\bar{z}_x)$	$-\frac{3i}{8}[z_{x-}-\bar{z}_{x-}+z_{x+}-\bar{z}_{x+}$ $+i(z_{y+}-\bar{z}_{y+}-z_{y-}+\bar{z}_{y-})]$	$\frac{3}{8}[z_{x-}-\bar{z}_{x-}-z_{x+}+\bar{z}_{x+}$ $-i(z_{y+}-\bar{z}_{y+}+z_{y-}-\bar{z}_{y-})]$	$\frac{3}{8}(z_y+\bar{z}_y)$
σ_3	0	0	0	0

$\otimes$	$\mathbb{I}$	σ_1	σ_2	σ_3
$\mathbb{I}$	$-\frac{3}{4}(z^0_x+\bar{z}^0_x)$ $-\frac{1}{4}(z'^0+\bar{z}'^0)$	$-\frac{3}{4}[z^0_{x-}+\bar{z}^0_{x-}+z^0_{x+}+\bar{z}^0_{x+}$ $+i(z^0_{y+}+\bar{z}^0_{y+}-z^0_{y-}-\bar{z}^0_{y-})]$ $+\frac{1}{4}(z'^0_-+z'^0_++\bar{z}'^0_-+\bar{z}'^0_+)$	$-\frac{3i}{4}[z^0_{x-}+\bar{z}^0_{x-}-z^0_{x+}-\bar{z}^0_{x+}$ $-i(z^0_{y+}+\bar{z}^0_{y+}+z^0_{y-}+\bar{z}^0_{y-})]$ $+\frac{i}{4}(z'^0_-+z'^0_+-\bar{z}'^0_--\bar{z}'^0_+)$	$-\frac{3i}{4}(z^0_y-\bar{z}^0_y)$ $+\frac{i\sqrt{3}}{4}(z'^0-\bar{z}'^0)$
σ_1	0	0	0	0
σ_2	0	0	0	0
σ_3	$-\frac{3}{4}(z^3_x+\bar{z}^3_x)$ $-\frac{1}{4}(z'^3+\bar{z}'^3)$	$-\frac{3}{4}[z^3_{x-}+\bar{z}^3_{x-}+z^3_{x+}+\bar{z}^3_{x+}$ $+i(z^3_{y+}+\bar{z}^3_{y+}-z^3_{y-}-\bar{z}^3_{y-})]$ $+\frac{1}{4}(z'^3_-+z'^3_++\bar{z}'^3_-+\bar{z}'^3_+)$	$-\frac{3i}{4}[z^3_{x-}+\bar{z}^3_{x-}-z^3_{x+}-\bar{z}^3_{x+}$ $-i(z^3_{y+}+\bar{z}^3_{y+}+z^3_{y-}+\bar{z}^3_{y-})]$ $+\frac{i}{4}(z'^3_-+z'^3_+-\bar{z}'^3_--\bar{z}'^3_+)$	$-\frac{3i}{4}(z^3_y-\bar{z}^3_y)$ $+\frac{i\sqrt{3}}{4}(z'^3-\bar{z}'^3)$

Fields in the tables

The lattice deformation fields can be expressed in terms of Cartesian coordinates: $\mathbf{z} = \frac{1}{2\pi}\sum_i z_{\hat{l}_i}\hat{k}_i$

$$z_x = (z_{\hat{2}} + z_{\hat{3}}), \qquad z_y = -\sqrt{3}(z_{\hat{2}} - z_{\hat{3}})$$

$\hat{k}_i$ are vectors of the dual basis

$$\begin{aligned}
z_x^0 &= \tfrac{1}{2}(z_{A\hat{1}\hat{2}} + z_{B\hat{1}\hat{2}} + z_{A\hat{1}\hat{3}} + z_{B\hat{1}\hat{3}}),\\
z_y^0 &= -\tfrac{\sqrt{3}}{2}(z_{A\hat{1}\hat{2}} + z_{B\hat{1}\hat{2}} - z_{A\hat{1}\hat{3}} - z_{B\hat{1}\hat{3}}),\\
z_x^3 &= \tfrac{1}{2}(z_{A\hat{1}\hat{2}} - z_{B\hat{1}\hat{2}} + z_{A\hat{1}\hat{3}} - z_{B\hat{1}\hat{3}}),\\
z_y^3 &= -\tfrac{\sqrt{3}}{2}(z_{A\hat{1}\hat{2}} - z_{B\hat{1}\hat{2}} - z_{A\hat{1}\hat{3}} + z_{B\hat{1}\hat{3}}),\\
z'^0 &= \tfrac{1}{2}(z_{A\hat{2}\hat{3}} + z_{B\hat{2}\hat{3}}),\\
z'^3 &= \tfrac{1}{2}(z_{A\hat{2}\hat{3}} - z_{B\hat{2}\hat{3}}).
\end{aligned}$$

Dirac algebra & Internal symmetry

Each matrix $\gamma^0 \cdot \sigma_I \otimes \sigma_J$ is proportional to either γ^μ, $\gamma^\mu \tau_a$, τ_a or 1: 16 in total

All products of sigma matrices after multiplication by γ^0: $\gamma^0 \cdot (\sigma_I \otimes \sigma_J)$, $I, J = 0, 1, 2, 3$, in terms of corresponding products of Dirac and isospin matrices:

$\otimes$	$\mathbb{I}$	σ_1	σ_2	σ_3
$\mathbb{I}$	$\mathrm{i}\sigma_3 \otimes \sigma_3 = -\gamma^0$	$-\sigma_3 \otimes \sigma_2 = \gamma^1 \tau_2$	$\sigma_3 \otimes \sigma_1 = \gamma^1 \tau_3$	$\mathrm{i}\sigma_3 \otimes \mathbb{I} = -\gamma^0 \tau_1$
σ_1	$-\sigma_2 \otimes \sigma_3 = \gamma^2 \tau_1$	$-\mathrm{i}\sigma_2 \otimes \sigma_2 = -\mathrm{i}\tau_3$	$\mathrm{i}\sigma_2 \otimes \sigma_1 = \mathrm{i}\tau_2$	$-\sigma_2 \otimes \mathbb{I} = \gamma^2$
σ_2	$\sigma_1 \otimes \sigma_3 = \gamma^1$	$\mathrm{i}\sigma_1 \otimes \sigma_2 = -\gamma_0 \tau_2$	$-\mathrm{i}\sigma_1 \otimes \sigma_1 = -\gamma^0 \tau_3$	$\sigma_1 \otimes \mathbb{I} = \gamma^1 \tau_1$
σ_3	$\mathrm{i}\mathbb{I} \otimes \sigma_3 = \mathrm{i}\tau_1$	$-\mathbb{I} \otimes \sigma_2 = \gamma^2 \tau_3$	$\mathbb{I} \otimes \sigma_1 = -\gamma^2 \tau_2$	$\mathrm{i}\mathbb{I} \otimes \mathbb{I} = \mathrm{i}1$

Once the identification is made...

Gauge and Yukawa coupling

We can express the deformations as interaction terms modifying the Lagrangian,

$$L = L_0 + \Phi\bar{\Psi}\Psi + U^a\bar{\Psi}\tau_a\Psi + A_\mu\bar{\Psi}\gamma^\mu\Psi + B^a_\mu\bar{\Psi}\gamma^\mu\tau_a\Psi$$

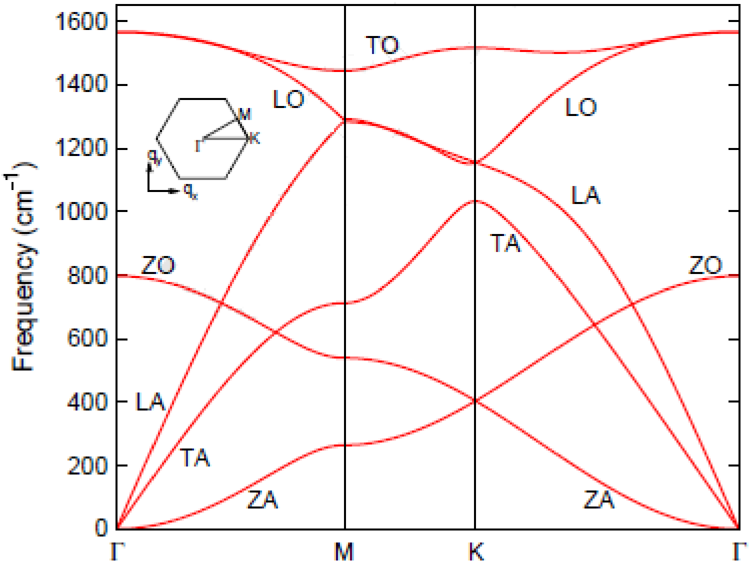
Gauge fields in terms of original fields

$$
\begin{aligned}
\Phi &= -\tfrac{3\mathrm{i}}{4}(z_y^3 - \bar z_y^3) + \tfrac{\mathrm{i}\sqrt{3}}{4}(z'^3 - \bar z'^3)\\
U^1 &= \tfrac{1}{2}(z_A - z_B) - \tfrac{3}{4}(z_x^3 + \bar z_x^3) - \tfrac{1}{4}(z'^3 + \bar z'^3),\\
U^2 &= -\tfrac{\mathrm{i}}{4}(z_{\hat 1 -} + \bar z_{\hat 1 -} - z_{\hat 1 +} - \bar z_{\hat 1 +}) - \tfrac{3\mathrm{i}}{8}[z_{x-} + \bar z_{x-} - z_{x+} - \bar z_{x+} - \mathrm{i}(z_{y+} + \bar z_{y+} + z_{y-} + \bar z_{y-})]\\
U^3 &= -\tfrac{1}{4}(z_{\hat 1 -} + \bar z_{\hat 1 -} + z_{\hat 1 +} + \bar z_{\hat 1 +}) + \tfrac{3}{8}[z_{x-} + \bar z_{x-} + z_{x+} + \bar z_{x+} + \mathrm{i}(z_{y+} + \bar z_{y+} - z_{y-} - \bar z_{y-})]\\
A_0 &= -\tfrac{1}{2}(z_A + z_B) + \tfrac{3}{4}(z_x^0 + \bar z_x^0) - \tfrac{1}{4}(z_x'^0 + \bar z_x'^0),\\
A_1 &= \tfrac{\mathrm{i}}{2}(z_{\hat 1} - \bar z_{\hat 1}) - \tfrac{3\mathrm{i}}{8}(z_x - \bar z_x),\\
A_2 &= -\tfrac{3\mathrm{i}}{8}(z_y - \bar z_y)\\
B_0^1 &= \tfrac{3\mathrm{i}}{4}(z_y^0 - \bar z_y^0) - \tfrac{\mathrm{i}\sqrt{3}}{4}(z'^0 - \bar z'^0), \quad B_1^1 = \tfrac{3}{8}(z_y + \bar z_y), \quad B_2^1 = \tfrac{1}{2}(z_{\hat 1} + \bar z_{\hat 1}) - \tfrac{3}{8}(z_x + \bar z_x)\\
B_0^2 &= -\tfrac{\mathrm{i}}{4}(z_{\hat 1 -} - \bar z_{\hat 1 -} + z_{\hat 1 +} - \bar z_{\hat 1 +}) + \tfrac{3\mathrm{i}}{8}[z_{x-} - \bar z_{x-} + z_{x+} - \bar z_{x+} + \mathrm{i}(z_{y+} - \bar z_{y+} - z_{y-} + \bar z_{y-})]\\
B_1^2 &= \tfrac{1}{4}(z_{A+} + z_{A-} + z_{B+} + z_{B-}) - \tfrac{3}{4}[z_{x-}^0 + \bar z_{x-}^0 + z_{x+}^0 + \bar z_{x+}^0 + \mathrm{i}(z_{y+}^0 + \bar z_{y+}^0 - z_{y-}^0 - \bar z_{y-}^0)] + \tfrac{1}{4}(z_-'^0 + z_+'^0 + \bar z_-'^0 + \bar z_+'^0)\\
B_2^2 &= -\tfrac{\mathrm{i}}{4}(z_{A-} - z_{B-} - z_{A+} + z_{B+}) + \tfrac{3\mathrm{i}}{4}[z_{x-}^3 + \bar z_{x-}^3 - z_{x+}^3 - \bar z_{x+}^3 - \mathrm{i}(z_{y+}^3 + \bar z_{y+}^3 + z_{y-}^3 + \bar z_{y-}^3)] + \tfrac{\mathrm{i}}{4}(z_-'^3 + z_+'^3 - \bar z_-'^3 - \bar z_+'^3)\\
B_0^3 &= \tfrac{1}{4}(z_{\hat 1 -} - \bar z_{\hat 1 -} - z_{\hat 1 +} + \bar z_{\hat 1 +}) - \tfrac{3}{8}[z_{x-} - \bar z_{x-} - z_{x+} + \bar z_{x+} - \mathrm{i}(z_{y+} - \bar z_{y+} + z_{y-} - \bar z_{y-})]\\
B_1^3 &= \tfrac{\mathrm{i}}{4}(z_{A-} - z_{A+} + z_{B-} - z_{B+}) - \tfrac{3\mathrm{i}}{4}[z_{x-}^0 + \bar z_{x-}^0 - z_{x+}^0 - \bar z_{x+}^0 - \mathrm{i}(z_{y+}^0 + \bar z_{y+}^0 + z_{y-}^0 + \bar z_{y-}^0)] + \tfrac{\mathrm{i}}{4}(z_-'^0 + z_+'^0 - \bar z_-'^0 - \bar z_+'^0)\\
B_2^3 &= \tfrac{1}{4}(z_{A-} - z_{B-} + z_{A+} - z_{B+}) - \tfrac{3}{4}[z_{x-}^3 + \bar z_{x-}^3 + z_{x+}^3 + \bar z_{x+}^3 + \mathrm{i}(z_{y+}^3 + \bar z_{y+}^3 - z_{y-}^3 - \bar z_{y-}^3)] + \tfrac{1}{4}(z_-'^3 + z_+'^3 + \bar z_-'^3 + \bar z_+'^3)
\end{aligned}
$$

Discussion

- We considered arbitrary deformations of the Hubbard model on a hexagonal lattice with the range up to next-to-nearest neighbor
- . . . and expressed these deformations in terms of couplings to U(2) Yukawa and gauge fields built out of the deformations
- The physical meaning of the deformations: phonon fields and topologial defects
- We got only Yukawa and gauge fields: no gravity on a membrane
- We did not discuss the dynamical part of deformations: can we have full fledged gauge interactions?

Phonon dispersion relations



Some reviews/books on graphene

There are many books/reviews on graphene

- AH Castro Neto, F Guinea, NMR Peres, KS Novoselov, & AK Geim, *The electronic properties of graphene* Reviews of Modern Physics, 2009, 81, 109-162 [0709.1163],
- *Carbon Nanotubes* Eds.: M Endo, S Iijima, MS Dresselhaus
- Vozmediano, M. A. H.; Katsnelson, M. I. & Guinea, F. *Gauge fields in graphene*, Physics Reports, 2010, 496,, 109 1003.5179
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