Modeling of JPA

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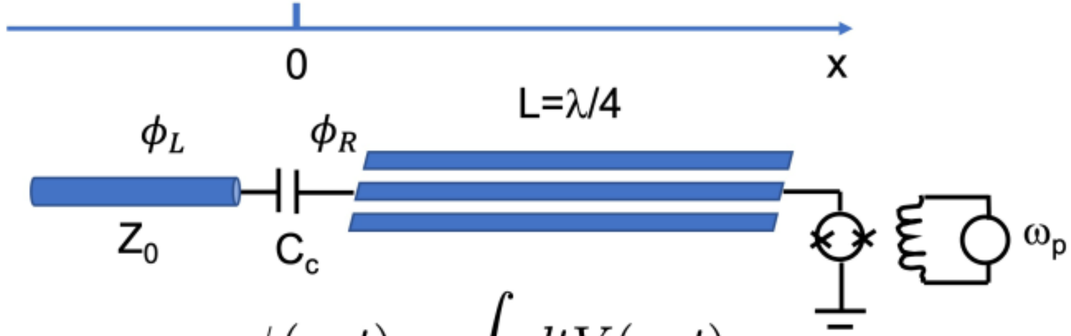
Aim: modelling of a JPA without simplifications of the nonlinear components

Model by Claudio Gatti



$$\phi(x,t) = \int dt V(x,t)$$

The Lagrangian density of the circuit in figure 1 is obtained by the sum of the capacitive energies minus the inductive energies:

$$\begin{aligned}\mathcal{L} \;=\; & \left(\frac{1}{2}C_L\dot{\phi_L}^2 - \frac{1}{2L_L}\phi_L'^2\right)\sigma(-x) + \left(\frac{1}{2}C_R\dot{\phi_R}^2 - \frac{1}{2L_R}\phi_R'^2\right)[\sigma(x) - \sigma(x-L)] \quad (1)\\ & + \left(\frac{1}{2}C_c(\dot{\phi}_L - \dot{\phi}_R)\right)\delta(x) + \left[\frac{C_J}{2}\dot{\phi}_J^2 + E_J(\phi_{ext})\cos\left(2\pi\frac{\phi_J}{\phi_0}\right)\right]\delta(x-L)\end{aligned}$$

where $\phi_{L,R}(x)$ is the flux variable defined as $\phi(x) = \int^t dt'V(t')$ in the left (right) transmission line and $C_{L,R}$ and $L_{L,R}$ are their capacitances and inductances per unit length, and $\phi_J = \phi_R(x=L)$. $\sigma(x)$ is the Heaviside (step) function. The (symmetric) SQUID Josephson energy is:

$$E_J(\phi_{ext}) = 2E_{0J}\cos\left(\pi\frac{\phi_{ext}}{\phi_0}\right) \quad (2)$$

where E_{0J} is the Josephson energy of the single junction.

Wave equations for ϕ_L and ϕ_R

$$\begin{aligned}\sigma(-x)\left[C_L\ddot{\phi}_L-\frac{1}{L_L}\phi''_L\right]+\delta(x)\left[C_c(\ddot{\phi}_L-\ddot{\phi}_R)+\frac{1}{L_L}\phi'_L\right] &= 0(4)\\ [\sigma(x)-\sigma(x-L)]\left[C_R\ddot{\phi}_R-\frac{1}{L_R}\phi''_R\right]+\delta(x)\left[-C_c(\ddot{\phi}_L-\ddot{\phi}_R)-\frac{1}{L_R}\phi'_R\right]+ & \\ \delta(x-L)\left[C_J\ddot{\phi}_R+\frac{1}{L_R}\phi'_R+\frac{2\pi}{\phi_0}E_J(\phi_{ext})\sin\left(2\pi\frac{\phi_R}{\phi_0}\right)\right] &= 0\end{aligned}$$

The solution is the sum of an outgoing and an incoming wave

$$\phi(x,t)=\phi^{in}(t-x/v_p)+\phi^{out}(t+x/v_p)$$

and $v_p=1/\sqrt{LC}$. The other equations give the boundary conditions:

$$-\frac{1}{L_L}\phi'_L=C_c(\ddot{\phi}_L-\ddot{\phi}_R)\qquad (x=0)$$

$$-\frac{1}{L_R}\phi'_R=C_c(\ddot{\phi}_L-\ddot{\phi}_R)\qquad (x=0)$$

$$C_J\ddot{\phi}_R+\frac{1}{L_R}\phi'_R+\frac{2\pi}{\phi_0}E_J(\phi_{ext})\sin\left(2\pi\frac{\phi_R}{\phi_0}\right)=0\qquad (x=L)$$

$$\begin{aligned}\phi' &= \frac{1}{v_p}\left(\dot{\phi}^{out}-\dot{\phi}^{in}\right)\\ \dot{\phi} &= \dot{\phi}^{in}+\dot{\phi}^{out}\end{aligned}$$

in and *out* fields.

$$2\ddot{\phi}_R^{in}+\frac{1}{CcZ_0}\dot{\phi}_R^{in}=\ddot{\phi}_L^{in}+\frac{1}{CcZ_0}\dot{\phi}_R^{out}\qquad (x=0)$$

$$C_J\ddot{\phi}_J+\frac{1}{Z_0}\dot{\phi}_J+\frac{2\pi}{\phi_0}E_J(\phi_{ext})\sin\left(2\pi\frac{\phi_J}{\phi_0}\right)=\frac{2}{Z_0}\dot{\phi}_R^{in}\qquad (x=L)\qquad \phi_R^{in}(t,x=L)=\phi_R^{in}(t-L/v_p,x=0)$$

$$\begin{aligned}\phi_R^{out}(t,x=L) &= \phi_R^{out}(t+L/v_p,x=0)=\phi_J(t)-\phi_R^{in}(t,x=L)\\ &= \phi_J(t)-\phi_R^{in}(t-L/v_p,x=0)\end{aligned}$$

$$\phi_L^{out}=\phi_R^{out}-\phi_R^{in}+\phi_L^{in}\qquad (x=0)$$

Transient Analysis of Lossless Transmission Lines

Manuscript received July 7, 1967; revised September 8, 1967.

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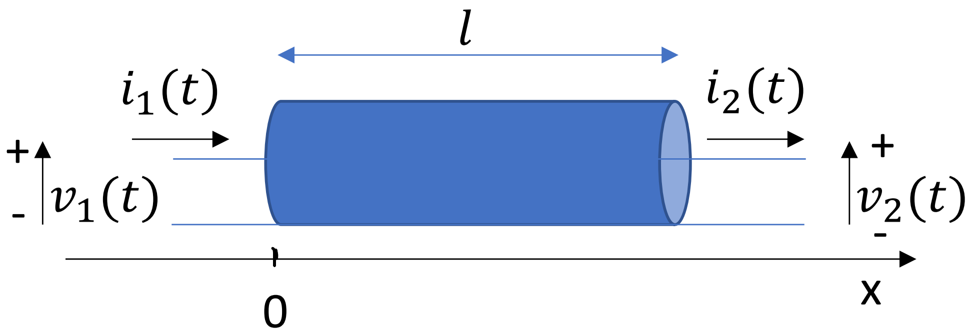

$\tau = \sqrt{LC}\, l$ line delay

$Z_0 = \sqrt{\frac{L}{C}}$ line impedance

$\Delta e = +Z_0 \Delta i, \qquad \Delta e = -Z_0 \Delta i$

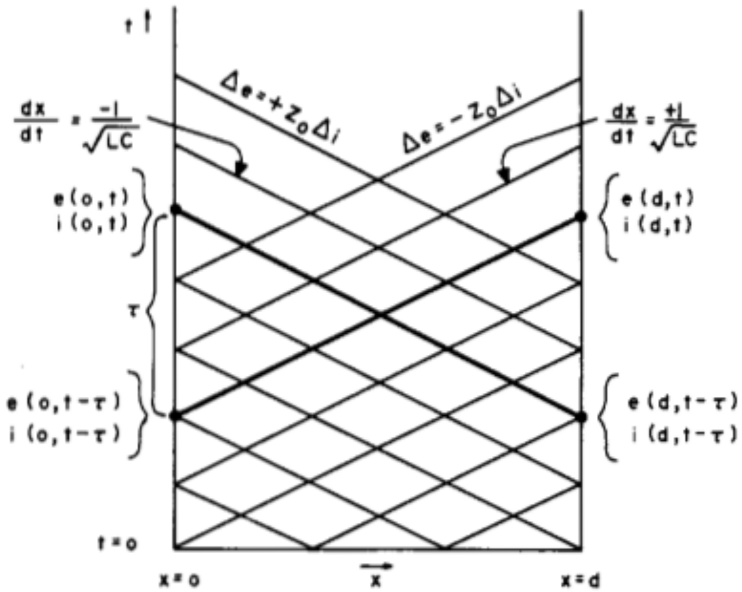

Fig. 1. Characteristic curves for a uniform transmission line.

$$v = v(x,t) \quad i = i(x,t)$$

$$L\frac{\partial i}{\partial t} + \frac{\partial v}{\partial x} = 0 \qquad\qquad i_{tt} - \frac{1}{L\,C} i_{xx} = 0$$

$$C\frac{\partial v}{\partial t} + \frac{\partial i}{\partial x} = 0 \qquad\qquad v_{tt} - \frac{1}{L\,C} v_{xx} = 0$$

The method of characteristics for solving these equations is based on a transformation in the $x-t$ plane which accomplishes the conversion of (1) and (2) into a pair of ordinary differential equations. Each of these two ordinary differential equations holds true along a different family of *characteristic curves* in the $x-t$ plane, one family corresponding to the forward or incident wave and the other to the backward or reflected wave. These two families of characteristic curves are, in effect, the basis for a new coordinate system instead of the canonical coordinates $x=$constant and $t=$constant.

$$dx/dt = 1/\sqrt{LC} \quad \text{and} \quad dx/dt = -1/\sqrt{LC}$$

$$\sqrt{\frac{L}{C}}\,di + \left(Ri + \sqrt{\frac{L}{C}}\,Ge\right)dx + de = 0 \tag{3}$$

which holds true only along the *forward* characteristics, defined as $dx/dt = 1/\sqrt{L/C}$, and

$$-\sqrt{\frac{L}{C}}\,di + \left(Ri - \sqrt{\frac{L}{C}}\,Ge\right)dx + de = 0 \tag{4}$$

which holds true only along the *backward* characteristics, defined as $dx/dt = -1/\sqrt{LC}$.

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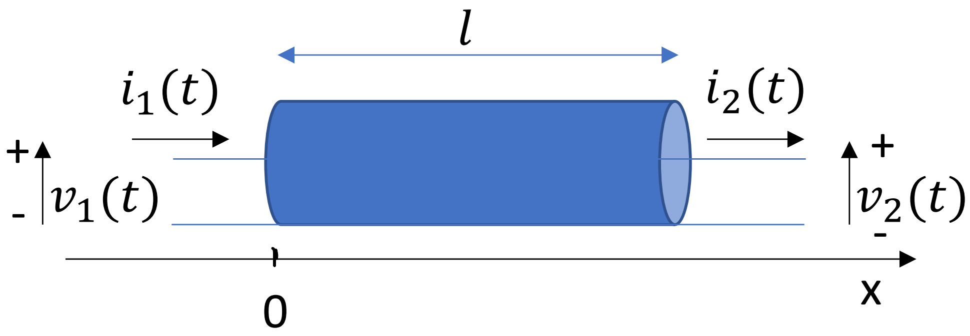
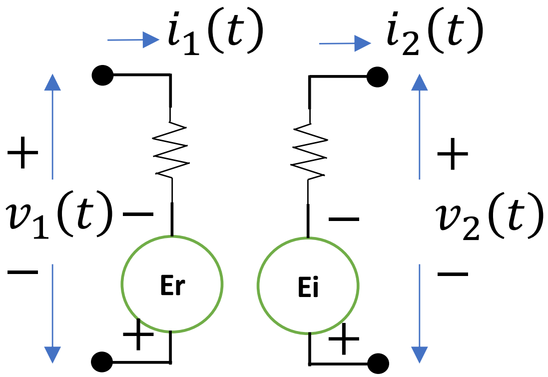


$v = v(x,t)$ $i = i(x,t)$

$\tau = \sqrt{LC}\, l$ line delay

$Z_0 = \sqrt{\frac{L}{C}}$ line impedance

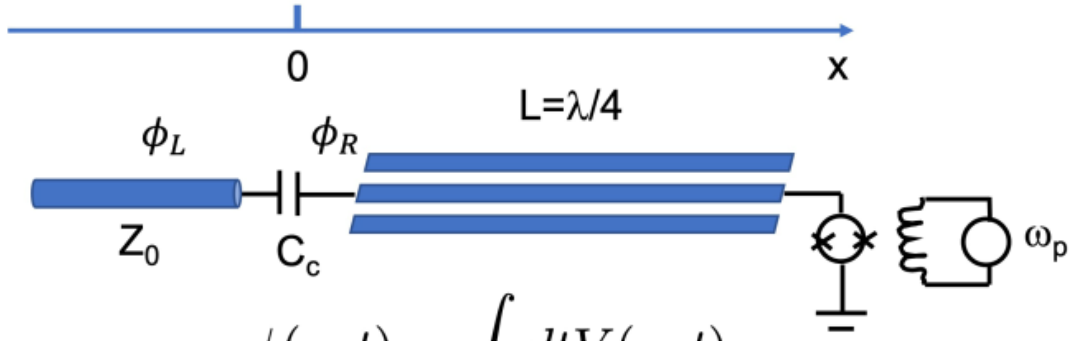


$$v(l,t) = -Z_0 i(l,t) + [v(0,t-\tau) + Z_0 i(0,t-\tau)]$$
$$v(0,t) = +Z_0 i(0,t) + [v(l,t-\tau) - Z_0 i(l,t-\tau)]$$

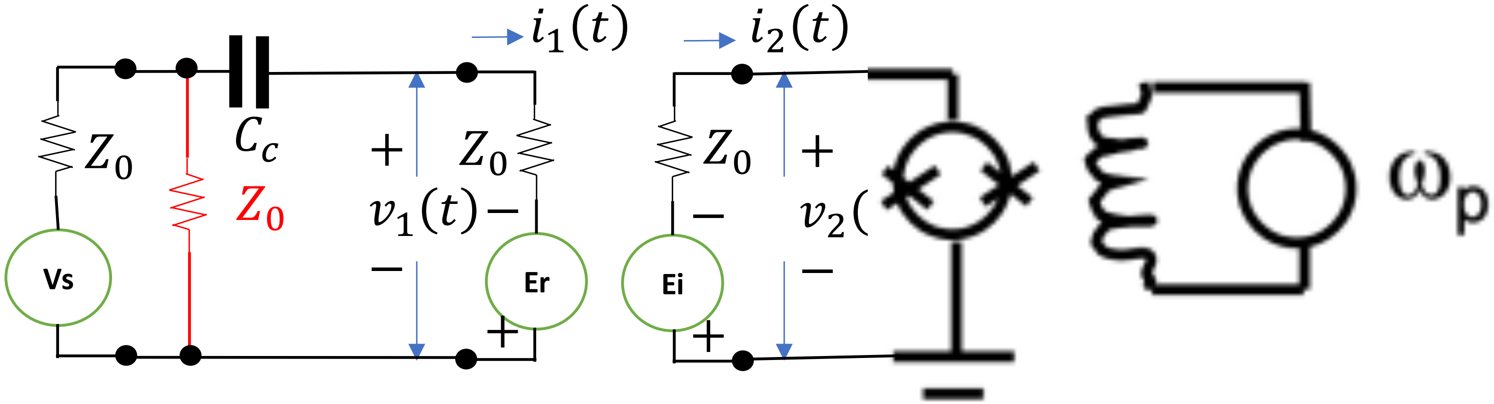
$$v_2(t) = -Z_0 i_2(t) + [v_1(t-\tau) + Z_0 i_1(t-\tau)]$$
$$v_1(t) = +Z_0 i_1(t) + [v_2(t-\tau) - Z_0 i_2(t-\tau)]$$

$$v_2(t) = -Z_0 i_2(t) - E_i(t-\tau)$$
$$v_1(t) = +Z_0 i_1(t) - E_r(t-\tau)$$

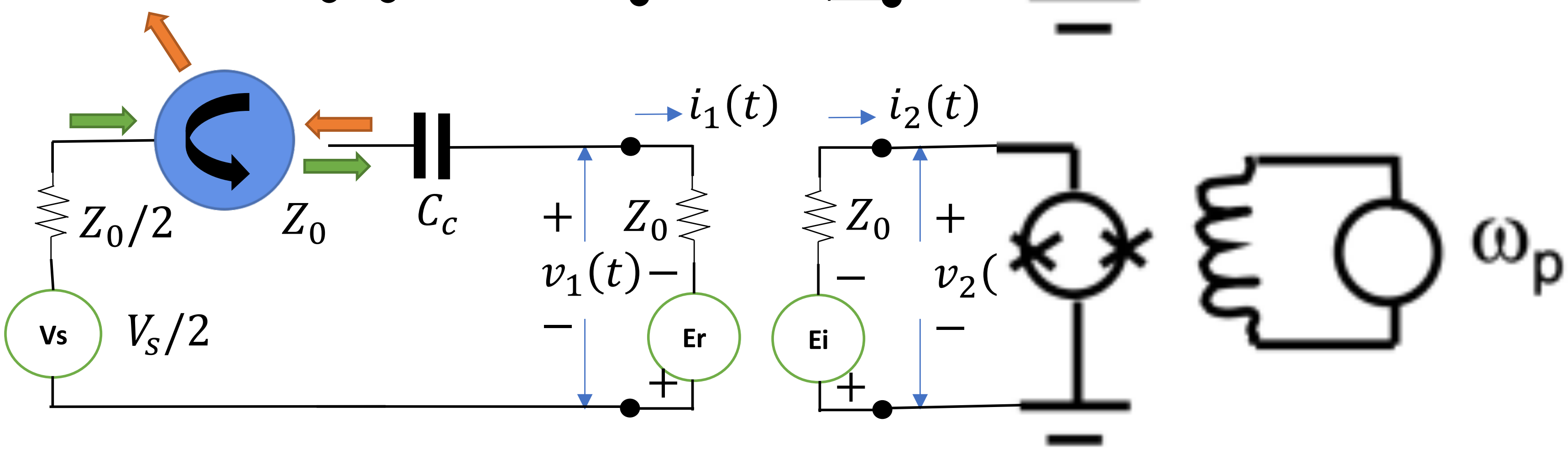
$$E_i(t) = -[2v_1(t) + E_r(t-\tau)]$$
$$E_r(t) = -[2v_2(t) + E_i(t-\tau)]$$



0
x
L=λ/4
ϕ_L
ϕ_R
Z_0
C_c
ω_p



$i_1(t)$
$i_2(t)$
Z_0
C_c
Z_0
Z_0
Z_0
$v_1(t)$
$v_2($
Vs
Er
Ei
ω_p



$i_1(t)$
$i_2(t)$
$Z_0/2$
Z_0
C_c
Z_0
Z_0
$v_1(t)$
$v_2($
Vs
$V_s/2$
Er
Ei
ω_p

$$\dot{v}_1 = -\frac{1}{2\,C_c\,Z_0}\,v_1 + \frac{1}{2}\dot{v}_a(t) - \frac{1}{2}\dot{E}_r(t-\tau) - \frac{1}{2\,C_c\,Z_0}\,E_r(t-\tau)$$

$$\dot{v}_2 = -\left(\frac{1}{C_j\,Z_0} + \frac{1}{C_j\,R_j}\right) v_2 - \frac{I_j}{C_j}\cos\left(\pi\frac{\phi_{ext}}{\phi_0}\right)\sin\varphi - \frac{1}{C_j\,Z_0}\,E_i(t-\tau)$$

$$\dot{\varphi} = \frac{2\,e}{\hbar}v_2$$

where

$$v_s(t) = v_{rf}\;\sin\omega t$$

$$\dot{v}_s(t) = v_{rf}\,\omega\;\cos\omega t$$

$$\phi_{ext}(t) = \phi_{dc} + \phi_{ac}\cos\omega_p\,t$$

$$Z_0 = 50\,\Omega\,;\quad \tau = \frac{T}{4} = \frac{\pi}{2\,\omega_{ris}}\;;$$

$I_j = 2I_{j0}, C_j = 2C_{j0}$ and $R_j = \frac{1}{2}R_{j0}$ are the SQUID critical current capacitance and resistance

We start with Ei and Er set to zero

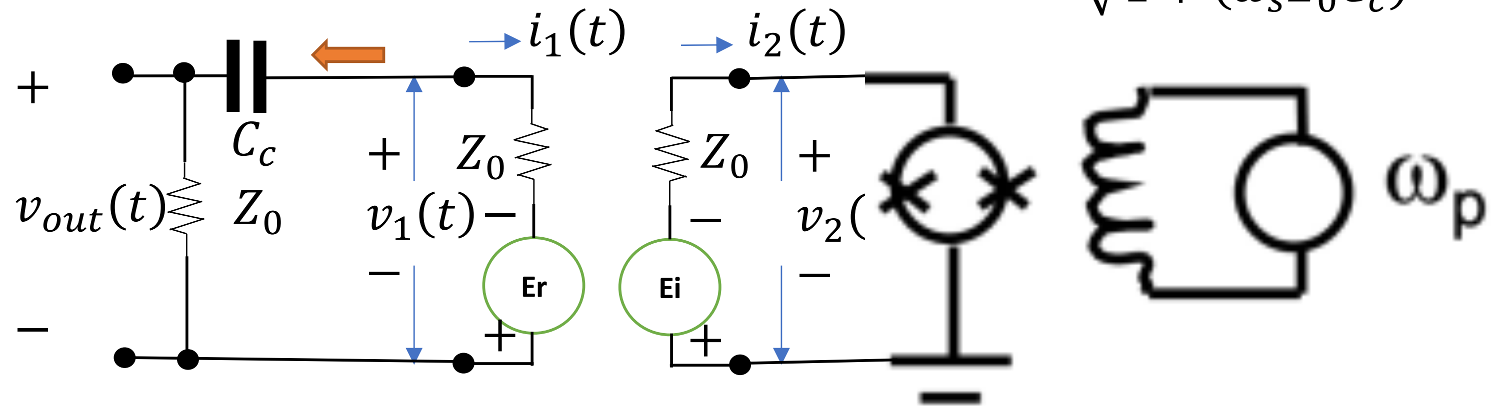
The values of Ei and Er are updated during the integration according to:

$$E_i(t) = -\,[2v_1(t) + E_r(t-\tau)]$$

$$E_r(t) = -[2v_2(t) + E_i(t-\tau)]$$

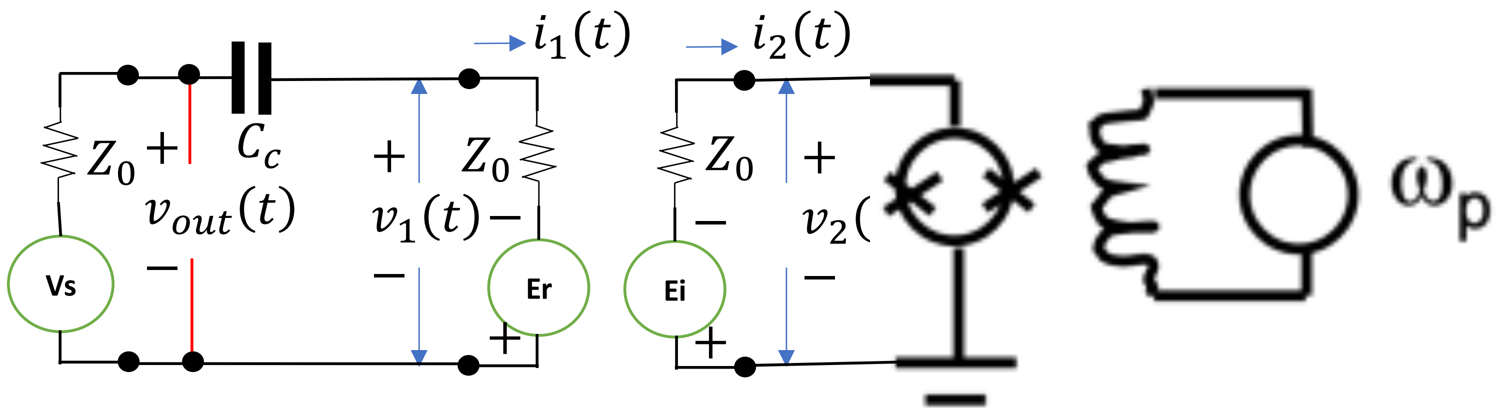
For Vout we have to decide how to simulate the circulator:
Assuming that the reflected wave do not see the signal source, its component at the signal frequency will see an RC circuit

$$v_{out} = v_1 \frac{\omega_s Z_0 C_c}{\sqrt{1+(\omega_s Z_0 C_c)^2}}$$



Alternatively, we can take the voltage at that the input of the coupling capacitor:

$$v_{out}(t) = V_s(t) - Z_0\, i_1(t)$$



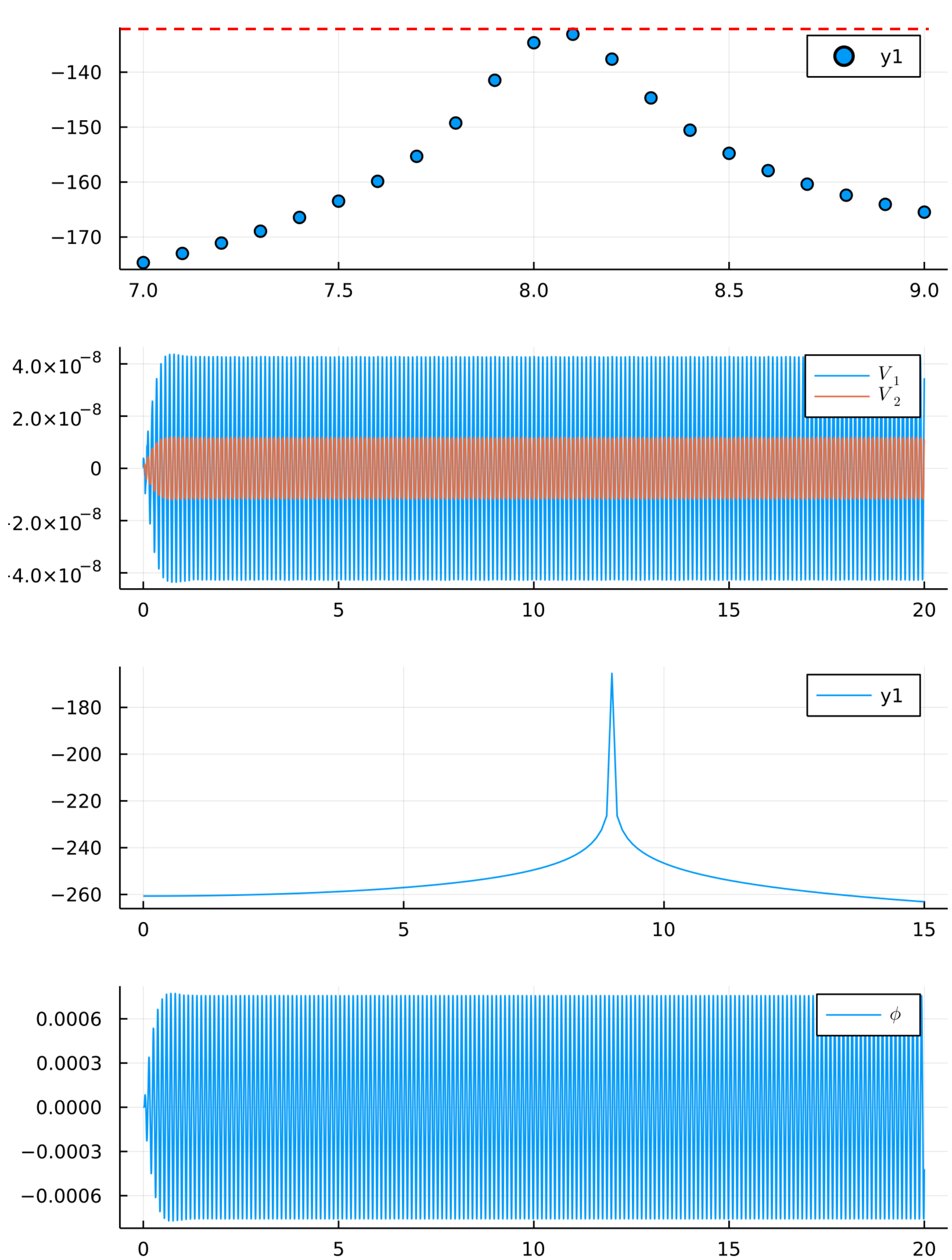
Simulation results

As parameter values the experimental data from LNF are used:

Resonator Length 2.62mm
Substrate silicon eps_eff=6.346
Bare resonator frequency **11.3 GHz**
Impedance **Z0=50 Ohm**
Coupling capacitor **14-18 fF**
S=12 µm e w=7 µm

SQUID Parameters:
Junction area 0.3-0.6 µm2
Junction **Ic=0.47 µA** (quella misurata)
Junction **Cj<0.5 pF**
Junction freq fjj>> 10 GHz
DC SQUID
Area 4x4 µm2
M=70 fH (misurata)

The effects of the SQUID load depend strongly on the critical current



Scan of pump frequency with fpump = 2 fsignal

The effects of the SQUID load depend strongly on the critical current

A resonance occurs for fsignal near 8 Ghz

fpump=2 fsignal

Zi = 50 Ohm   Cc = 0.015 pF   Z0 = 50 Ohm
I0 = 2.0 uA   Cj = 0.5 pF   Rj = 100000 Ohm
Fplasma = 110.24538380186623 GHz

Lambda/4 Resonator frequency = 11.3 GHz   Time delay = 22.12 ps

input RF signal = 0.1 uV = -130.0 dBm

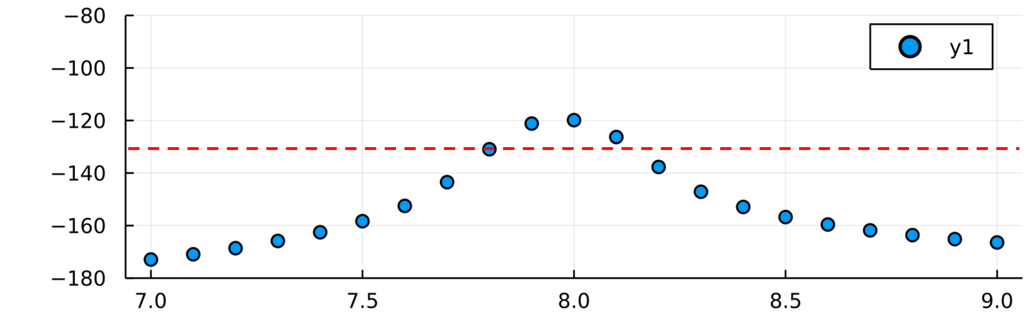
Fstart = 7.0 GHz   Fend = 9.0 GHz Fstep = 100.0 MHz

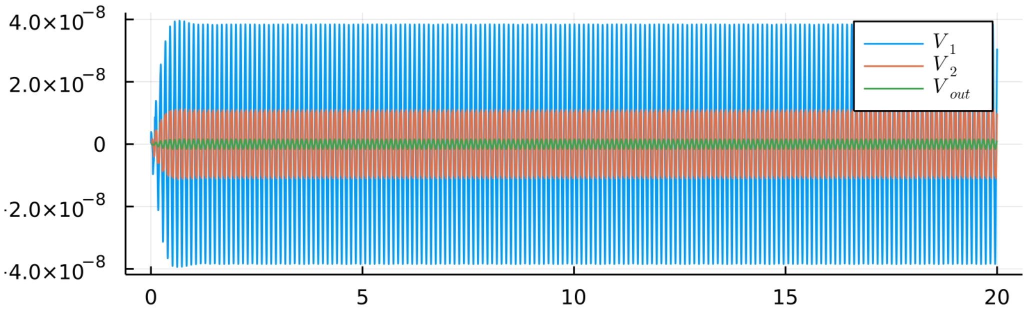
Magnetic flux amplitude **DC = 0.29** Phi0 **RF = 0.15** Phi0   Frequency = 10.2 GHz

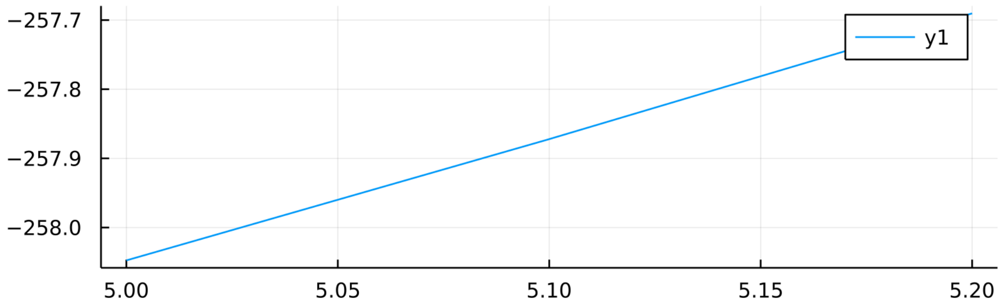
Integration time = 20.0 ns   time step = 0.1 ps
integrating 200000 time steps with 221 lag steps

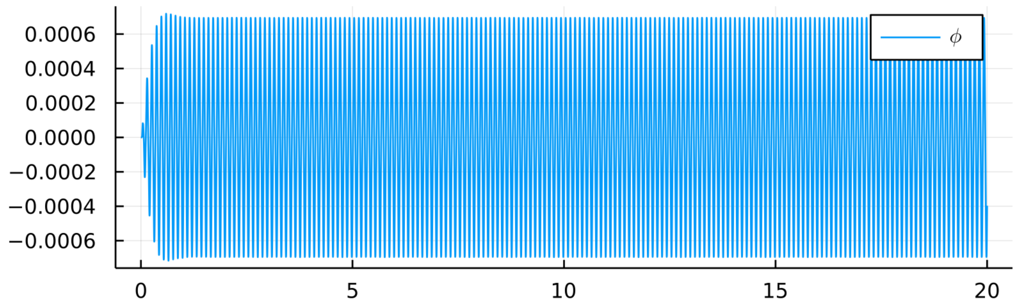
increase of dc flux bias

Effect of dc flux bias









fpump=2 fsignal

Zi = 50 Ohm   Cc = 0.015 pF   Z0 = 50 Ohm
I0 = 2.0 uA   Cj = 0.5 pF   Rj = 100000 Ohm

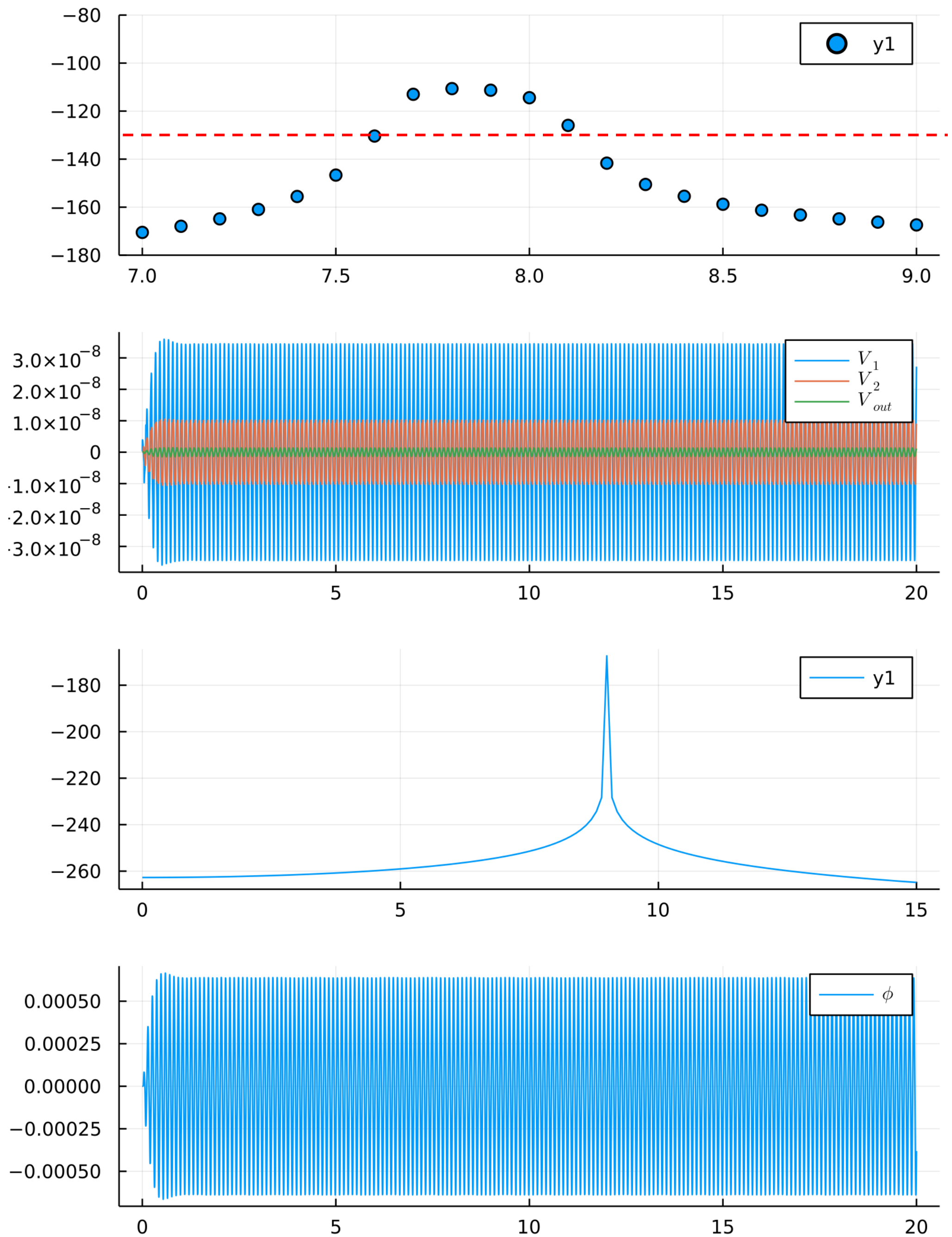
Fplasma = 110 GHz

Lambda/4 Resonator frequency = 11.3 GHz   Time delay = 22.12 ps

input RF signal = 0.1 uV   = -130.0 dBm

Fstart = 7.0 GHz   Fend = 9.0 GHz   Fstep = 100.0 MHz

Magnetic flux amplitude **DC = 0.3** Phi0 RF = 0.15 Phi0
Frequency = 10.2 GHz
Integration time = 20.0 ns   time step = 0.1 ps
integrating 200000 time steps with 221 lag steps



Effect of dc flux bias

Ic=2uA

fpump=2 fsignal

Zi = 50 Ohm   Cc = 0.015 pF   Z0 = 50 Ohm
I0 = 2.0 uA   Cj = 0.5 pF   Rj = 100000 Ohm

Fplasma = 110.24 GHz

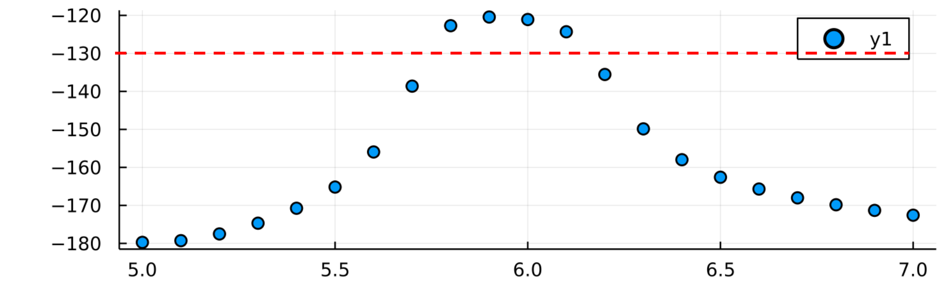
Lambda/4 Resonator frequency = 11.3 GHz   Time delay = 22.12 ps

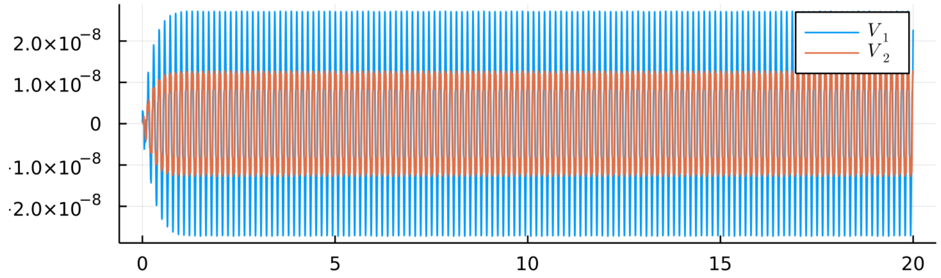
input RF signal = 0.1 uV = -130.0 dBm

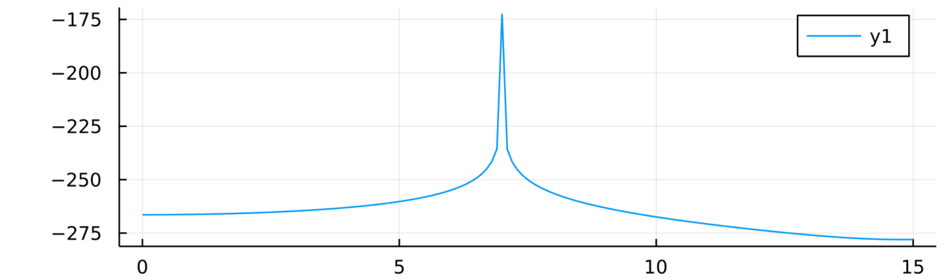
Fstart = 7.0 GHz   Fend = 9.0 GHz Fstep = 100.0 MHz

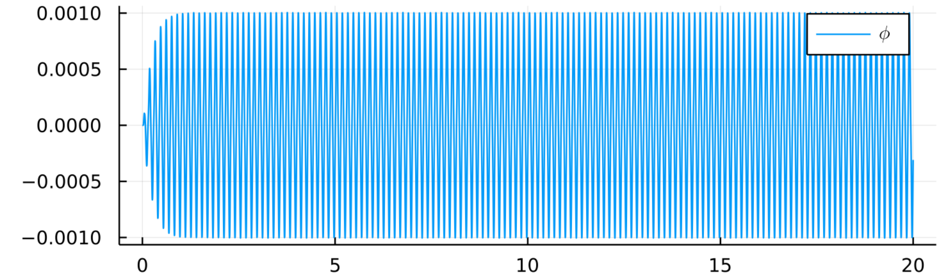
Magnetic flux amplitude **DC = 0.31** Phi0 RF = 0.15 Phi0
Frequency = 10.2 GHz

Integration time = 20.0 ns   time step = 0.1 ps
integrating 200000 time steps with 221 lag steps









Now we reduce the SQUID critical current: Ic=1uA

It is necessary more pump power

The resonance moves to lower frequency

fpump=2 fsignal

Zi = 50 Ohm   Cc = 0.015 pF   Z0 = 50 Ohm
I0 = 1.0 uA   Cj = 0.5 pF   Rj = 100000 Ohm
Fplasma = 77.95 GHz

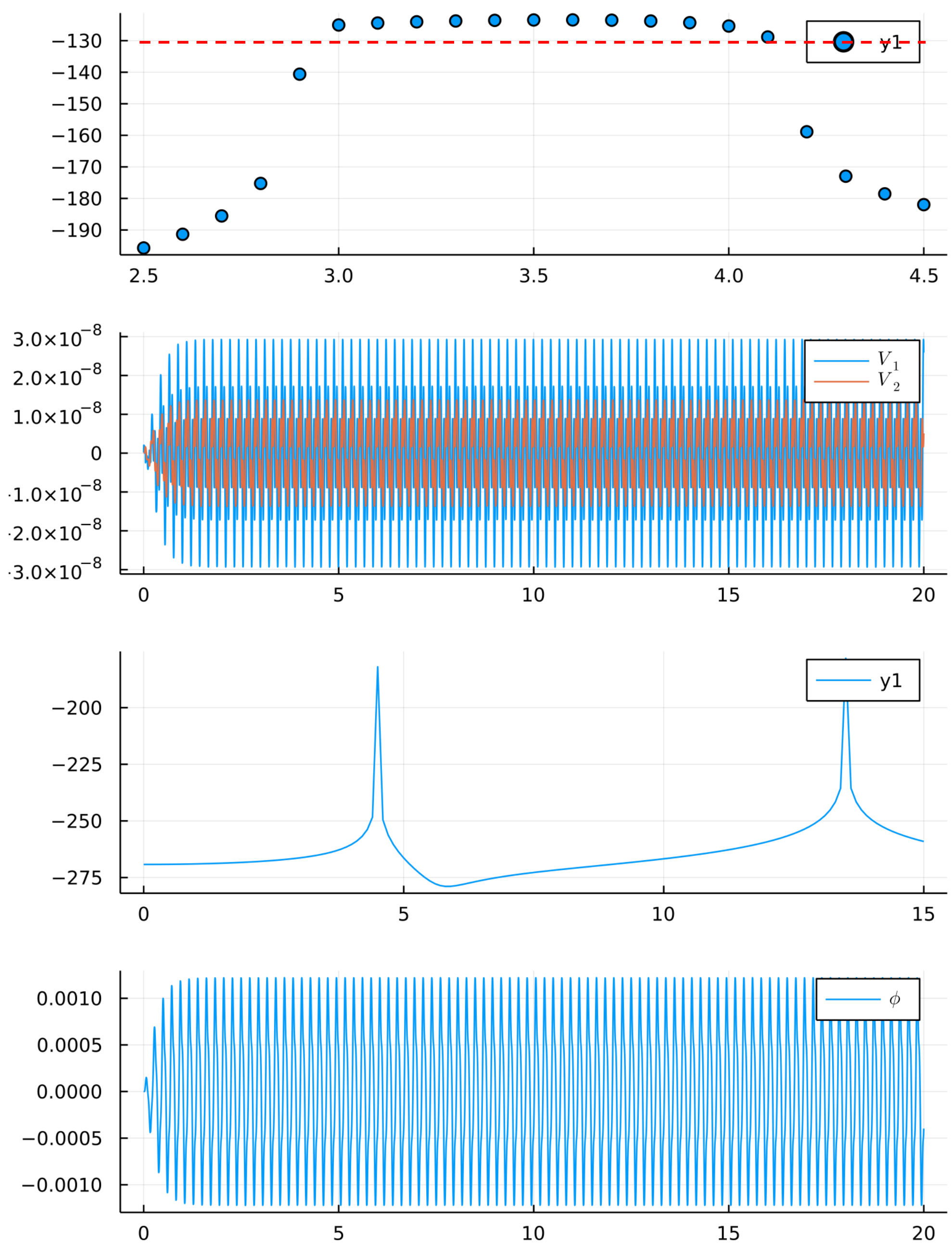
Lambda/4 Resonator frequency = 11.3 GHz   Time delay = 22.12 ps

input RF signal = 0.1 uV = -130.0 dBm

Fstart = 5.0 GHz   Fend = 7.0 GHz Fstep = 100.0 MHz

Magnetic flux amplitude **DC = 0.33 Phi0 RF = 0.25 Phi0**
Frequency = 10.2 GHz

Integration time = 20.0 ns   time step = 0.1 ps
integrating 200000 time steps with 221 lag steps



Reducing furtherthe SQUID critical current: Ic=0.5uA

More pump power is necessary

The resonance frequency is further reduced

fpump=2 fsignal

Zi = 50 Ohm   Cc = 0.015 pF   Z0 = 50 Ohm I0 = 0.5 uA   Cj = 0.5 pF   Rj = 100000 Ohm

Fplasma = 55.12 GHz

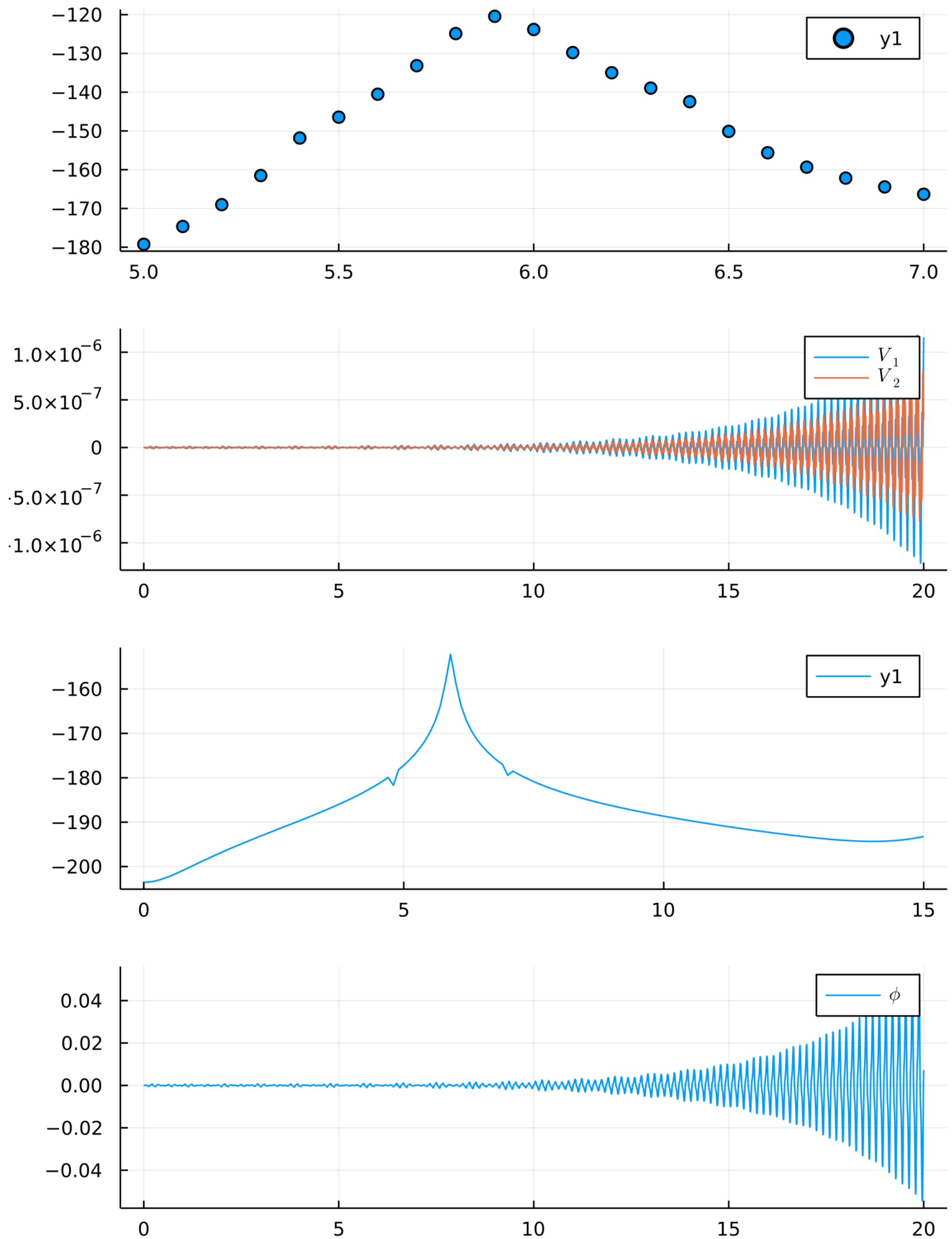
Lambda/4 Resonator frequency = 11.3 GHz   Time delay = 22.12 ps

input RF signal = 0.1 uV = -130.0 dBm

Fstart = 2.5 GHz   Fend = 4.5 GHz Fstep = 100.0 MHz

Magnetic flux **amplitude DC = 0.33 Phi0 RF = 0.3 Phi0**
Frequency = 10.2 GHz

Integration time = 20.0 ns   time step = 0.1 ps
integrating 200000 time steps with 221 lag steps



Back to Ic=1uA

Fix the pump frequency to 11.8 GHz
and sweep the signal frequency

After 20 ns the system is not stationary

Zi = 50 Ohm   Cc = 0.015 pF   Z0 = 50 Ohm
I0 = 1.0 uA   Cj = 0.5 pF   Rj = 100000 Ohm
Fplasma = 77.95 GHz

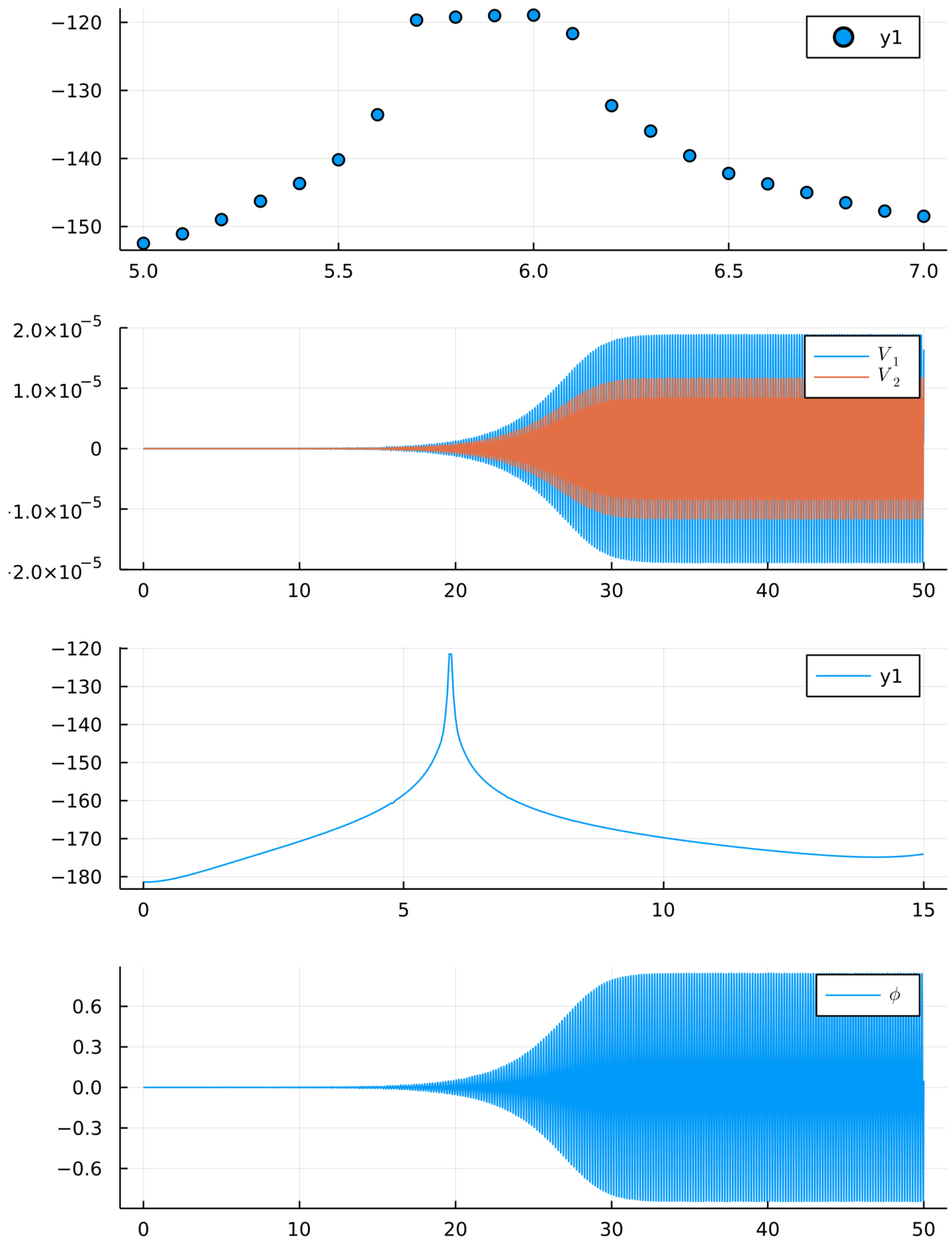
Lambda/4 Resonator frequency = 11.3 GHz   Time delay = 22.12 ps
input RF signal = 0.1 uV = -130.0 dBm

Fstart = 5.0 GHz   Fend = 7.0 GHz Fstep = 100.0 MHz

Magnetic flux amplitude DC = 0.33 Phi0 RF = 0.25 Phi0
Frequency = 11.8 GHz

Integration time = 20.0 ns   time step = 0.1 ps
integrating 200000 time steps with 221 lag steps

Increase to 50 ns



Zi = 50 Ohm   Cc = 0.015 pF   Z0 = 50 Ohm
I0 = 1.0 uA   Cj = 0.5 pF   Rj = 100000 Ohm

Fplasma = 77.95 GHz

Lambda/4 Resonator frequency = 11.3 GHz   Time delay = 22.12 ps
input RF signal = 0.1 uV = -130.0 dBm

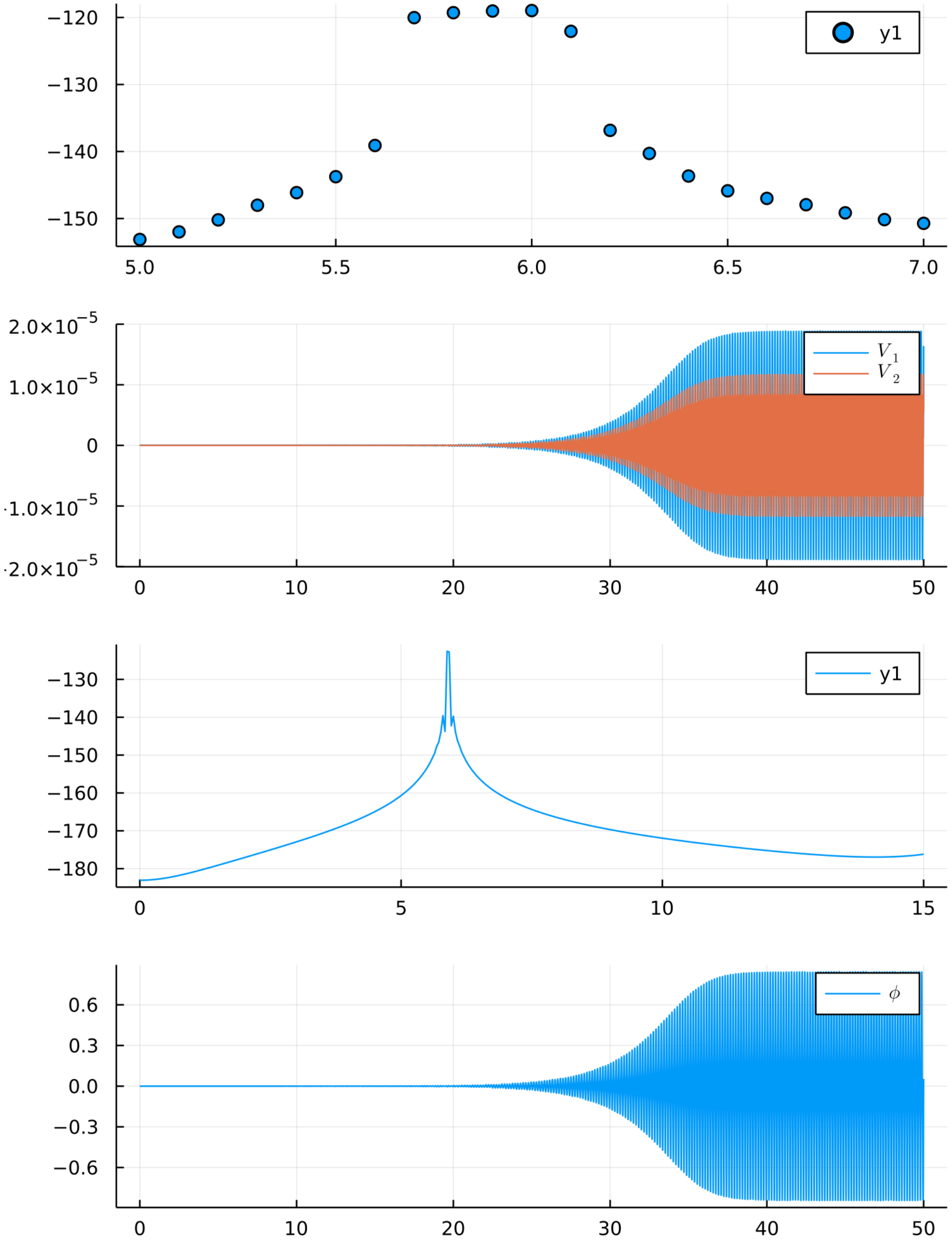
Fstart = 5.0 GHz   Fend = 7.0 GHz Fstep = 100.0 MHz

Magnetic flux amplitude DC = 0.33 Phi0 RF = 0.25 Phi0
Frequency = 11.8 GHz

Integration time = **50.0 ns**   time step = 0.1 ps
integrating 500000 time steps with 221 lag steps

Lowering the signal does not change the result

The system is self oscillating, triggered by the signal



Zi = 50 Ohm   Cc = 0.015 pF   Z0 = 50 OhmI0 = 1.0 uA   Cj = 0.5 pF   Rj = 100000 Ohm

Fplasma = 77.95 GHz
Lambda/4 Resonator frequency = 11.3 GHz   Time delay = 22.12 ps
input RF signal = 0.01 uV **= -150.0 dBm**

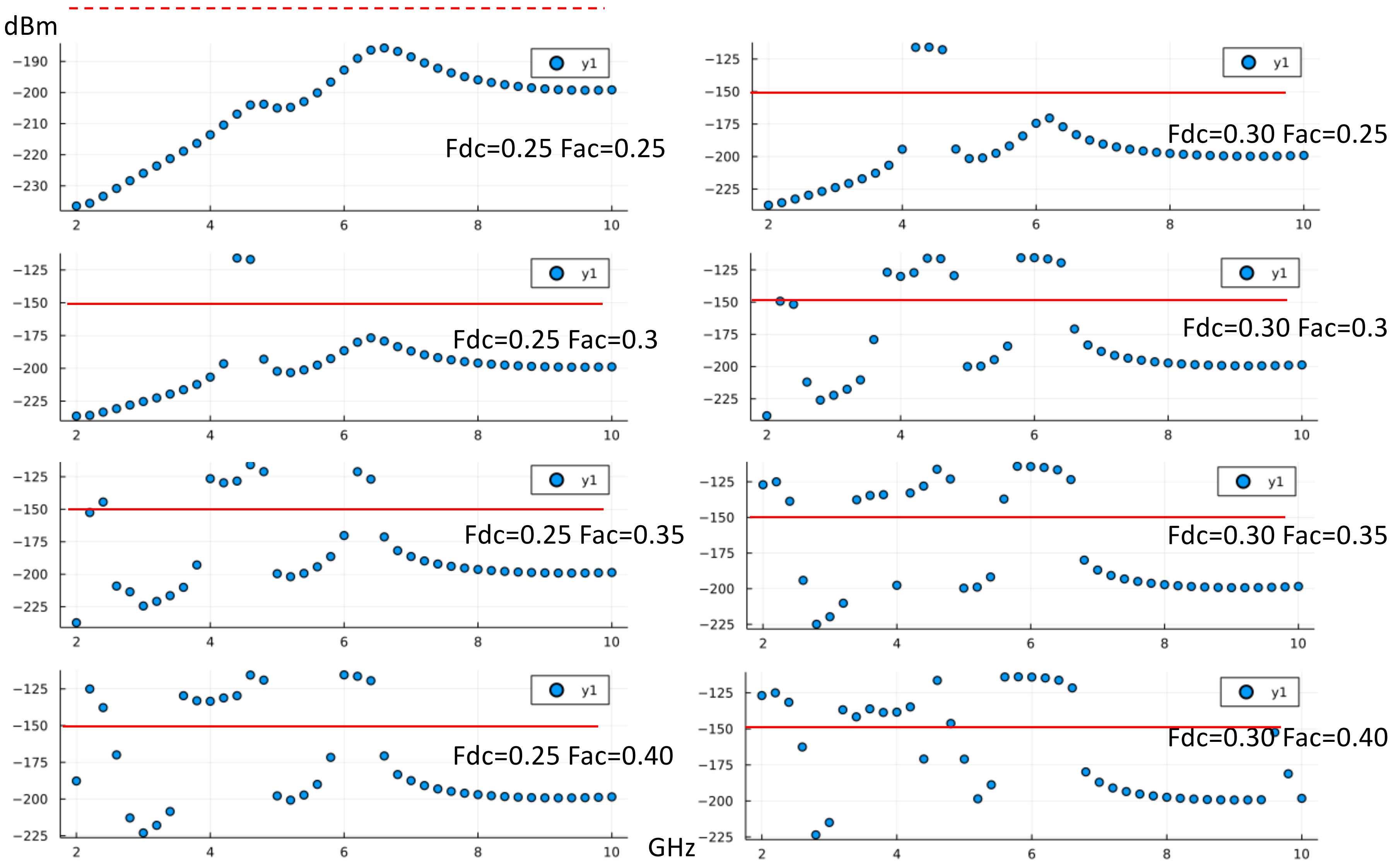
Fstart = 5.0 GHz   Fend = 7.0 GHz Fstep = 100.0 MHz

Magnetic flux amplitude DC = 0.33 Phi0 RF = 0.25 Phi0
Frequency = 11.8 GHz

Integration time = 50.0 ns   time step = 0.1 ps
integrating 500000 time steps with 221 lag steps

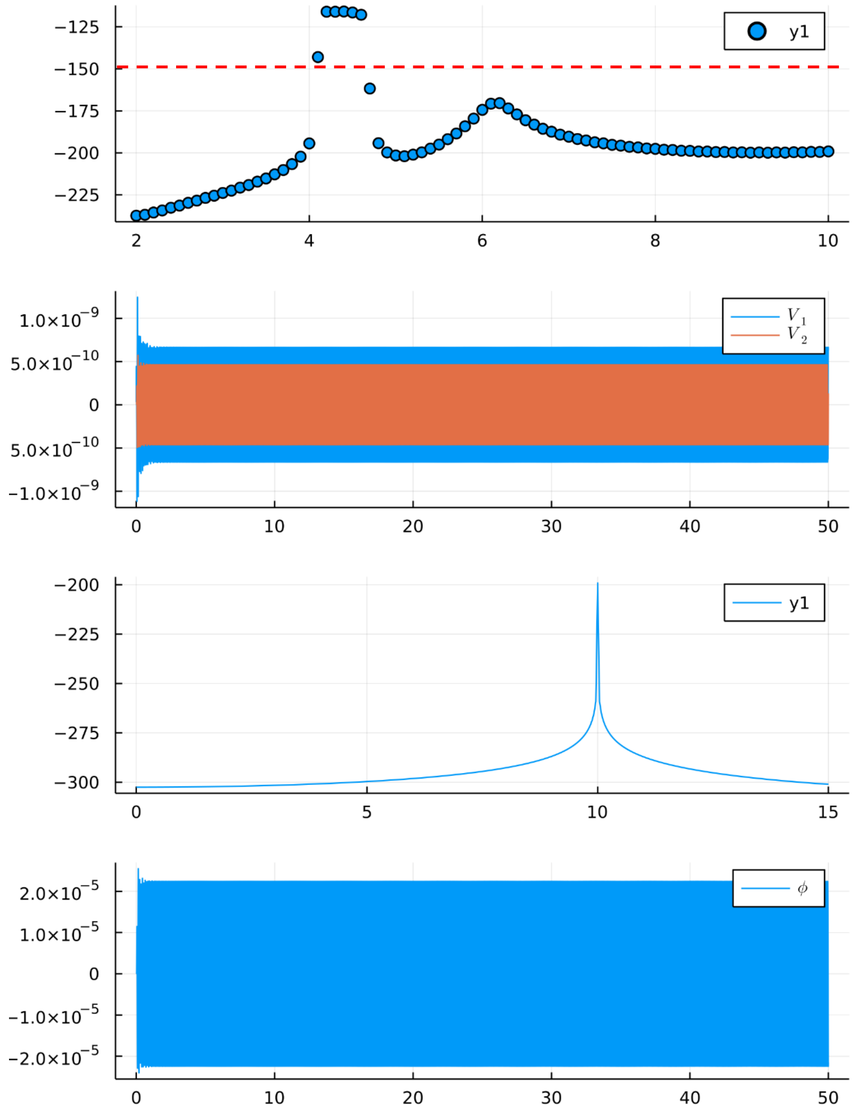
Scan of different dc and rf amplitudes of magnetic flux

Fpump = 2 Fsignal   Asignal = -150dBm



Effect of increasing the dc magnetic flux

Fdc = 0.30



Zi = 50 Ohm Cc = 0.015 pF Z0 = 50 Ohm I0 = 1.0 uA Cj = 0.5 pF Rj = 100000 Ohm

Fplasma = 77.95525848081317 GHz

Lambda/4 Resonator frequency = 11.3 GHz Time delay = 22.123893805309738 ps

input RF signal = 0.01 uV = -150.0 dBm

Fstart = 2.0 GHz Fend = 10.0 GHz Fstep = 100.0 MHz

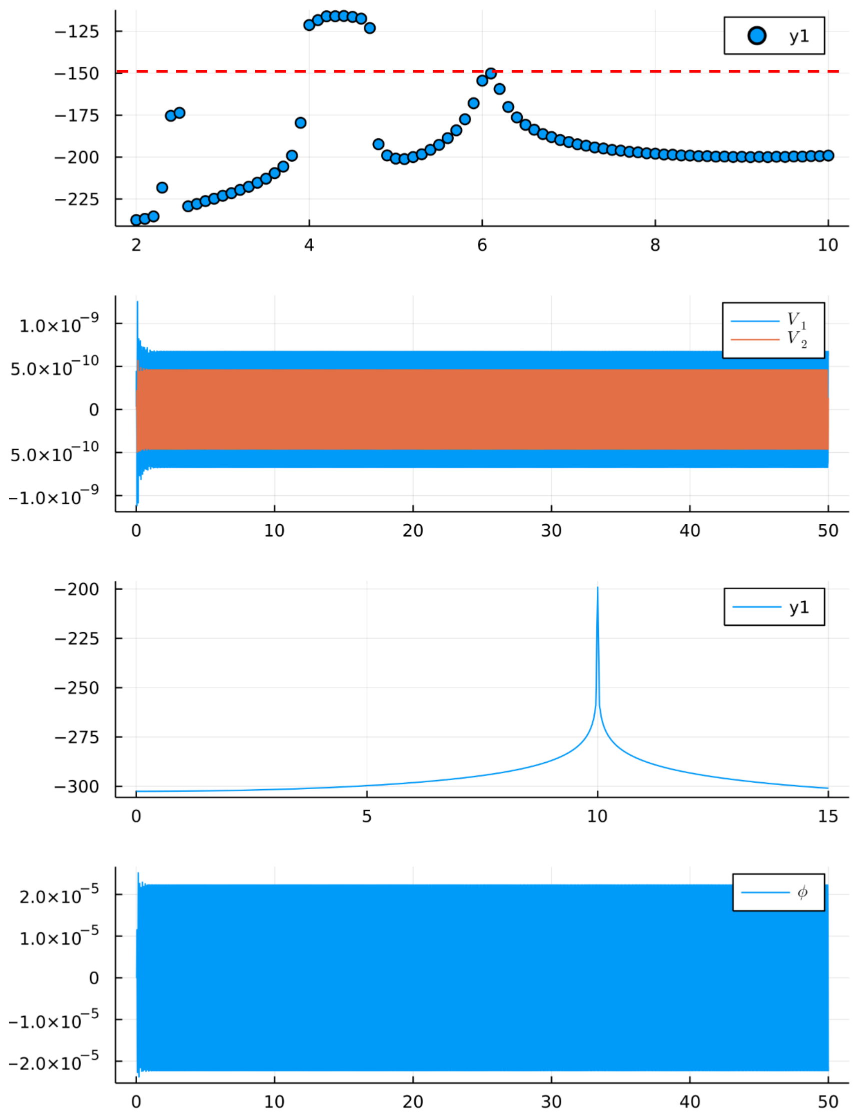
Magnetic flux amplitude DC = 0.30 Phi0 RF = 0.25 Phi0 Frequency = 11.3 GHz

Integration time = 50.0 ns time step = 0.1 ps

integrating 500000 time steps with 221 lag steps

Effect of increasing the dc magnetic flux

Fdc = 0.31



Zi = 50 Ohm Cc = 0.015 pF Z0 = 50 Ohm I0 = 1.0 uA Cj = 0.5 pF Rj = 100000 Ohm

Fplasma = 77.95525848081317 GHz

Lambda/4 Resonator frequency = 11.3 GHz Time delay = 22.123893805309738 ps

input RF signal = 0.01 uV = -150.0 dBm

Fstart = 2.0 GHz Fend = 10.0 GHz Fstep = 100.0 MHz

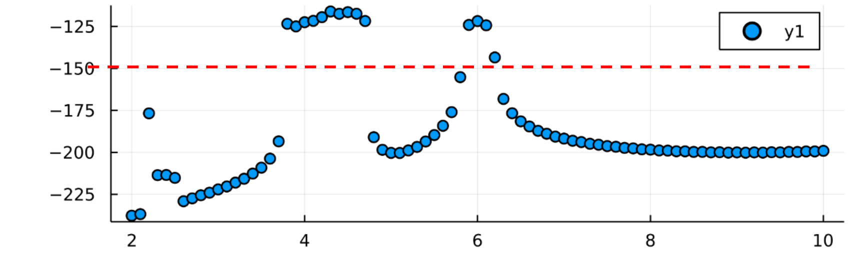
Magnetic flux amplitude DC = 0.31 Phi0 RF = 0.25 Phi0 Frequency = 11.3 GHz

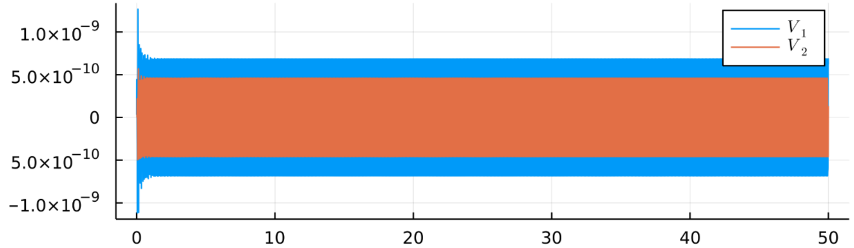
Integration time = 50.0 ns time step = 0.1 ps

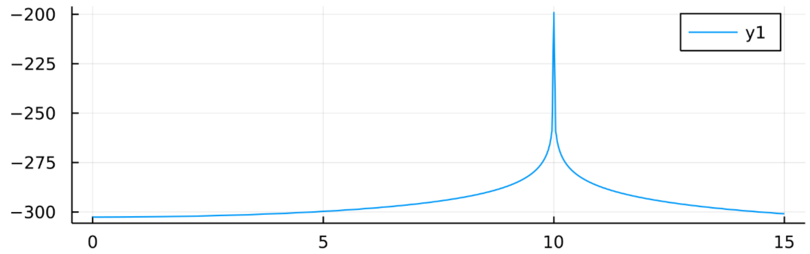
integrating 500000 time steps with 221 lag steps

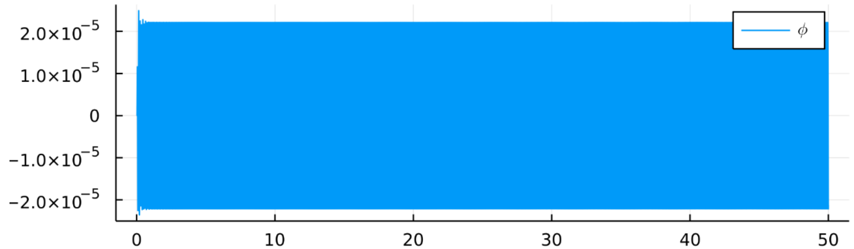
Effect of increasing the dc magnetic flux

Fdc = 0.32









Zi = 50 Ohm Cc = 0.015 pF Z0 = 50 Ohm
I0 = 1.0 uA Cj = 0.5 pF Rj = 100000 Ohm
Fplasma = 77.95 GHz

Lambda/4 Resonator frequency = 11.3 GHz Time delay = 22.123893805309738 ps

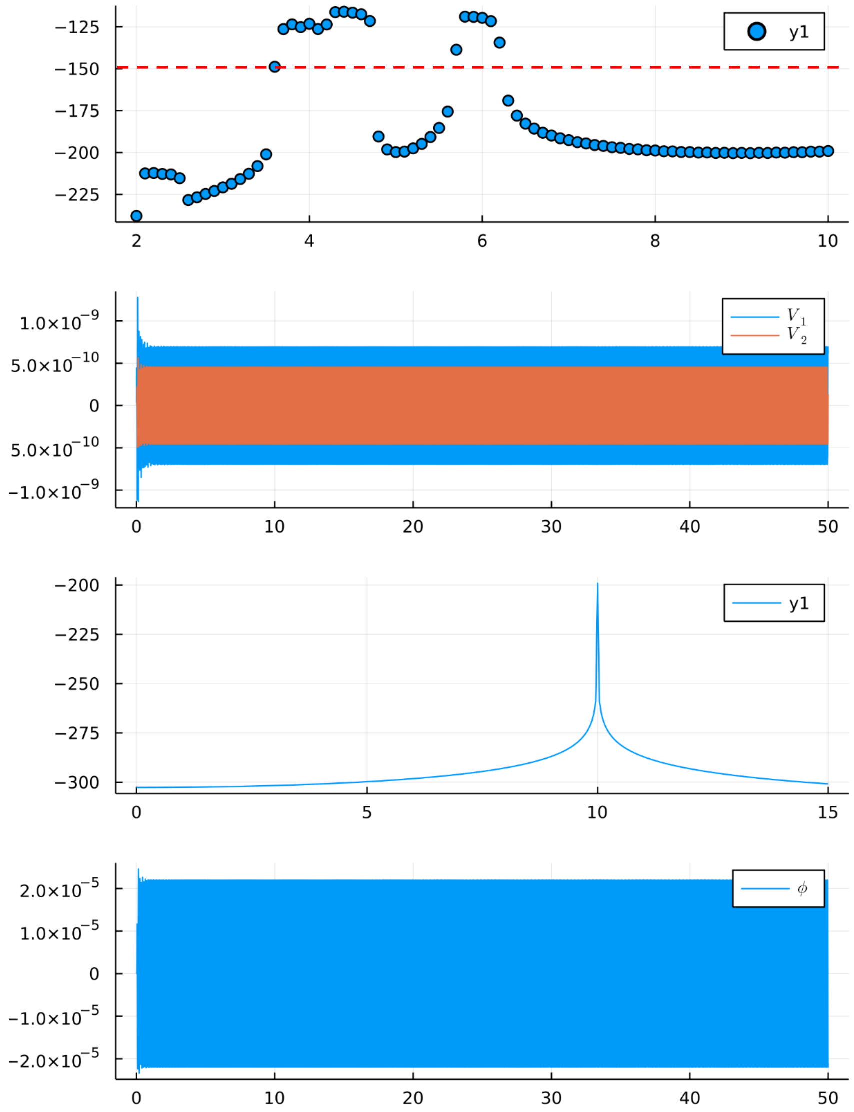
input RF signal = 0.01 uV = -150.0 dBm

Fstart = 2.0 GHz Fend = 10.0 GHz Fstep = 100.0 MHz

Magnetic flux amplitude DC = 0.32 Phi0 RF = 0.25 Phi0 Frequency = 11.3 GHz

Integration time = 50.0 ns time step = 0.1 ps

integrating 500000 time steps with 221 lag steps



Effect of increasing the dc magnetic flux

Fdc = 0.33

Zi = 50 Ohm Cc = 0.015 pF Z0 = 50 Ohm
I0 = 1.0 uA Cj = 0.5 pF Rj = 100000 Ohm

Fplasma = 77.95 GHz

Lambda/4 Resonator frequency = 11.3 GHz Time delay = 22.123893805309738 ps

input RF signal = 0.01 uV = -150.0 dBm

Fstart = 5.0 GHz Fend = 7.0 GHz Fstep = 100.0 MHz

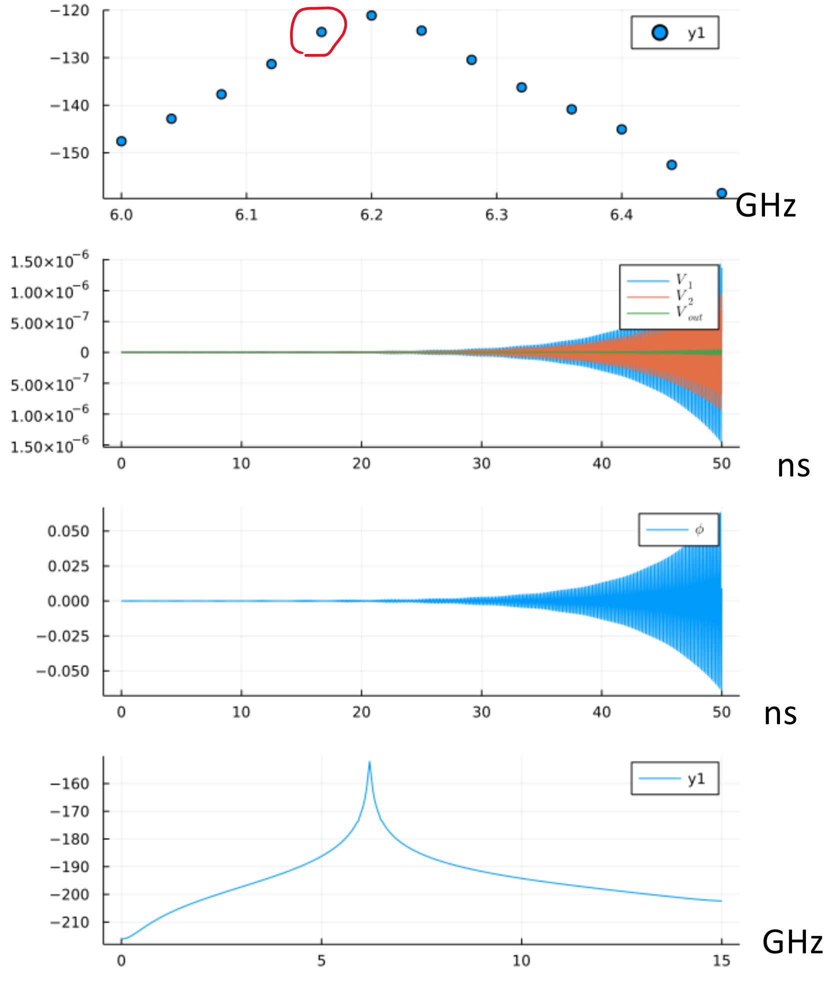
Magnetic flux amplitude DC = 0.33 Phi0 RF = 0.25 Phi0 Frequency = 11.8 GHz

Integration time = 50.0 ns time step = 0.1 ps

integrating 500000 time steps with 221 lag steps

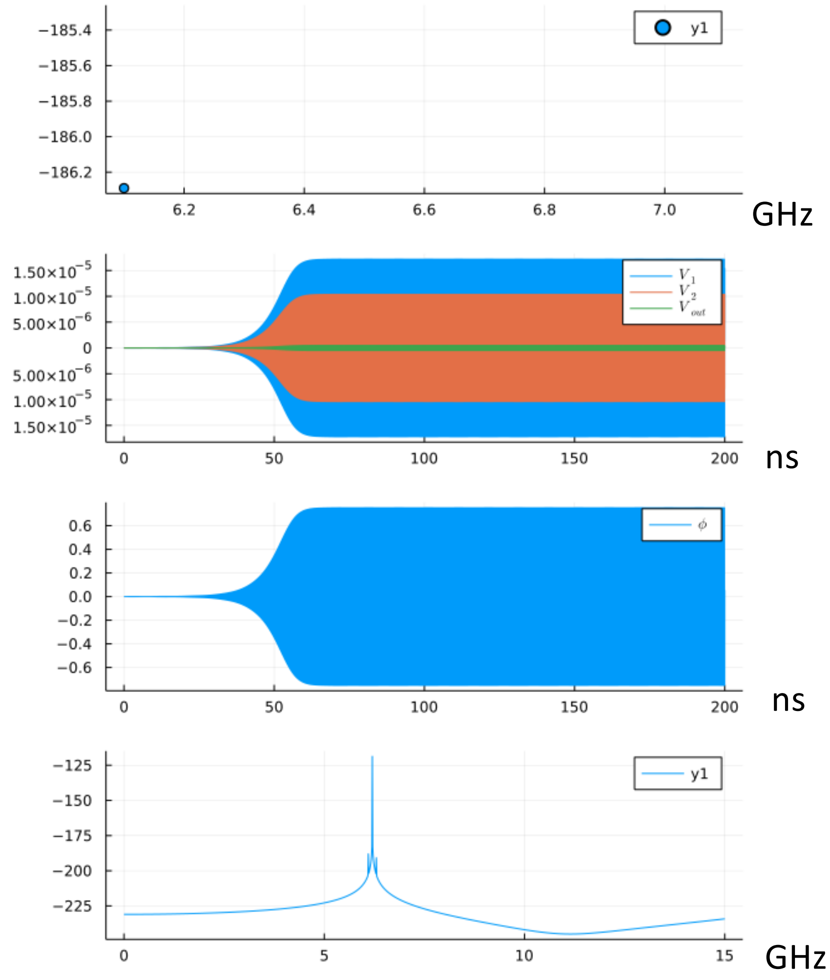
Scan of signal frequency for fixed pump

Scan of signal frequency
Fpump=12.4 GHz   Asignal = -150 dBm



Detail of dynamics

fixed signal frequency at 6.18 GHz
Fpump=12.4 GHz   Asignal = -150 dBm



Conclusions (for now)

1) the numerical model seems to work

2) Extension to a real SQUID is in progress

3) The value of experimental parameters are important (Cc and Io)

4) The JPA simulation show expected responses but not amplifications

5) Self oscillations are triggered by input signal

6) More tests are necessary to fully validate the model

7) The role of circulator needs to be clarified