# Revisiting evolution of GPDs

Valerio Bertone

IRFU, CEA, Université Paris-Saclay

In collaboration with:

R.F. del Castillo, M.G. Echevarria, O. del Rio, and S. Rodini

November 1, 2023, EINN 2023, Paphos

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement № 824093

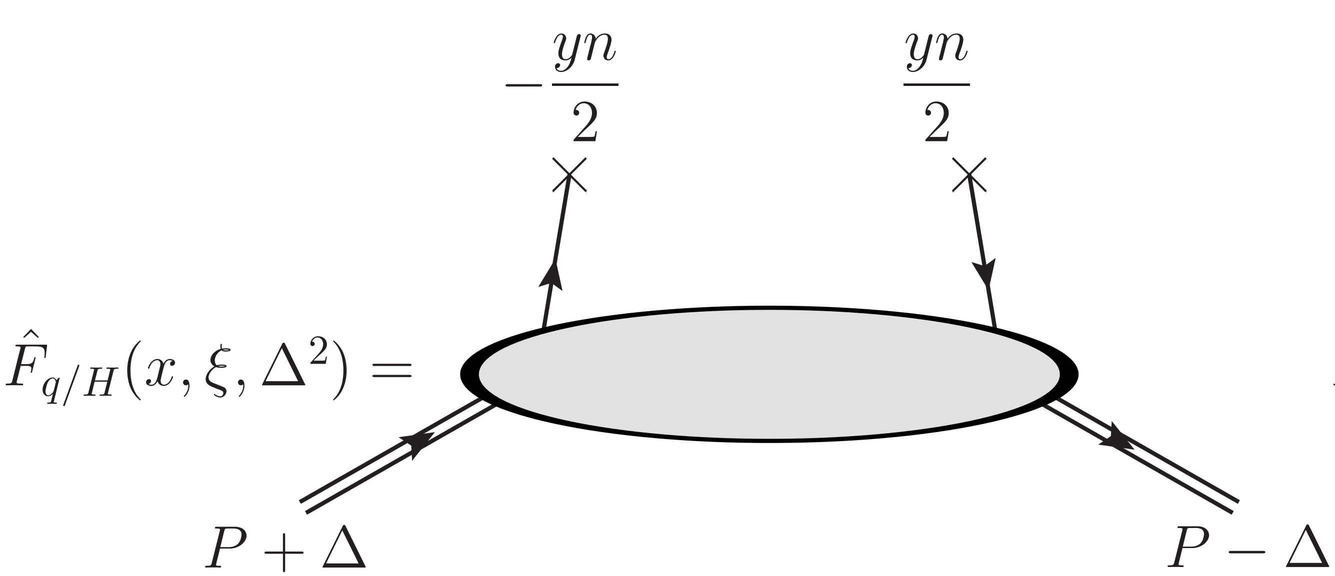
# GPD definition

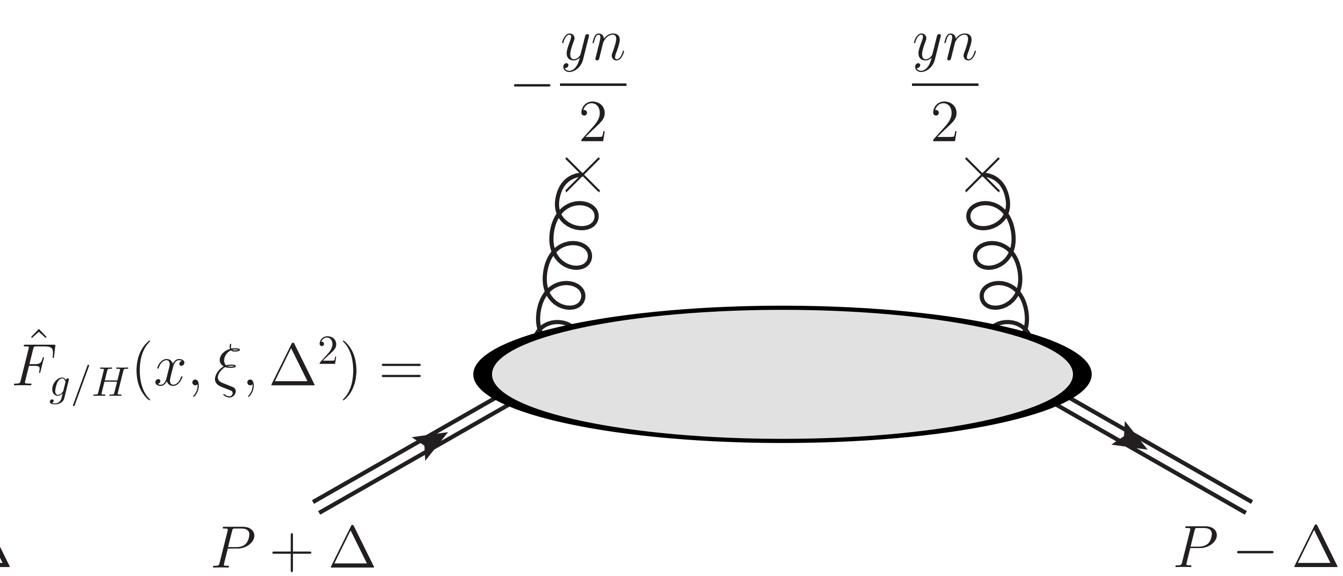
- Generalised parton distributions (GPDs) are a "byproduct" of factorisation of *amplitudes* for **exclusive** processes such as deeply-virtual Compton scattering.
  [Collins, Freund, *Phys.Rev.D* 59 (1999) 074009] [Ji, *Phys.Rev.D* 55 (1997) 7114-7125]

- An operator definition of the GPDs in the **light-cone gauge** ($n \cdot A = 0$) reads:

$$\hat{F}^{ij}_{q/H}(x,\xi,\Delta^2) = \int \frac{dy}{2\pi} e^{-ix(n\cdot P)y} \left\langle P-\Delta \left| \overline{\psi}^i_q\left(\frac{yn}{2}\right) \psi^j_q\left(-\frac{yn}{2}\right) \right| P+\Delta \right\rangle \qquad \xi = \frac{\Delta^+}{P^+}$$

$$\hat{F}^{\mu\nu}_{g/H}(x,\xi,\Delta^2) = \frac{n_\alpha n_\beta}{x(n\cdot P)} \int \frac{dy}{2\pi} e^{-ix(n\cdot P)y} \left\langle P-\Delta \left| F^{\mu\alpha}_a\left(\frac{yn}{2}\right) F^{\nu\beta}_a\left(-\frac{yn}{2}\right) \right| P+\Delta \right\rangle$$



# GPD definition

- GPD correlators are obtained by projection:

$$\hat{F}^{[\Gamma]}_{q/H}(x,\xi,\Delta^2) = \frac{1}{2}\Gamma^{ij}_{q}\hat{F}^{ij}_{q/H}(x,\xi,\Delta^2)$$

$$\hat{F}^{[\Gamma]}_{g/H}(x,\xi,\Delta^2) = \Gamma_{g,\mu\nu}\hat{F}^{\mu\nu}_{g/H}(x,\xi,\Delta^2)$$

- Projectors are parameterised in terms of a basis of four four-vectors:
  - $n$ and $\bar{n}$ parameterise the **longitudinal** directions,
  - $R$ and $L$ parameterise the **transverse** directions,
  - all scalar products are zero except: $(n\bar{n}) = -(RL) = 1$.
  - A typical realisation in Sudakov decomposition is:

$$n^\mu = (0,1,\mathbf{0}_T)\,, \quad \overline{n}^\mu = (1,0,\mathbf{0}_T)\,, \quad R^\mu = \left(0,0,-\frac{1}{\sqrt{2}}(1,i)\right)\,, \quad L^\mu = \left(0,0,-\frac{1}{\sqrt{2}}(1,-i)\right)$$

- The relevant **twist-2** projectors are:

$$\Gamma_q \in \{\not{n}, \not{n}\gamma_5, i\sigma^{\alpha+}\gamma_5\}$$

$$\Gamma^{\mu\nu}_g \in \left\{-g^{\mu\nu}_T \equiv -g^{\mu\nu} + n^\mu\overline{n}^\nu + \overline{n}^\mu n^\nu,\ -i\epsilon^{\mu\nu}_T \equiv -i\epsilon^{\alpha\beta\mu\nu}\overline{n}_\alpha n_\beta,\ -R^\mu R^\nu - L^\mu L^\nu\right\}$$

# GPD definition

- GPD correlators are typically parameterised in terms of **eight** independent GPDs for quarks ($i = q$) and as many for gluons ($i = g$):
  - labelling $\Gamma_{q/g} \in \{U, L, T\}$.

$$\hat{F}^{[U]}_{i/H}(x,\xi,\Delta^2) = \frac{1}{n\cdot P}\overline{u}(P{-}\Delta)\left[\hat{H}_{i/H}(x,\xi,\Delta^2)\frac{\not{n}}{2} + \hat{E}_{i/H}(x,\xi,\Delta^2)\frac{i\sigma^{\mu\nu}n_\mu\Delta_\nu}{4M}\right]u(P{+}\Delta)$$

$$\hat{F}^{[L]}_{i/H}(x,\xi,\Delta^2) = \frac{1}{n\cdot P}\overline{u}(P{-}\Delta)\left[\hat{\tilde{H}}_{i/H}(x,\xi,\Delta^2)\frac{\not{n}\gamma^5}{2} + \hat{\tilde{E}}_{i/H}(x,\xi,\Delta^2)\frac{n^\mu\Delta_\mu\gamma^5}{4M}\right]u(P{+}\Delta)$$

$$\begin{aligned}\hat{F}^{[T]}_{i/H}(x,\xi,\Delta^2) &= \frac{1}{n\cdot P}\overline{u}(P-\Delta)\left[\hat{H}^{[T]}_{i/H}(x,\xi,\Delta^2)\frac{n_\mu\sigma^{\mu j}\gamma^5}{2} + \hat{\tilde{H}}^{[T]}_{i/H}(x,\xi,\Delta^2)\frac{n_\mu\epsilon^{\mu j\alpha\beta}\Delta_\alpha P_\beta}{2M^2}\right.\\ &+ \left.\hat{E}^{[T]}_{i/H}(x,\xi,\Delta^2)\frac{n_\mu\epsilon^{\mu j\alpha\beta}\Delta_\alpha\gamma_\beta}{4M} + \hat{\tilde{E}}^{[T]}_{i/H}(x,\xi,\Delta^2)\frac{n_\mu\epsilon^{\mu j\alpha\beta}P_\alpha\gamma_\beta}{4M}\right]u(P+\Delta)\end{aligned}$$

[Diehl, *Eur.Phys.J.C* 19 (2001) 485-492]

- All the GPDs with the same polarisation label evolve in the same way.

# GPD evolution

- Using dimensional regularisation in $4-2\varepsilon$ dimensions, the **UV** renormalisation of GPDs can be implemented in a multiplicative fashion:

$$F_{i/H}^{[\Gamma]}(x,\xi,\Delta^2;\mu) = \lim_{\varepsilon\to 0}\sum_{j=q,g}\int_{-1}^{1}\frac{dy}{|y|}Z_{ij}^{[\Gamma]}\left(\frac{x}{y},\frac{\xi}{x},\alpha_s(\mu),\varepsilon\right)\hat{F}_{j/H}^{[\Gamma]}(y,\xi,\Delta^2;\varepsilon,\mu^{-\varepsilon})$$

- In the $\overline{\text{MS}}$ scheme, renormalisation constants have the following structure:

$$Z_{ij}^{[\Gamma]}(z,\kappa,\alpha_s,\varepsilon) = \delta_{ij}\delta(1-z) + \sum_{n=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^n\sum_{p=1}^{n}\frac{1}{\varepsilon^p}Z_{ij}^{[\Gamma],[n,p]}(z,\kappa)$$

- Exploiting the independence of the bare GPDs on $\mu$ (for $\varepsilon\to 0$), one can derive a **RGE** governing the evolution of renormalised GPDs w.r.t. $\mu$:

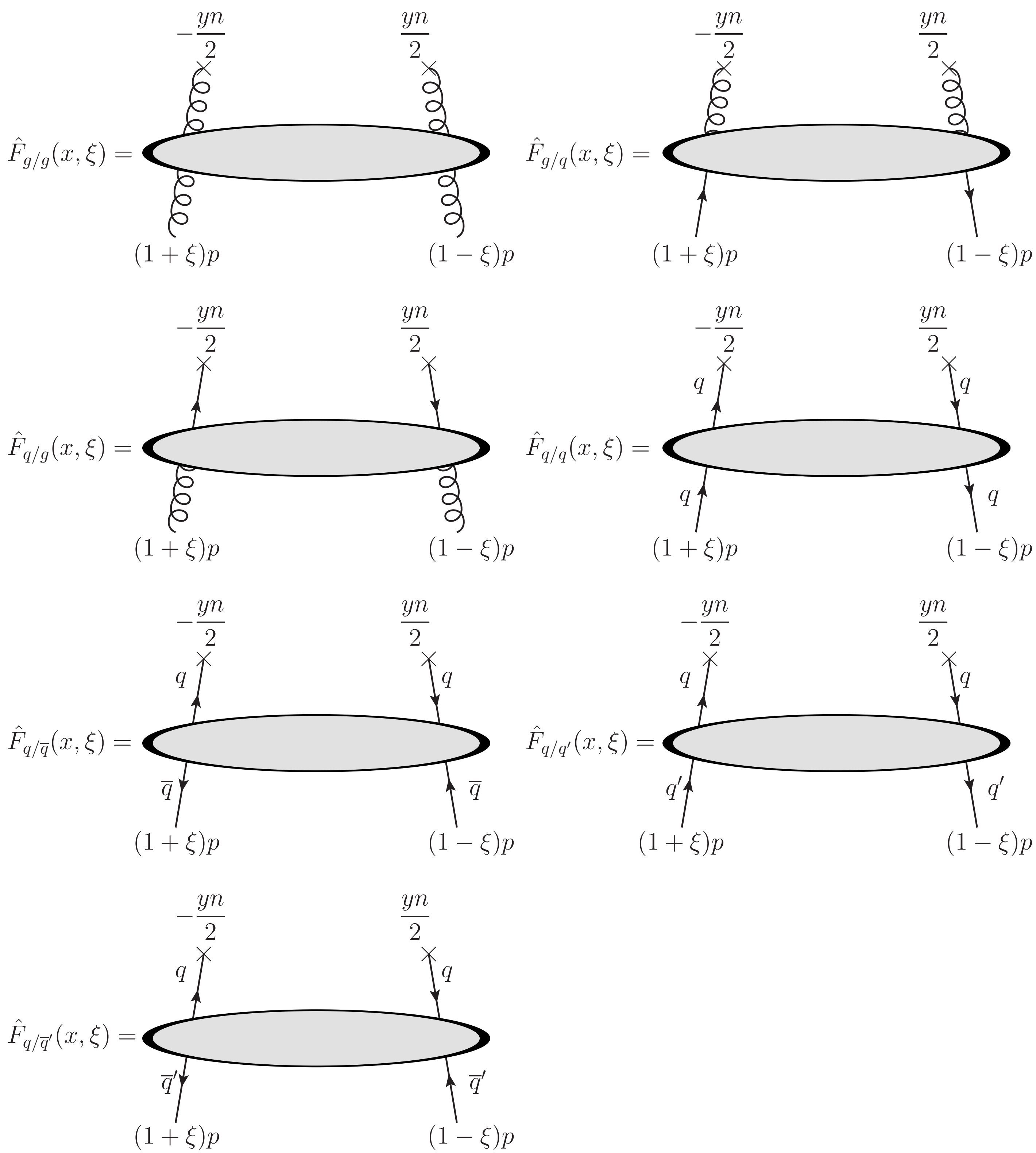
$$\frac{dF_{i/H}^{[\Gamma]}(x,\xi,\Delta^2;\mu)}{d\ln\mu^2} = \sum_{k=q,g}\int_{-1}^{1}\frac{dz}{|z|}\mathcal{P}_{ik}^{[\Gamma]}\left(\frac{x}{z},\frac{\xi}{x},\alpha_s(\mu)\right)F_{k/H}^{[\Gamma]}(z,\xi,\Delta^2;\mu)$$

- The evolution kernels $\mathscr{P}$ are related to the normalisation constants $Z$:

$$\mathcal{P}_{ik}^{[\Gamma]}\left(\frac{x}{z},\frac{\xi}{x},\alpha_s\right) = \lim_{\varepsilon\to 0}\sum_{j}\int_{-1}^{1}\frac{dy}{|y|}\frac{dZ_{ij}^{[\Gamma]}\left(\frac{x}{y},\frac{\xi}{x},\alpha_s,\varepsilon\right)}{d\ln\mu^2}Z_{jk}^{[\Gamma]-1}\left(\frac{y}{z},\frac{\xi}{y},\alpha_s,\varepsilon\right)$$

# Parton-in-parton GPDs

- The renormalisation constants $Z$ are extracted by means of **parton-in-parton** GPDs, *i.e.* GPDs where the *hadronic* states are replaced by *partonic* states.




- Dependence on $\Delta^2$ can be neglected at twist-2.

# Parton-in-parton GPDs

- In light-cone gauge:

$$\hat{F}^{[\Gamma]}_{g/g,q}(x,\xi) = \frac{(n\cdot p)(x^2-\xi^2)}{2(N_c^2-1)x}\int\frac{dy}{2\pi}e^{-ix(n\cdot p)y}\left\langle(1-\xi)p,s'\left|A^\mu_a\left(\frac{yn}{2}\right)\Gamma_{g,\mu\nu}A^\nu_a\left(-\frac{yn}{2}\right)\right|(1+\xi)p,s\right\rangle_{g,q}\Lambda^{[\Gamma]}_{s's}$$

$$\hat{F}^{[\Gamma]}_{q/g,q,\overline{q},q',\overline{q}'}(x,\xi) = \frac{1}{2N_c}\int\frac{dy}{2\pi}e^{-ix(n\cdot p)y}\left\langle(1-\xi)p,s'\left|\overline{\psi}^i_q\left(\frac{yn}{2}\right)\Gamma^{ij}_q\psi^j_q\left(-\frac{yn}{2}\right)\right|(1+\xi)p,s\right\rangle_{g,q,\overline{q},q',\overline{q}'}\Lambda^{[\Gamma]}_{s's}$$

- The projectors $\Lambda_{s's}$ are introduced for convenience to project out the physical partonic spin/helicity states that contribute to the amplitude:

$$\Lambda^{[\Gamma]}_{s's}\overline{u}_{q,s'}((1-\xi)p)u_{q,s}((1+\xi)p) = \Lambda^{[\Gamma]}_q = \sqrt{1-\xi^2}\ \{\not{p},\not{p}\gamma^5, i\sigma^{\mu\nu}P_\nu\gamma^5\}$$

$$\Lambda^{[\Gamma]}_{s's}e^{\mu *}_{s'}((1-\xi)p)e^\nu_s((1+\xi)p) = \Lambda^{[\Gamma]\mu\nu}_g = \{-g^{\mu\nu}_T, -i\varepsilon^{\mu\nu}_T, -R^\mu R^\nu - L^\mu L^\nu\}$$

$$\Gamma\in\{U,L,T\}$$

# Evolution kernels at one loop

- The general structure is for **all channels**:

$$\mathcal{P}_{ij}^{[\Gamma],[0]}(y,\kappa) = \theta(1-y)\left[\theta(1+\kappa)p^{\Gamma}_{i/j}(y,\kappa)+\theta(1-\kappa)p^{\Gamma}_{i/j}(y,-\kappa)\right]$$

$$+ \delta_{ij}\delta(1-y)C_i\left[K_i - \ln\left(\left|1-\kappa^2\right|\right) - 2\int_0^1 \frac{dz}{1-z}\right] \qquad \kappa = \frac{\xi}{x}$$

- with $C_q = C_F$ and $C_g = C_A$, and:

$$K_q = \frac{3}{2} \qquad K_g = \frac{11C_A - 4n_f T_R}{6C_A}$$

- In [Eur. Phys. J. C 82 (2022) 10, 888] we have computed the full set of $p^U_{i/j}$:

$$p^U_{q/q}\left(x,\frac{\xi}{x}\right) = C_F\frac{(x+\xi)(1-x+2\xi)}{\xi(1+\xi)(1-x)}$$

$$p^U_{q/g}\left(x,\frac{\xi}{x}\right) = T_R\frac{(x+\xi)(1-2x+\xi)}{\xi(1+\xi)(1-\xi^2)}$$

$$p^U_{g/q}\left(x,\frac{\xi}{x}\right) = C_F\frac{(x+\xi)(2-x+\xi)}{\xi x(1+\xi)}$$

$$p^U_{g/g}\left(x,\frac{\xi}{x}\right) = -C_A\frac{x^2-\xi^2}{\xi x(1-\xi^2)}\left[1-\frac{2\xi}{1-x}-\frac{2(1+x^2)}{(x-\xi)(1+\xi)}\right]$$

# Evolution kernels at one loop

- We have computed these functions also in the **longitudinally polarised** case:

$$p^L_{q/q}\left(x, \frac{\xi}{x}\right) = -C_F \frac{(x+\xi)(x-1-2\xi)}{(1+\xi)\xi(1-x)}$$

$$p^L_{q/g}\left(x, \frac{\xi}{x}\right) = -2n_f T_R \frac{x+\xi}{\xi(1+\xi)^2}$$

$$p^L_{g/q}\left(x, \frac{\xi}{x}\right) = C_F \frac{(x+\xi)^2}{x\xi(1+\xi)}$$

$$p^L_{g/g}\left(x, \frac{\xi}{x}\right) = \frac{C_A(\xi+x)\left(-\xi^2(2\xi+1)+\xi+(\xi-3)x^2+\left(\xi^2+3\right)x\right)}{(1-\xi^2)\xi(1+\xi)(1-x)x}$$

- and in the **transversely polarised** case:

$$p^T_{q/q}\left(x, \frac{\xi}{x}\right) = 2C_F \frac{x+\xi}{(1+\xi)(1-x)}$$

$$p^T_{q/g}\left(x, \frac{\xi}{x}\right) = p^T_{g/q}\left(x, \frac{\xi}{x}\right) = 0$$

$$p^T_{g/g}\left(x, \frac{\xi}{x}\right) = 2C_A \frac{(x+\xi)^2}{(1+\xi)^2(1-x)x}$$

- Paper in preparation.

# Evolution equations

- Defining the **anti-quark** distributions as:

$$F^{[U,T]}_{\bar{q}/H}(x,\xi,\Delta^2;\mu) = -F^{[U,T]}_{q/H}(-x,\xi,\Delta^2;\mu)$$

$$F^{[L]}_{\bar{q}/H}(x,\xi,\Delta^2;\mu) = +F^{[L]}_{q/H}(-x,\xi,\Delta^2;\mu)$$

- one can construct **non-singlet** and **singlet** combinations:

$$F^{[\Gamma],-} = F^{[\Gamma]}_{q/H} - F^{[\Gamma]}_{\bar{q}/H} \qquad F^{[\Gamma],+} = \begin{pmatrix} \sum_{q=1}^{n_f} F^{[\Gamma]}_{q/H} + F^{[\Gamma]}_{\bar{q}/H} \\ F^{[\Gamma]}_{g/H} \end{pmatrix}$$

- The evolution equations **decouple** and can be written in a **DGLAP-like** fashion:

$$\frac{dF^{[\Gamma],\pm}(x,\xi,\mu)}{d\ln\mu^2} = \frac{\alpha_s(\mu)}{4\pi}\int_x^\infty \frac{dy}{y}\mathcal{P}^{[\Gamma]\pm,[0]}\left(y,\frac{\xi}{x}\right)F^{[\Gamma],\pm}\left(\frac{x}{y},\xi,\mu\right)$$

$$\mathcal{P}^{[\Gamma]\pm,[0]}\left(y,\frac{\xi}{x}\right) = \theta(1-y)\mathcal{P}_1^{[\Gamma]\pm,[0]}\left(y,\frac{\xi}{x}\right) + \theta(\xi-x)\mathcal{P}_2^{[\Gamma]\pm,[0]}\left(y,\frac{\xi}{x}\right)$$

DGLAP region — ERBL contribution

- $\mathscr{P}^{[\Gamma]\pm,[0]}_{1,2}$ are appropriate combinations of the functions $p^{\Gamma}_{i/j}$ presented before.

# Numerical setup

- The evolution kernels for *all polarisations* are now implemented in **APFEL++** and will soon be available through **PARTONS** allowing for LO GPD evolution in momentum space.

- We achieved a stable numerical implementation over the full range $\mathbf{0 \leq \xi \leq 1}$:

  - numerical check that both the **DGLAP** and **ERBL** limits are recovered,

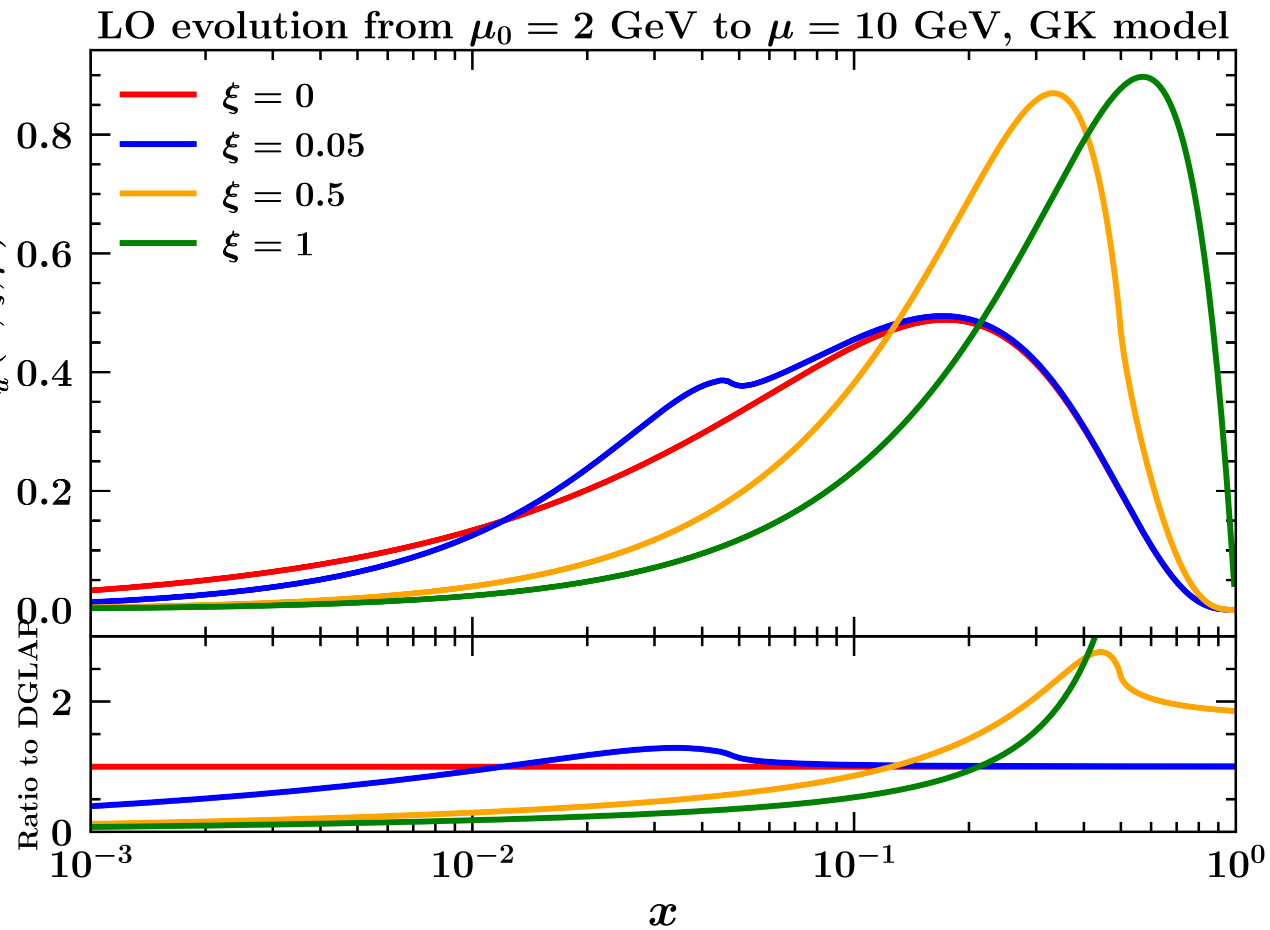
  - numerical check of **polynomiality** conservation.

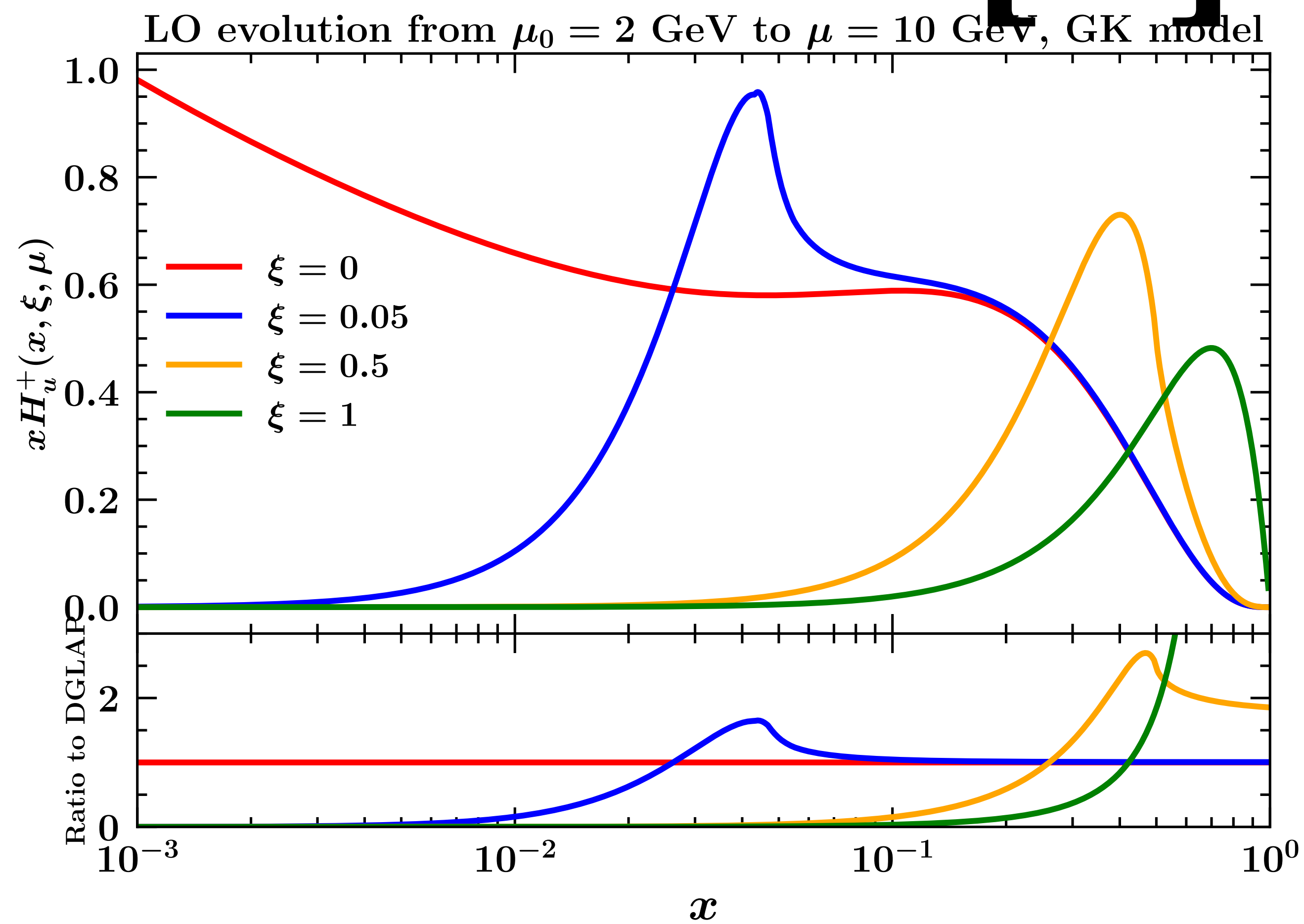
- Numerical tests use the *realistic* Goloskokov-Kroll (GK) model for proton GPDs [*Eur. Phys. J. C* 53 (2008) 367-384] as implemented in **PARTONS** as an initial-scale set of distributions:

  - we consistently used $H_{i/H}$ for unpolarised, $\widetilde{H}_{i/H}$ for longitudinally polarised, and $H^T_{i/H}$ for transversely polarised evolution.

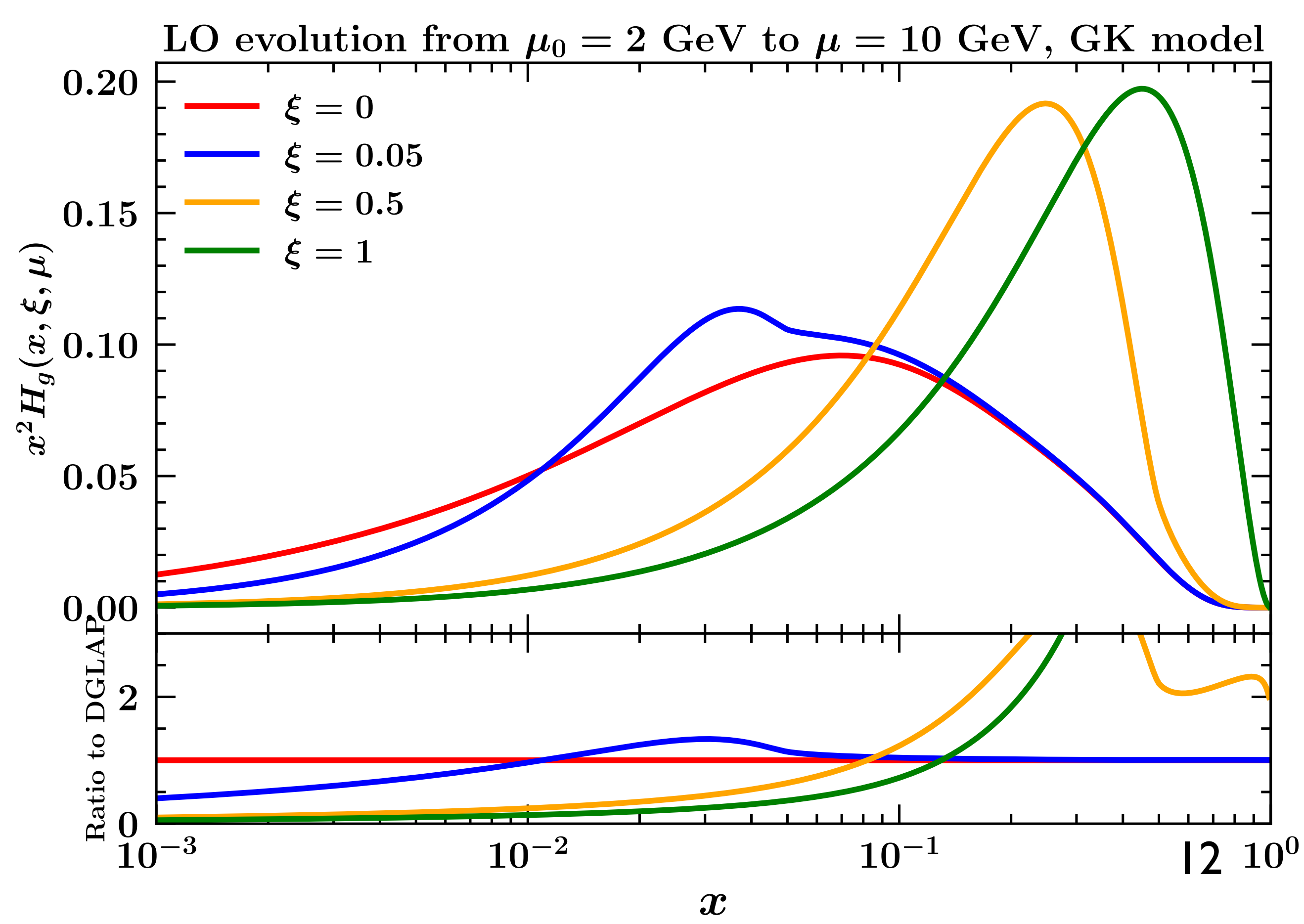
  - GPDs are evolved from 2 to 10 GeV in the **variable-flavour-number scheme**, *i.e.* accounting for heavy-quark-threshold crossing, at $\Delta^2 = -0.1$ GeV$^2$.

# Evolution and DGLAP limit [U]

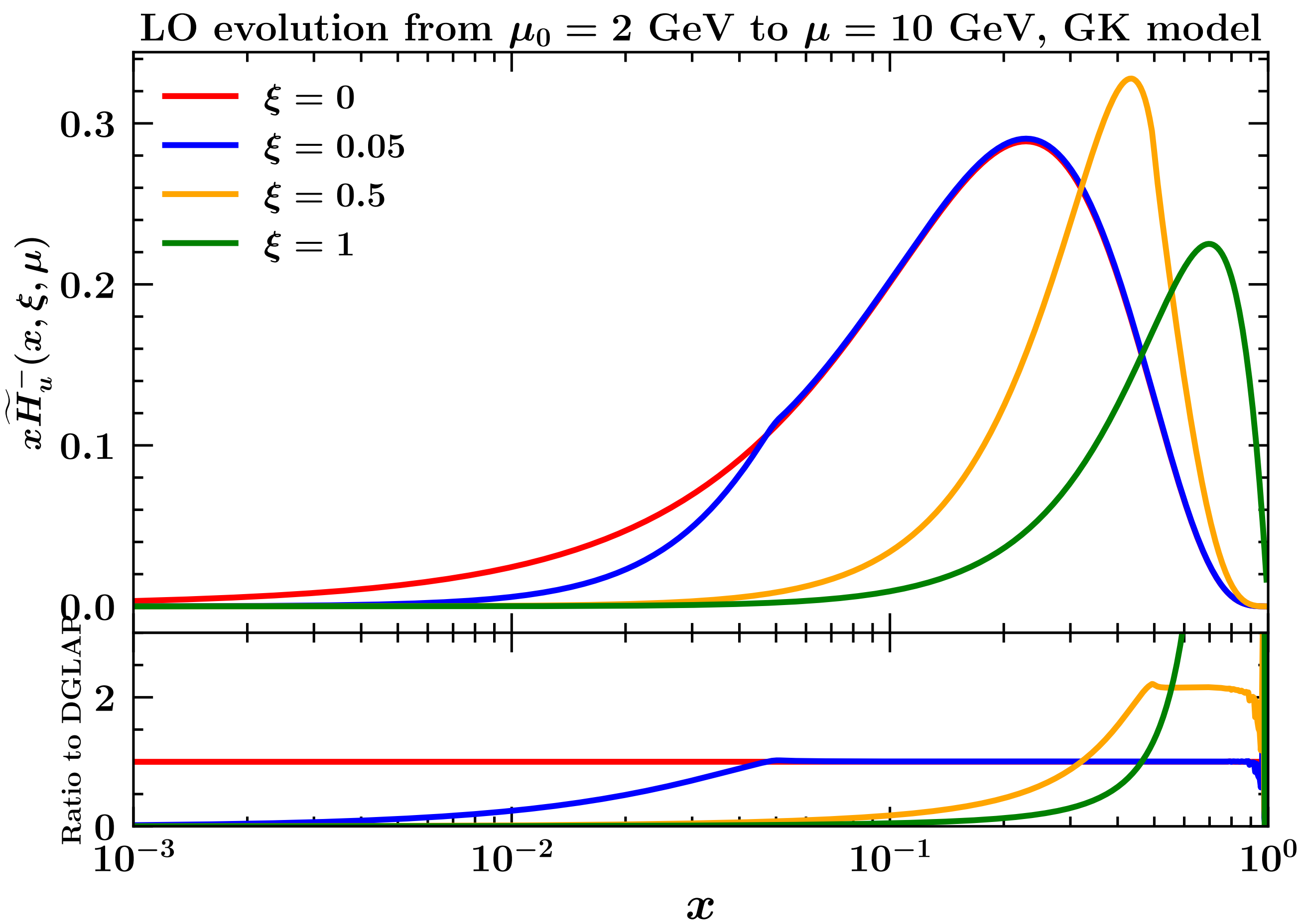


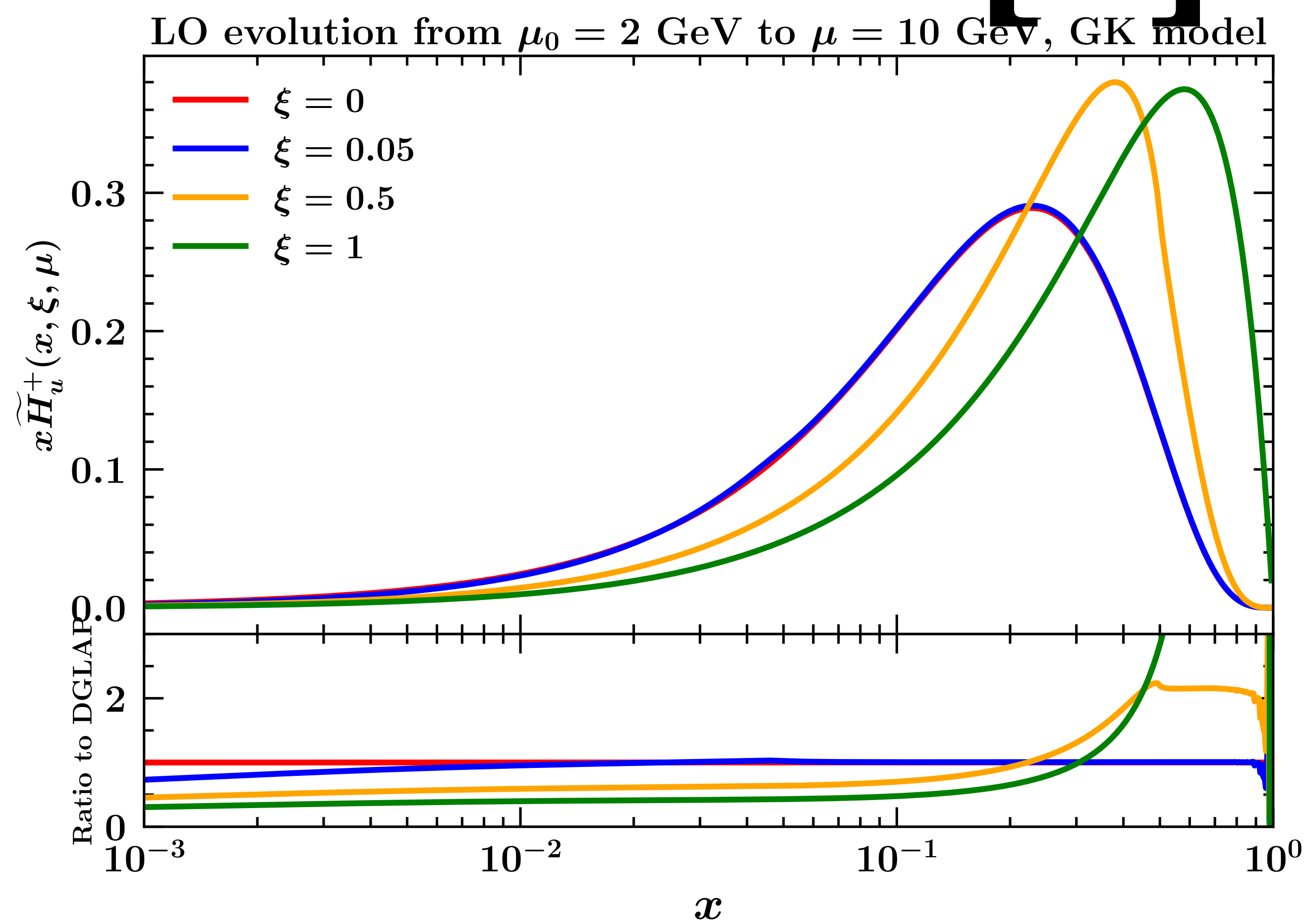




- **DGLAP limit** reproduced within $10^{-5}$ relative accuracy.
- GPD evolution may significantly deviate from DGLAP evolution.
- The evolution generates a cusp at $x = \xi$ but the distribution remains **continuous** at this point.

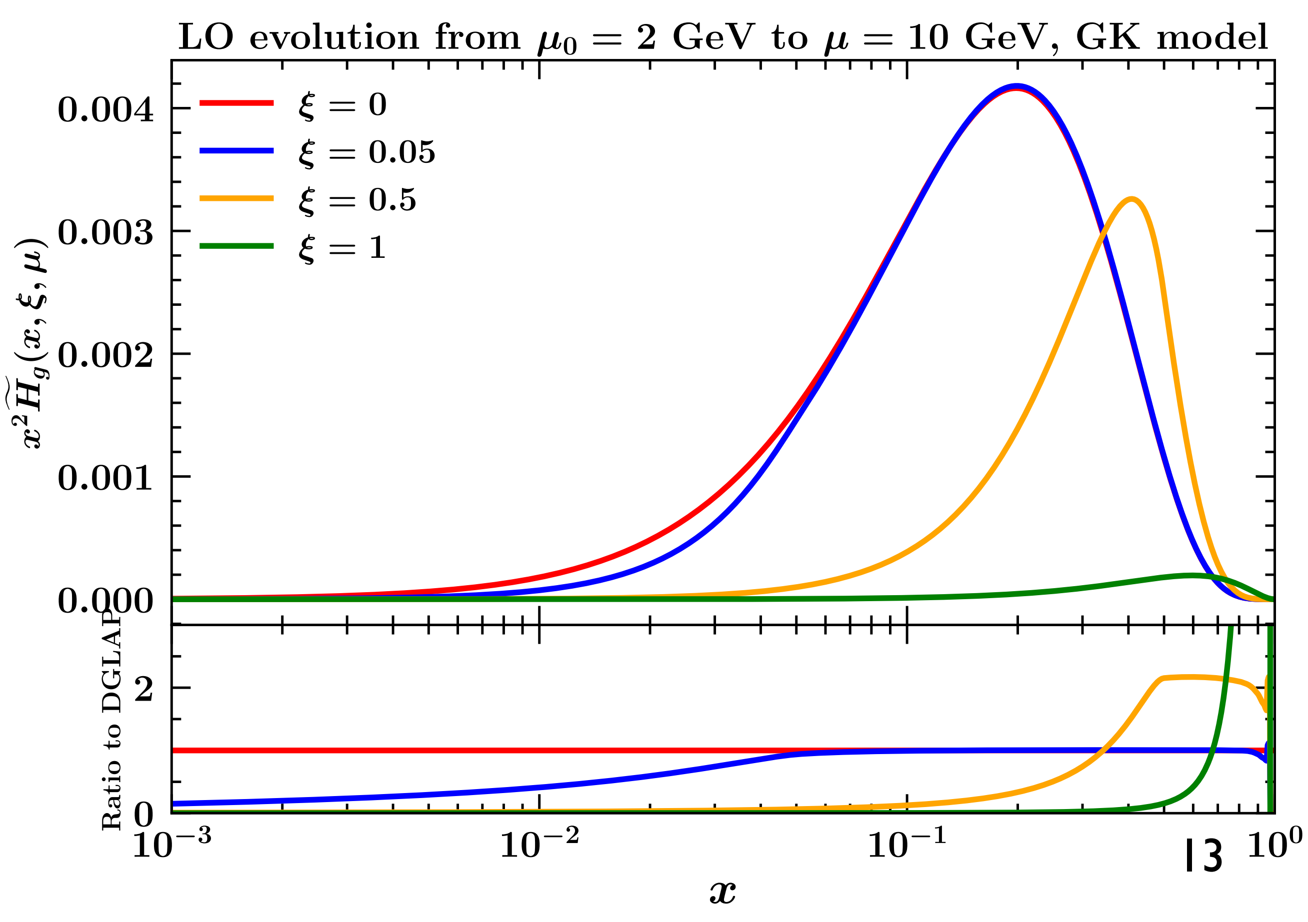


# Evolution and DGLAP limit [L]

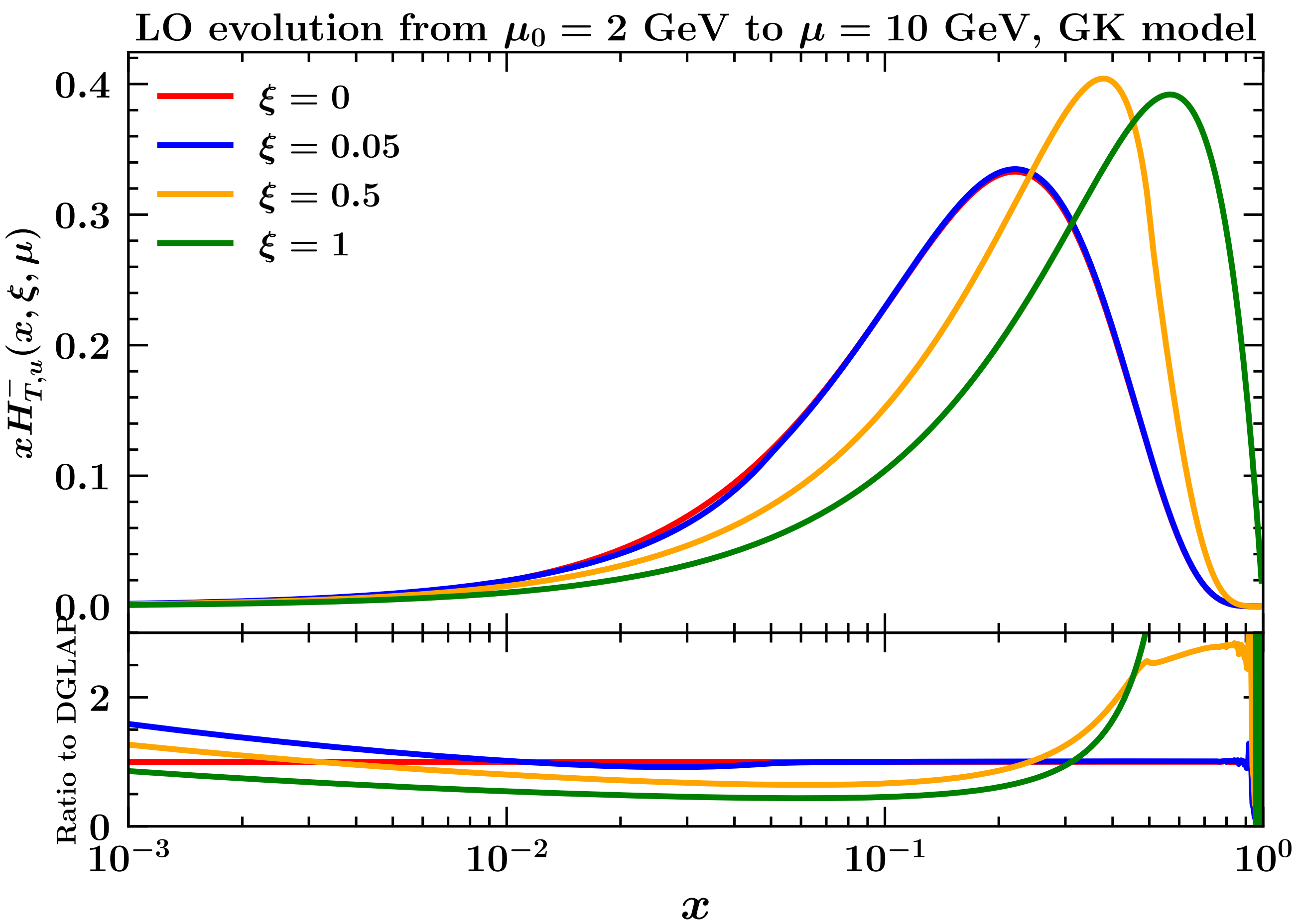


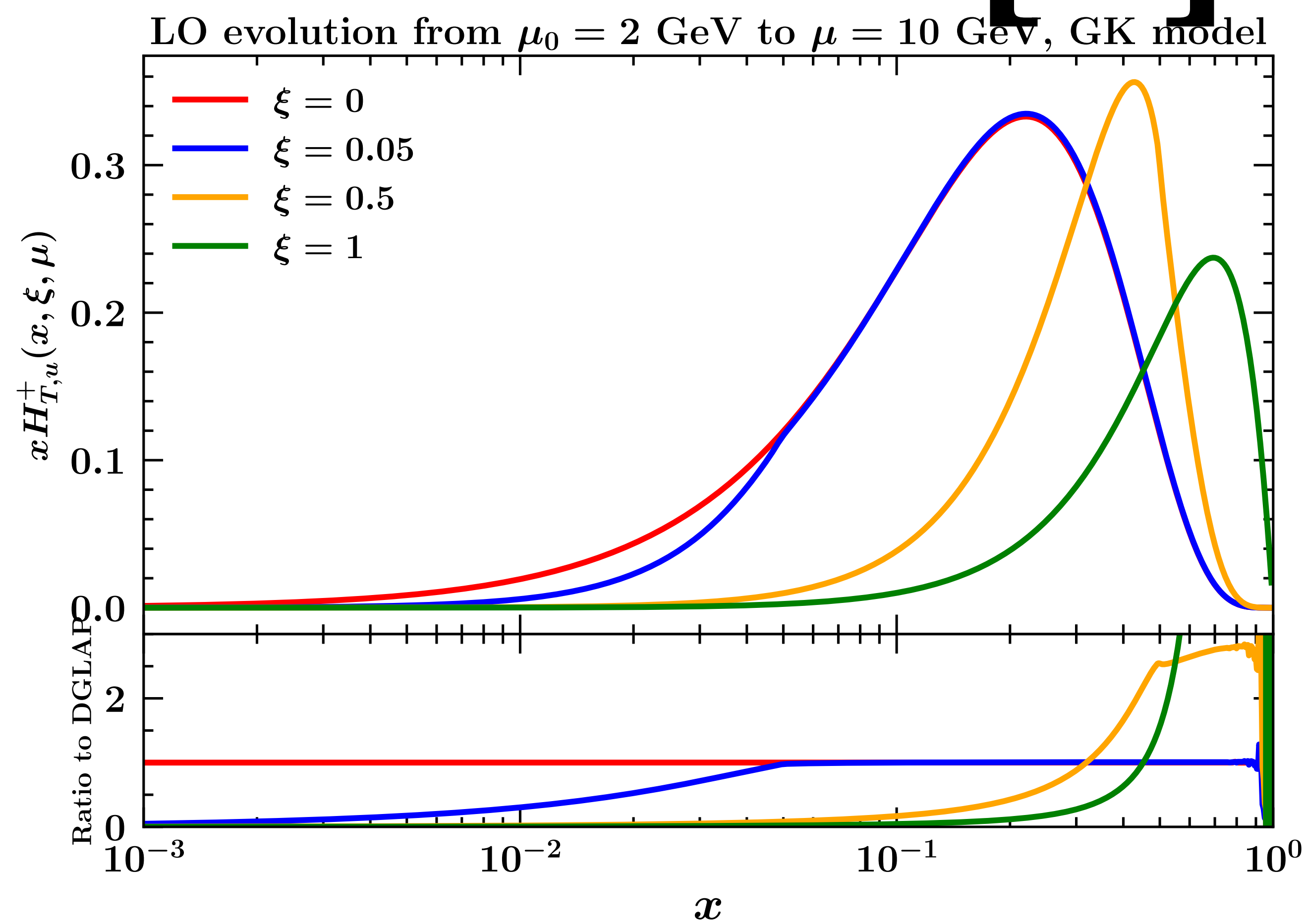




- **DGLAP limit** reproduced within $10^{-5}$ relative accuracy.
- GPD evolution may significantly deviate from DGLAP evolution.
- The evolution generates a cusp at $x = \xi$ but the distribution remains **continuous** at this point.

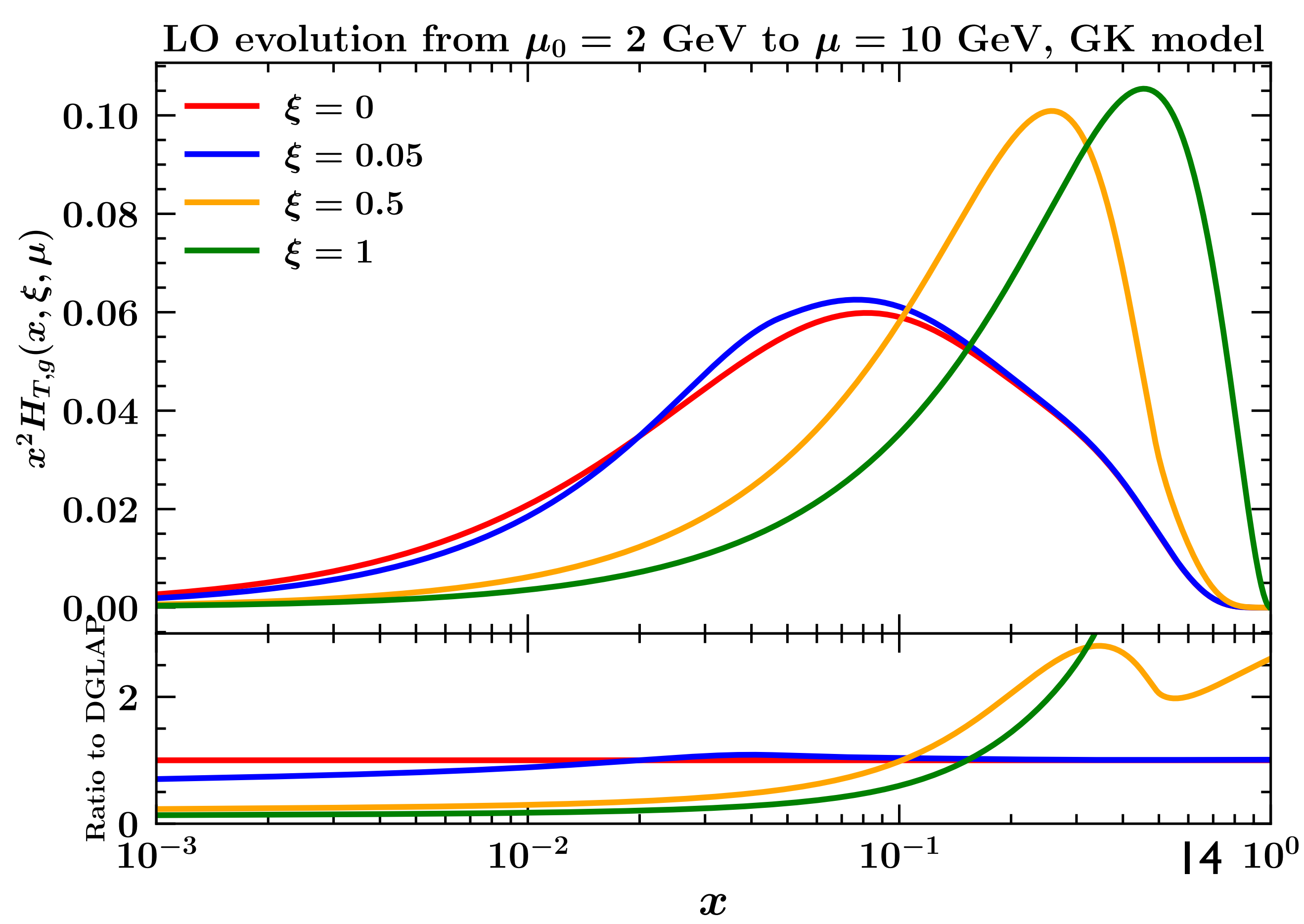


# Evolution and DGLAP limit [T]




- **DGLAP limit** reproduced within $10^{-5}$ relative accuracy.
- GPD evolution may significantly deviate from DGLAP evolution.
- The evolution generates a cusp at $x = \xi$ but the distribution remains **continuous** at this point.



# Polynomiality

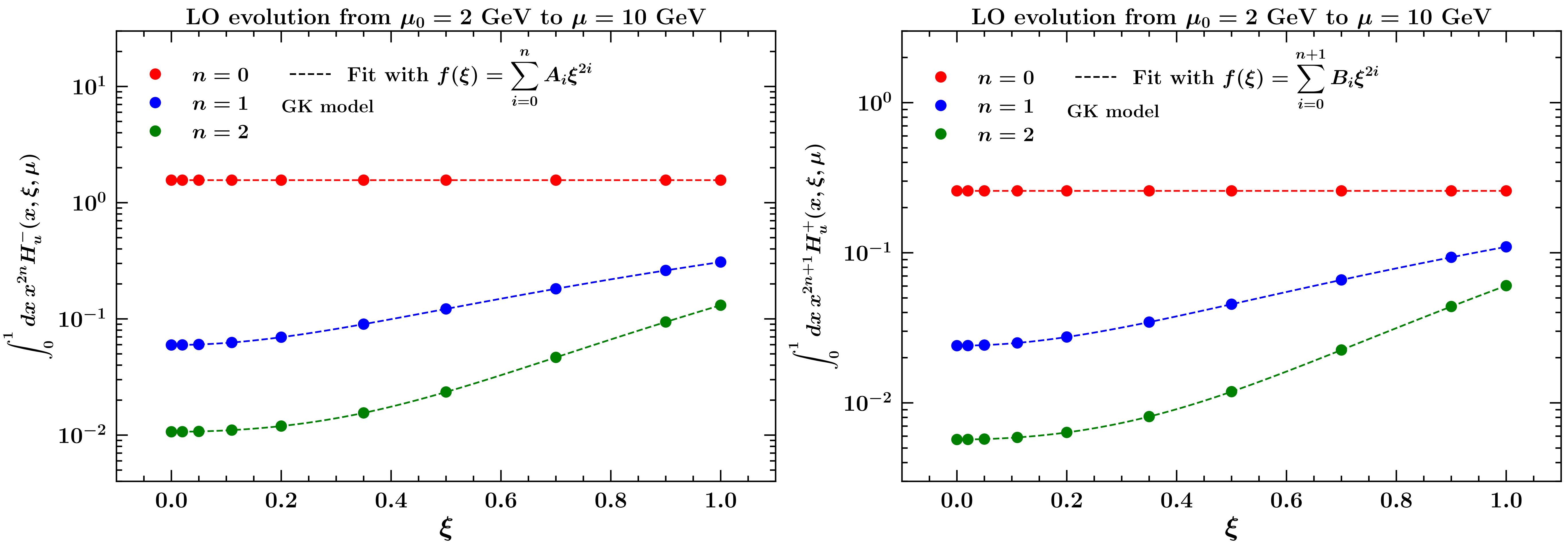
- GPD evolution should preserve **polynomiality**.
  [Xiang-Dong Ji, *J.Phys.G* 24 (1998) 1181-1205] [A.V. Radyushkin, *Phys.Lett.B* 449 (1999) 81-88]
- The following relations for the Mellin moments must hold at **all scales**:

$$\int_0^1 dx\, x^{2n} F_q^{[\Gamma]-}(x,\xi,\mu) = \sum_{k=0}^{n} A_k^{[\Gamma]}(\mu)\xi^{2k}$$

$$\int_0^1 dx\, x^{2n+1} F_q^{[\Gamma]+}(x,\xi,\mu) = \sum_{k=0}^{n(+1)} B_k^{[\Gamma]}(\mu)\xi^{2k}$$

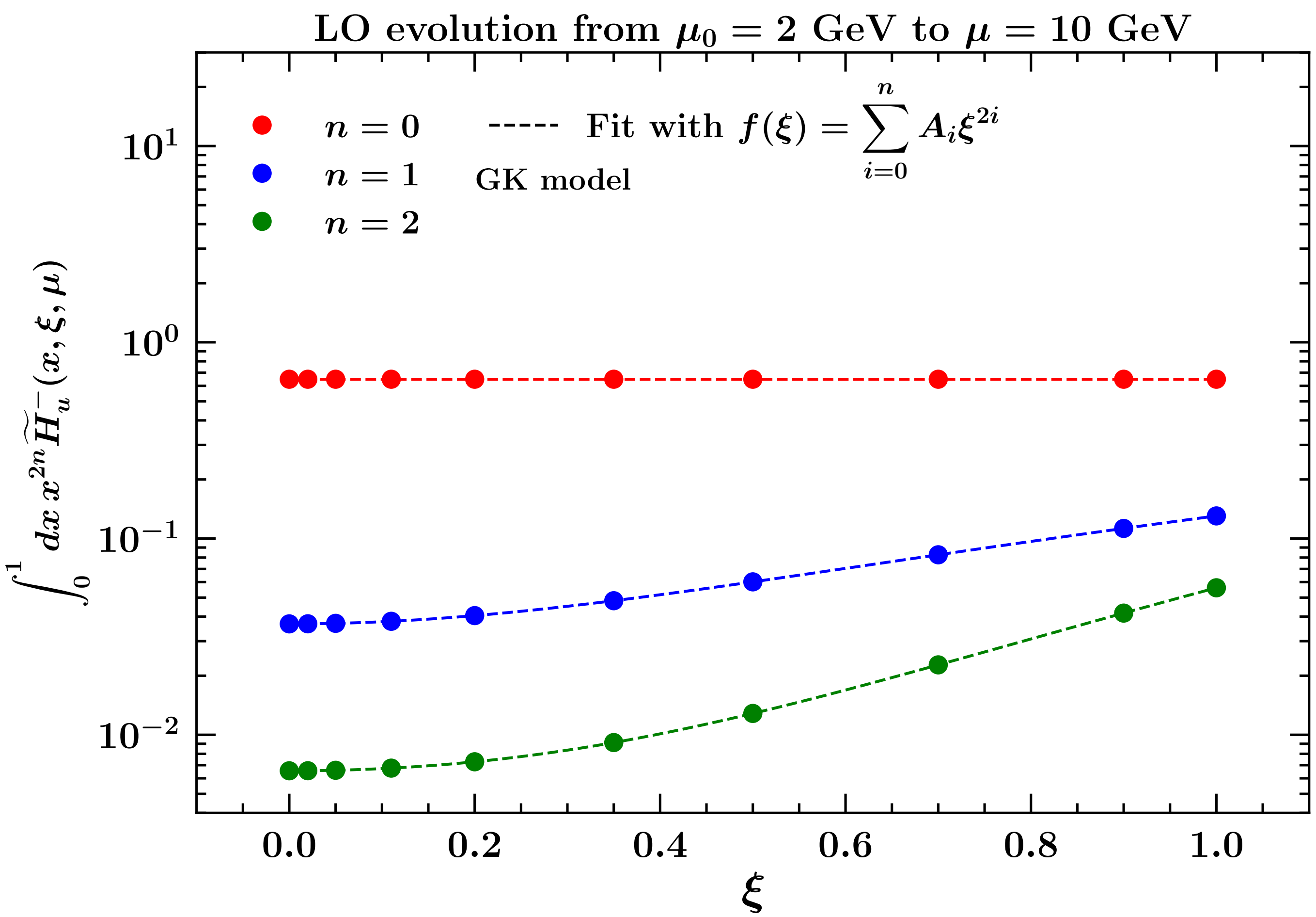
- Polynomiality predicts that the first moment ($n = 0$) of the *non-singlet* distribution is **constant** in $\xi$.
- The coefficient of the $\xi^{2n+2}$ term of the *singlet* (D-term), only allowed in the unpolarised case, is absent in the GK model and is *not* generated by evolution:
  - also the first moment of the singlet is expected to be **constant** in $\xi$.
- For larger values of $n$, one can just **fit** the behaviour in $\xi$ and check that it follows the **expected power law in** $\xi$.

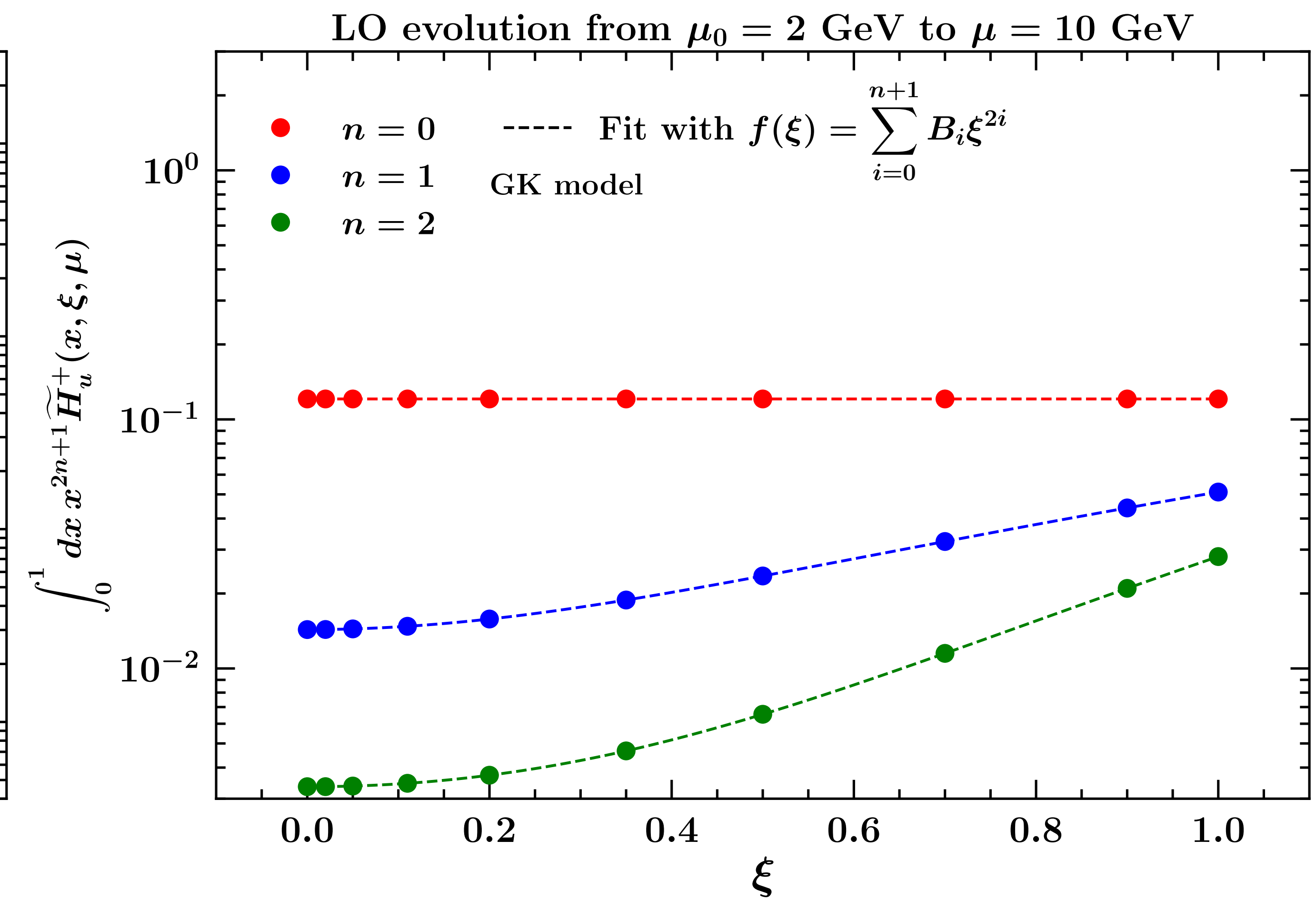
# Polynomiality [U]




- **First moment** for both singlet and non-singlet is indeed **constant** in $\xi$:
  - this was expected and the expectation is very nicely fulfilled.
- **Second and third moments** follow the expected law:
  - including odd-power terms in the fit gives coefficients very close to zero.
  - $B_{n+1}$ in the singlet is consistently found to be compatible with zero (no D-term).

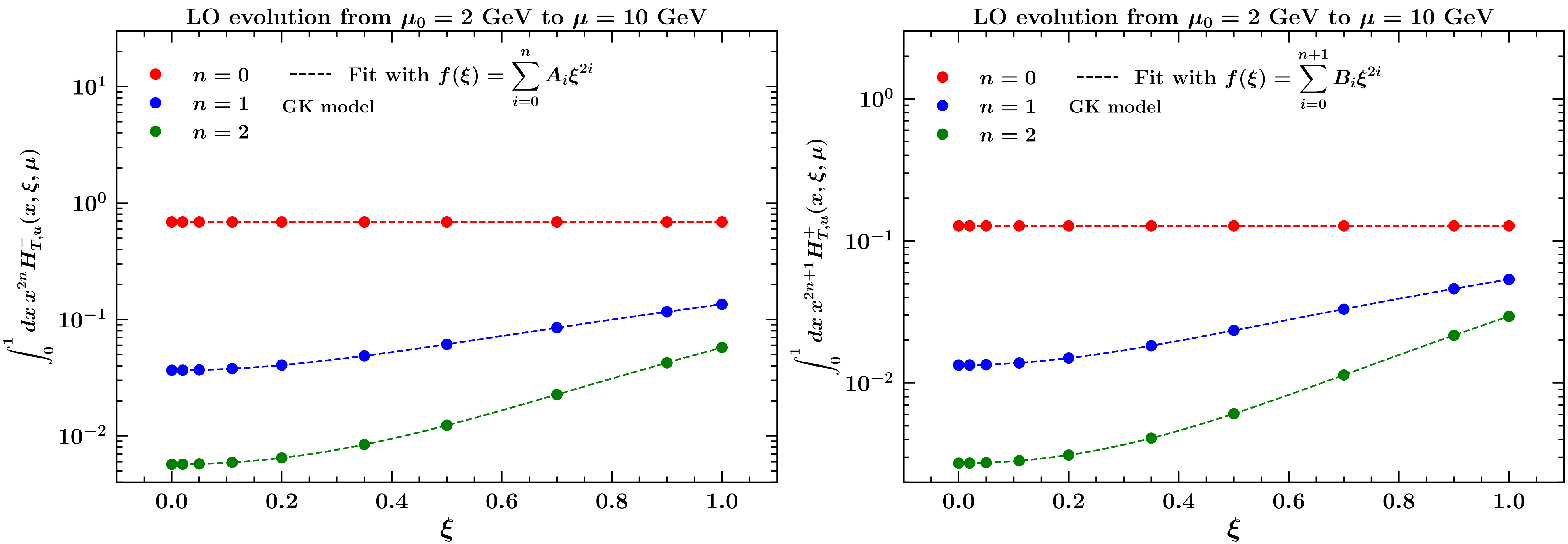
# Polynomiality [L]



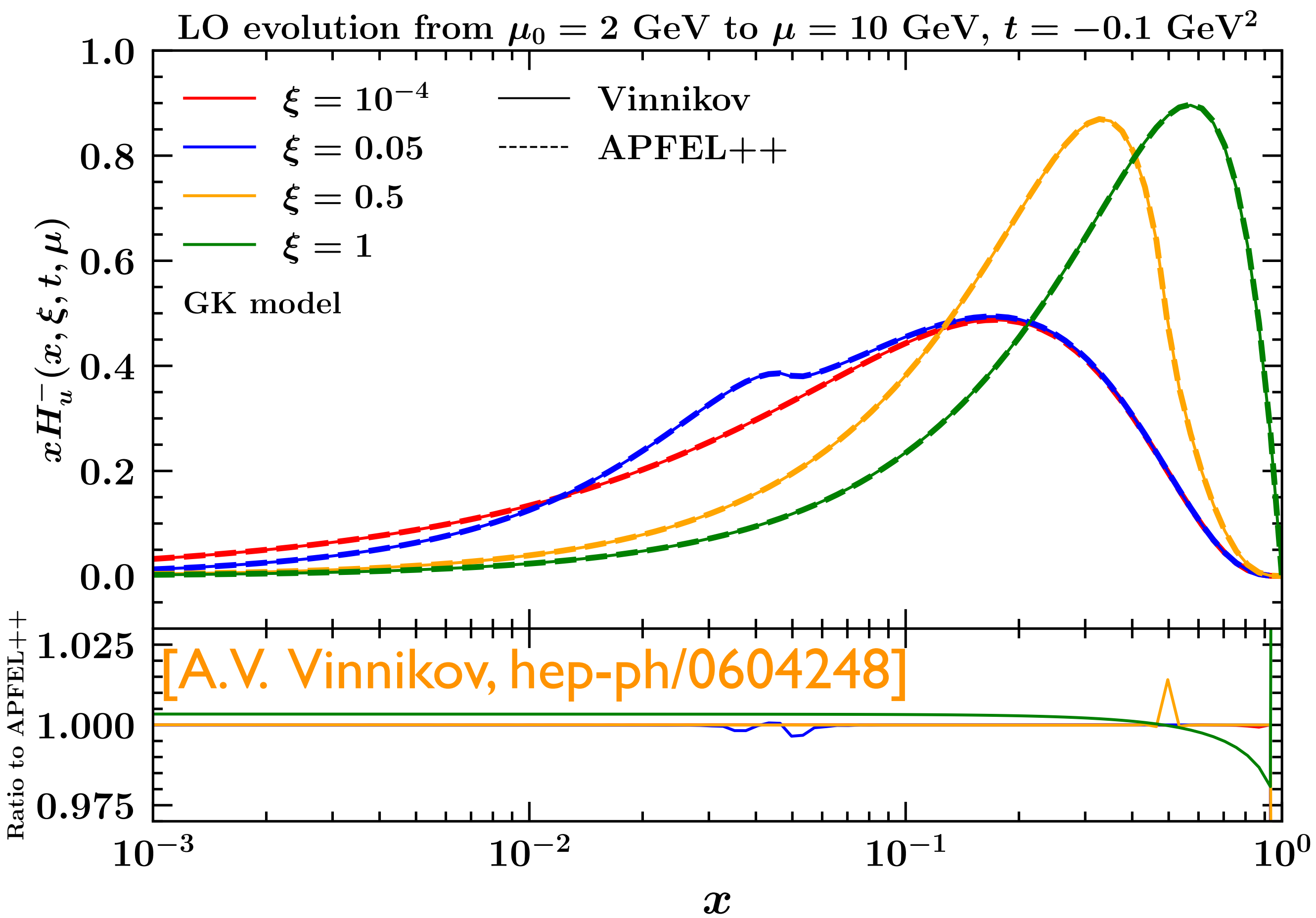

- **First moment** for both singlet and non-singlet is indeed **constant** in $\xi$:
  - this was expected and the expectation is very nicely fulfilled.
- **Second and third moments** follow the expected law:
  - including odd-power terms in the fit gives coefficients very close to zero.
  - $B_{n+1}$ in the singlet is consistently found to be compatible with zero (no D-term).

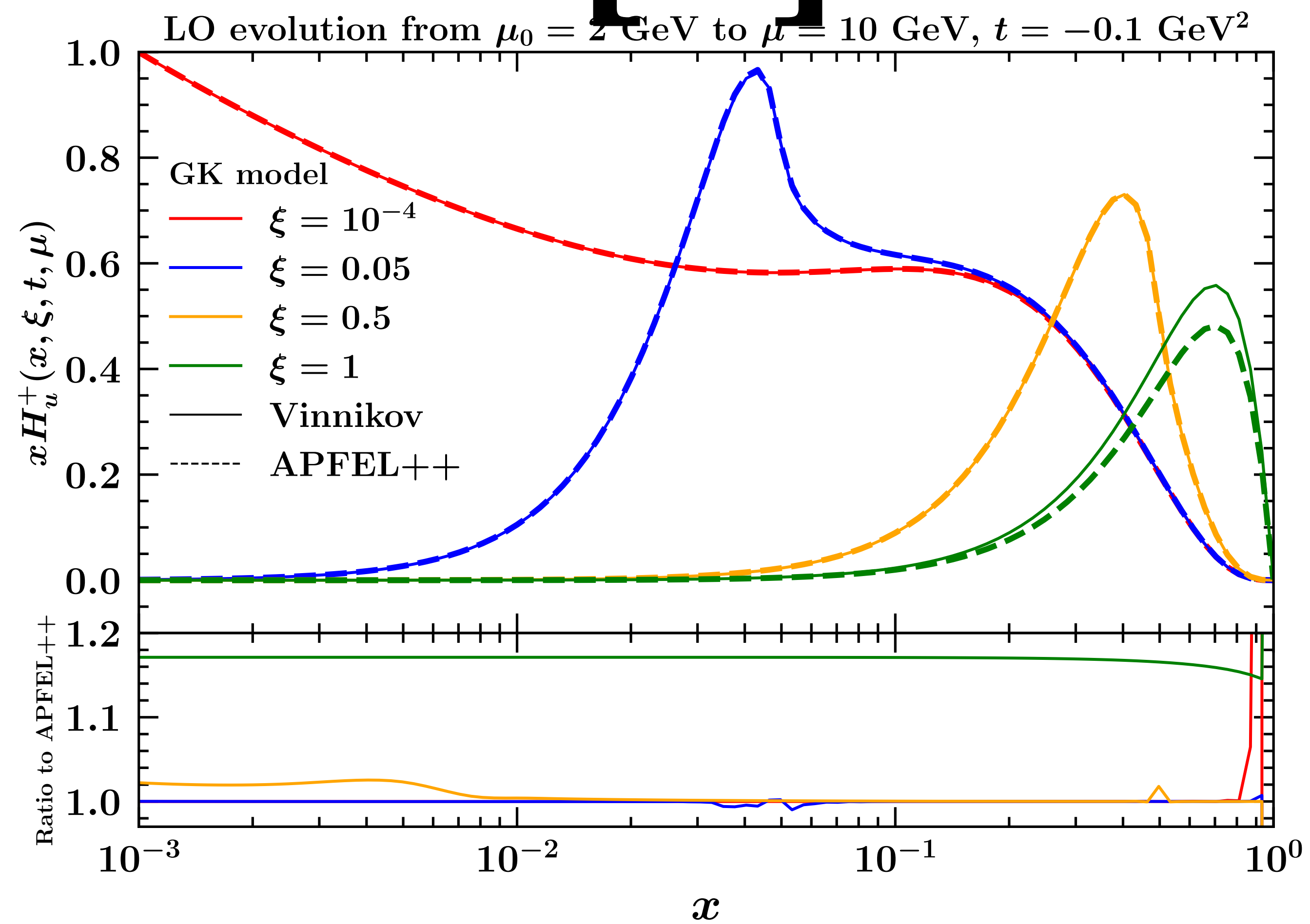
# Polynomiality [T]



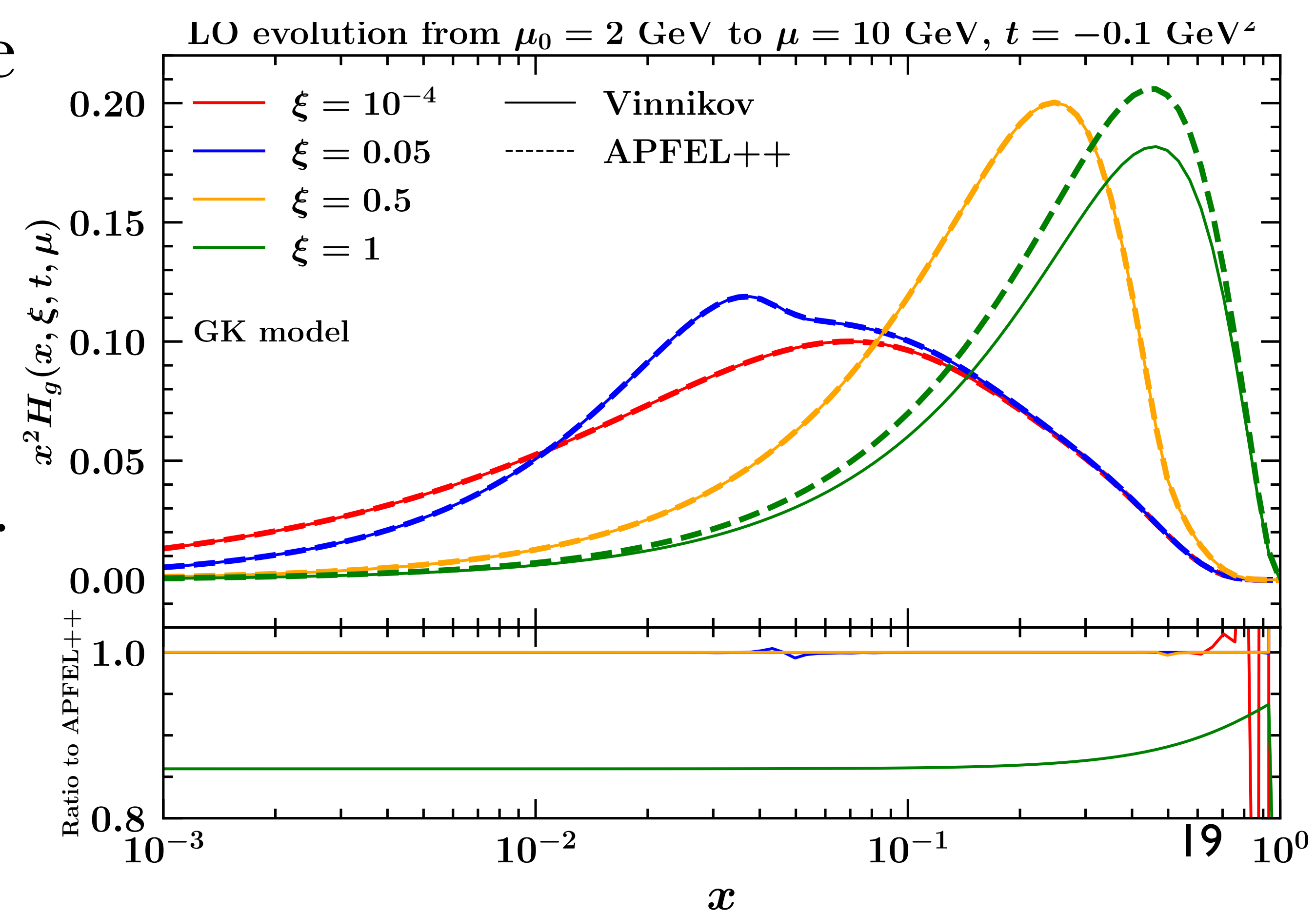

- **First moment** for both singlet and non-singlet is indeed **constant** in $\xi$:
  - this was expected and the expectation is very nicely fulfilled.
- **Second and third moments** follow the expected law:
  - including odd-power terms in the fit gives coefficients very close to zero.
  - $B_{n+1}$ in the singlet is consistently found to be compatible with zero (no D-term).

# APFEL++ vs. Vinnikov [U]

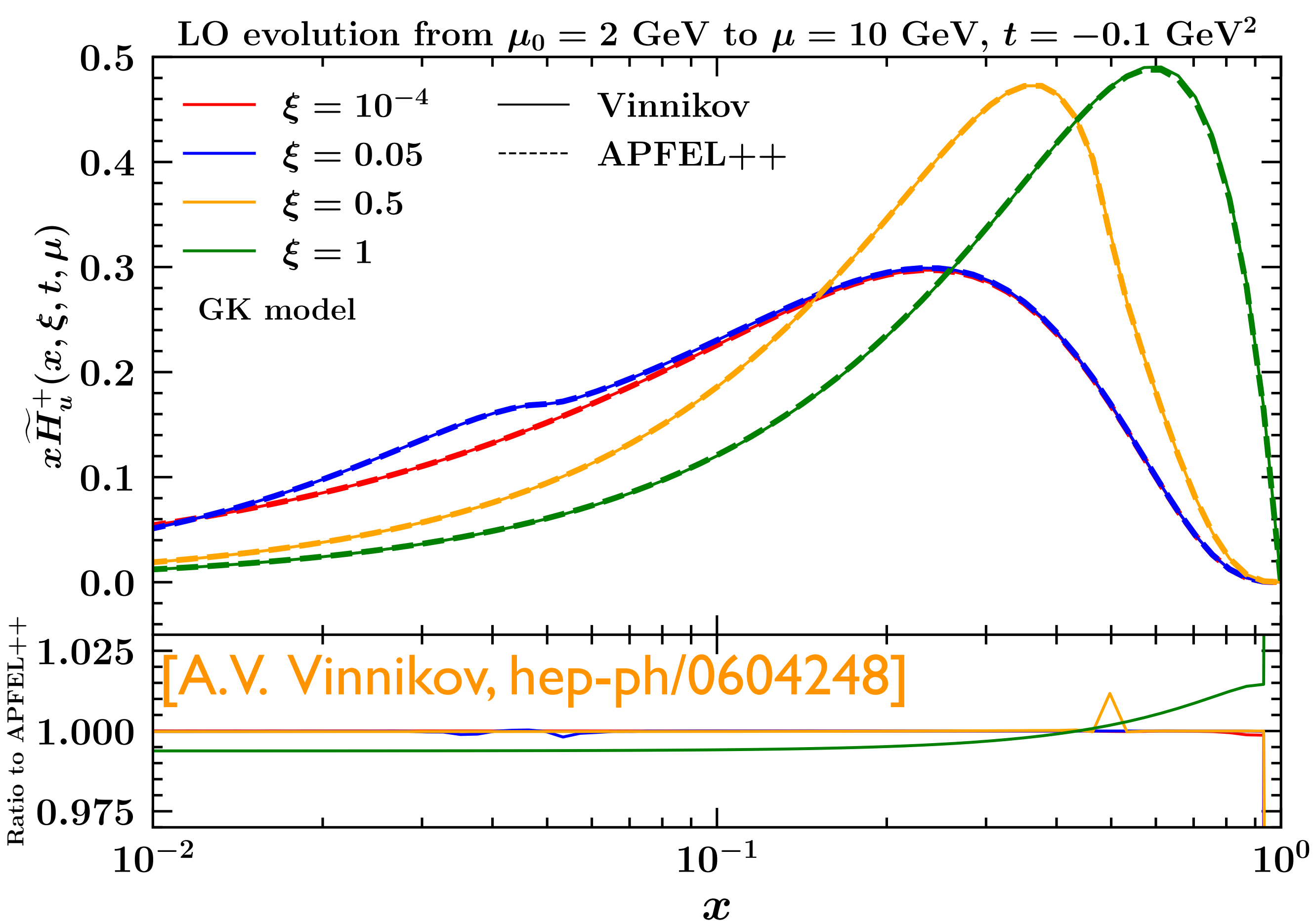


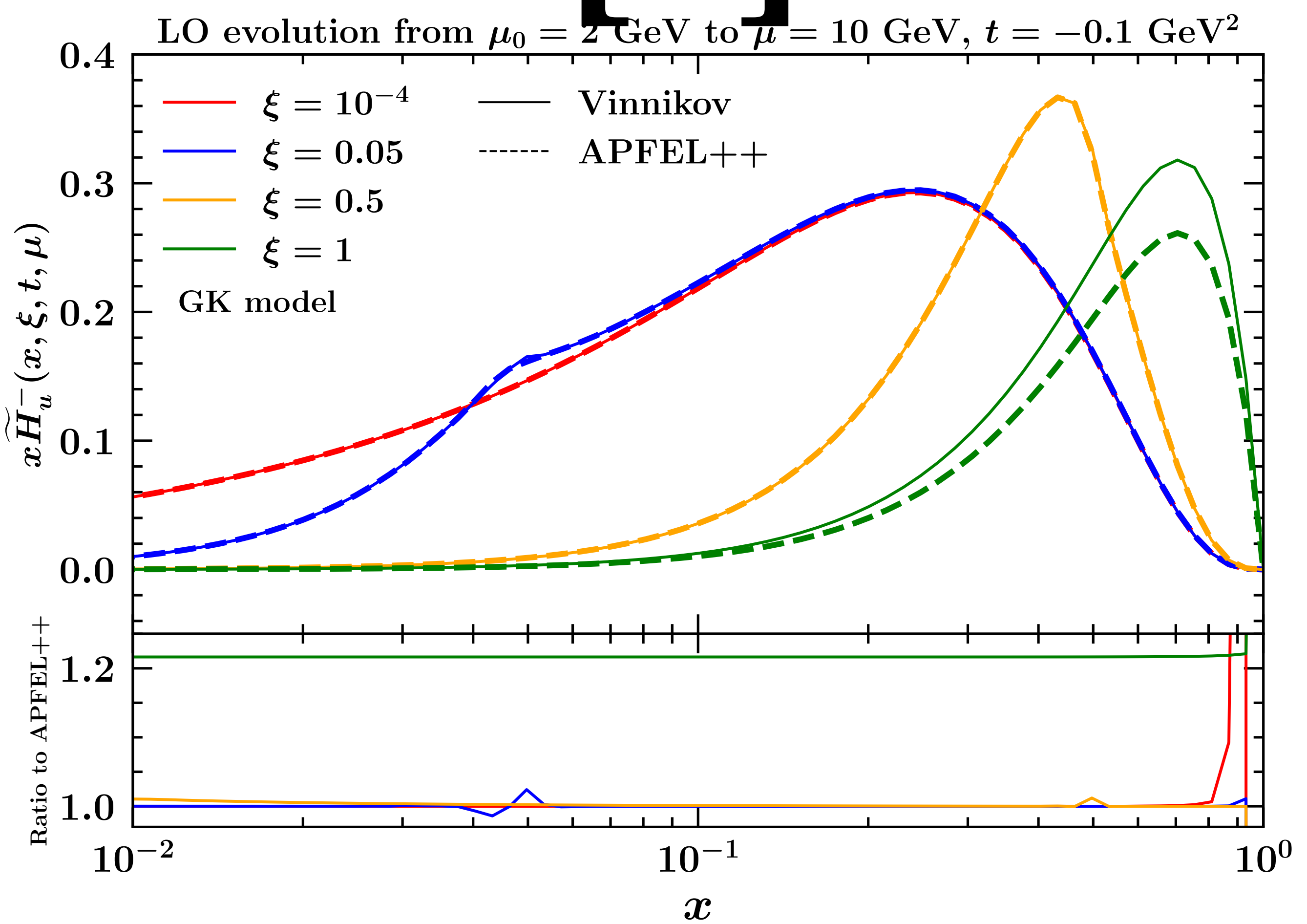




- **Excellent agreement** between the two code for $\xi \lesssim 0.6$.
- Agreement deteriorates for $\xi \gtrsim 0.6$:
  - discrepancy larger for the singlets (~20%) than for the non-singlet (~1%).
  - possible numerical instabilities of Vinnikov's code?
  - Inability to check the ERBL limit.

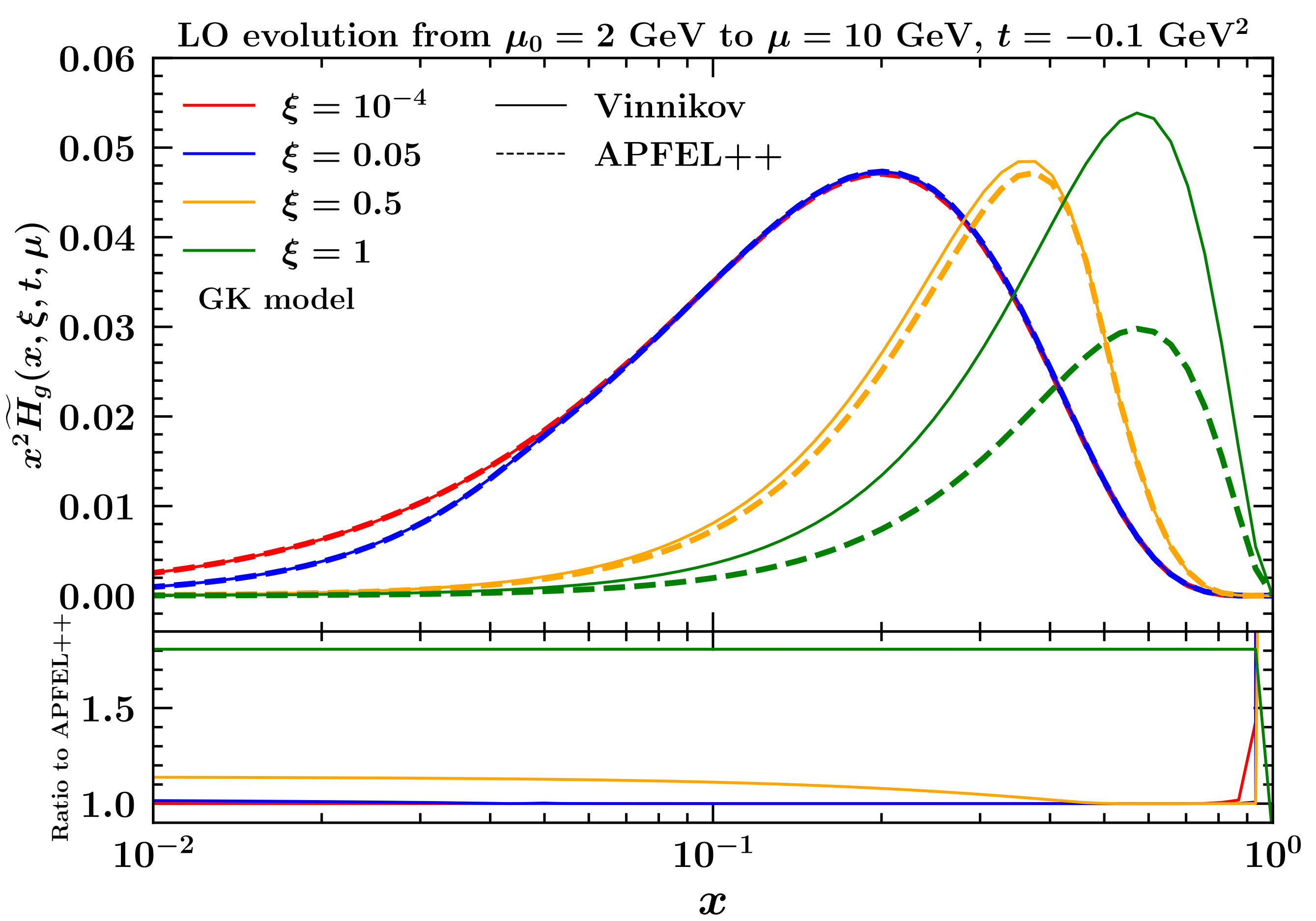


# APFEL++ vs. Vinnikov [L]




- **Excellent agreement** between the two code for $\xi \lesssim 0.6$.
- Agreement deteriorates for $\xi \gtrsim 0.6$:
  - discrepancy larger for the singlets ($\sim$20%) than for the non-singlet ($\sim$1%).
  - possible numerical instabilities of Vinnikov's code?
  - Inability to check the ERBL limit.



# Conclusions and outlook

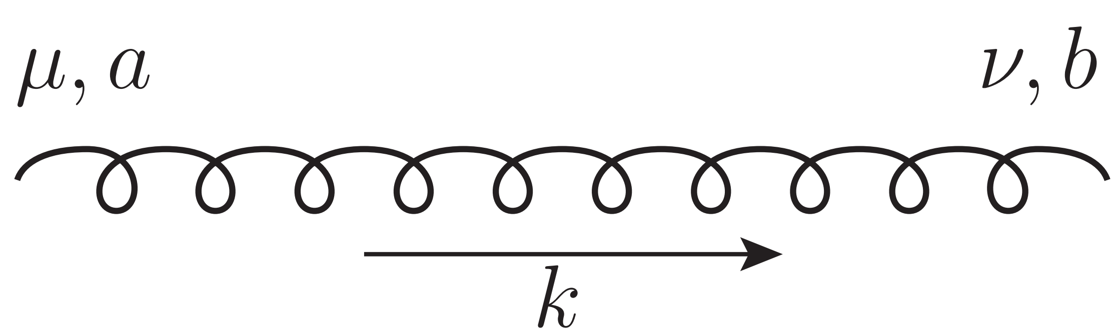
- We have **revisited LO GPD evolution** in momentum space:
  - *Ab-initio* calculation of the LO unpolarised splitting kernels based on Feynman diagrams in light-cone gauge for **all twist-2 operators**.
  - GPD evolution equations recasted in a DGLAP-like form convenient for implementation.
  - Various analytical properties of the kernels highlighted and numerically checked.
  - DGLAP (and ERBL) limit correctly recovered within excellent accuracy.
  - Evolution conserves polynomiality (and agrees with conformal-space evolution).
  - the code (APFEL++) will be made public and available within PARTONS
    https://github.com/vbertone/apfelxx
    http://partons.cea.fr/partons/doc/html/index.html

- **Next steps**:
  - **middle term:** benchmark of public evolution codes (discussion already started),
  - **longer term:** (re)calculation and implementation of NLO corrections (already on the way).

# Back up

# GPD definition

- The use of light-cone gauge implies:
  - the **absence of the Wilson line**,
  - a simpler gluon GPD written in terms of the **gluon field** and not the field strength,
  - the **absence of ghosts** in perturbative calculations,
  - **more complicated** gluon propagator:

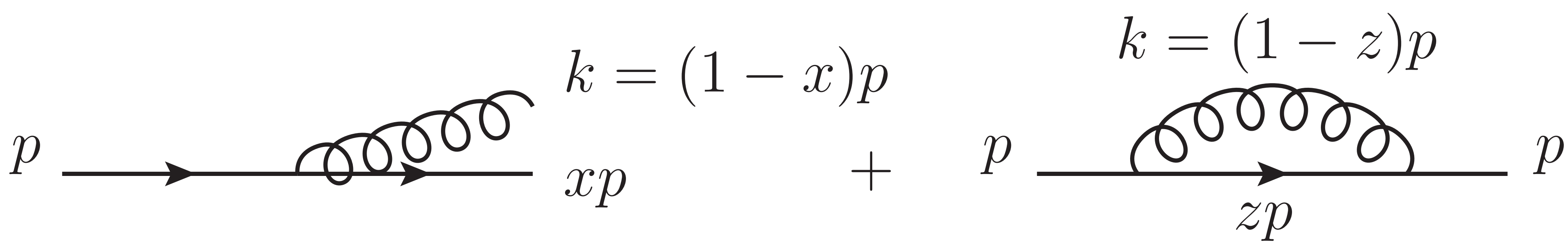



$$\mathcal{D}^{\mu\nu}_{ab}(k) = \frac{i\delta_{ab} d^{\mu\nu}(k)}{k^2 + i0}\,, \qquad d^{\mu\nu}(k) = -g^{\mu\nu} + \frac{k^\mu n^\nu + k^\nu n^\mu}{(nk)_{\mathrm{Reg.}}}$$

# GPD definition

$$\mathcal{D}^{\mu\nu}(k) = \frac{i d^{\mu\nu}(k)}{k^2 + i0}\,, \qquad d^{\mu\nu}(k) = -g^{\mu\nu} + \frac{k^\mu n^\nu + k^\nu n^\mu}{(nk)_{\text{Reg.}}}$$

- The linear (eikonal) propagator $(nk)^{-1}$ needs to be **regularised**:
  - it separately gives rise to non-integrable end-point singularities in real-emission graphs and to plain divergences in virtual graphs,
  - the two **cancel** giving an **integrable** result.




$$\frac{1}{(nk)} \sim \frac{1}{1-x} \qquad + \qquad \delta(1-x)\int \frac{dk}{(nk)} \sim \delta(1-x)\int \frac{dz}{1-z}$$

$$\sim \left(\frac{1}{1-x}\right)_+$$

# GPD evolution

- Using dimensional regularisation in $4-2\varepsilon$ dimensions, the **UV** renormalisation of GPDs can be implemented in a multiplicative fashion:

$$F_{i/H}^{[\Gamma]}(x,\xi,\Delta^2;\mu) = \lim_{\varepsilon\to 0}\sum_{j=q,g}\int_{-1}^{1}\frac{dy}{|y|}Z_{ij}^{[\Gamma]}\left(\frac{x}{y},\frac{\xi}{x},\alpha_s(\mu),\varepsilon\right)\hat{F}_{j/H}^{[\Gamma]}(y,\xi,\Delta^2;\varepsilon,\mu^{-\varepsilon})$$

- In the $\overline{\mathrm{MS}}$ scheme, renormalisation constants have the following structure:

$$Z_{ij}^{[\Gamma]}(z,\kappa,\alpha_s,\varepsilon) = \delta_{ij}\delta(1-z) + \sum_{n=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^n\sum_{p=1}^{n}\frac{1}{\overline{\varepsilon}^p}Z_{ij}^{[\Gamma],[n,p]}(z,\kappa)$$

- with:

$$\frac{1}{\overline{\varepsilon}} = \frac{S_\varepsilon}{\varepsilon} = \frac{1}{\varepsilon} + \ln 4\pi - \gamma_{\mathrm{E}} + \mathcal{O}(\varepsilon)$$

- Exploiting the independence of the bare GPDs on $\mu$ (for $\varepsilon \to 0$), one can derive a **RGE** governing the evolution of renormalised GPDs w.r.t. $\mu$:

$$\frac{dF_{i/H}^{[\Gamma]}(x,\xi,\Delta^2;\mu)}{d\ln\mu^2} = \sum_{k=q,g}\int_{-1}^{1}\frac{dz}{|z|}\mathcal{P}_{ik}^{[\Gamma]}\left(\frac{x}{z},\frac{\xi}{x},\alpha_s(\mu)\right)F_{k/H}^{[\Gamma]}(z,\xi,\Delta^2;\mu)$$

# GPD evolution

- The evolution kernels $\mathscr{P}$ are related to the normalisation constants $Z$ as follows:

$$\mathcal{P}_{ik}^{[\Gamma]}\left(\frac{x}{z},\frac{\xi}{x},\alpha_s\right)=\lim_{\varepsilon\to 0}\sum_j\int_{-1}^{1}\frac{dy}{|y|}\frac{dZ_{ij}^{[\Gamma]}\left(\frac{x}{y},\frac{\xi}{x},\alpha_s,\varepsilon\right)}{d\ln\mu^2}Z_{jk}^{[\Gamma]-1}\left(\frac{y}{z},\frac{\xi}{y},\alpha_s,\varepsilon\right)$$

- where the inverse of the renormalisation constant $Z^{-1}$ is defined as:

$$\sum_j\int_{-1}^{1}\frac{dw}{|w|}Z_{ij}^{[\Gamma]}\left(\frac{w}{x},\frac{\xi}{w},\alpha_s,\varepsilon\right)Z_{jk}^{[\Gamma]-1}\left(\frac{z}{w},\frac{\xi}{z},\alpha_s,\varepsilon\right)=\delta_{ik}\delta\left(1-\frac{z}{x}\right)$$

- If factorisation holds, the evolution kernels $\mathscr{P}$ **must be finite**:
  - consider the factorisation of a Compton form factor: $\mathcal{F}(Q)=C(\mu/Q,\alpha_s(\mu))\otimes F(\mu)$
  - Being $\mathscr{F}$ a physical observable, it has to be independent of $\mu$ order by order in $\alpha_s$:

$$C^{-1}\otimes\frac{d\mathcal{F}}{d\ln\mu^2}=0=\left[\frac{d\ln C(\mu,\alpha_s(\mu))}{d\ln\mu^2}+\mathcal{P}(\alpha_s(\mu))\right]\otimes F(\mu)$$

  - Since the coefficient function $C$ is finite, so must be $\mathscr{P}$.

# GPD evolution

- The finiteness of the evolution kernels $\mathscr{P}$ has important consequences on the structure of the renormalisation constants $Z$:

$$\mathcal{P} = \frac{d \ln Z}{d \ln \mu^2} = \overline{\beta}(\alpha_s, \varepsilon) \frac{\partial \ln Z}{\partial \alpha_s}$$

- but:

$$Z = 1 + \sum_{n=1}^{\infty} \alpha_s^n \sum_{p=1}^{n} \frac{1}{\varepsilon^p} Z^{[n,p]} = 1 + \sum_{p=1}^{\infty} \frac{1}{\varepsilon^p} \sum_{n=p}^{\infty} \alpha_s^n Z^{[n,p]} = 1 + \sum_{p=1}^{\infty} \frac{1}{\varepsilon^p} Z^{[p]}(\alpha_s)$$

- so that:

$$\frac{\partial \ln Z}{\partial \alpha_s} = Z^{-1} \frac{\partial Z}{\partial \alpha_s} = \frac{1}{\varepsilon} \frac{\partial Z^{[1]}}{\partial \alpha_s} + \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$$

- Since $\overline{\beta}(\alpha_s, \varepsilon) = -\varepsilon\alpha_s + \beta(\alpha_s)$, it follows that:

$$\mathcal{P} = -\alpha_s \frac{\partial Z^{[1]}}{\partial \alpha_s} + \mathcal{O}\left(\frac{1}{\varepsilon}\right)$$

- The evolution kernels are extracted from the **single pole** of the renormalisation constants **to all orders** in $\alpha_s$.

- The finiteness of $\mathscr{P}$ implies that the residual $\mathscr{O}(1/\varepsilon)$ has to be **identically zero**:

  - higher-order-pole coefficients $Z^{[n]}$, $n > 1$, are related to $Z^{[1]}$ and $\beta$.

# GPD evolution

- The kernels $\mathscr{P}$ admit the **perturbative expansion**:

$$\mathcal{P}(\alpha_s) = \alpha_s \sum_{n=0}^{\infty} \alpha_s^n \mathcal{P}^{[n]}$$

- At one loop, *i.e.* the leading order, one simply finds:

$$\mathcal{P}^{[0]} = -Z^{[1,1]}$$

- At two loops:

$$\mathcal{P}^{[1]} = -2Z^{[2,1]}$$

- But with the additional constraints that:

$$Z^{[2,2]} = \frac{1}{2}\beta_0 Z^{[1,1]} + \frac{1}{2} Z^{[1,1]} \otimes Z^{[1,1]}$$

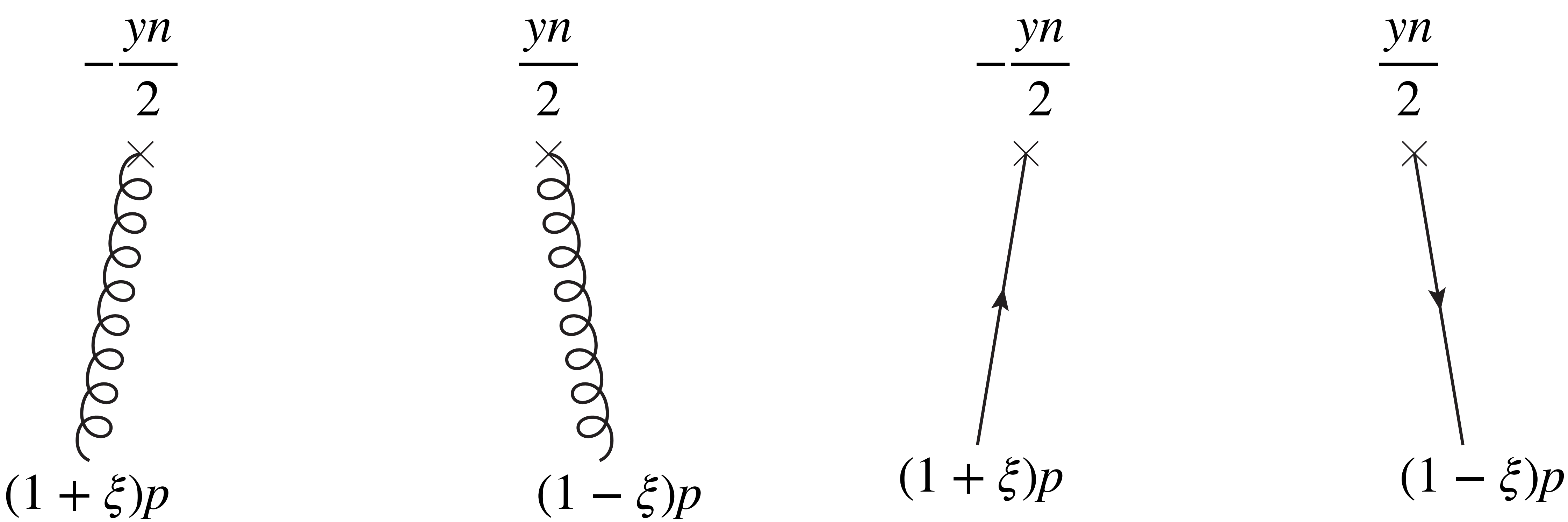
- An explicit two-loop calculation must fulfil this identity, thus providing a **strong check of the calculation** itself.

# Parton-in-parton GPDs at LO

- At $\mathcal{O}(1)$:

$$\psi_q(x) = \psi_q^{(0)}(x) \qquad A_a^j(x) = A_a^{(0),j}(x)$$

- One immediately finds that the only non-zero GPDs are $g/g$ and $q/q$:



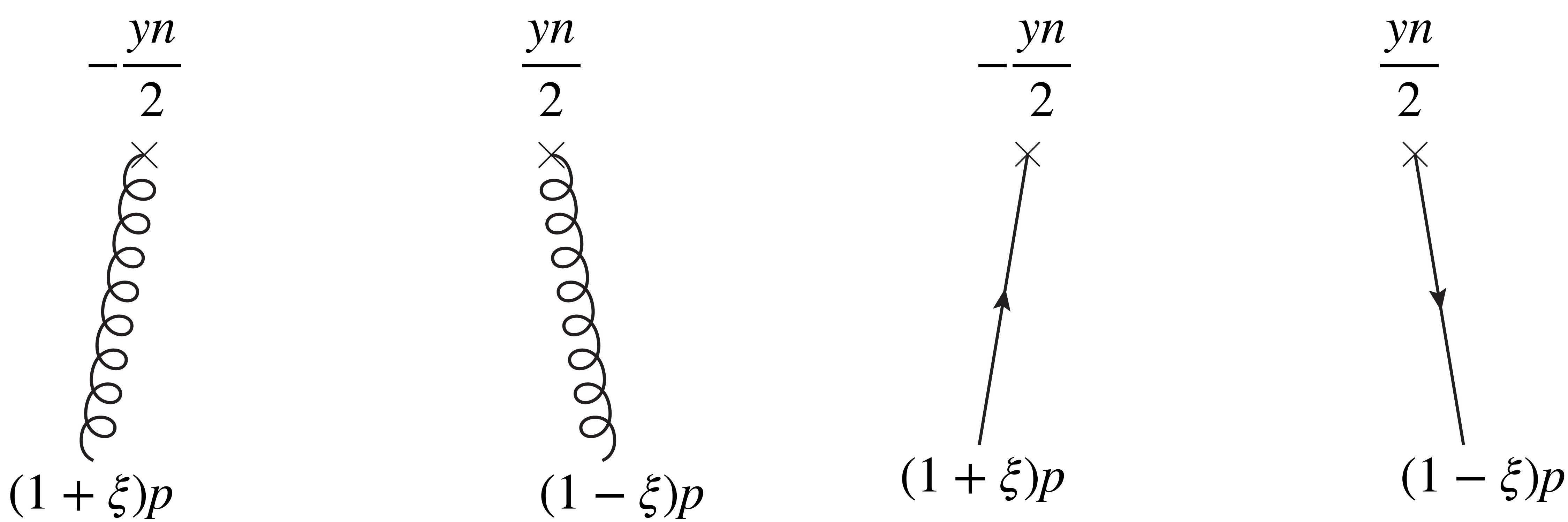

$$F_{g/g}^{[U][U],[0]}(x,\xi) = F_{g/g}^{[L][L],[0]}(x,\xi) = F_{g/g}^{[T][T],[0]}(x,\xi) = (1-\xi^2)\delta(1-x)$$

$$F_{q/q}^{[U][U],[0]}(x,\xi) = F_{q/q}^{[L][L],[0]}(x,\xi) = F_{q/q}^{[T][T],[0]}(x,\xi) = \sqrt{1-\xi^2}\delta(1-x)$$

- No divergences at this order and thus **no need for renormalisation**.
- This calculation sets the **normalisation** of GPDs.

# Parton-in-parton GPDs at LO

- At $\mathcal{O}(1)$ one immediately finds that the only non-zero GPDs are $g/g$ and $q/q$:



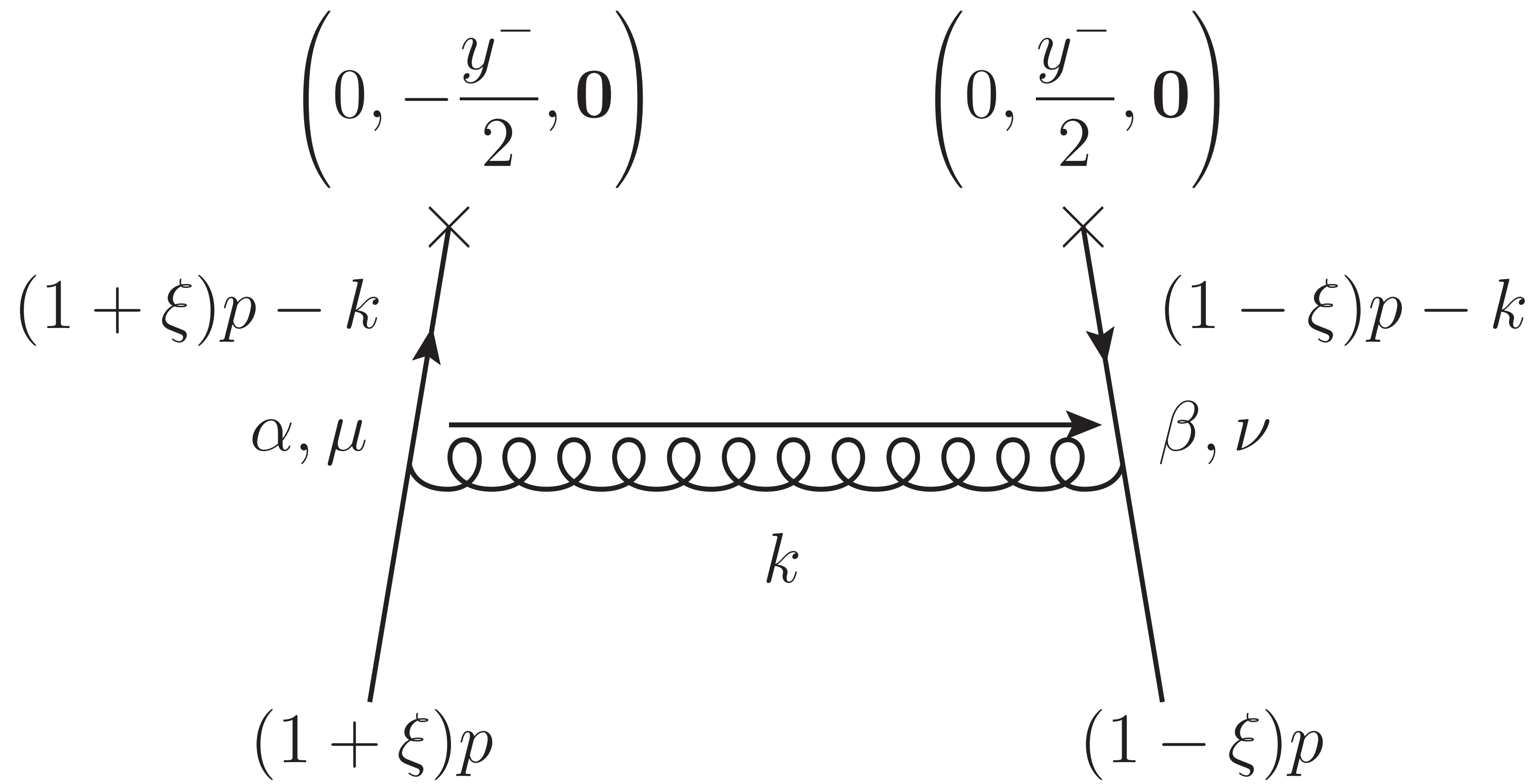

$$F_{g/g}^{[U],[0]}(x,\xi) = F_{g/g}^{[L],[0]}(x,\xi) = F_{g/g}^{[T],[0]}(x,\xi) = (1-\xi^2)\delta(1-x)$$

$$F_{q/q}^{[U],[0]}(x,\xi) = F_{q/q}^{[L],[0]}(x,\xi) = F_{q/q}^{[T],[0]}(x,\xi) = \sqrt{1-\xi^2}\,\delta(1-x)$$

- No divergences at this order and thus **no need for renormalisation**.
- This calculation sets the **normalisation** of GPDs.

# Parton-in-parton GPDs at NLO

- At $\mathcal{O}(\alpha_s)$ for the $q/q$ channel one has to compute *one single* "real" diagram:




- This produces:

$$\frac{\alpha_s}{4\pi}\hat{F}^{[\Gamma],[1],\text{real}}_{q/q}(x,\xi,\varepsilon) = \sqrt{1-\xi^2}\int_{-\infty}^{\infty}\frac{dy_-}{2\pi}e^{i(1-x)p_+y_-}\,\text{Tr}\left[R^{[\Gamma]}_{qq}(y_-,\xi,\varepsilon)\Lambda^{[\Gamma]}_q\right]$$

- with:

$$R^{[\Gamma]}_{qq}(y_-,\xi,\varepsilon) = \frac{\alpha_s}{4\pi}iC_F\int\frac{d^{4-2\varepsilon}k}{(2\pi)^{2-2\varepsilon}}e^{-ik_+y_-}\frac{\gamma^\mu[(1+\xi)\not{p}-\not{k}]\,\Gamma_q\,[(1-\xi)\not{p}-\not{k}]\gamma^\nu d_{\mu\nu}(k)}{[((1+\xi)p-k)^2+i0][((1-\xi)p-k)^2+i0]}$$

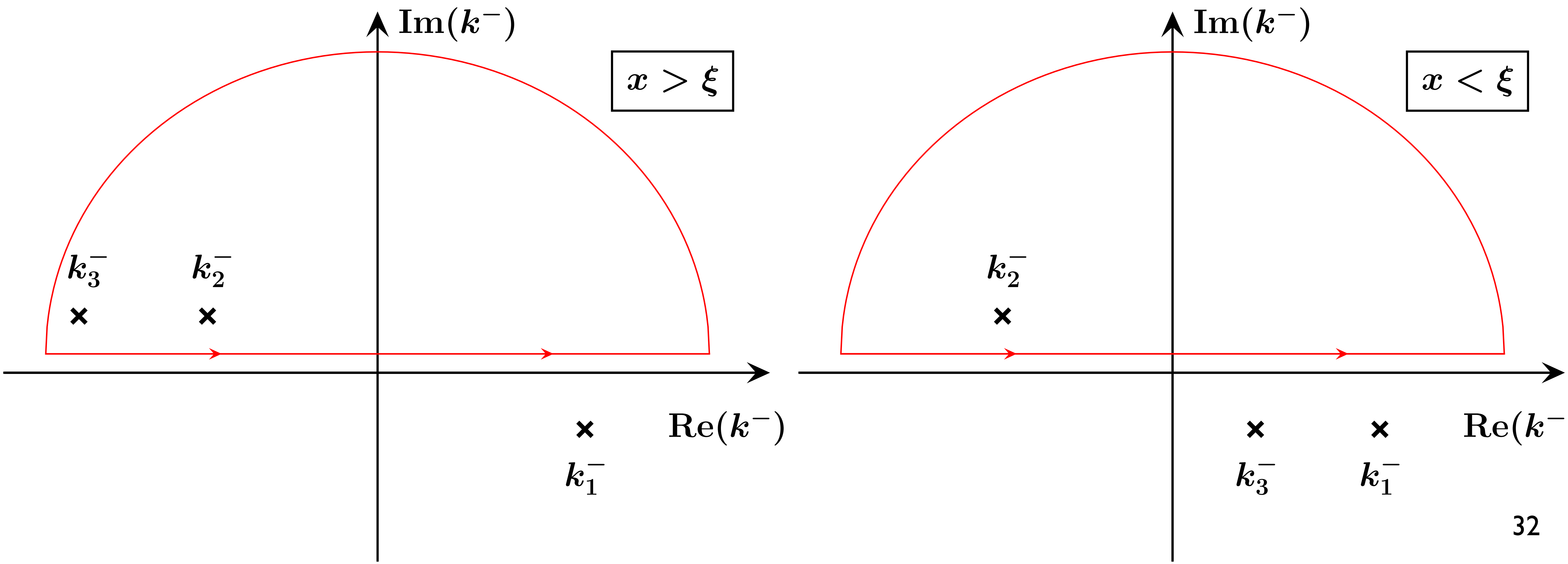
# Parton-in-parton GPDs at NLO

- After the trivial integration over $k^+$ and the evaluation of contractions and traces, one finds:

$$\hat{F}_{q/q}^{[\Gamma],[1],\mathrm{real}}(x,\xi,\varepsilon) = \int \frac{d^{2-2\varepsilon}\mathbf{k}_T}{(2\pi)^{2-2\varepsilon}}\mathbf{k}_T^2 \int_{-\infty}^{+\infty} dk^- \frac{A(x,\xi)+B(x,\xi)p^+k^-/\mathbf{k}_T^2}{(k^- - k_1^-)(k^- - k_2^-)(k^- - k_3^-)}$$

$$k_1^- = \frac{\mathbf{k}_T^2}{2(1-x)p^+} - i(1-x)\eta \quad k_2^- = -\frac{\mathbf{k}_T^2}{2(x+\xi)p^+} + i(x+\xi)\eta \quad k_3^- = -\frac{\mathbf{k}_T^2}{2(x-\xi)p^+} + i(x-\xi)\eta$$

- Assuming $x, \xi > 0$, the pole structure depends on the sign of $x - \xi$:



# Parton-in-parton GPDs at NLO

- The final result looks like this:

$$\hat{F}_{q/q}^{[\Gamma],[1],\mathrm{real}}(x,\xi,\varepsilon) = \sqrt{1-\xi^2}\,\theta(1-x)\left[\theta(x+\xi)p_{q/q}^{\Gamma}\left(x,\frac{\xi}{x}\right) + \theta(x-\xi)p_{q/q}^{\Gamma}\left(x,-\frac{\xi}{x}\right)\right]\mu^{2\varepsilon}S_\varepsilon\int_0^\infty \frac{dk_T^2}{k_T^{2+2\epsilon}}$$

- Strictly speaking:

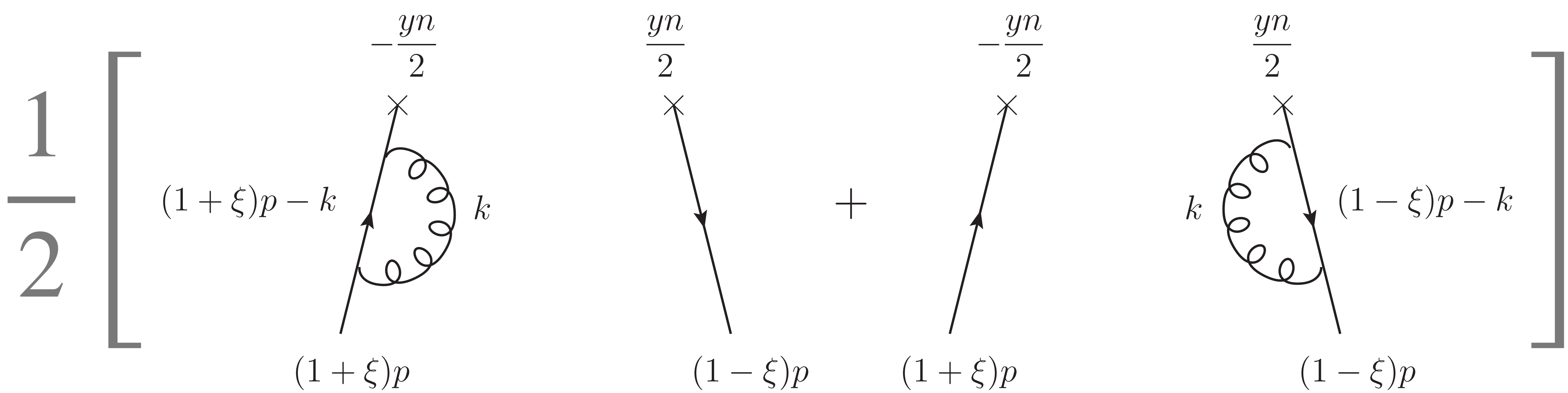
$$\int_0^\infty \frac{dk_T^2}{k_T^{2+2\epsilon}} = \frac{1}{\varepsilon_{\mathrm{UV}}} - \frac{1}{\varepsilon_{\mathrm{IR}}} = 0 \quad \Rightarrow \quad \hat{F}_{q/q}^{[\Gamma][\Lambda],[1],\mathrm{real}}(x,\xi,\varepsilon) = 0$$

- We are only concerned with the **UV part**: the IR one has to cancel against the partonic cross section when computing a physical observable (**IR safety**).

$$\hat{F}_{q/q}^{[\Gamma],[1],\mathrm{real}}(x,\xi,\varepsilon) = \sqrt{1-\xi^2}\,\theta(1-x)\left[\theta(x+\xi)p_{q/q}^{\Gamma}\left(x,\frac{\xi}{x}\right) + \theta(x-\xi)p_{q/q}^{\Gamma}\left(x,-\frac{\xi}{x}\right)\right]\frac{\mu^{2\varepsilon}}{\overline{\varepsilon}} + \mathrm{IR}$$

# Evolution kernels at one loop

- The **virtual** contribution (common to all polarisations) is computed as:




- The final result is:

$$\hat{F}^{[\Gamma],[1]}_{q/q}(x,\xi,\varepsilon) = \sqrt{1-\xi^2}\left\{\theta(1-x)\left[\theta(x+\xi)p^{\Gamma}_{q/q}\left(x,\frac{\xi}{x}\right)+\theta(x-\xi)p^{\Gamma}_{q/q}\left(x,-\frac{\xi}{x}\right)\right]\right.$$

$$\left. + \;\delta(1-x)C_F\left[\frac{3}{2}-\ln\left(\left|1-\frac{\xi^2}{x^2}\right|\right)-2\int_0^1\frac{dz}{1-z}\right]\right\}\frac{\mu^{2\varepsilon}}{\overline{\varepsilon}} + \mathrm{IR}$$

- The resulting evolution kernel is:

$$\mathcal{P}^{[\Gamma],[0]}_{qq}(y,\kappa) = \theta(1-y)\left[\theta(1+\kappa)p^{\Gamma}_{q/q}(y,\kappa)+\theta(1-\kappa)p^{\Gamma}_{q/q}(y,-\kappa)\right]$$

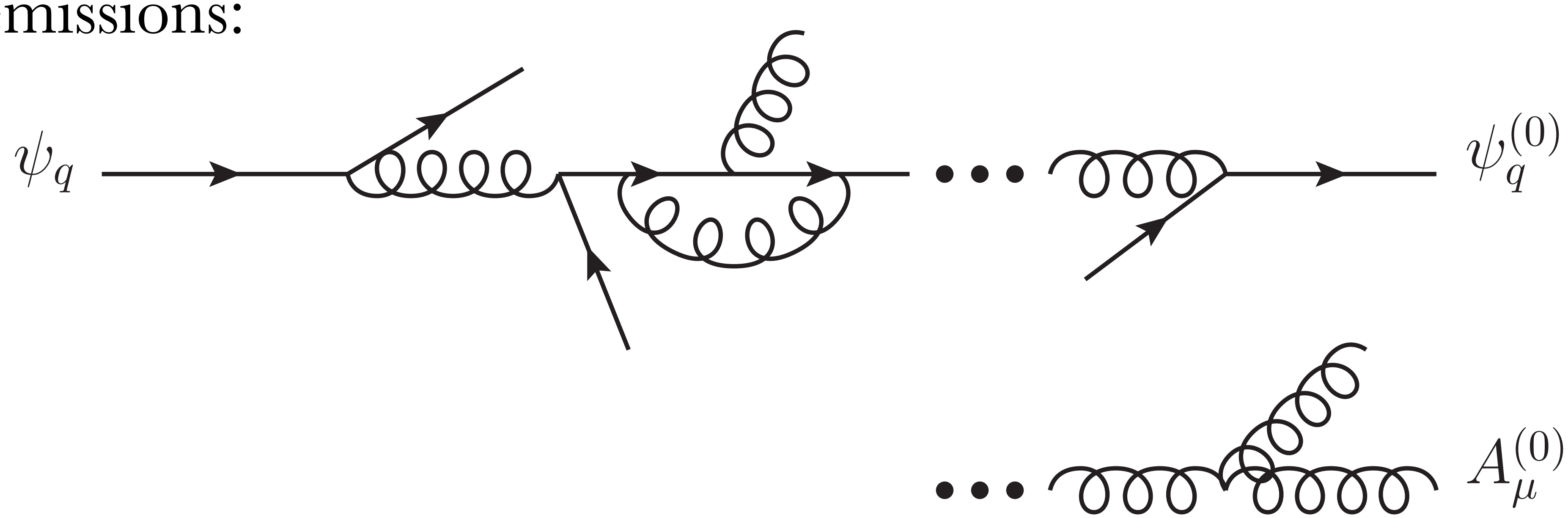
$$+ \;\delta(1-y)C_F\left[\frac{3}{2}-\ln\left(\left|1-\kappa^2\right|\right)-2\int_0^1\frac{dz}{1-z}\right]$$

$$\kappa = \frac{\xi}{x}$$

# Parton-in-parton GPDs

- The partonic fields that appear in the operator definition of the GPD correlators are **interacting fields**.

- Interacting fields reduce to **free fields** after an arbitrary number of *real* and *virtual* emissions:




- Additional radiation gives rise to **perturbative corrections** and the **need for renormalisation**.

- Free partonic fields eventually **annihilate** the appropriate partonic states:

$$\psi_q^{(0)}(x)|k,s\rangle_q = e^{-ik\cdot x}u_{q,s}(k)|0\rangle$$

$$\psi_q^{(0)}(x)|k,s\rangle_{\overline{q}} = e^{ik\cdot x}v_{q,s}(k)|0\rangle$$

$$A_a^{(0),j}(x)|k,s\rangle_g = e^{-ik\cdot x}e^j_{a,s}(k)|0\rangle$$

- All other combinations give zero.

# Parton-in-parton GPDs

- In light-cone gauge:

$$\hat{F}^{[\Gamma][\Lambda]}_{g/g,q}(x,\xi) = \frac{(n\cdot p)(x^2-\xi^2)}{2(N_c^2-1)x}\int \frac{dy}{2\pi} e^{-ix(n\cdot p)y} \left\langle (1-\xi)p, s' \left| A_a^{\mu}\left(\frac{yn}{2}\right) \Gamma_{g,\mu\nu} A_a^{\nu}\left(-\frac{yn}{2}\right)\right| (1+\xi)p, s \right\rangle_{g,q} \Lambda_{s's}$$

$$\hat{F}^{[\Gamma][\Lambda]}_{q/g,q,\overline{q},q',\overline{q}'}(x,\xi) = \frac{1}{2N_c}\int \frac{dy}{2\pi} e^{-ix(n\cdot p)y} \left\langle (1-\xi)p, s' \left| \overline{\psi}_q^i\left(\frac{yn}{2}\right) \Gamma_q^{ij} \psi_q^j\left(-\frac{yn}{2}\right)\right| (1+\xi)p, s \right\rangle_{g,q,\overline{q},q',\overline{q}'} \Lambda_{s's}$$

- The projectors $\Lambda_{s's}$ are introduced for *convenience* to project out the physical partonic spin/helicity states that contribute to the amplitude:

$$\left.\begin{aligned} \Lambda_{s's}\overline{u}_{q,s'}((1-\xi)p)u_{q,s}((1+\xi)p) &= \Lambda_q = \sqrt{1-\xi^2}\,\{\not{p}, \not{p}\gamma^5, i\sigma^{\mu\nu}P_\nu\gamma^5\} \\ \Lambda_{s's}e^{\mu *}_{s'}((1-\xi)p)e^{\nu}_{s}((1+\xi)p) &= \Lambda_g^{\mu\nu} = \{-g_T^{\mu\nu}, -i\varepsilon_T^{\mu\nu}, -R^\mu R^\nu - L^\mu L^\nu\} \end{aligned}\right\} \in \{U, L, T\}$$

- These quark-in-quark combinations:

$$\hat{F}^{\mathrm{NS},\pm}_{q/q} = (\hat{F}_{q/q} - \hat{F}_{q/q'}) \pm (\hat{F}_{q/\overline{q}} - \hat{F}_{q/\overline{q}'})$$

$$\hat{F}^{\mathrm{NS,V}}_{q/q} = \hat{F}^{\mathrm{NS},-}_{q/q} + n_f(\hat{F}_{q/q'} - \hat{F}_{q/\overline{q}'})$$

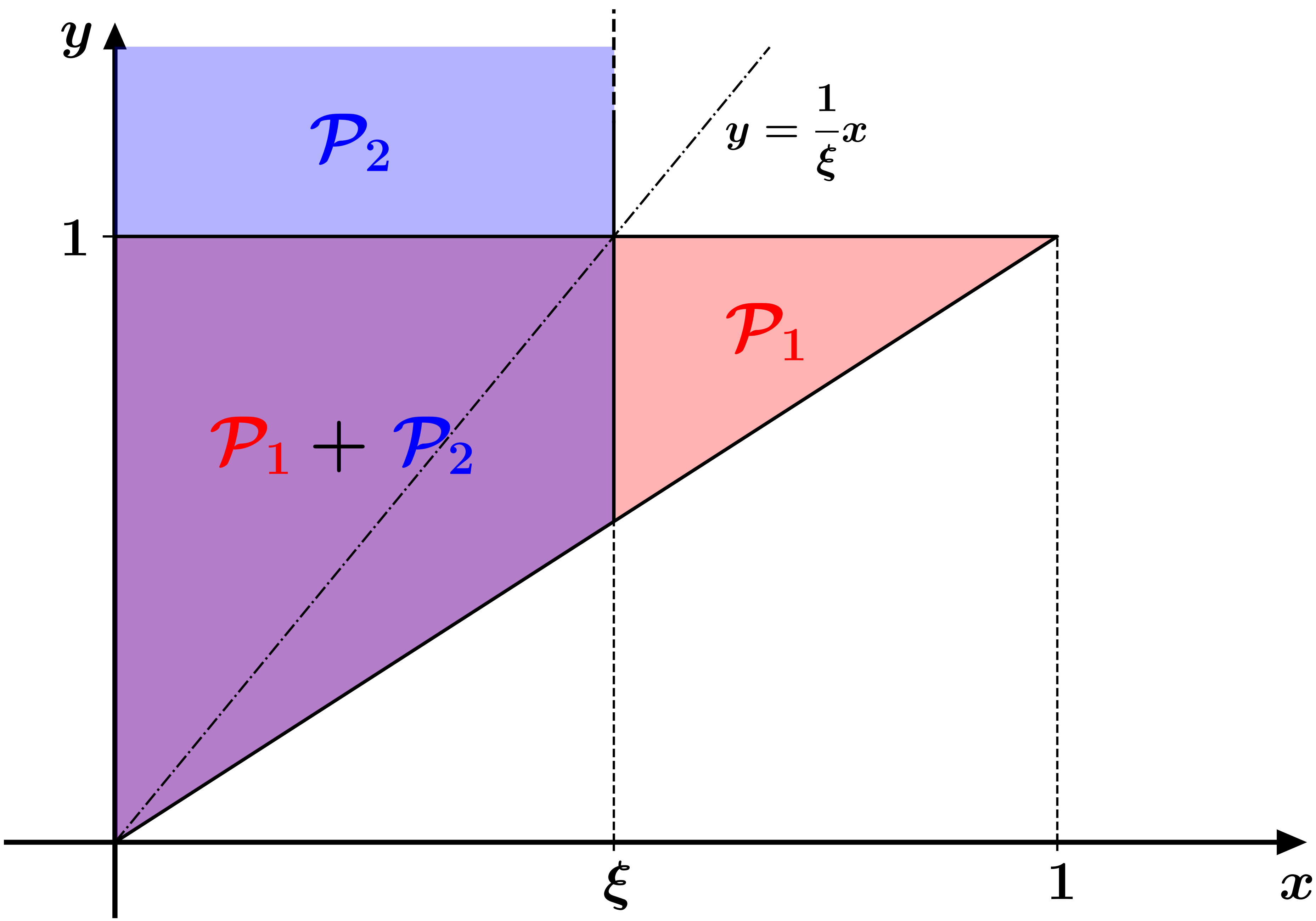
$$\hat{F}^{\mathrm{SG}}_{q/q} = \hat{F}^{\mathrm{NS},+}_{q/q} + n_f(\hat{F}_{q/q'} + \hat{F}_{q/\overline{q}'})$$

- are particularly convenient when implementing the evolution.

# Properties of the kernels

$$\frac{dF^{[\Gamma],\pm}(x,\xi,\mu)}{d\ln\mu^2} = \frac{\alpha_s(\mu)}{4\pi}\int_{x}^{\infty}\frac{dy}{y}\mathcal{P}^{[\Gamma]\pm,[0]}\left(y,\frac{\xi}{x}\right)F^{[\Gamma],\pm}\left(\frac{x}{y},\xi,\mu\right)$$

$$\mathcal{P}^{[\Gamma]\pm,[0]}\left(y,\frac{\xi}{x}\right) = \theta(1-y)\mathcal{P}_1^{[\Gamma]\pm,[0]}\left(y,\frac{\xi}{x}\right) + \theta(\xi-x)\mathcal{P}_2^{[\Gamma]\pm,[0]}\left(y,\frac{\xi}{x}\right)$$



# Properties of the kernels

$$\mathcal{P}^{[\Gamma]\pm,[0]}(y,\kappa) = \theta(1-y)\mathcal{P}_1^{[\Gamma]\pm,[0]}(y,\kappa) + \theta(\kappa-1)\mathcal{P}_2^{[\Gamma]\pm,[0]}(y,\kappa) \qquad \kappa = \frac{\xi}{x}$$

- In the limit $\kappa \to 0$ the **DGLAP** splitting functions are recovered:

$$\lim_{\kappa\to 0} \mathcal{P}^{[\Gamma]\pm,[0]}(y,\kappa) = \theta(1-y)P^{[\Gamma]\pm,[0]}(y)$$

- In the limit $\kappa \to 1/x$ the **ERBL** non-singlet kernel in the unpolarised case is recovered:
  e.g. [Mikhailov, Radyushkin, *Nucl.Phys.B* 254 (1985) 89-126]
  or [Blümlein, Geyer, Robaschik, *Phys.Lett.B* 406 (1997) 161-170]

$$\frac{1}{2u-1}\mathcal{P}^{[U]-,[0]}\left(\frac{2t-1}{2u-1},\frac{1}{2t-1}\right) = C_F\left[\theta(u-t)\left(\frac{t-1}{u}+\frac{1}{u-t}\right) - \theta(t-u)\left(\frac{t}{1-u}+\frac{1}{u-t}\right)\right]_+$$

with $[f(t,u)]_+ \equiv f(t,u) - \delta(u-t)\int_0^1 du'\, f(t,u')$

We have also derived singlet and non-singlet ERBL kernels for the other polarisations.

- **Continuity** of GPDs at the crossover point $x = \xi$ ($\kappa = 1$) guaranteed:

$$\lim_{\kappa\to 1} \mathcal{P}_1^{[\Gamma]\pm,[0]}(y,\kappa) = \text{finite} \qquad \mathcal{P}_2^{[\Gamma]\pm,[0]}(y,\kappa) \propto (1-\kappa)$$

# Properties of the kernels

$$\mathcal{P}^{[\Gamma]\pm,[0]}(y,\kappa) = \theta(1-y)\mathcal{P}_1^{[\Gamma]\pm,[0]}(y,\kappa) + \theta(\kappa-1)\mathcal{P}_2^{[\Gamma]\pm,[0]}(y,\kappa) \qquad \kappa = \frac{\xi}{x}$$

- Valence **sum rule** (polynomiality of the first moment of the **unpolarised non-singlet**):

$$\int_0^1 dx\, F^{[U],-}(x,\xi,\Delta^2;\mu) = \mathrm{FF}(\Delta^2) \quad \Rightarrow \quad \int_0^1 dz \left[\mathcal{P}_1^{[U]-,[0]}\left(z,\frac{\xi}{yz}\right) + \frac{\xi}{y}\mathcal{P}_2^{[U]-,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right)\right] = 0$$

- As consequence of the **Ji's sum rule** one also finds: [Ji, *Phys. Rev. Lett.* 78 (1997) 610-613]

$$\int_0^1 dx\, x\left[F_q^{[U],+}(x,\xi,\Delta^2;\mu) + F_g^{[U],+}(x,\xi,\Delta^2;\mu)\right] = \text{constant in } \xi \text{ and } \mu$$

- that leads to:

$$\int_0^1 dz\, z\left[\mathcal{P}_{1,qq}^{[U]+,[0]}\left(z,\frac{\xi}{yz}\right) + \mathcal{P}_{1,gq}^{[U]+,[0]}\left(z,\frac{\xi}{yz}\right) + \frac{\xi^2}{y^2}\left(\mathcal{P}_{2,qq}^{[U]+,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right) + \mathcal{P}_{2,gq}^{[U]+,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right)\right)\right] = 0$$

$$\int_0^1 dz\, z\left[\mathcal{P}_{1,qg}^{[U]+,[0]}\left(z,\frac{\xi}{yz}\right) + \mathcal{P}_{1,gg}^{[U]+,[0]}\left(z,\frac{\xi}{yz}\right) + \frac{\xi^2}{y^2}\left(\mathcal{P}_{2,qg}^{[U]+,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right) + \mathcal{P}_{2,gg}^{[U]+,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right)\right)\right] = 0$$

- These identities were analytically verified in [*Eur. Phys. J. C* 82 (2022) 10, 888].

# Properties of the kernels

$$\mathcal{P}^{[\Gamma]\pm,[0]}(y,\kappa) = \theta(1-y)\mathcal{P}_1^{[\Gamma]\pm,[0]}(y,\kappa) + \theta(\kappa-1)\mathcal{P}_2^{[\Gamma]\pm,[0]}(y,\kappa) \qquad \kappa = \frac{\xi}{x}$$

- The $\xi$-independence of the **1st moment of longitudinally polarised** GPDs implies:

$$\int_0^1 dz \left[\mathcal{P}_{1,ij}^{L,+,[0]}\left(z,\frac{\xi}{yz}\right) + \frac{\xi}{y}\mathcal{P}_{2,ij}^{L,+,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right)\right] = \text{constant in } \xi$$

- This is true and we also find that the $q/q$ and and $q/g$ channels are identically zero, *i.e.* the first moment of $F_{q/H}^{[L],+}$ is **scale independent**:
  - physical observable connected with the anti-symmetric part of the EMT.
- The $\xi$-independence of the **2nd moment of longitudinally polarised** GPDs implies:

$$\int_0^1 dz\, z \left[\mathcal{P}_1^{L,-,[0]}\left(z,\frac{\xi}{yz}\right) + \frac{\xi^2}{y^2}\mathcal{P}_2^{L,-,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right)\right] = \text{constant in } \xi$$

- Similar arguments apply to **transversely pol.** GPDs and lead to the verified constraints:

$$\int_0^1 dz \left[\mathcal{P}_1^{T,-,[0]}\left(z,\frac{\xi}{yz}\right) + \frac{\xi}{y}\mathcal{P}_2^{T,-,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right)\right] = \text{constant in } \xi$$

$$\int_0^1 dz\, z \left[\mathcal{P}_{1,qq}^{T,+,[0]}\left(z,\frac{\xi}{yz}\right) + \frac{\xi^2}{y^2}\mathcal{P}_{2,qq}^{T,+,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right)\right] = \text{constant in } \xi$$

$$\int_0^1 dz\, z \left[\mathcal{P}_{1,gg}^{T,+,[0]}\left(z,\frac{\xi}{yz}\right) + \frac{\xi^2}{y^2}\mathcal{P}_{2,gg}^{T,+,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right)\right] = \text{constant in } \xi$$

# The ERBL limit

- The limit $\xi \to \mathbf{1}$ ($\kappa \to 1/x$) we should reproduce the **ERBL equation**.

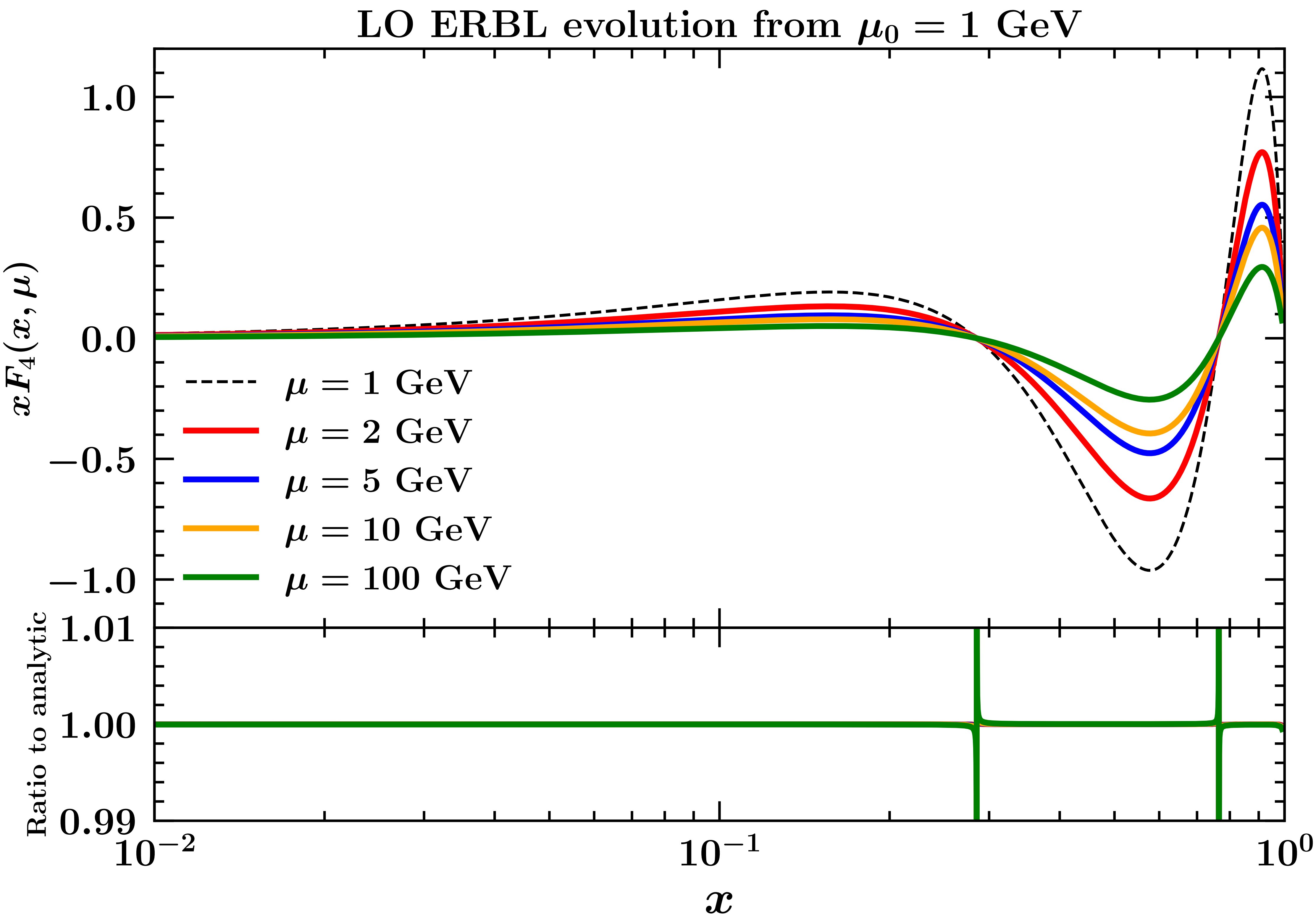
- It is well known that in this limit **Gegenbauer polynomials** decouple upon LO evolution, such that:

$$F_{2n}(x,\mu_0) = (1-x^2)C_{2n}^{(3/2)}(x) \quad \Rightarrow \quad F_{2n}(x,\mu) = \exp\left[\frac{V_{2n}^{[0]}}{4\pi}\int_{\mu_0}^{\mu} d\ln\mu^2 \alpha_s(\mu)\right] F_{2n}(x,\mu_0)$$

- where the kernels $V_{2n}^{[0]}$ can be read off, for example, from [Brodsky, Lepage, *Phys.Rev.D* 22 (1980) 2157] or [Efremov, Radyushkin, *Phys.Lett.B* 94 (1980) 245-250].

- We have compared this expectation with the numerical results for GPD evolution by setting $\kappa = 1/x$ and using a Gegenbauer polynomial as an initial-scale GPD.

# The ERBL limit




- **ERBL limit** reproduced within less than $10^{-5}$ relative accuracy,
- Same accuracy for **higher-degree** Gegenbauer polynomials.

# Conformal-space evolution

- In order to check that LO GPD evolution ($\xi \neq 0$) in conformal space is diagonal in a **realistic** case, we have considered the RDDA:

$$H_q(x,\xi,\mu_0) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha \delta\left(x-\beta-\xi\alpha\right) q(|\beta|)\pi(\beta,\alpha)$$

with:

$$q(x) = \frac{35}{32} x^{-1/2}(1-x)^3, \quad \pi(\beta,\alpha) = \frac{3}{4}\frac{\left((1-|\beta|)^2-\alpha^2\right)}{(1-|\beta|)^3}$$

We have evolved the 4th moment:

$$\mathcal{C}_4^-(\xi,\mu) = \xi^4 \int_{-1}^{1} dx\, C_4^{(3/2)}\left(\frac{x}{\xi}\right) H_q(x,\xi,\mu)$$

from $\mu_0 = 1$ GeV using the (analytic) conformal-space evolution and the (numerical) momentum-space evolution.

we found excellent agreement.