Theory of TMDs

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Corigliano-Rossano
Italy

Plan of the lectures

✓ Review the idea of structure functions for DIS and introduce them for semi-inclusive DIS

✓ Introduce the idea of quark-quark correlation functions

✓ Parametrize correlation functions in terms of PDFs or Transverse Momentum Distributions (TMDs)

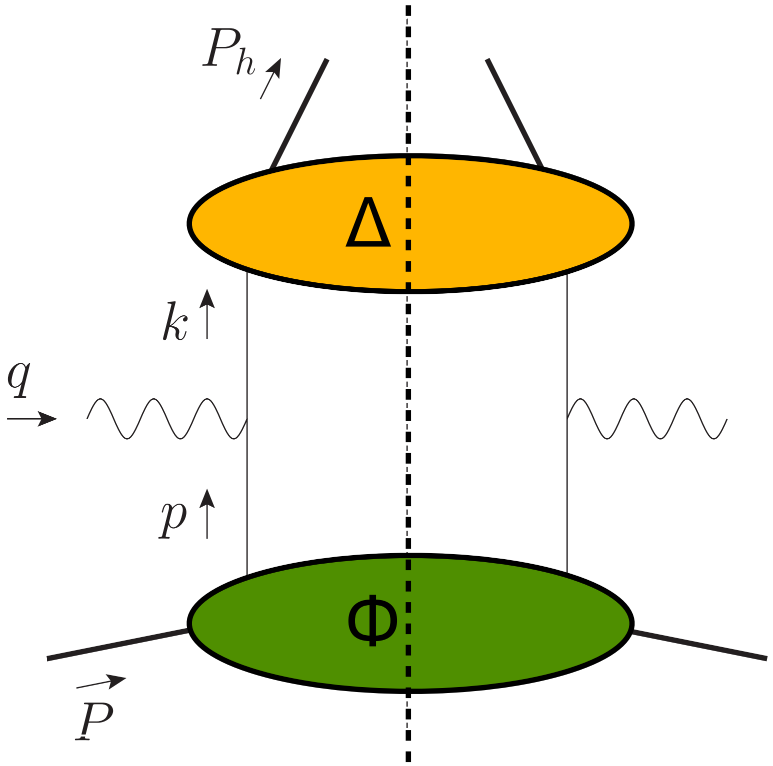
✓ Obtain the expression of structure functions for semi-inclusive DIS in terms of TMDs

- Discuss concept of TMD factorization and TMD evolution

- Discuss a bit of phenomenology

Final formula for hadronic tensor

$$2MW^{\mu\nu}(q,P,S,P_h) = \frac{2z_h}{x_B}\,\mathcal{C}\Big[\mathrm{Tr}(\Phi(x_B,\boldsymbol{p}_T,S)\,\gamma^\mu\,\Delta(z_h,\boldsymbol{K}_T)\,\gamma^\nu)\Big]$$



$$\mathcal{C}\big[wfD\big] = \sum_a x\,e_a^2 \int d^2\boldsymbol{p}_T\,d^2\boldsymbol{K}_T\,\delta^{(2)}\big(z\boldsymbol{p}_T - \boldsymbol{K}_T - \boldsymbol{P}_{h\perp}\big)\,w(\boldsymbol{p}_T,\boldsymbol{K}_T)\,f^a(x,p_T^2)\,D^a(z,K_T^2),$$

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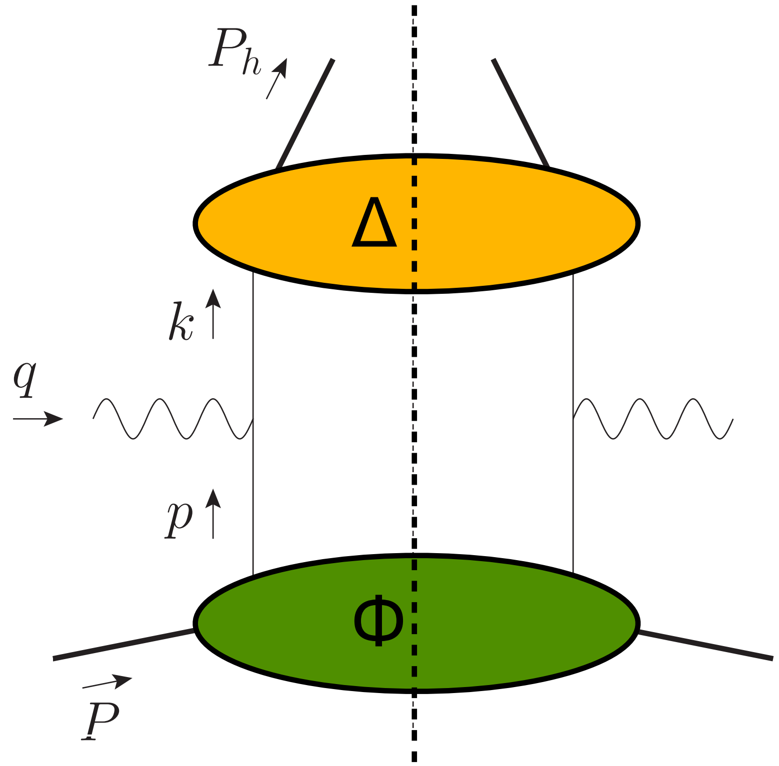

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Only at low transverse momentum $\boldsymbol{P}_{h\perp}^2 \ll Q^2$

Quark-quark correlation functions

$$\Phi_{ij}(x,S) = \int d^2\boldsymbol{p}_T\ \Phi_{ij}(x,\boldsymbol{p}_T)$$
$$= \int \frac{d\xi^-}{2\pi}\ e^{ip\cdot\xi}\langle P,S\big|\,\bar{\psi}_j(0)\,\psi_i(\xi)\,\big|P,S\rangle\Big|_{\xi^+=\boldsymbol{\xi}_T=0}$$

$$\Phi_{ij}(x,\boldsymbol{p}_T,S) = \int dp^-\ \Phi(p,P,S)\Big|_{p^+=xP^+}$$
$$= \int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{(2\pi)^3}\ e^{ip\cdot\xi}\langle P,S\big|\,\bar{\psi}_j(0)\,\psi_i(\xi)\,\big|P,S\rangle\Big|_{\xi^+=0}$$

ξ_T

ξ^-

Quark-quark correlation functions

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$$\Phi_{ij}(x,\boldsymbol{p}_T,S) = \int dp^- \;\, \Phi(p,P,S)\Big|_{p^+=xP^+}$$

$$= \int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{(2\pi)^3} \; e^{ip\cdot\xi} \langle P,S \big| \bar{\psi}_j(0)\,\psi_i(\xi)\,\big| P,S\rangle \Big|_{\xi^+=0}$$

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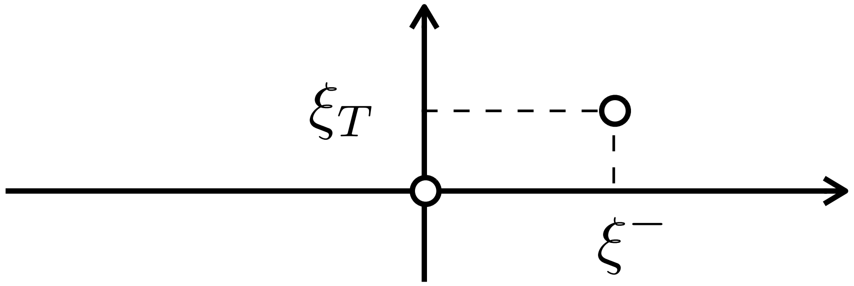

Gauge link

Need of a gauge link

$$\Phi_{ij}(p,P,S) = \frac{1}{(2\pi)^4} \int d^4\xi \; e^{ip\cdot\xi} \big\langle P,S \big| \,\bar{\psi}_j(0)\,\psi_i(\xi)\,\big| P,S \big\rangle$$

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not invariant under $\psi(\xi) \rightarrow e^{i\alpha(\xi)}\, \psi(\xi)$

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⇓

$$\Phi_{ij}(p,P,S) = \frac{1}{(2\pi)^4}\int d^4\xi \; e^{ip\cdot\xi}\langle P,S\big|\,\bar{\psi}_j(0)\,U_{[0,\xi]}\,\psi_i(\xi)\,\big|P,S\rangle$$

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$$U(\xi_1,\xi_2) \to e^{i\alpha(\xi_1)}\,U(\xi_1,\xi_2)\,e^{-i\alpha(\xi_2)}.$$

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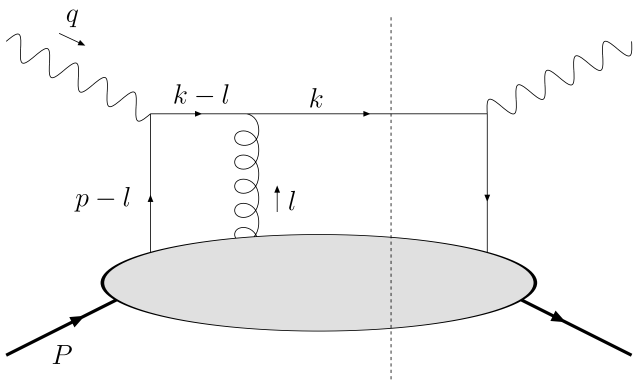
$$U(\xi_1,\xi_2) \to e^{i\alpha(\xi_1)}\,U(\xi_1,\xi_2)\,e^{-i\alpha(\xi_2)}.$$

$$U_{[a,b]} = \mathcal{P}\exp\left[-ig\int_a^b d\eta^\mu A_\mu(\eta)\right]$$

Origin of gauge link



(a)

Ji, Yuan, PLB 543 (02); Belitsky, Ji, Yuan, NPB656 (03)

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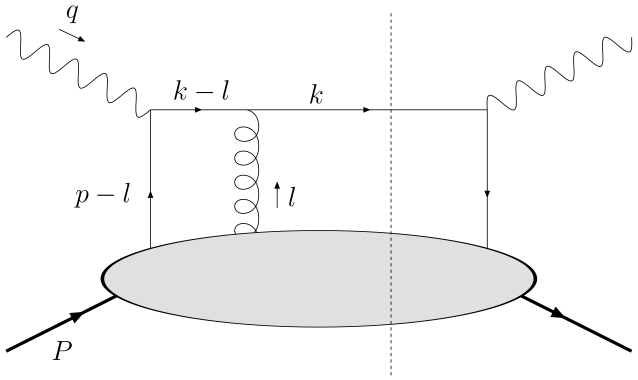

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$$2MW_{\mu\nu}^{(a)} \sim \int d^4l \int \frac{d^4\eta}{(2\pi)^4} e^{il\cdot(\eta-\xi)} \langle P,S|\overline{\psi}(0)\gamma_\mu\gamma^+\gamma_\alpha \frac{\not k - \not l}{(k-l)^2+i\epsilon}\gamma_\nu g A^\alpha(\eta)\psi(\xi)|P,S\rangle$$

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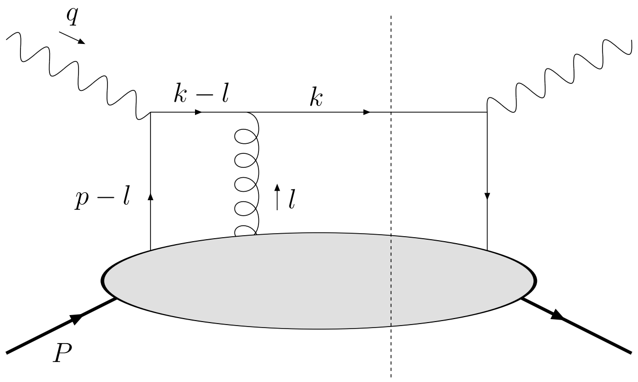

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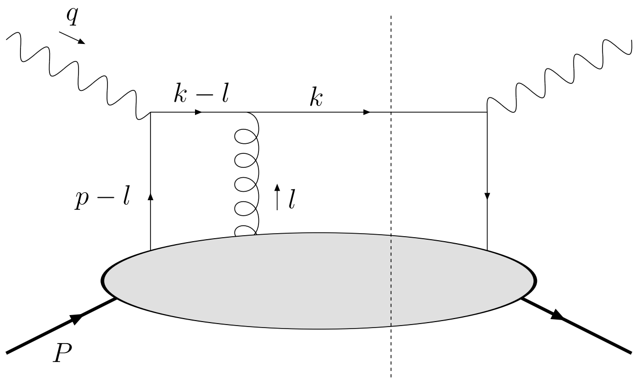

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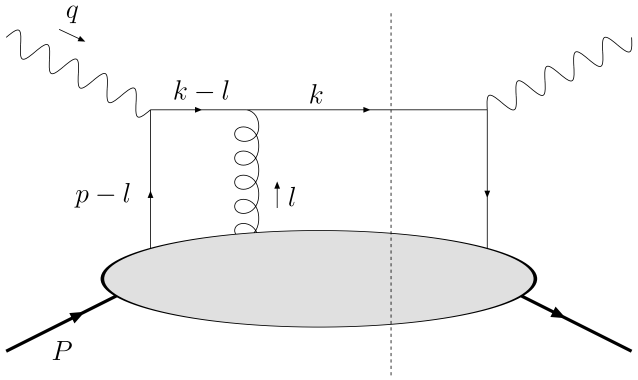

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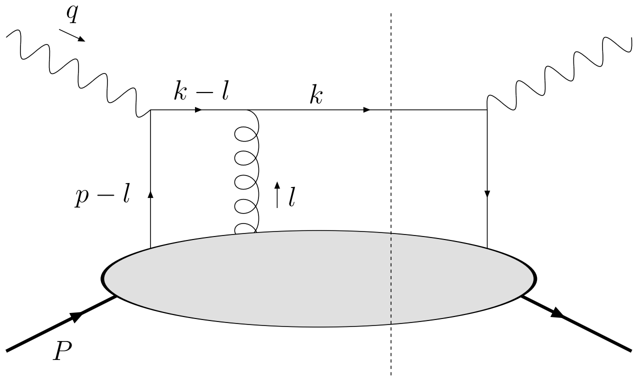

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eikonal approximation

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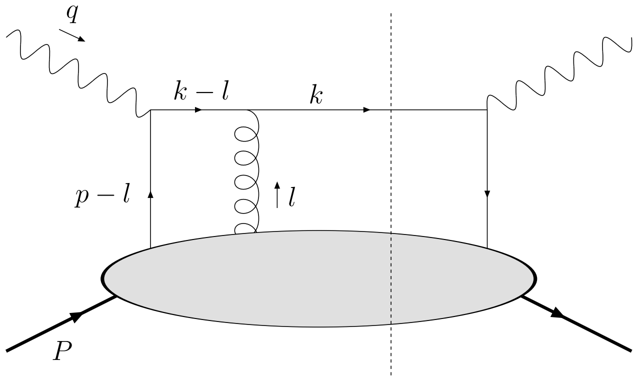

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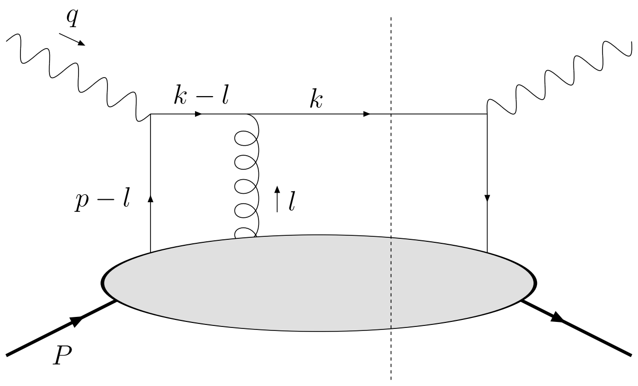

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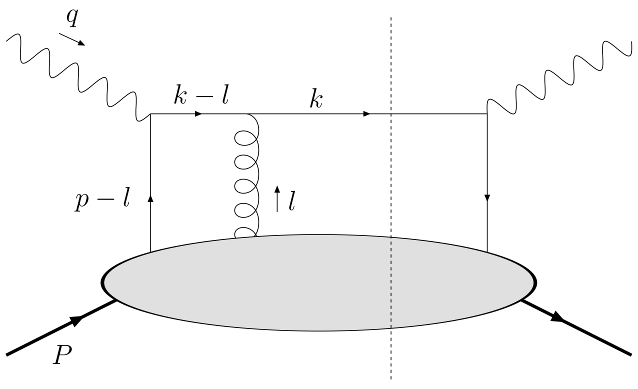

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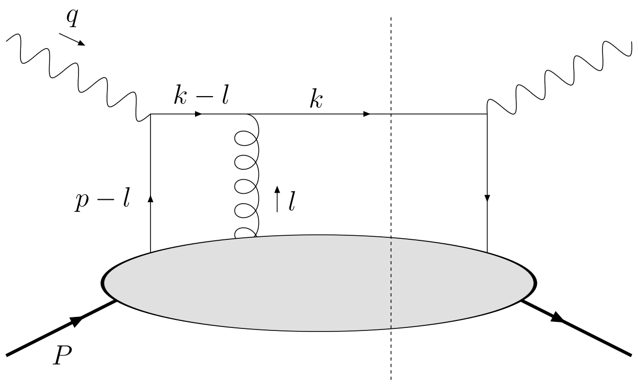

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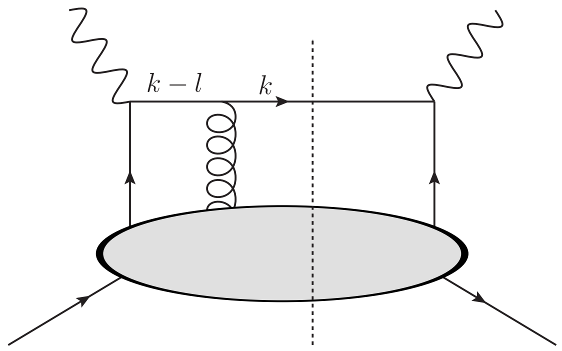
First contribution to gauge link



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◦

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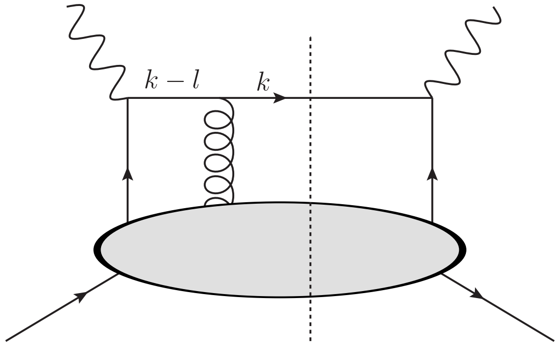

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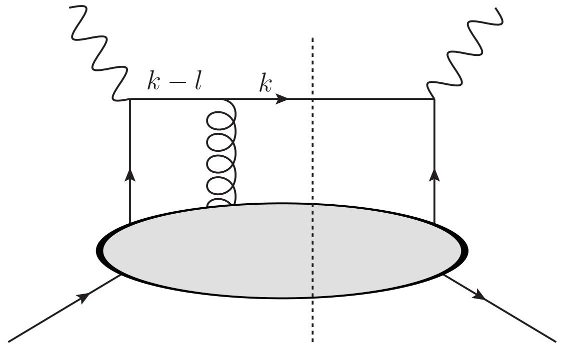
compare with:

$$2MW^{\mu\nu}(q,P,S) \approx \sum_q e_q^2\,\frac{1}{2}\mathrm{Tr}\left[\Phi(x_B,S)\,\gamma^\mu\gamma^+\gamma^\nu\right].$$

○

First contribution to gauge link



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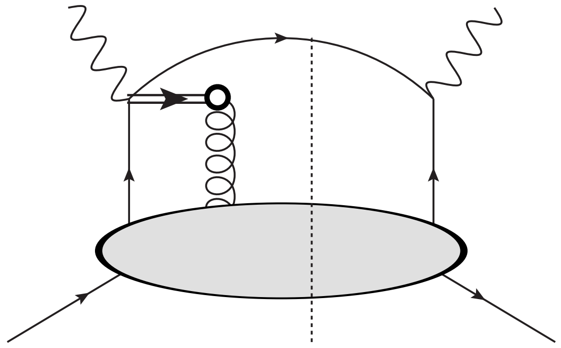
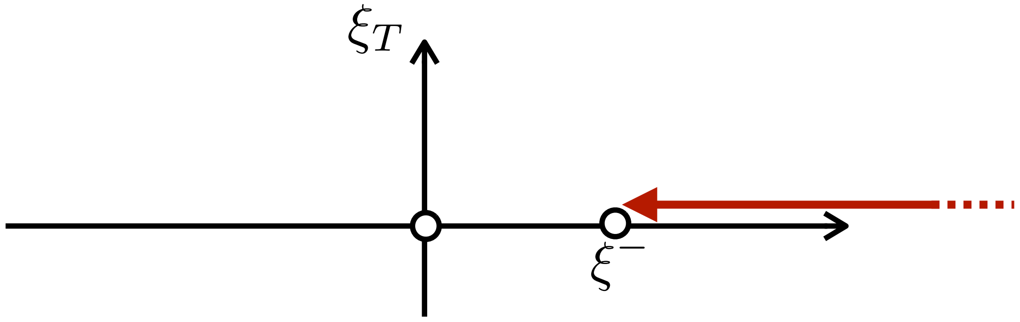


$$\Phi^{(a)}(x,S) \sim \big\langle P,S\big|\,\bar{\psi}(0)\,(-ig)\int_{\infty^-}^{\xi^-} \mathrm{d}\eta^-\;A^+(\eta)\;\psi(\xi)\,\big|P,S\big\rangle$$



Feynman rules for eikonal lines

Buffing, Diehl, Kasemets, arxiv:1708.03528

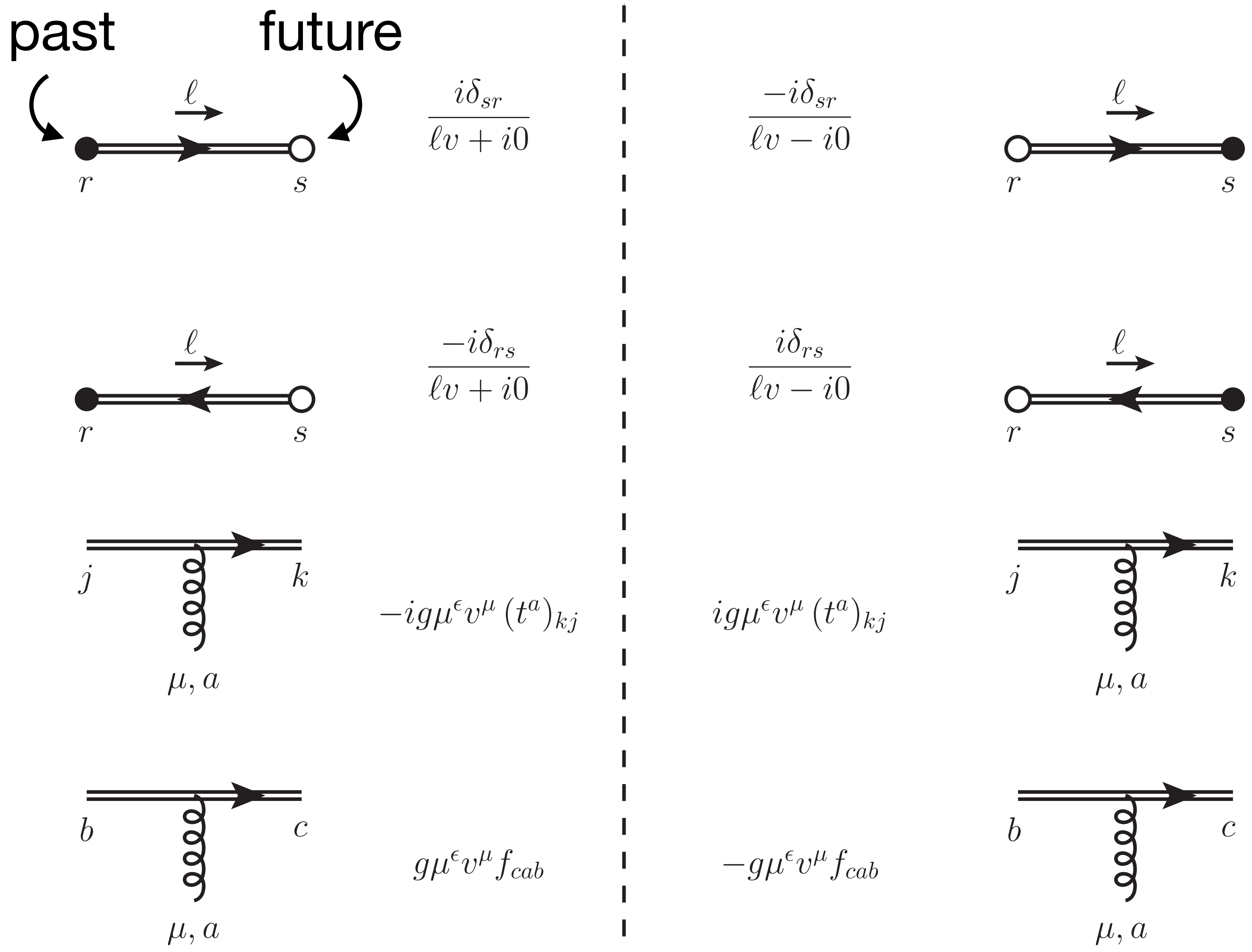
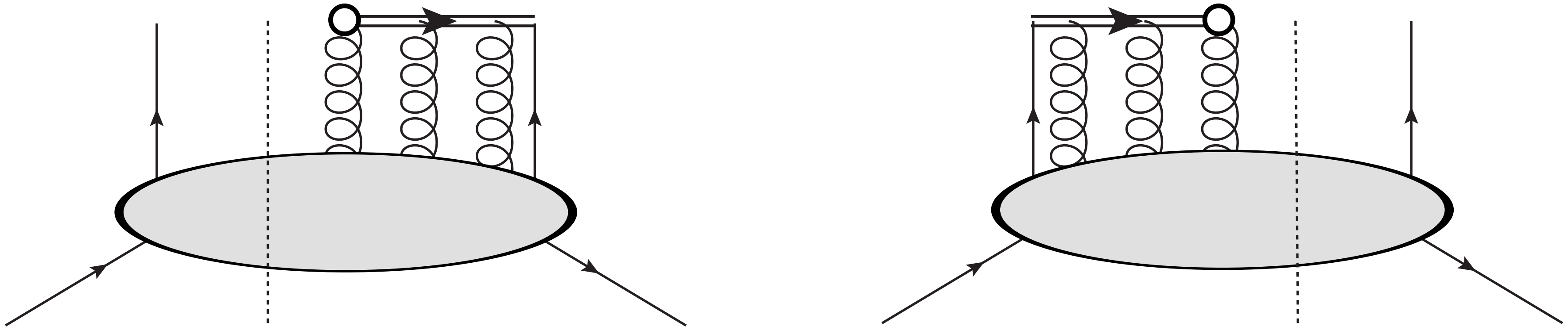


Figure 13. Feynman rules involving eikonal lines along the direction v. The colour indices r and s refer to either the fundamental or the adjoint representation, whereas j and k are colour triplet and a, b and c are colour octet indices.

Gauge link in collinear PDFs

$$\Phi(x,S) \sim \big\langle P,S \big| \overline{\psi}(0)\, U_{[0,\infty^-]}\, U_{[\infty^-,\xi^-]} \psi(\xi) \big| P,S \big\rangle$$



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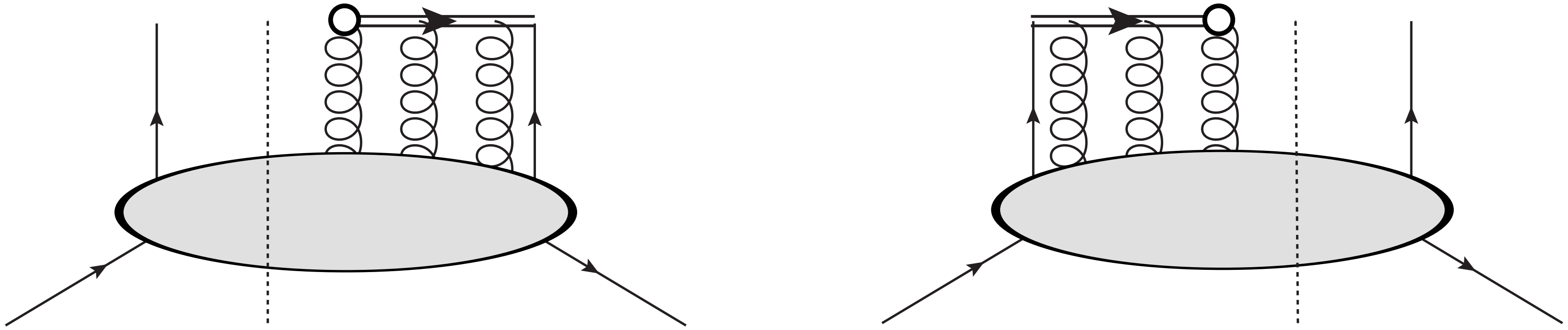
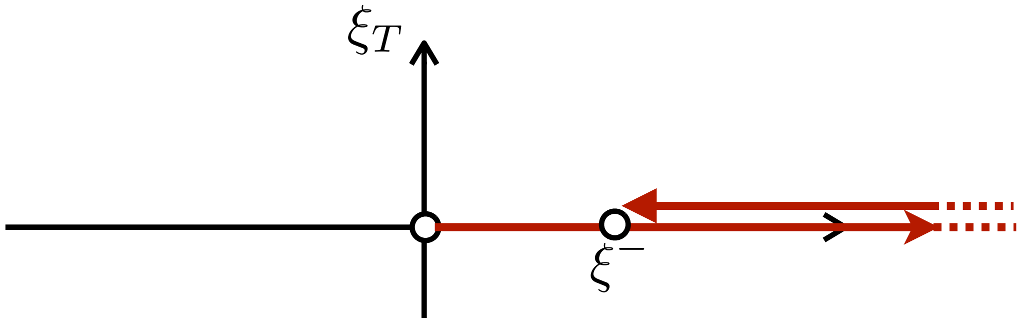
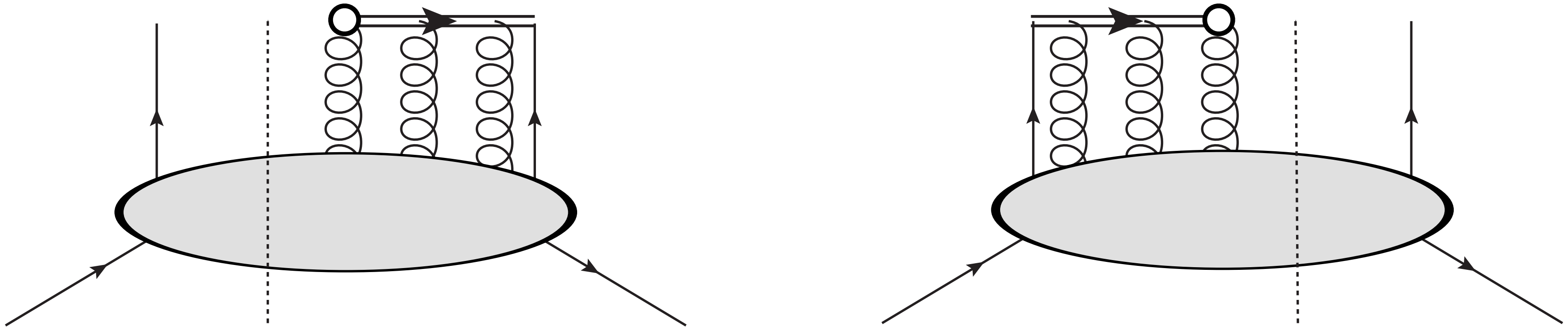
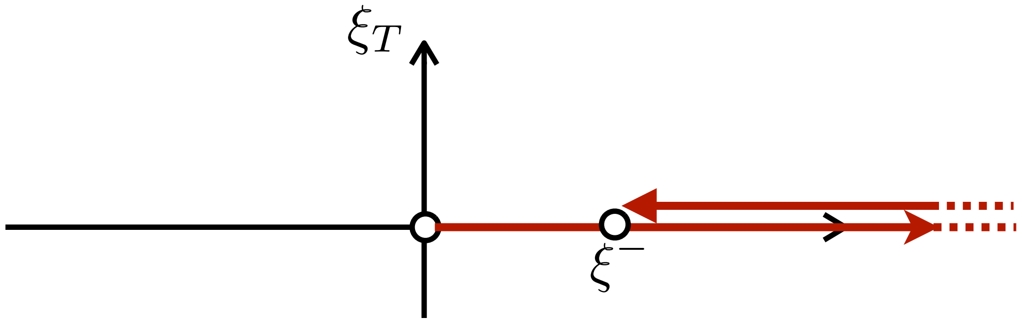


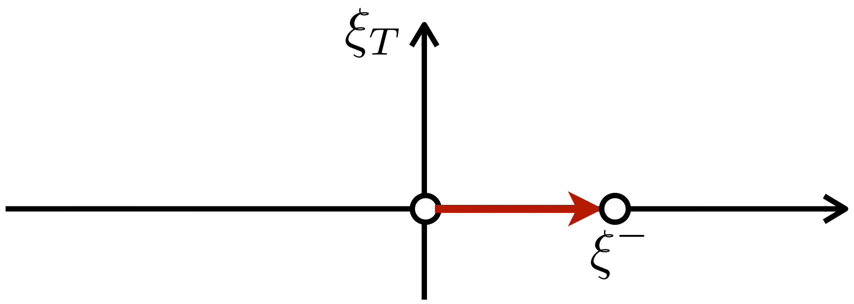


Gauge link in collinear PDFs

$$\Phi(x, S) \sim \langle P, S \big| \, \overline{\psi}(0) \, U_{[0,\infty^-]} \, U_{[\infty^-,\xi^-]} \psi(\xi) \, \big| P, S \rangle$$







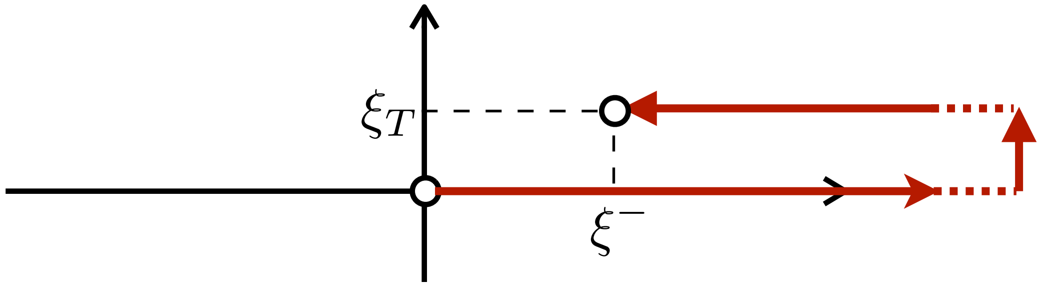
Gauge link for TMDs

$$\Phi_{ij}(x, p_T) = \int \frac{d\xi^- d^2\xi_T}{8\pi^3}\, e^{ip\cdot\xi} \langle P|\bar{\psi}_j(0) U_{[0,\xi]} \psi_i(\xi)|P\rangle\Big|_{\xi^+=0}$$

Gauge link for TMDs

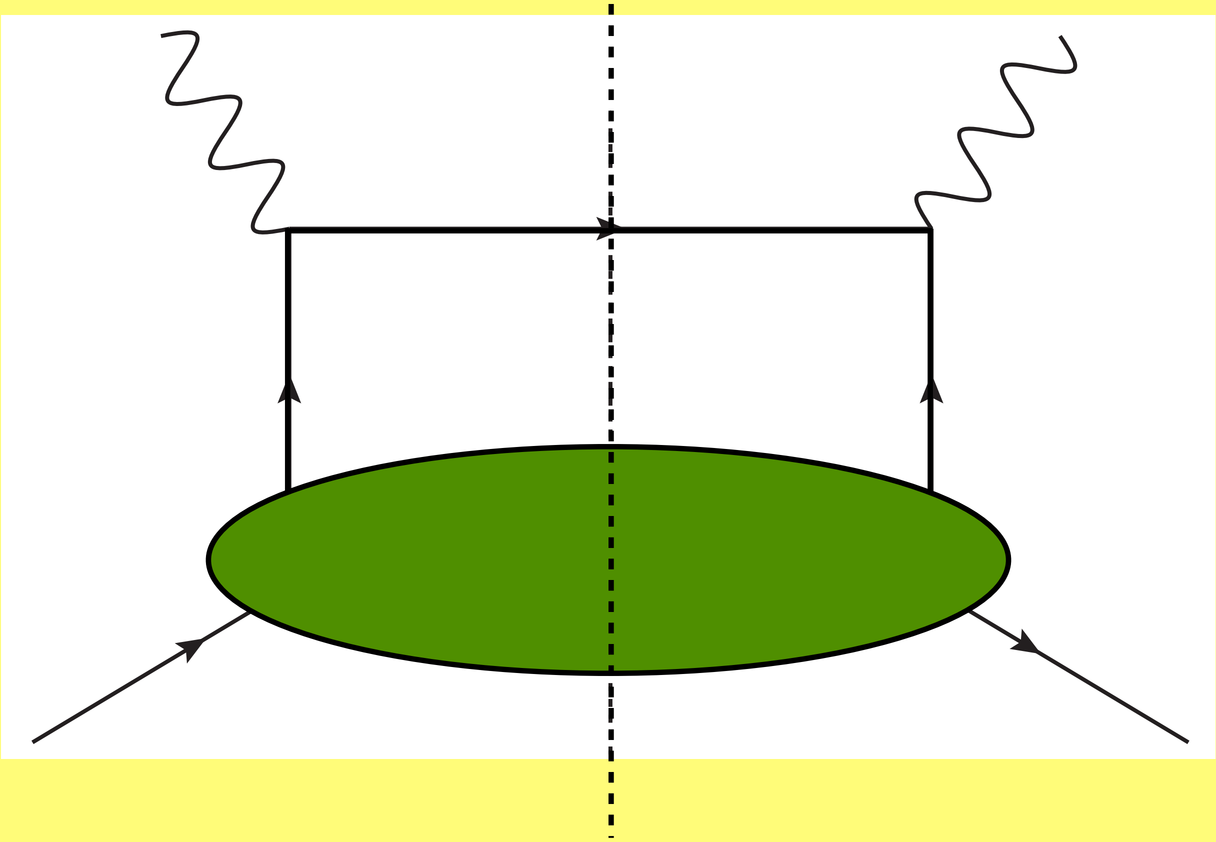
$$\Phi_{ij}(x, p_T) = \int \frac{d\xi^- d^2\xi_T}{8\pi^3} e^{ip\cdot\xi} \langle P|\bar{\psi}_j(0) U_{[0,\xi]} \psi_i(\xi)|P\rangle\Big|_{\xi^+=0}$$

SIDIS

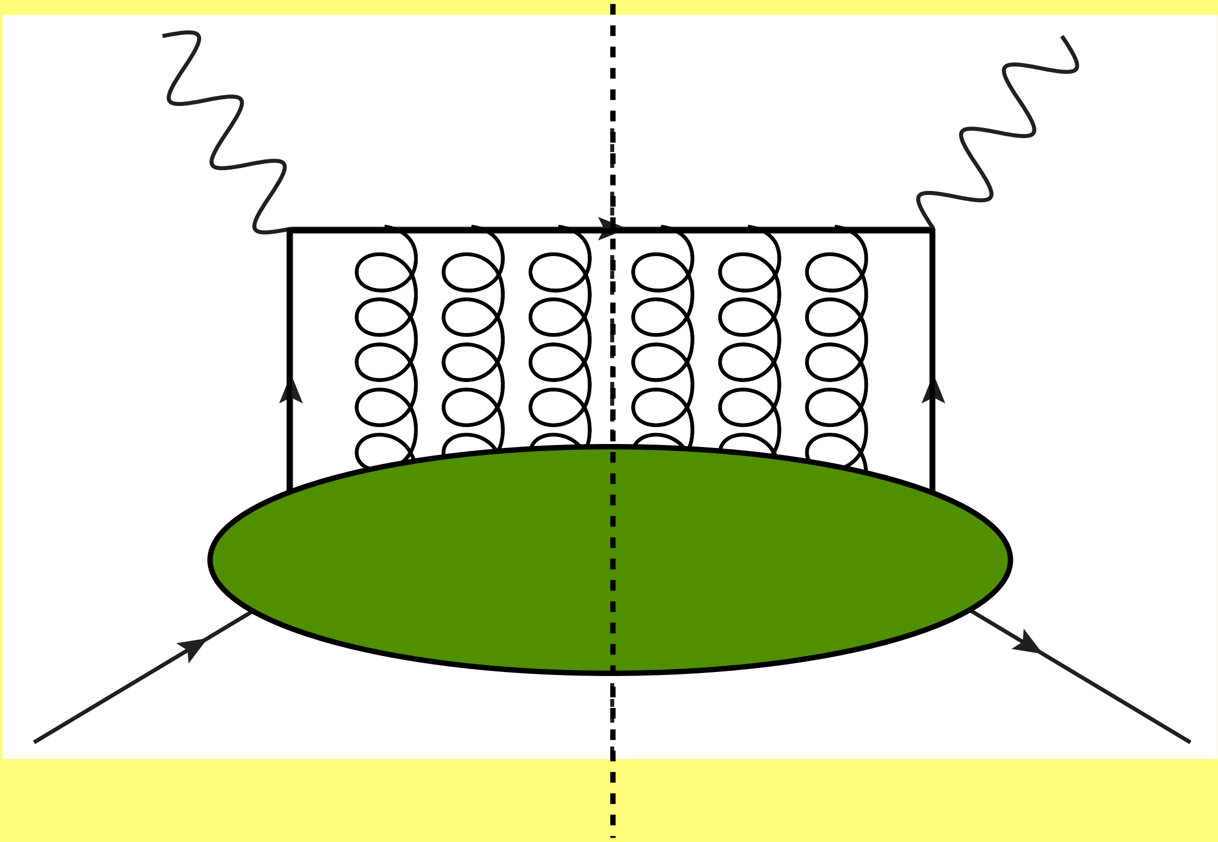


$U_{[+]}$

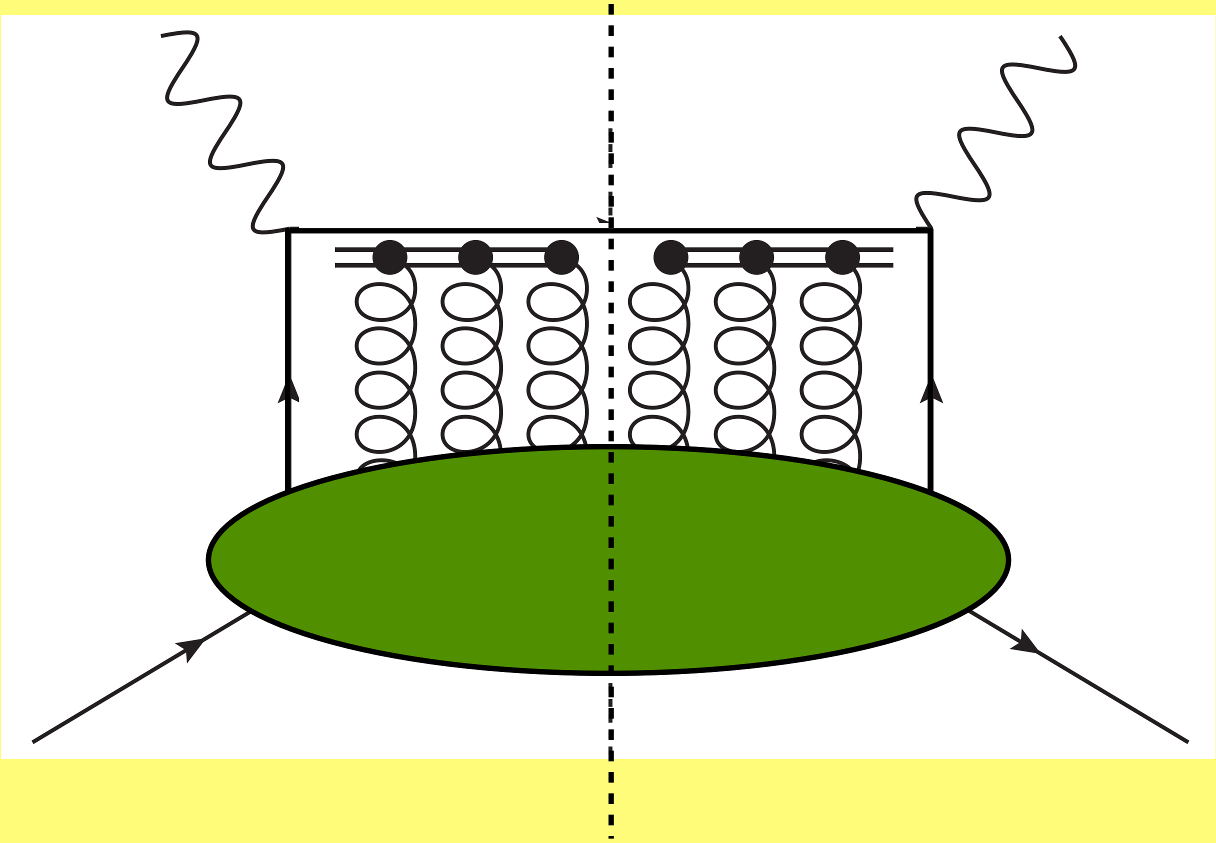
Key point (graphically)



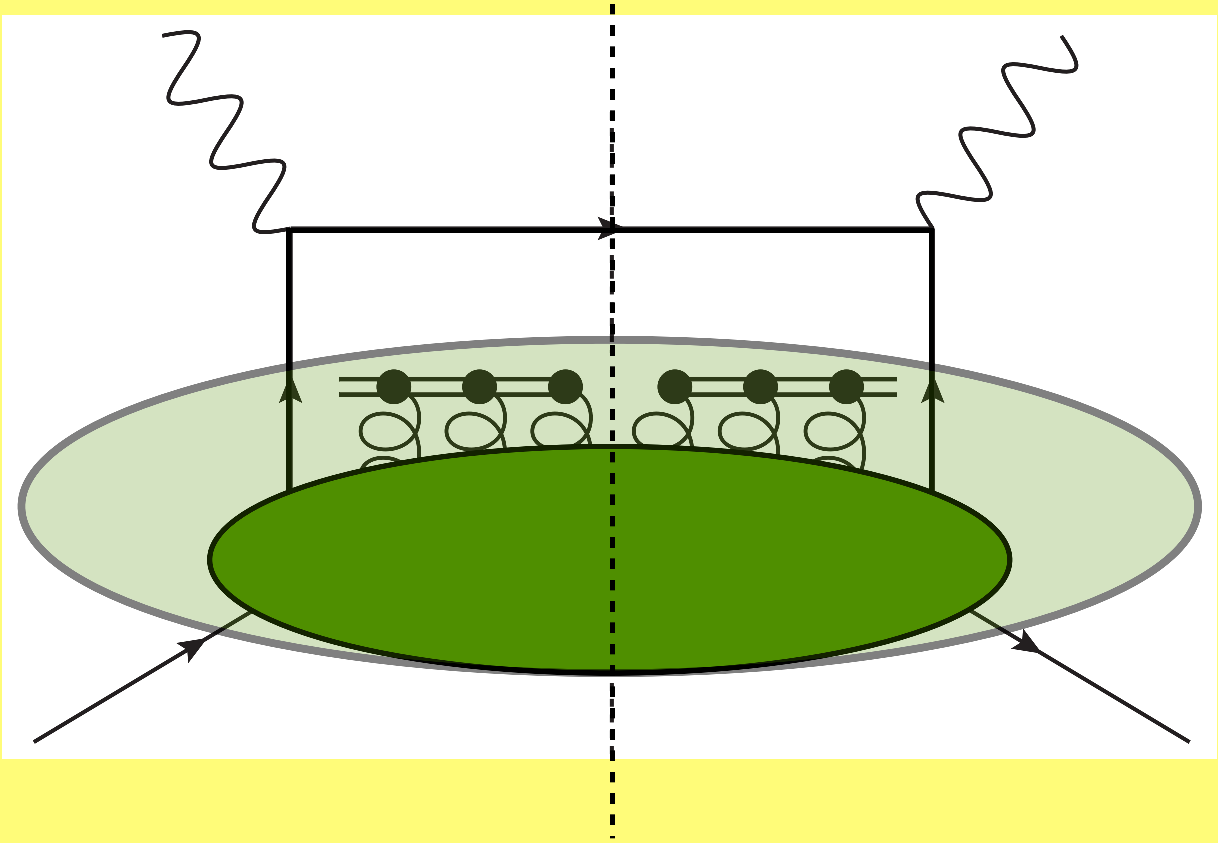
Key point (graphically)



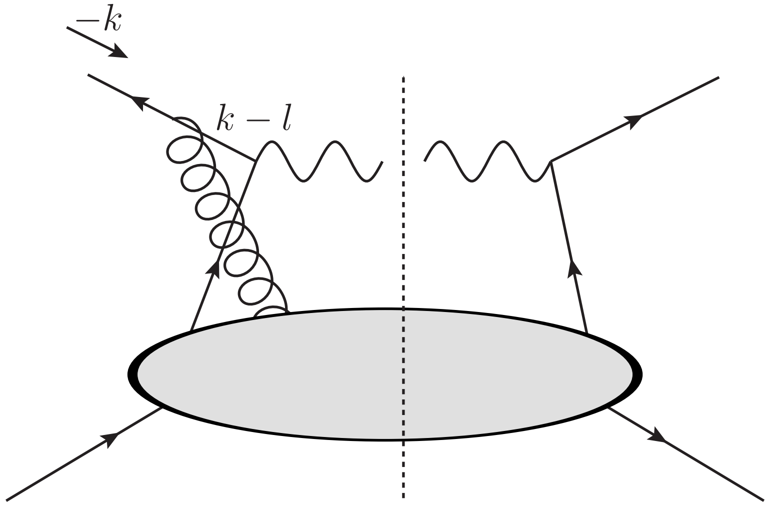
Key point (graphically)



Key point (graphically)



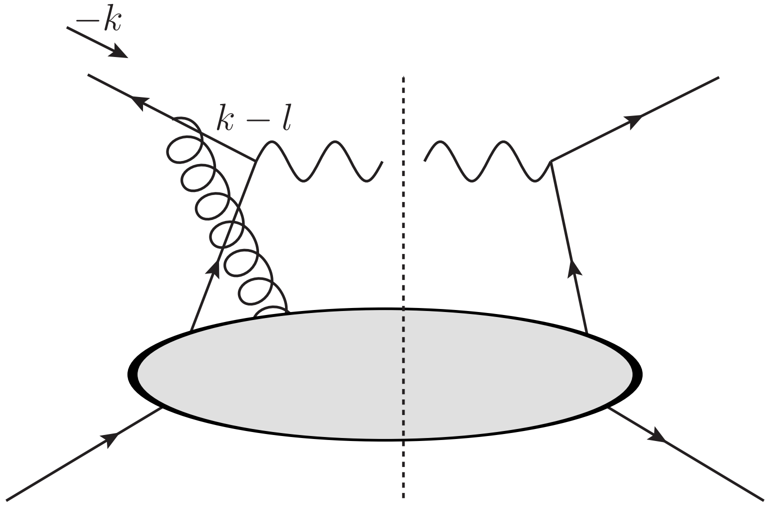
Gauge links in Drell-Yan

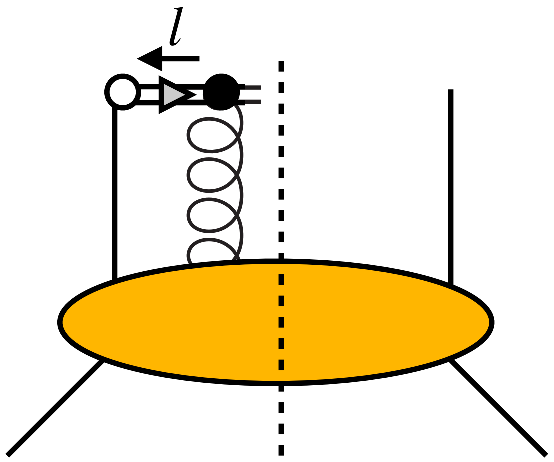


$$2MW_{\mu\nu}^{(a)} \sim \int d^4l \int \frac{d^4\eta}{(2\pi)^4}\, e^{il\cdot(\eta-\xi)} \langle P,S|\overline{\psi}(0)\gamma_\mu\gamma^+\gamma_\alpha\, \frac{\not{k}-\not{l}}{(k-l)^2+i\epsilon}\, \gamma_\nu g A^\alpha(\eta)\psi(\xi)|P,S\rangle$$

Collins, PLB 536 (02)

Gauge links in Drell-Yan

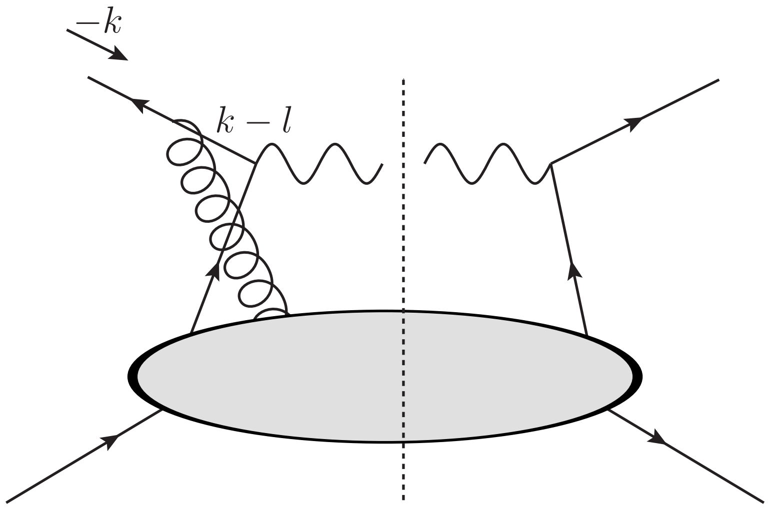


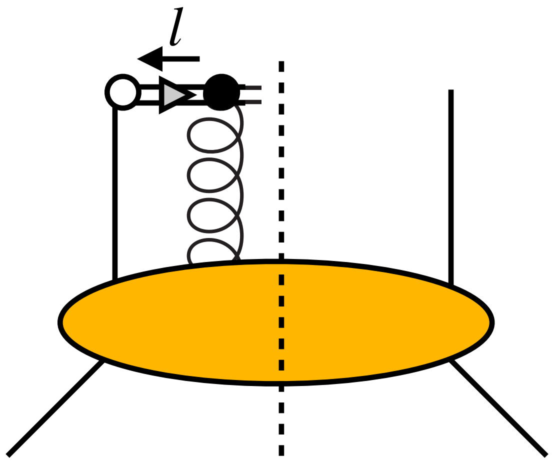


$$2MW^{(a)}_{\mu\nu} \sim \int d^4 l \int \frac{d^4\eta}{(2\pi)^4}\, e^{il\cdot(\eta-\xi)} \langle P,S|\overline{\psi}(0)\gamma_\mu\gamma^+\gamma_\alpha \,\frac{\not{k}-\not{l}}{(k-l)^2+i\epsilon}\, \gamma_\nu g A^\alpha(\eta)\psi(\xi)|P,S\rangle$$

Collins, PLB 536 (02)

Gauge links in Drell-Yan



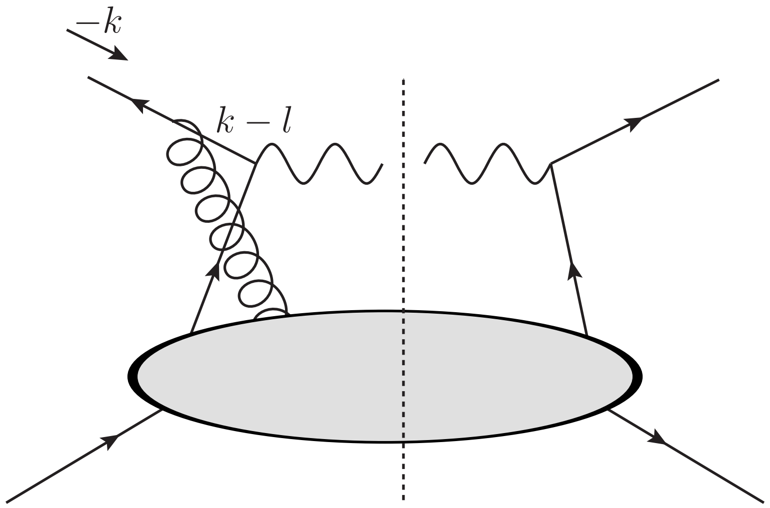


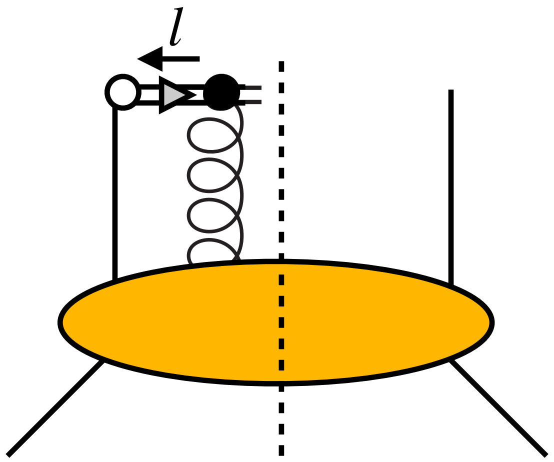
$$2MW_{\mu\nu}^{(a)} \sim \int d^4l \int \frac{d^4\eta}{(2\pi)^4}\, e^{il\cdot(\eta-\xi)} \langle P,S|\overline{\psi}(0)\gamma_\mu\gamma^+\gamma_\alpha \frac{\not{k} - \not{l}}{(k-l)^2+i\epsilon}\gamma_\nu g A^\alpha(\eta)\psi(\xi)|P,S\rangle$$

$$i\,\frac{\not{k} - \not{l} + m}{(k-l)^2 - m^2 + i\epsilon} \approx i\,\frac{-(-k)^-\gamma^+}{2l^+(-k)^- + i\epsilon} \approx \frac{i}{2}\,\frac{\gamma^+}{-l^+ - i\epsilon}$$

Collins, PLB 536 (02)

Gauge links in Drell-Yan



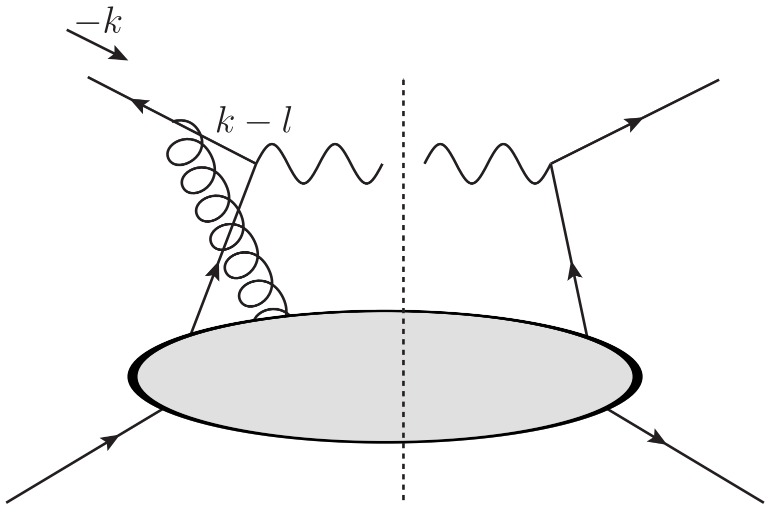


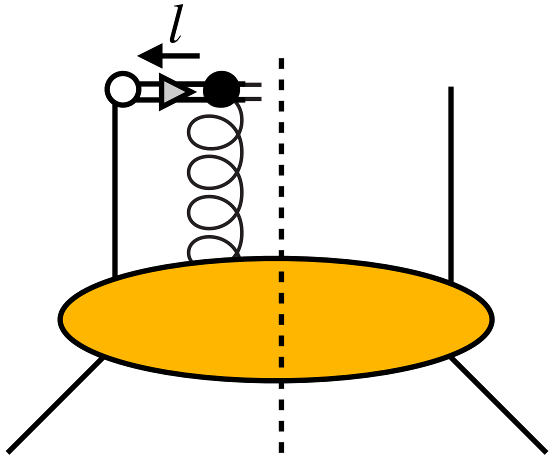
$$2MW^{(a)}_{\mu\nu} \sim \int d^4l \int \frac{d^4\eta}{(2\pi)^4}\, e^{il\cdot(\eta-\xi)} \langle P,S|\overline{\psi}(0)\gamma_\mu\gamma^+\gamma_\alpha \frac{\not k - \not l}{(k-l)^2+i\epsilon}\gamma_\nu g A^\alpha(\eta)\psi(\xi)|P,S\rangle$$

$$i\,\frac{\not k - \not l + m}{(k-l)^2 - m^2 + i\epsilon} \approx i\,\frac{-(-k)^-\gamma^+}{2l^+(-k)^- + i\epsilon} \approx \frac{i}{2}\,\frac{\gamma^+}{-l^+ - i\epsilon}$$

Collins, PLB 536 (02)

Gauge links in Drell-Yan





$$2MW^{(a)}_{\mu\nu} \sim \int d^4 l \int \frac{d^4\eta}{(2\pi)^4}\, e^{il\cdot(\eta-\xi)} \langle P,S|\overline{\psi}(0)\gamma_\mu\gamma^+\gamma_\alpha \frac{\not{k}-\not{l}}{(k-l)^2+i\epsilon}\gamma_\nu g A^\alpha(\eta)\psi(\xi)|P,S\rangle$$

$$i\,\frac{\not{k}-\not{l}+m}{(k-l)^2-m^2+i\epsilon} \approx i\,\frac{-(-k)^-\gamma^+}{2l^+(-k)^-+i\epsilon} \approx \frac{i}{2}\,\frac{\gamma^+}{-l^+-i\epsilon}$$

$$2MW^{(a)}_{\mu\nu} \sim \langle P,S|\overline{\psi}(0)\,\gamma_\mu\gamma^+\,\gamma_\nu\,(-ig)\int_{-\infty^-}^{\xi^-} \mathrm{d}\eta^-\;A^+(\eta)\;\psi(\xi)|P,S\rangle\Bigg|_{\eta^+=0;\;\boldsymbol{\eta}_T=\boldsymbol{\xi}_T}$$

Collins, PLB 536 (02)

Gauge link for TMDs

$$\Phi_{ij}(x, p_T) = \int \frac{d\xi^- d^2\xi_T}{8\pi^3}\, e^{ip\cdot\xi} \langle P|\bar{\psi}_j(0) U_{[0,\xi]} \psi_i(\xi)|P\rangle \Big|_{\xi^+=0}$$

Gauge link for TMDs

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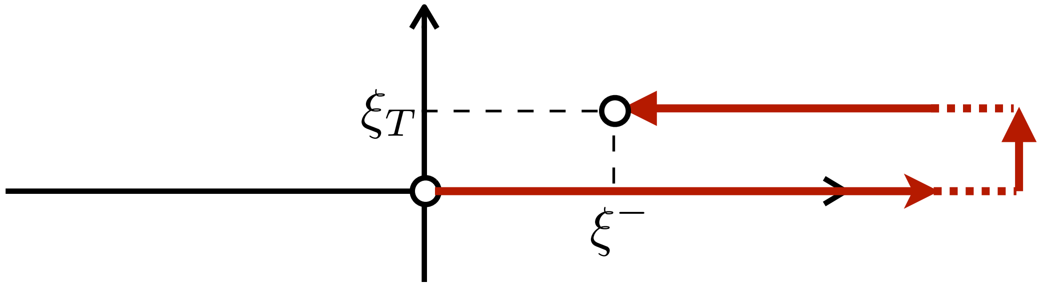
SIDIS

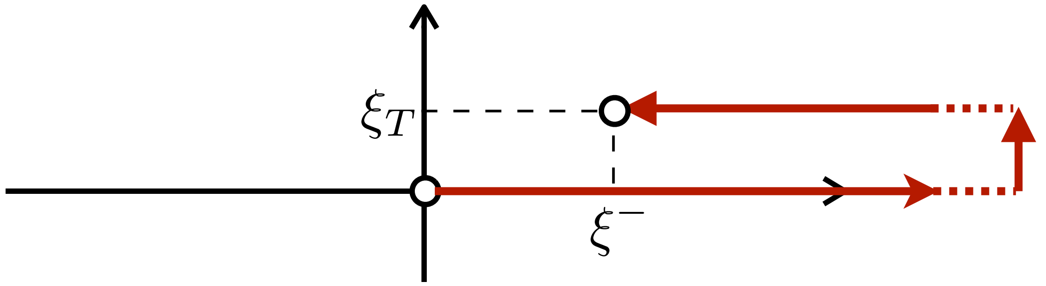


$U_{[+]}$

Gauge link for TMDs

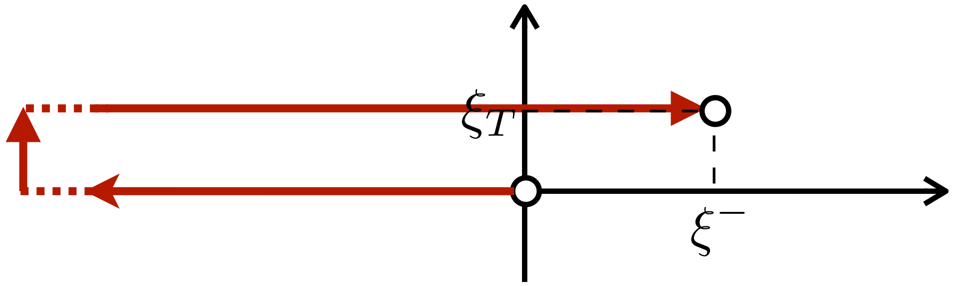
$$\Phi_{ij}(x,p_T) = \int \frac{d\xi^- d^2\xi_T}{8\pi^3}\, e^{ip\cdot\xi} \langle P|\bar{\psi}_j(0)U_{[0,\xi]}\psi_i(\xi)|P\rangle\Big|_{\xi^+=0}$$

SIDIS



$U_{[+]}$

Drell-Yan



$U_{[-]}$

Gauge link and (naive) T-odd functions

- Time reversal determines whether the gauge link goes to + or – infinity

- In the collinear case, where the gauge link turns out to be the same, this means that a certain class of PDFs (called “T-odd” or “naive T-odd”) has to vanish

- In the TMD case, however, the gauge links are different and this means that it is possible to have T-odd functions (Boer-Mulders and Sivers)

Key point

- Gauge links have a staple-like shape
- Different processes have different gauge links
- Gauge links are there also for collinear PDFs, but they are “trivial” and universal
- The difference in the gauge links makes it possible to have T-odd TMDs

Basic ideas about factorization

Factorization for Drell-Yan

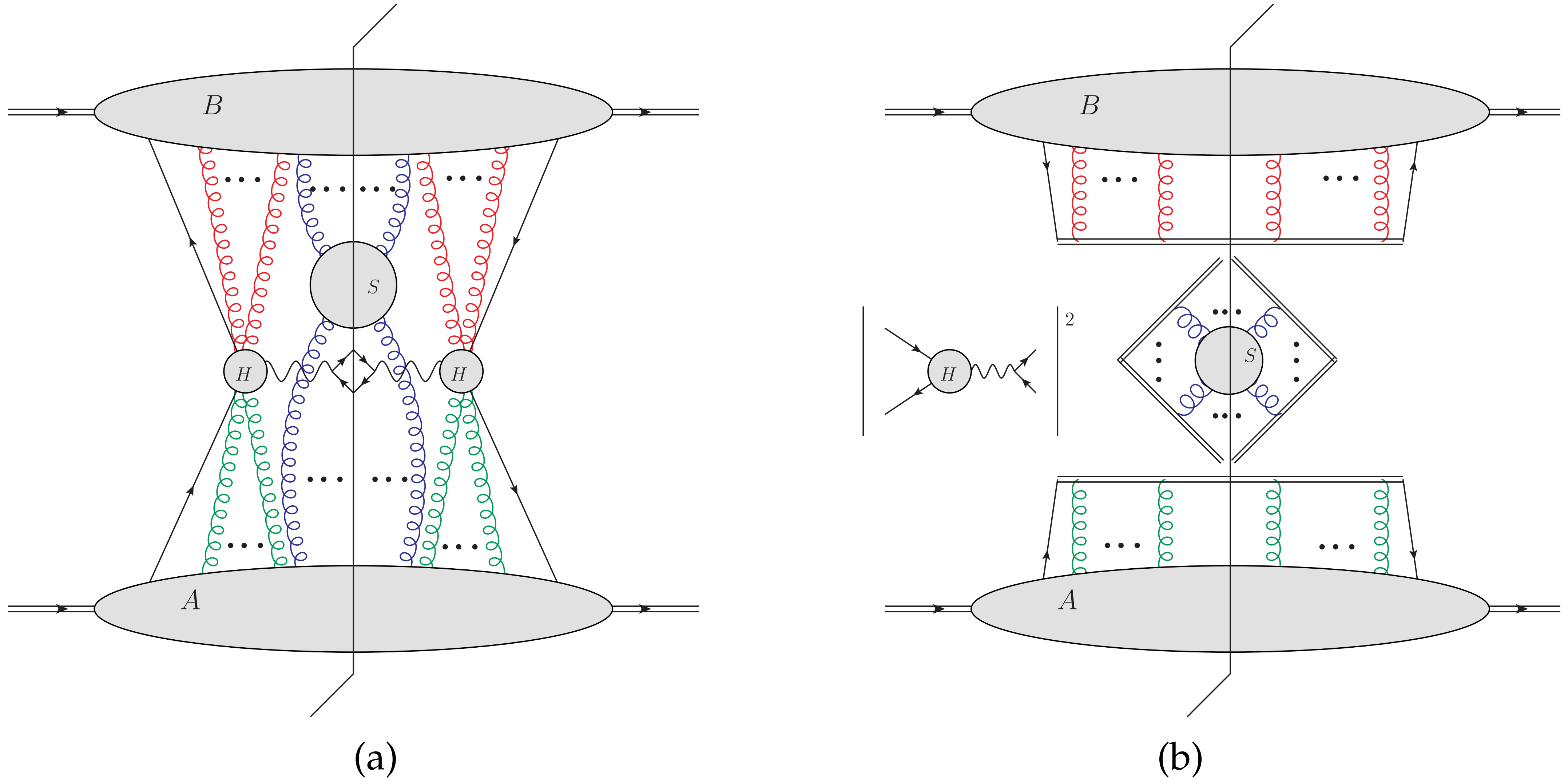
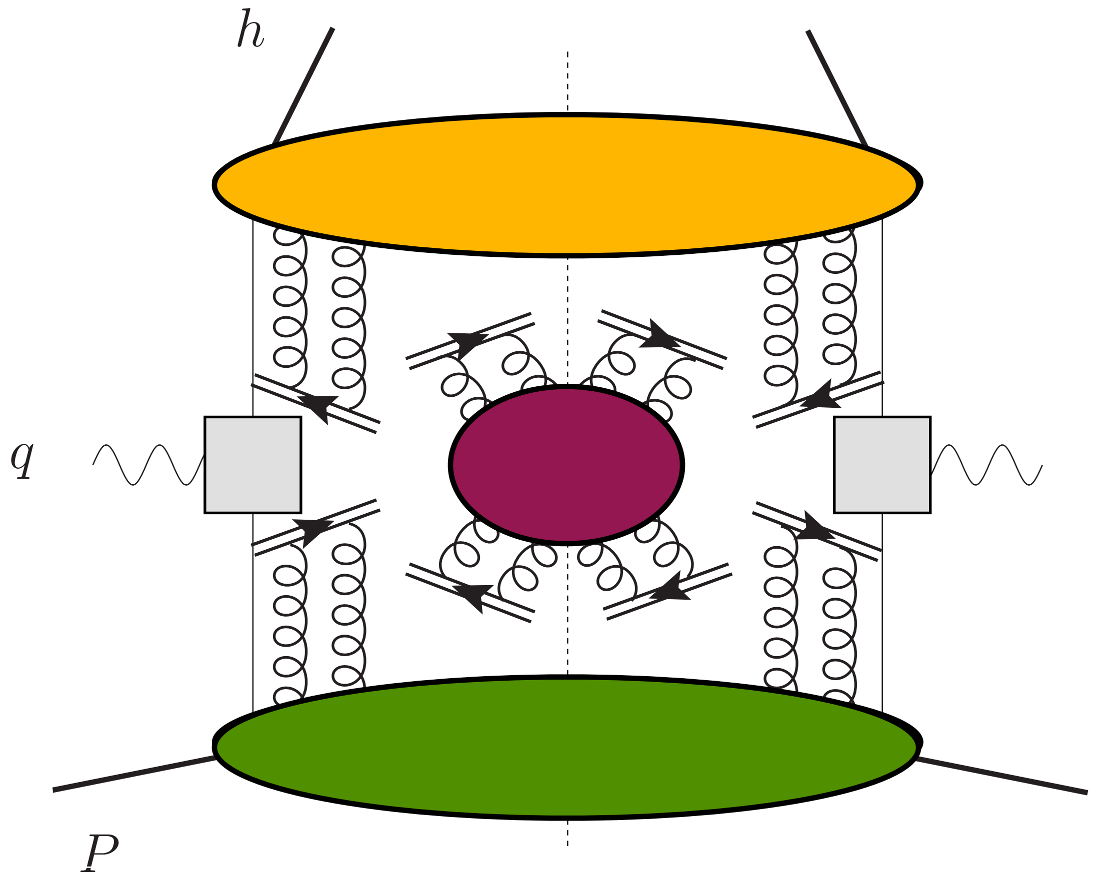


Figure 3.1: (a) Graphical structure corresponding to leading regions in Drell-Yan scattering, before factorization. Green gluons are collinear to lines in the *A*-blob, red gluons are collinear to lines in the *B*-blob, and blue gluons have nearly zero momentum (soft). (b) Separation into hard, soft, and collinear parts after approximations and Ward identities—see Sec. 3.2.5.

Factorization for Drell-Yan

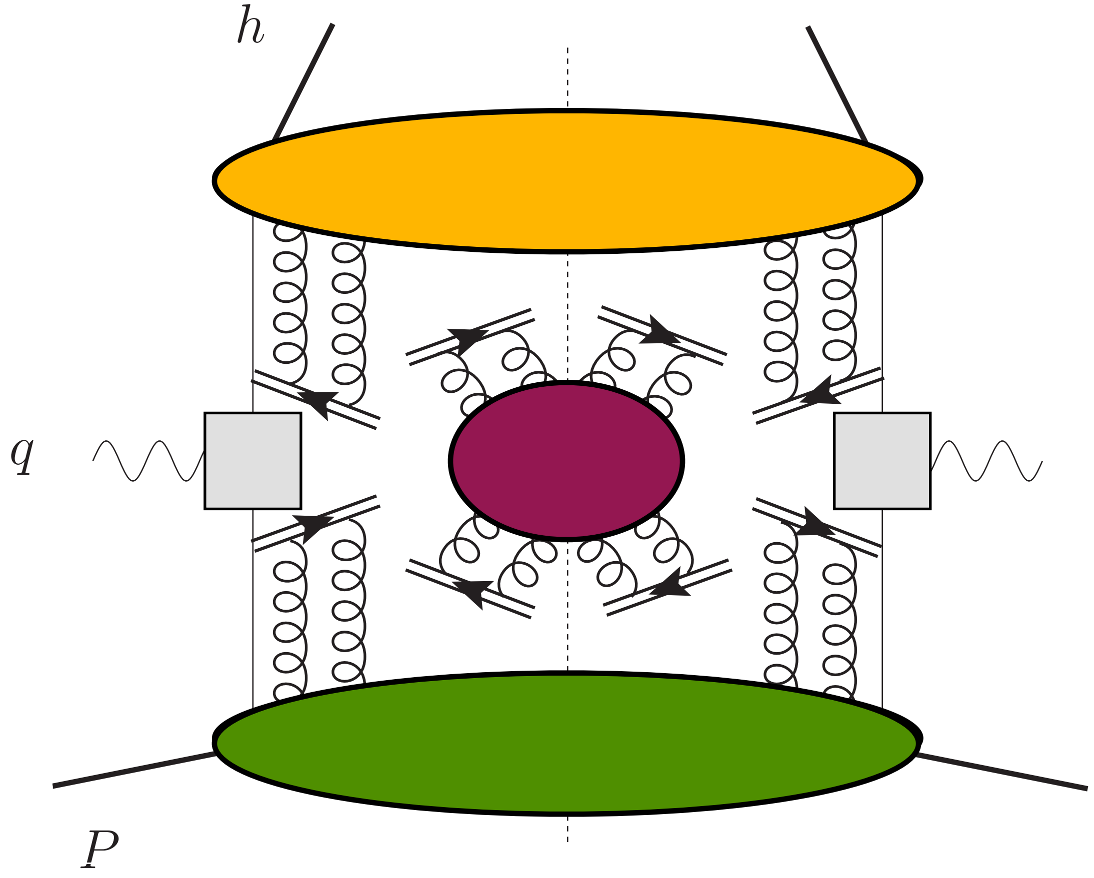
$$\frac{\mathrm{d}\sigma^{\mathrm{W}}}{\mathrm{d}Q\,\mathrm{d}Y\,\mathrm{d}^2\mathbf{q}_T} = \sum_{\text{flavors } i} H_{i\bar{i}}(Q^2, \mu) \int \mathrm{d}^2\mathbf{b}_T \, e^{i\mathbf{b}_T\cdot\mathbf{q}_T} \, \tilde{f}_{i/p}(x_a, \mathbf{b}_T, \mu, \zeta_a) \, \tilde{f}_{\bar{i}/p}(x_b, \mathbf{b}_T, \mu, \zeta_b)$$

Factorization for SIDIS



$$
\begin{aligned}
F_{UU,T}(x,z,P_{h\perp}^2,Q^2) &= \mathcal{C}'\big[f_1 D_1\big] \\
&= H(Q^2,\mu^2)\int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, d^2\boldsymbol{l}_T\, \delta^{(2)}\big(\boldsymbol{p}_T-\boldsymbol{k}_T+\boldsymbol{l}_T-\boldsymbol{P}_{h\perp}/z\big) \\
&\quad x\sum_a e_a^2\, f_1^a(x,p_T^2,\mu^2,\zeta)\, D_1^a(z,k_T^2,\mu^2,\zeta_h)\, U(l_T^2,\mu^2,\zeta\zeta_h)
\end{aligned}
$$

Factorization for SIDIS



$$\begin{aligned} F_{UU,T}(x,z,P_{h\perp}^2,Q^2) &= \mathcal{C}'\big[f_1 D_1\big] \\ &= H(Q^2,\mu^2)\int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, d^2\boldsymbol{l}_T\, \delta^{(2)}\big(\boldsymbol{p}_T - \boldsymbol{k}_T + \boldsymbol{l}_T - \boldsymbol{P}_{h\perp}/z\big) \\ &\quad x\sum_a e_a^2\, f_1^a(x,p_T^2,\mu^2,\zeta)\, D_1^a(z,k_T^2,\mu^2,\zeta_h)\, U(l_T^2,\mu^2,\zeta\zeta_h) \end{aligned}$$

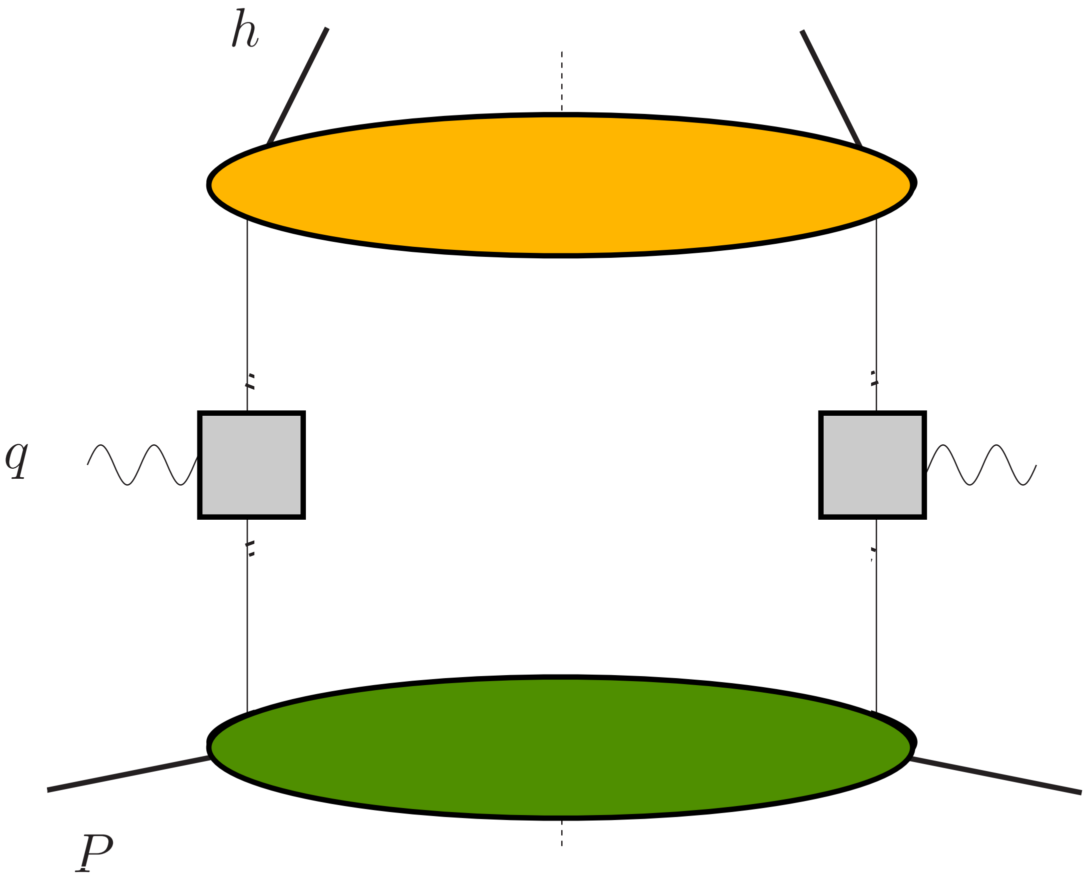
Hard part

unsubtracted TMD PDF

unsubtracted TMD FF

Soft factor

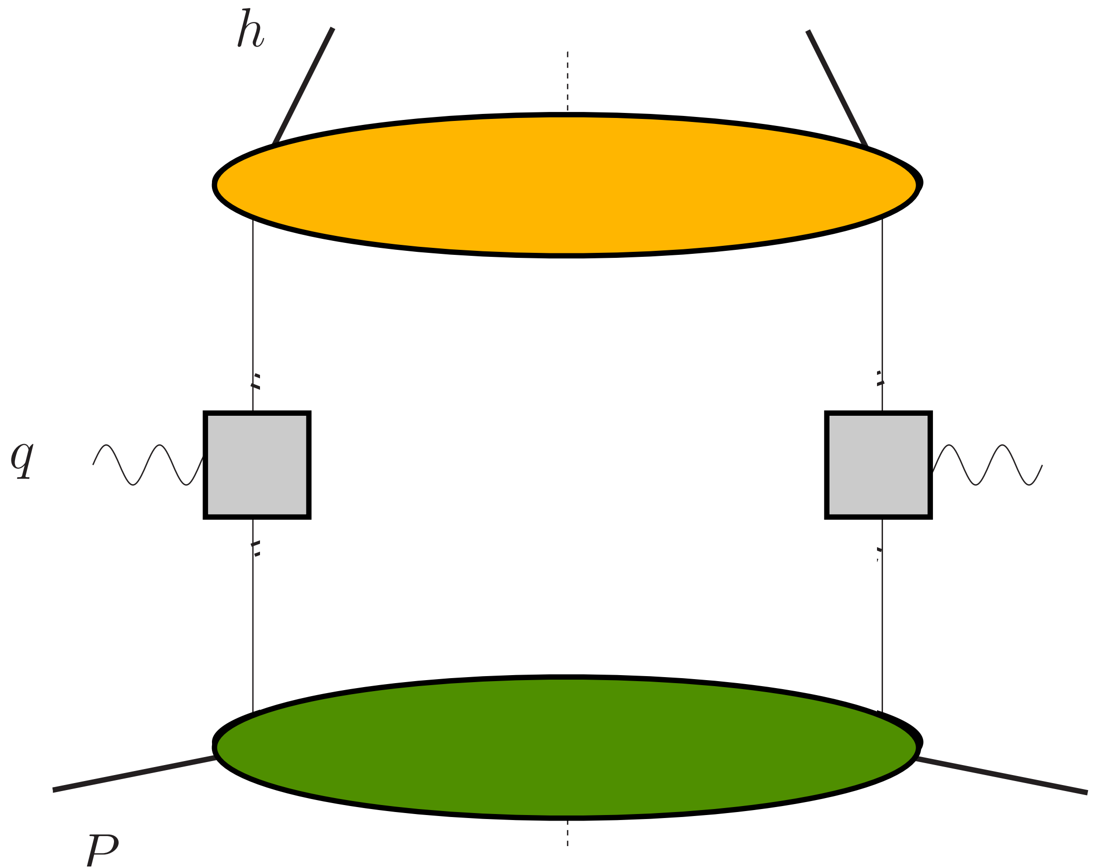
Factorization for SIDIS



$$F_{UU,T}(x,z,\boldsymbol{P}_{hT}^2,Q^2)$$

$$= x\sum_a \mathcal{H}^q_{UU,T}(Q^2,\mu)\int db_T b_T J_0(b_T|\boldsymbol{P}_{h\perp}|)\hat{f}_1^q\big(x,z^2b_\perp^2;\mu,\zeta\big)\,\hat{D}_1^{a\to h}\big(z,b_\perp^2;\mu,\zeta_h\big)$$

Factorization for SIDIS



$$F_{UU,T}(x,z,\boldsymbol{P}_{hT}^2,Q^2)$$

$$= x\sum_a \mathcal{H}^q_{UU,T}(Q^2,\mu)\int db_T b_T J_0(b_T|\boldsymbol{P}_{h\perp}|)\hat{f}_1^q(x,z^2b_\perp^2;\mu,\zeta)\hat{D}_1^{a\to h}(z,b_\perp^2;\mu,\zeta_h)$$

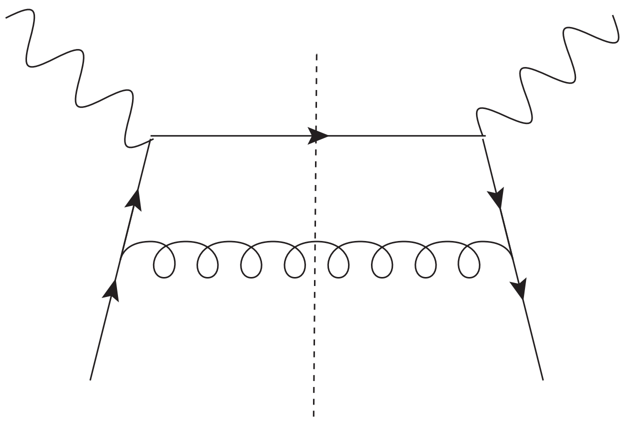
renormalized TMD PDF

renormalized TMD FF

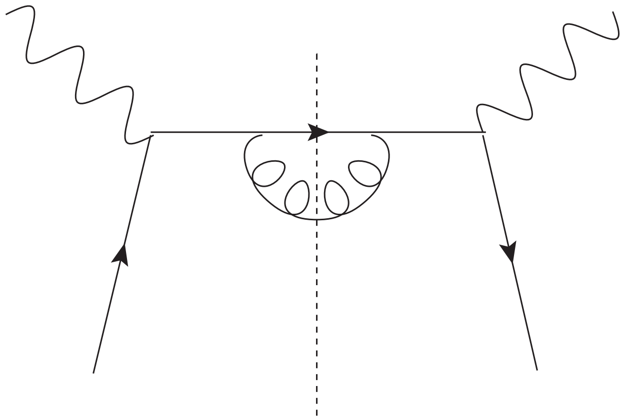
Key point

The parton-model results are still valid, but we the additional dependence on two scales and the addition of a hard factor

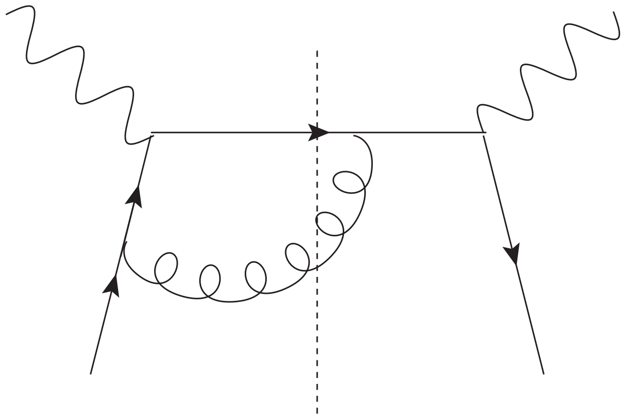
One loop analysis (quark-in-quark case)



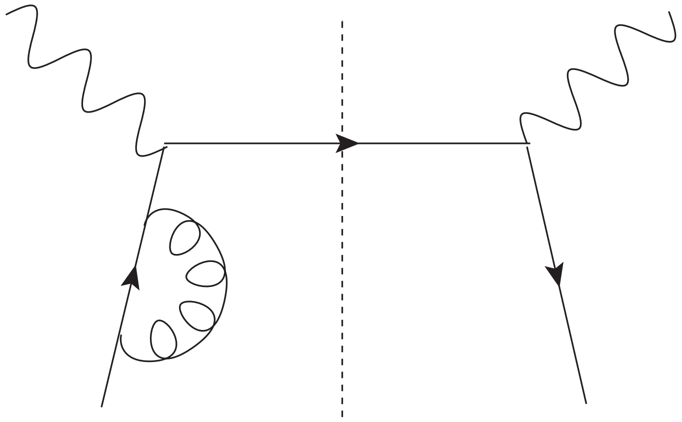
(r1)



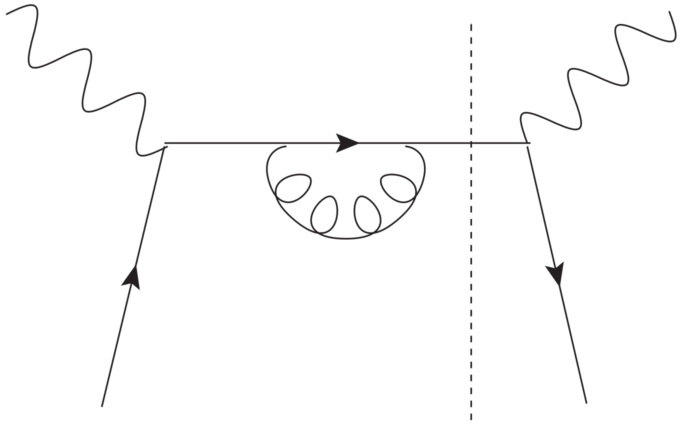
(r2)



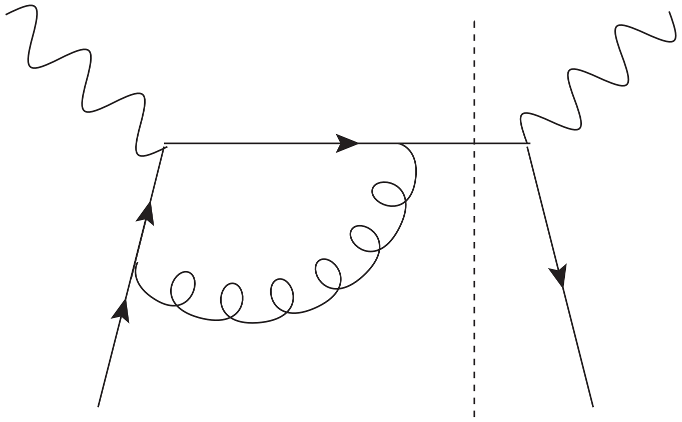
(r3)



(v1)

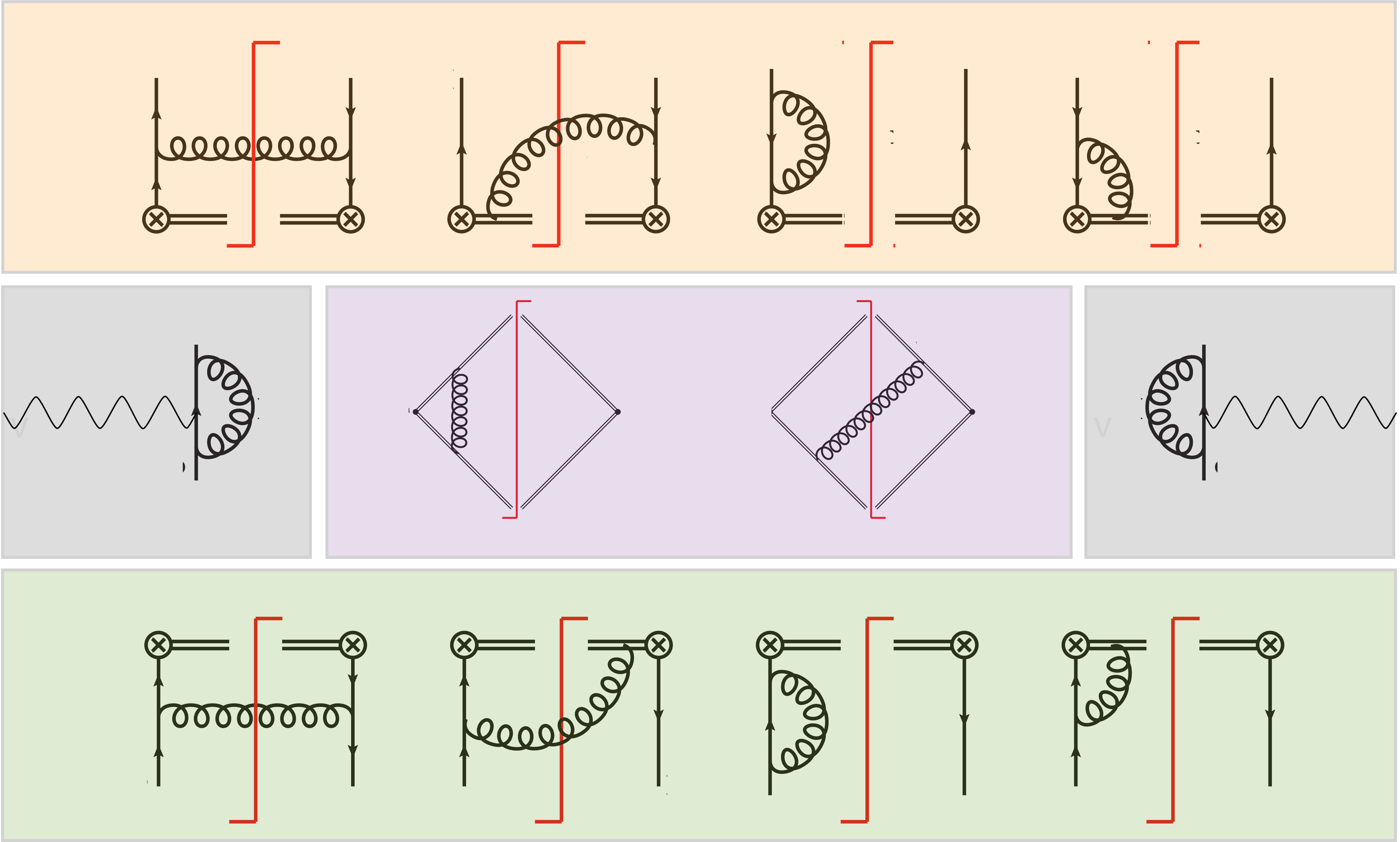


(v2)



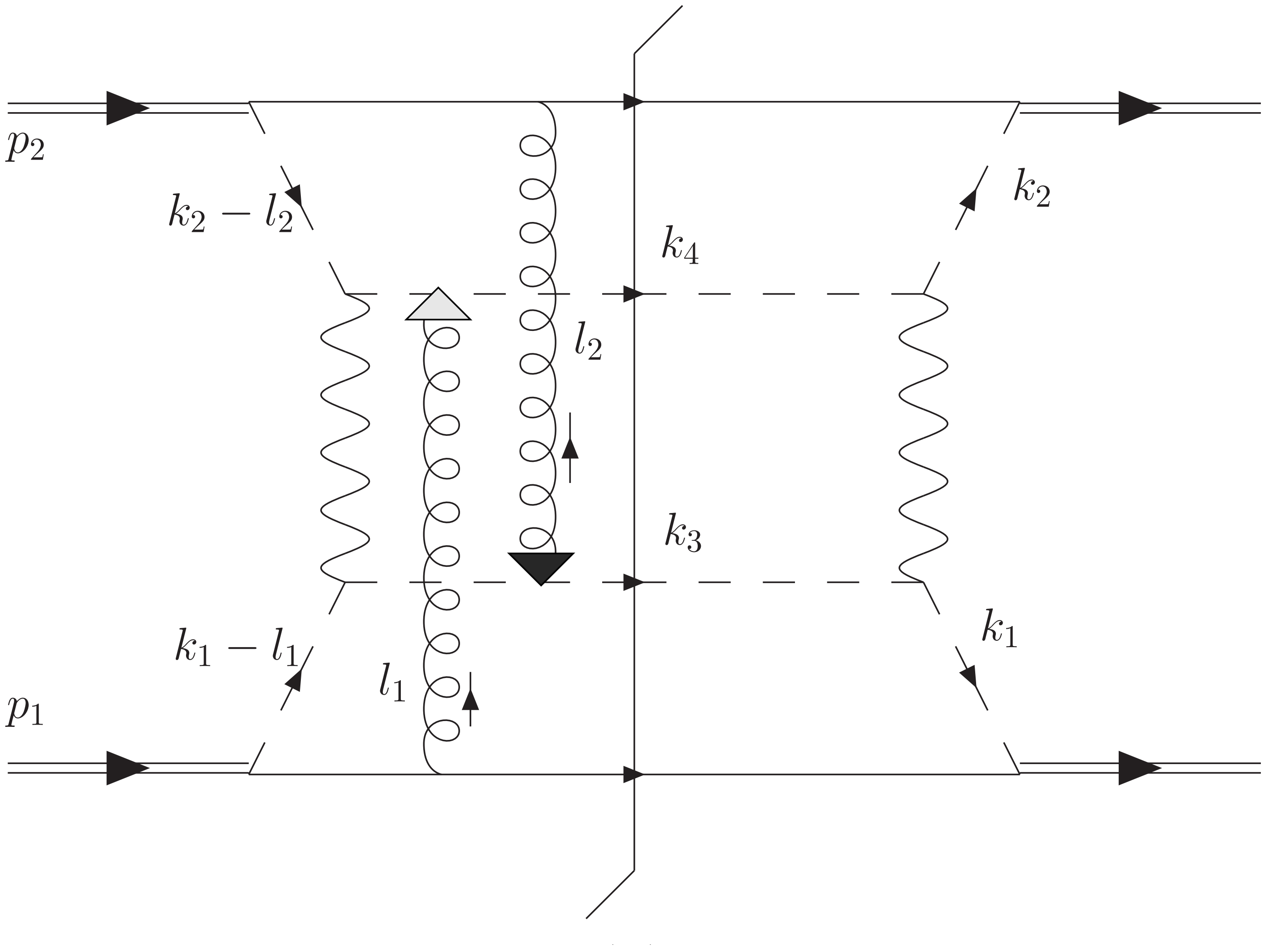
(v3)

Separation into various regions



Factorization breaking in pp collisions

Rogers, Mulders, arXiv:1001.2977



Soft factor at one loop

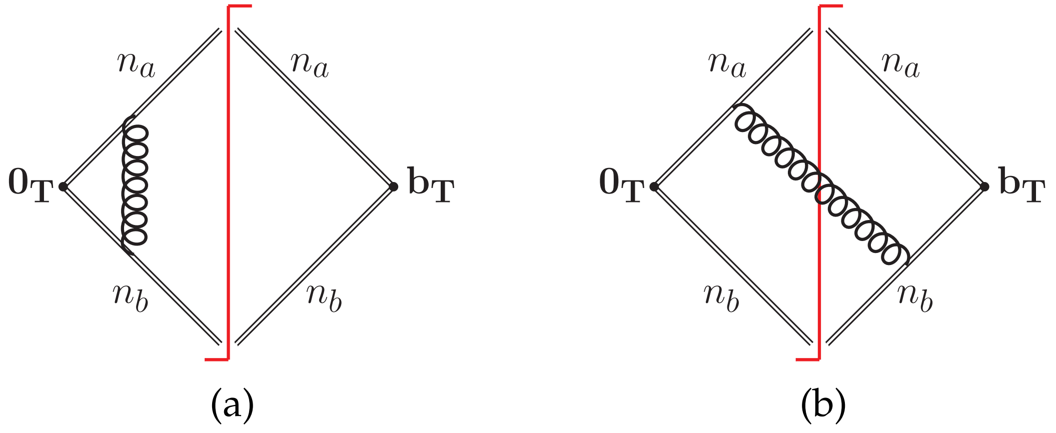
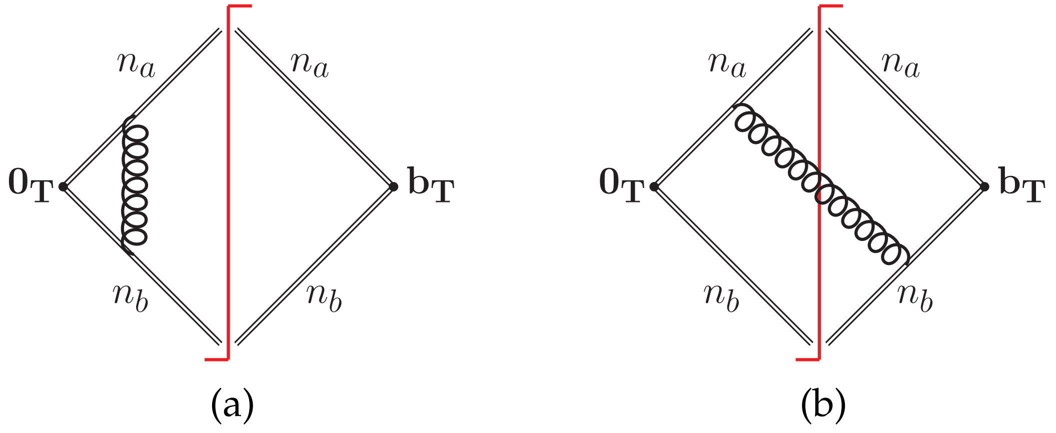


Figure 2.3: One-loop contributions to the soft function, with mirror diagrams obtained by a left-right swap of the exchanged gluon not shown. The double lines denote the Wilson lines from the transverse positions $\mathbf{0_T}$ and $\mathbf{b}_T$ stretching to light-cone infinity as indicated. The red line denotes the on-shell cut. Diagram (a) is scaleless and vanishes in pure dimensional regularization.

$$\mathcal{M}_S = 2g_0^2 C_F \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{i\mathbf{b}_T\cdot\mathbf{k}_T} \frac{-i}{(2k^+k^- - \mathbf{k}_T^2 + i0)} \frac{1}{(k^+ - i0)(-k^- + i0)}$$

Soft factor at one loop



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$$\frac{1}{k^2 + i0} \to 2\,\mathrm{Im}\left(\frac{1}{k^2 + i0}\right) = -2\pi i\theta(k^0)\delta(k^2) \equiv -2\pi i\delta_+(k^2)\,.$$

Soft factor at one loop

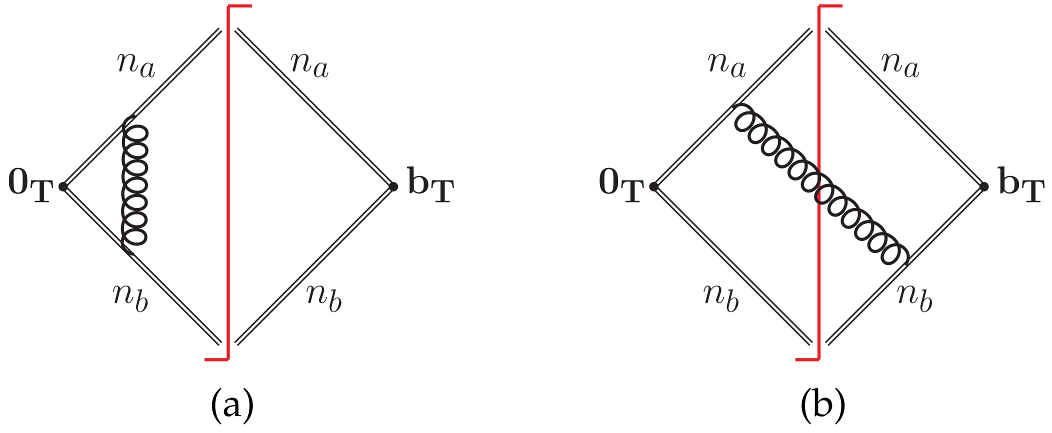
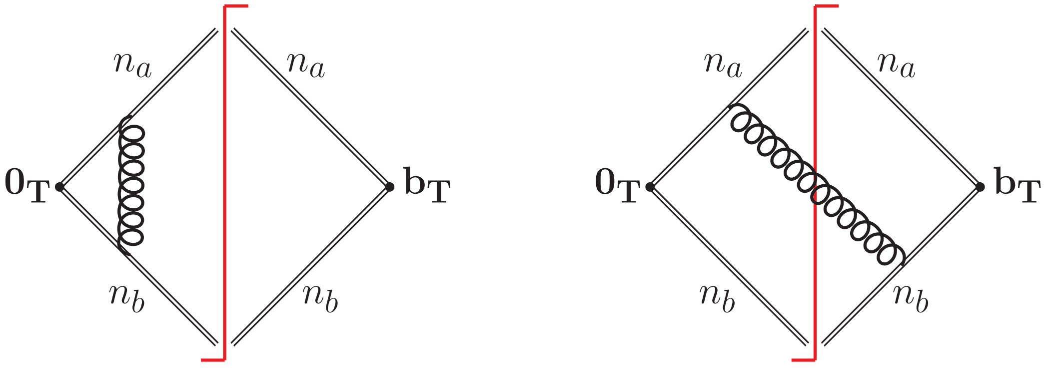


Figure 2.3: One-loop contributions to the soft function, with mirror diagrams obtained by a left-right swap of the exchanged gluon not shown. The double lines denote the Wilson lines from the transverse positions $\mathbf{0_T}$ and $\mathbf{b}_T$ stretching to light-cone infinity as indicated. The red line denotes the on-shell cut. Diagram (a) is scaleless and vanishes in pure dimensional regularization.

$$\begin{aligned}\mathcal{M}_S &= 2g_0^2 C_F \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{i\mathbf{b}_T\cdot\mathbf{k}_T} \frac{-i}{(2k^+k^- - \mathbf{k}_T^2 + i0)} \frac{1}{(k^+ - i0)(-k^- + i0)} \\ &= 2g_0^2 C_F \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{i\mathbf{b}_T\cdot\mathbf{k}_T} (2\pi)\delta_+(k^2) \frac{1}{k^+k^-} \\ &= \frac{g_0^2 C_F}{\pi} \int \frac{\mathrm{d}^{2-2\epsilon}\mathbf{k}_T}{(2\pi)^{d-2}} \frac{e^{i\mathbf{b}_T\cdot\mathbf{k}_T}}{k_T^2} \int_0^\infty \frac{\mathrm{d}k^-}{k^-}\,.\end{aligned}$$

$$\frac{1}{k^2+i0} \to 2\,\mathrm{Im}\Big(\frac{1}{k^2+i0}\Big) = -2\pi i\theta(k^0)\delta(k^2) \equiv -2\pi i\delta_+(k^2)\,.$$

Soft factor at one loop



(a) (b)

Figure 2.3: One-loop contributions to the soft function, with mirror diagrams obtained by a left-right swap of the exchanged gluon not shown. The double lines denote the Wilson lines from the transverse positions $\mathbf{0_T}$ and $\mathbf{b}_T$ stretching to light-cone infinity as indicated. The red line denotes the on-shell cut. Diagram (a) is scaleless and vanishes in pure dimensional regularization.

$$
\begin{aligned}
\mathcal{M}_S &= 2g_0^2 C_F \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{i\mathbf{b}_T\cdot\mathbf{k}_T} \frac{-i}{(2k^+k^- - \mathbf{k}_T^2 + i0)} \frac{1}{(k^+ - i0)(-k^- + i0)} \\
&= 2g_0^2 C_F \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{i\mathbf{b}_T\cdot\mathbf{k}_T} (2\pi)\delta_+(k^2) \frac{1}{k^+k^-} \\
&= \frac{g_0^2 C_F}{\pi} \int \frac{\mathrm{d}^{2-2\epsilon}\mathbf{k}_T}{(2\pi)^{d-2}} \frac{e^{i\mathbf{b}_T\cdot\mathbf{k}_T}}{k_T^2} \int_0^\infty \frac{\mathrm{d}k^-}{k^-}\,.
\end{aligned}
$$

$$
\frac{1}{k^2 + i0} \to 2\,\mathrm{Im}\Big(\frac{1}{k^2 + i0}\Big) = -2\pi i \theta(k^0)\delta(k^2) \equiv -2\pi i \delta_+(k^2)\,.
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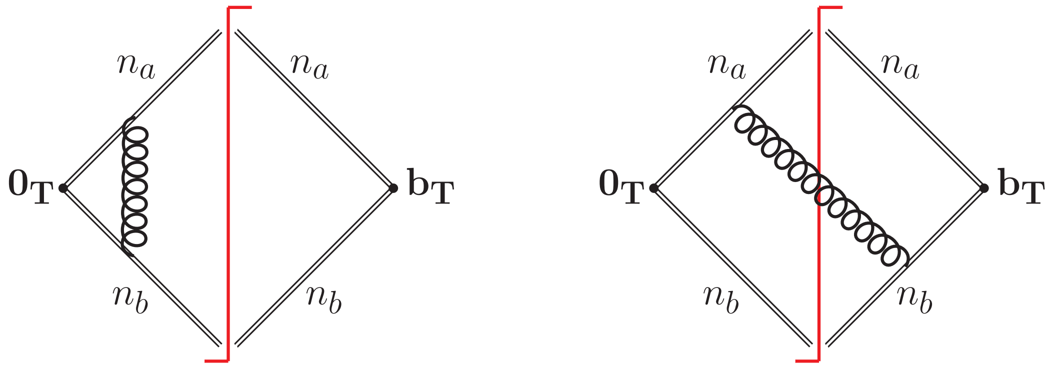
UV and IR divergences

Soft factor at one loop



(a) (b)

Figure 2.3: One-loop contributions to the soft function, with mirror diagrams obtained by a left-right swap of the exchanged gluon not shown. The double lines denote the Wilson lines from the transverse positions $\mathbf{0_T}$ and $\mathbf{b}_T$ stretching to light-cone infinity as indicated. The red line denotes the on-shell cut. Diagram (a) is scaleless and vanishes in pure dimensional regularization.

$$\begin{aligned}\mathcal{M}_S &= 2g_0^2 C_F \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{i\mathbf{b}_T\cdot\mathbf{k}_T} \frac{-i}{(2k^+k^- - \mathbf{k}_T^2 + i0)} \frac{1}{(k^+ - i0)(-k^- + i0)} \\ &= 2g_0^2 C_F \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{i\mathbf{b}_T\cdot\mathbf{k}_T} (2\pi)\delta_+(k^2) \frac{1}{k^+k^-} \\ &= \frac{g_0^2 C_F}{\pi} \int \frac{\mathrm{d}^{2-2\epsilon}\mathbf{k}_T}{(2\pi)^{d-2}} \frac{e^{i\mathbf{b}_T\cdot\mathbf{k}_T}}{k_T^2} \int_0^\infty \frac{\mathrm{d}k^-}{k^-}\,.\end{aligned}$$

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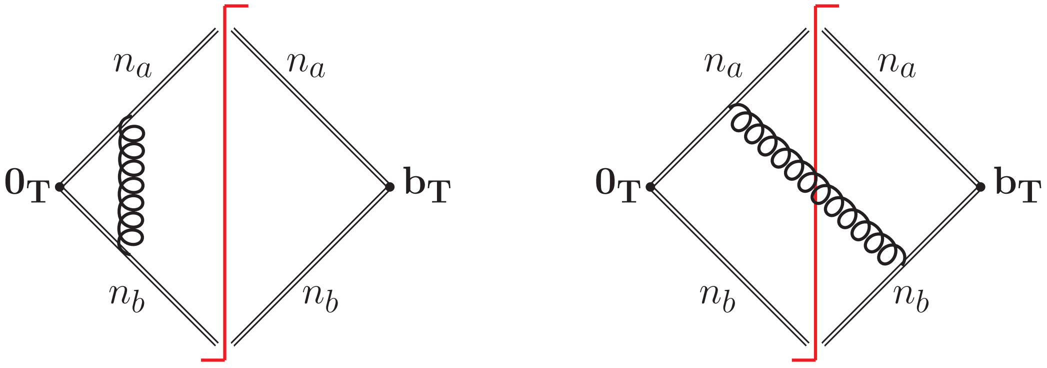

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UV and IR divergences

rapidity divergence

$$y_k = \frac{1}{2}\ln\frac{k^+}{k^-}$$

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regulate the rapidity divergence (different prescriptions are used)

$$\int_0^\infty \frac{\mathrm{d}k^-}{k^-} \to w^2\left(\frac{\nu}{\sqrt{2}}\right)^\tau \int_0^\infty \frac{\mathrm{d}k^-}{k^-}\left|\frac{\mathbf{k}_T^2}{2k^-} - k^-\right|^{-\tau} = \frac{\nu^\tau k_T^{-\tau}}{2^\tau\sqrt{\pi}}\Gamma\left(\frac{1}{2} - \frac{\tau}{2}\right)\Gamma\left(\frac{\tau}{2}\right)$$

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$$\tilde{S}_q^{0\,(1)}(b_T, \epsilon, \tau) = \frac{\alpha_s(\mu) C_F}{2\pi} \left[\frac{2}{\epsilon^2} + 4\left(\frac{1}{\epsilon} + L_b\right)\left(-\frac{1}{\tau} + \ln\frac{\mu}{\nu}\right) - L_b^2 - \frac{\pi^2}{6}\right] + \mathcal{O}(\tau) + \mathcal{O}(\epsilon)\,. \tag{2.77}$$

Renormalized TMD for quark-in-quark

Renormalized TMD for quark-in-quark

The final “renormalized” or subtracted TMD depends on two arbitrary scales, introduced in the procedure of regularizing the divergences.

The physical TMDs will be of course different from the perturbative ones, but the dependence on these two scales should be the same as the perturbative

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$$L_b = \ln\frac{\mathbf{b}_T^2\mu^2}{b_0^2}, \qquad \text{with} \quad b_0 = 2e^{-\gamma_E}$$

Rapidity scale dependence

$$\tilde{f}^{(1)}_{q'/q}(x, b_T, \mu, \zeta) = \delta_{q'q}\delta(1-x) + \delta_{q'q}\frac{\alpha_S(\mu)C_F}{2\pi}\left[-\left(\frac{1}{\epsilon} + L_b\right)[P_{qq}(x)]_+ + (1-x)\right.$$
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$$L_b = \ln\frac{\mu^2 \boldsymbol{b}_T^2}{b_0^2} \qquad b_0 = 2e^{-\gamma_E} = 1.123\ \mathrm{GeV}^{-1}$$

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Collins-Soper kernel
or rapidity anomalous dimension

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Cusp anomalous dimension

UV scale dependence

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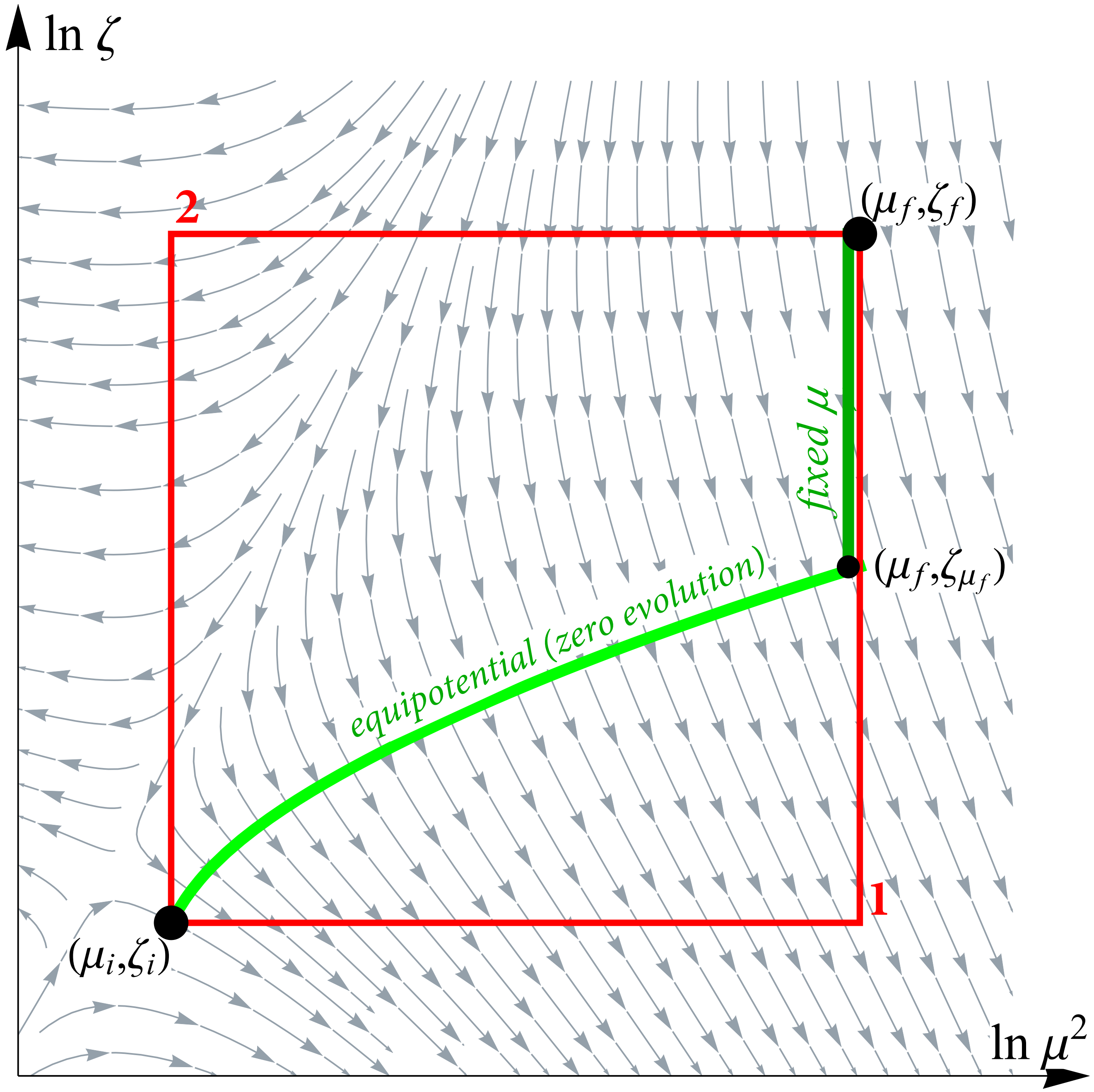
TMD two-scale evolution

rapidity evolution



UV evolution

Solution of evolution equation

$$\tilde{f}_{i/P}(x,\mathbf{b}_T,\mu,\zeta) = \tilde{f}_{i/P}(x,\mathbf{b}_T,\mu_0,\zeta_0)\ \exp\left\{\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_q\left[\alpha_s(\mu');\zeta_0/\mu'^2\right]\right\}\exp\left\{\tilde{K}(b_T;\mu)\ln\sqrt{\frac{\zeta}{\zeta_0}}\right\}\ , \tag{4.17}$$

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NLL $\alpha_S^n \ln^{2n}\left(\frac{\mu^2}{\mu_0^2}\right), \quad \alpha_S^n \ln^{2n-1}\left(\frac{\mu^2}{\mu_0^2}\right)$

Final TMD Structure

$$\hat{f}_1^a(x, |\boldsymbol{b}_T|; \mu, \zeta) = \int d^2\boldsymbol{k}_\perp \, e^{i\boldsymbol{b}_T \cdot \boldsymbol{k}_\perp} \, f_1^a(x, \boldsymbol{k}_\perp^2; \mu, \zeta)$$

see, e.g., Collins, "Foundations of Perturbative QCD" (11)
TMD collaboration, "TMD Handbook," arXiv:2304.03302

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collinear PDF

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(perturbative and
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nonperturbative part
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What about extractions?

TMD tables: quark, leading twist

Mulders-Tangerman, NPB 461 (96)
Boer-Mulders, PRD 57 (98)

quark pol.

nucleon pol.

	U	L	T
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L		g_{1L}	$h_{1L}^{\perp}$
T	$f_{1T}^{\perp}$	g_{1T}	$h_1,\ h_{1T}^{\perp}$

TMDs in **black** survive integration over transverse momentum

TMDs in **red** are time-reversal odd

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Quark, subleading twist

Mulders-Tangerman, NPB 461 (96)
Boer-Mulders, PRD 57 (98)
Bacchetta, Mulders, Pijlman, hep-ph/0405154
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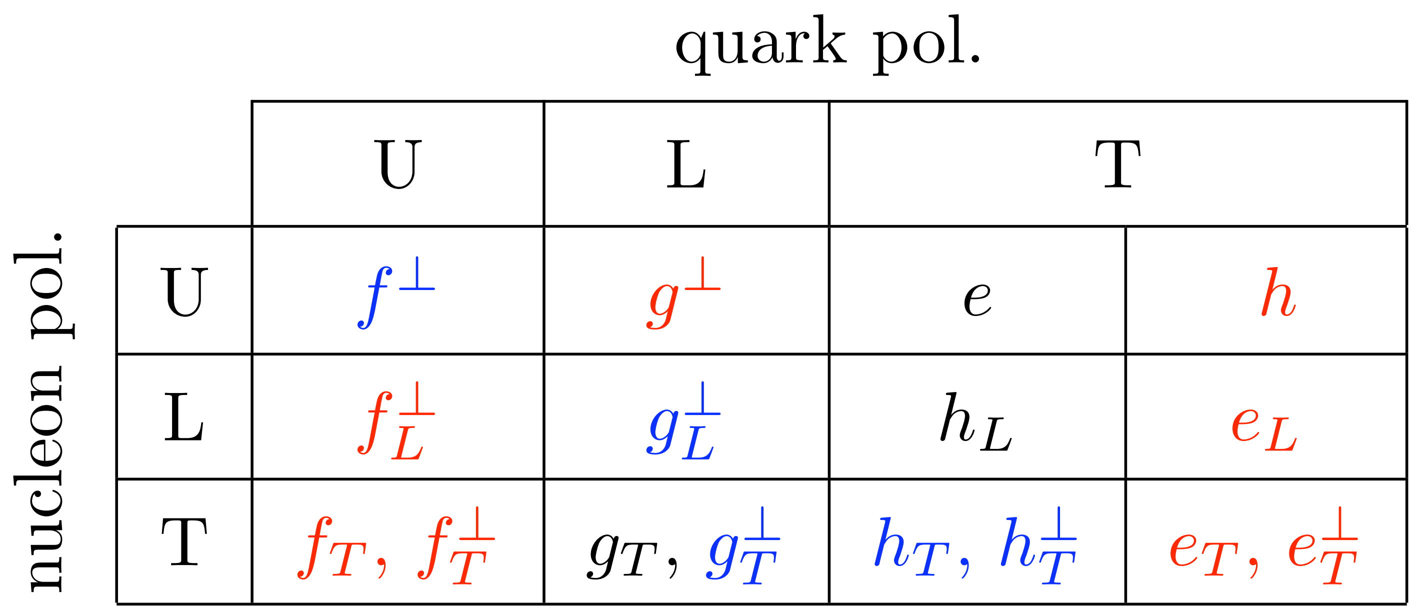

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- First hints about e x-dependence
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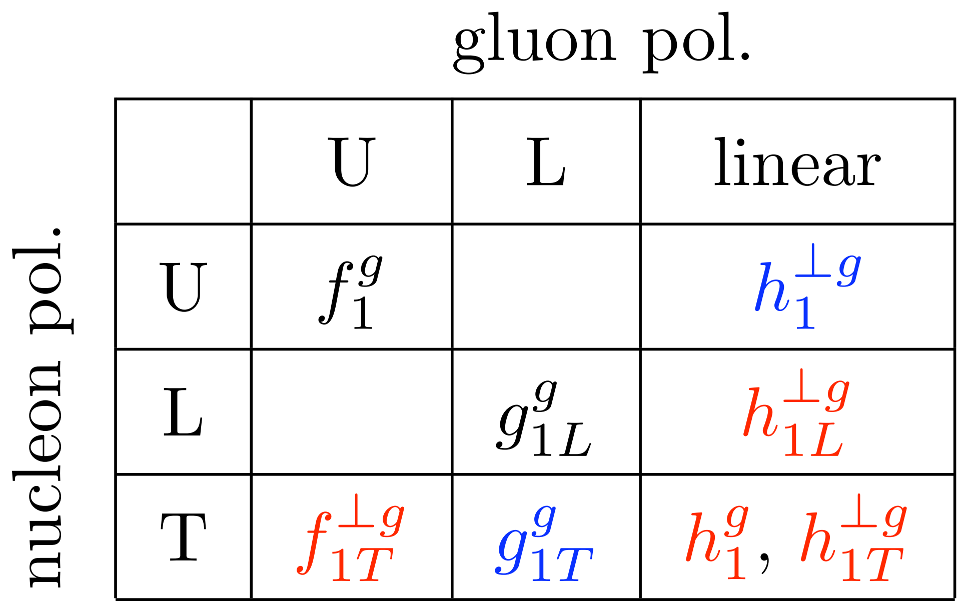
Gluons, leading twist

Mulders, Rodrigues, PRD63, 2001



gluon pol.

nucleon pol.

	U	L	linear
U	f_1^g		$h_1^{\perp g}$
L		g_{1L}^g	$h_{1L}^{\perp g}$
T	$f_{1T}^{\perp g}$	g_{1T}^g	h_1^g, $h_{1T}^{\perp g}$

TMDs in **black** survive integration over transverse momentum

TMDs in **red** are time-reversal odd

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TMDs in **black** survive integration over transverse momentum

TMDs in **red** are time-reversal odd

“Old” fits for unpolarized TMD f_1

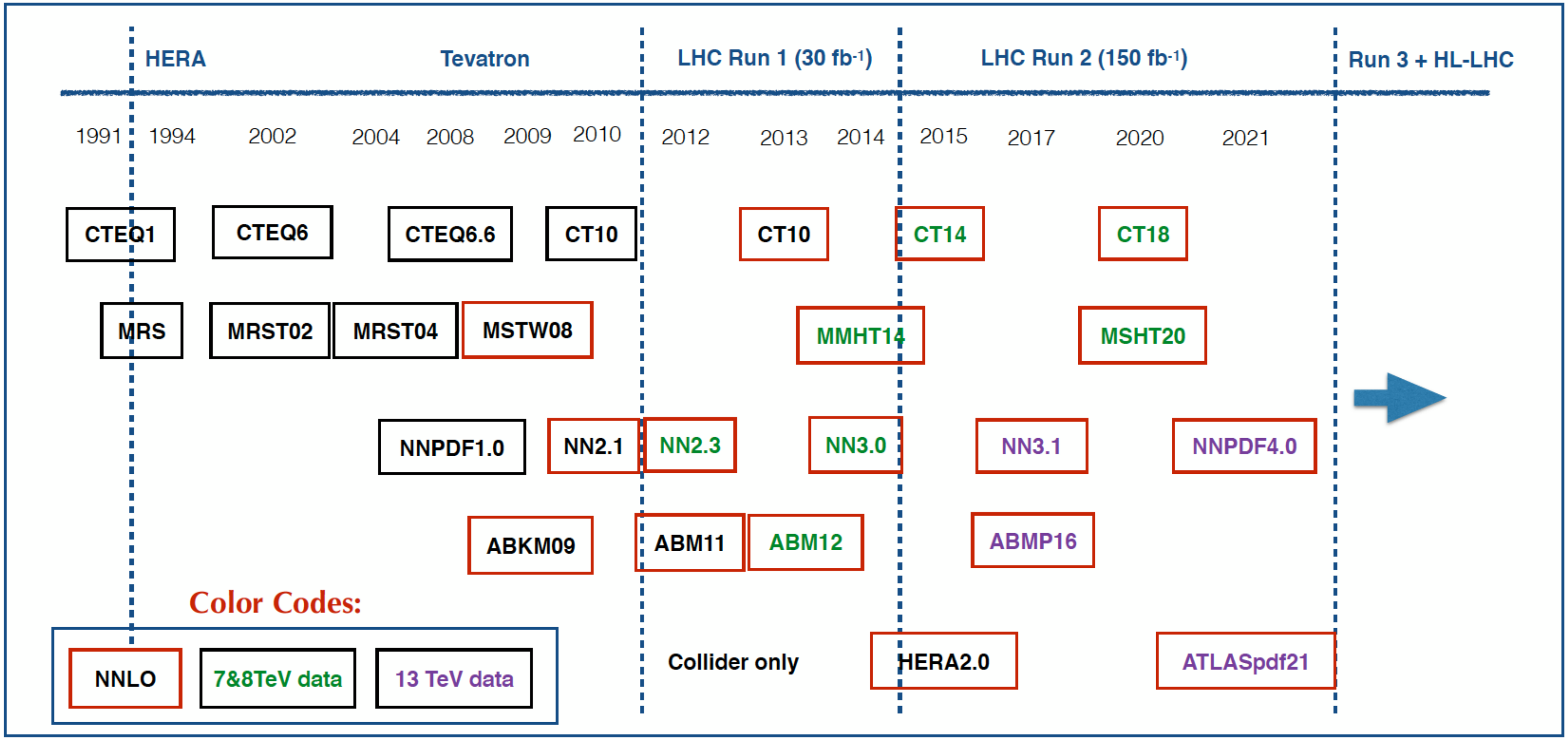
	Framework	HERMES	COMPASS	DY	Z production	N of points
KN 2006 *hep-ph/0506225*	NLL’	✘	✘	✔	✔	98
Pavia 2013 *arXiv:1309.3507*	LO	✔	✘	✘	✘	1538
Torino 2014 *arXiv:1312.6261*	LO	✔ (separately)	✔ (separately)	✘	✘	576 (H) 6284 (C)
DEMS 2014 *arXiv:1407.3311*	NNLL’	✘	✘	✔	✔	223
EIKV 2014 *arXiv:1401.5078*	NLL	1 (x,Q^2) bin	1 (x,Q^2) bin	✔	✔	500 (?)
SIYY *arXiv:1406.3073*	NLL’	qualitative	qualitative	✔	✔	140
Pavia 2017 *arXiv:1703.10157*	NLL	✔	✔	✔	✔	8059
SV 2017 *arXiv:1706.01473*	NNLL’	✘	✘	✔	✔	309

2017: the dawn of TMD global fits era

Available fits for unpolarized TMD f_1

	Accuracy	SIDIS HERMES	SIDIS COMPASS	DY fixed target	DY collider	N of points	χ^2/N_{points}
Pavia 2017 arXiv:1703.10157	NLL	✔	✔	✔	✔	8059	1.55
SV 2019 arXiv:1912.06532	N^3LL^-	✔	✔	✔	✔	1039	1.06
MAP22 arXiv:2206.07598	N^3LL^-	✔	✔	✔	✔	2031	1.06
ART23 arXiv:2305.07473	N^4LL^-			✔	✔	627	0.96

Comparison with PDFs



HERA Tevatron LHC Run 1 (30 fb-1) LHC Run 2 (150 fb-1) Run 3 + HL-LHC

1991 1994 2002 2004 2008 2009 2010 2012 2013 2014 2015 2017 2020 2021

CTEQ1 CTEQ6 CTEQ6.6 CT10 CT10 CT14 CT18

MRS MRST02 MRST04 MSTW08 MMHT14 MSHT20

NNPDF1.0 NN2.1 NN2.3 NN3.0 NN3.1 NNPDF4.0

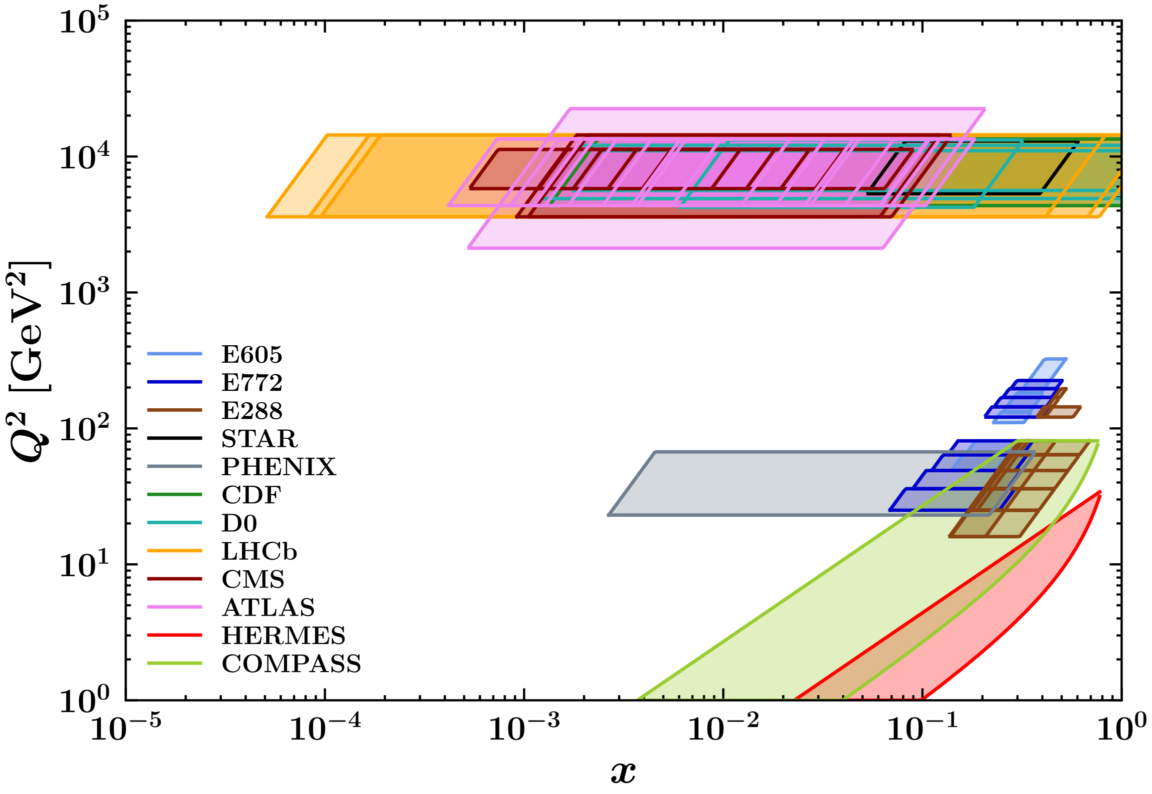
ABKM09 ABM11 ABM12 ABMP16

Color Codes:

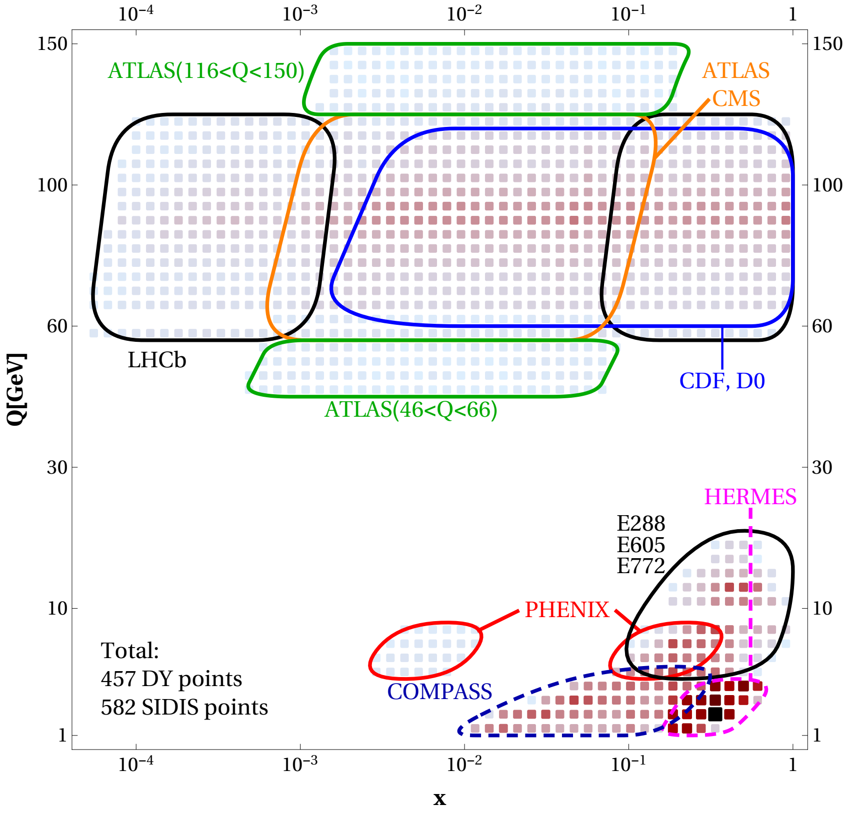
NNLO 7&8TeV data 13 TeV data

Collider only HERA2.0 ATLASpdf21

x-Q^2 coverage

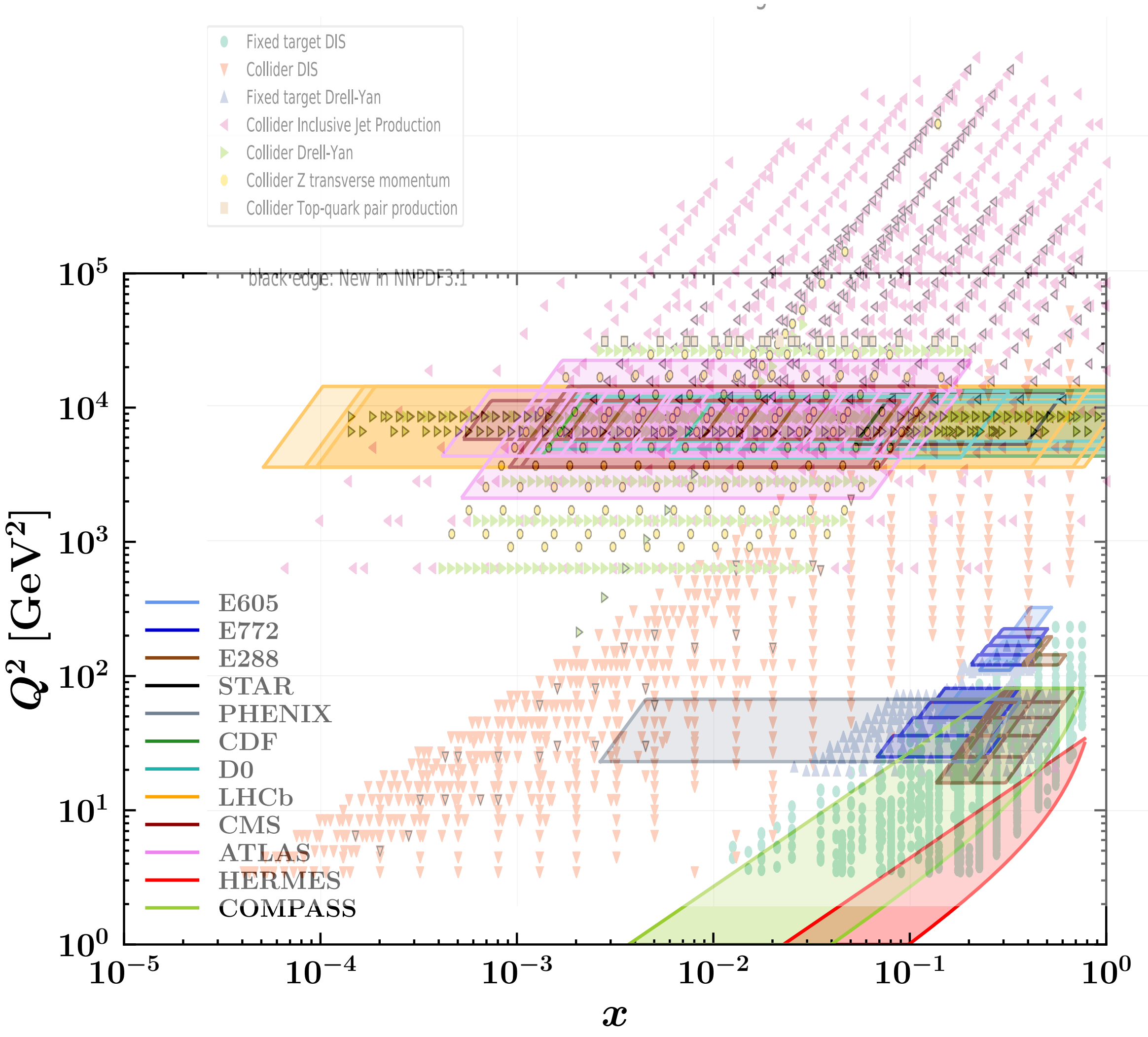


[MAP Collaboration](#)
[Bacchetta, Bertone, Bissolotti,](#)



[Scimemi, Vladimirov, arXiv:1912.06532](#)

x-Q^2 coverage



MAP Collaboration
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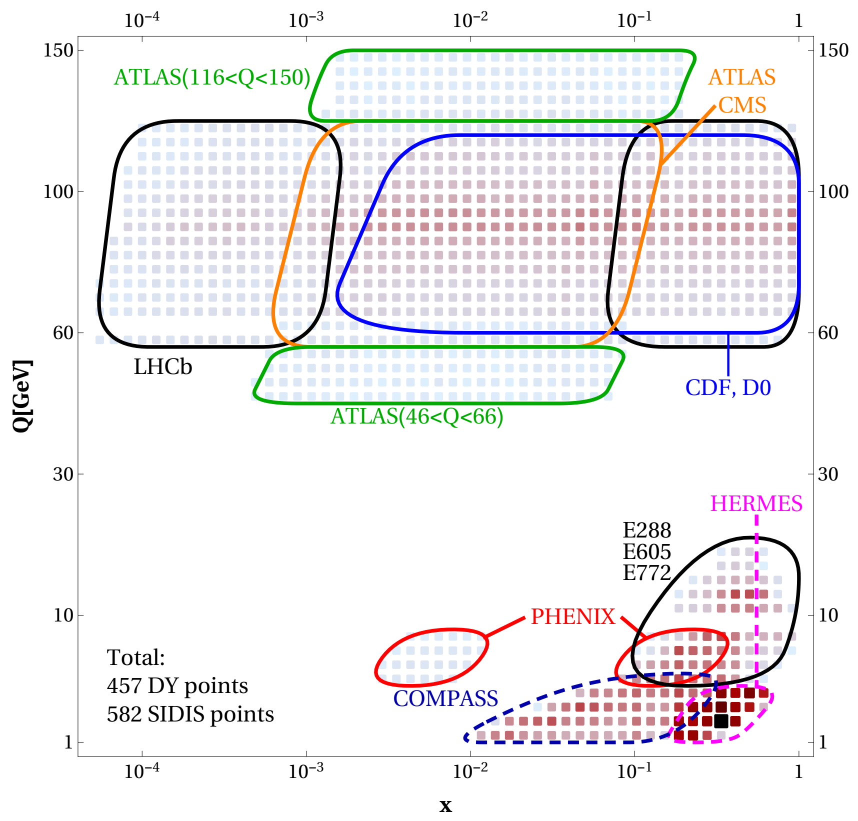
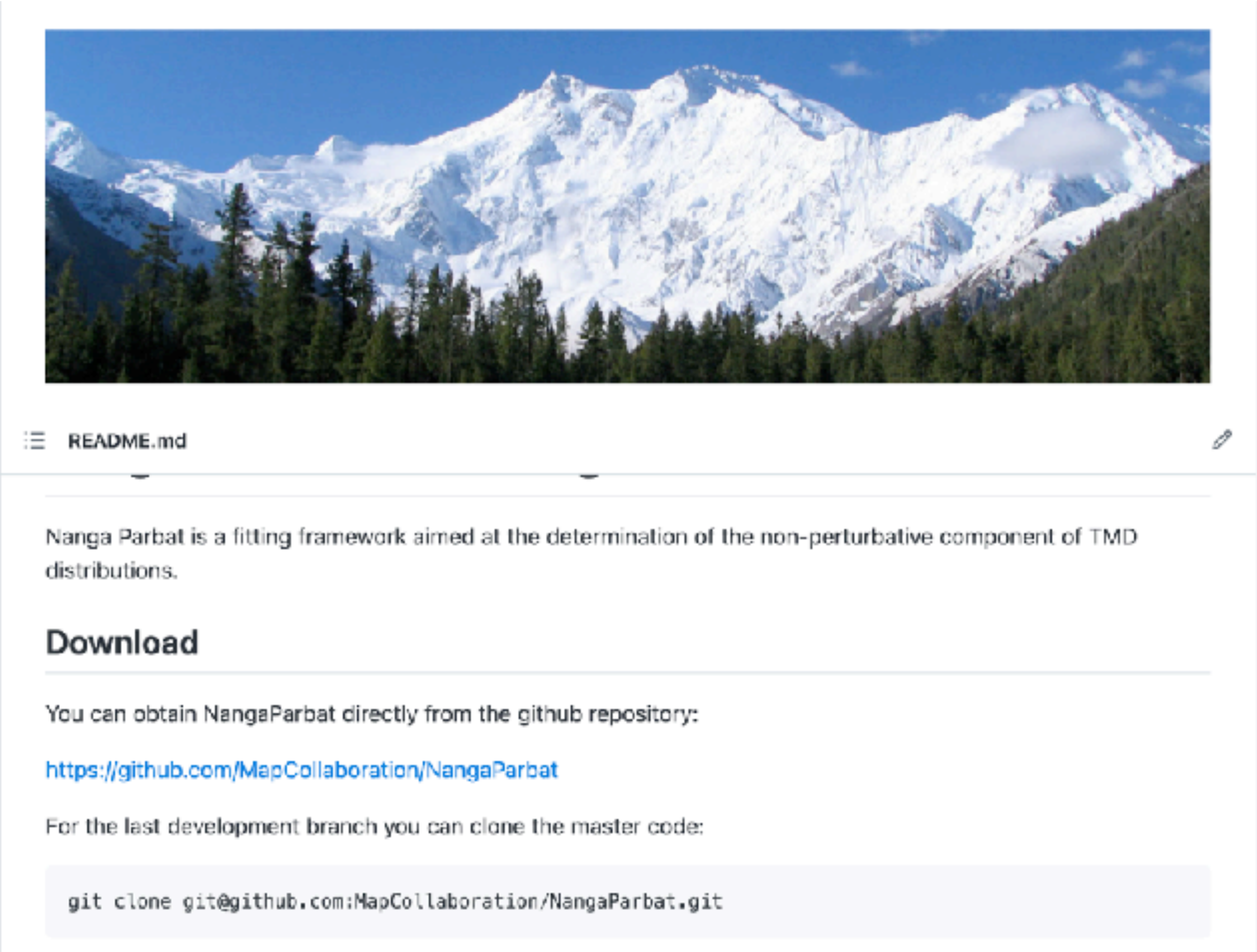


Scimemi, Vladimirov,
arXiv:1912.06532

Available tools

https://github.com/MapCollaboration/NangaParbat

see Matteo’s cooking session



Nanga Parbat is a fitting framework aimed at the determination of the non-perturbative component of TMD distributions.

Download

You can obtain NangaParbat directly from the github repository:

https://github.com/MapCollaboration/NangaParbat

For the last development branch you can clone the master code:

```
git clone git@github.com:MapCollaboration/NangaParbat.git
```

Available tools: artemide

https://teorica.fis.ucm.es/artemide/

arTeMiDe

News

12 Dec 2019: Version 2.02 released (+manual update).

23 Feb 2019: Version 1.4 released (+manual update).

21 Jan 2019: Artemide now has a repository.

Archive of older links/news.

Articles, presentations & supplementary materials

Extra pictures for the paper arXiv:1902.08474

Seminar of A.Vladimirov in Pavia 2018 on TMD evolution.

Link to the text in Inspire.

Archive of older links/news.

Download

Recent version/release can be found in repository

About us & Contacts

If you have found mistakes, or have suggestions/questions, please, contact us.

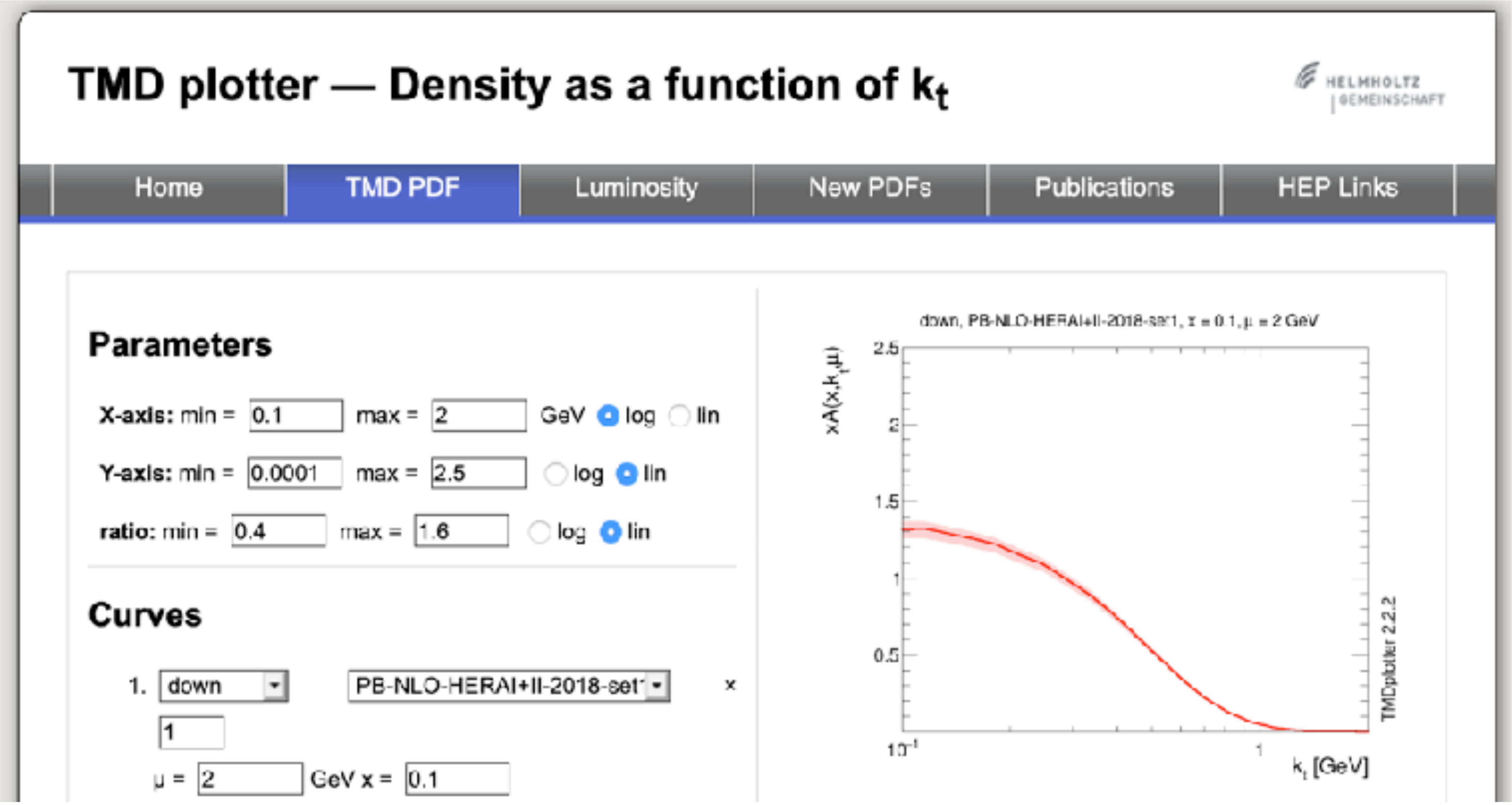
Some extra materials can be found on Alexey's web-page

Alexey Vladimirov Alexey.Vladimirov@physik.uni-regensburg.de

Ignazio Scimemi ignazios@fis.ucm.es

Available tools: TMDlib and TMDplotter

https://tmdlib.hepforge.or



TMD plotter — Density as a function of k_t

Parameters

X-axis: min = 0.1 max = 2 GeV ● log ○ lin

Y-axis: min = 0.0001 max = 2.5 ○ log ● lin

ratio: min = 0.4 max = 1.6 ○ log ● lin

Curves

1. down PB-NLO-HERAI+II-2018-set1 ×

1

μ = 2 GeV x = 0.1

Backup slides

Ingredients and accuracy

Accuracy
LL
NLL
NLL$'$
NNLL
NNLL$'$
N^{3}LL

K and γ_F	γ_K	α_s evolution
-	1	-
1	2	LO
1	2	NLO
2	3	NLO
2	3	NNLO
3	4	NNLO

$\gamma_K\left(\alpha_s(\mu)\right)$	$\beta[\alpha_s(\mu)]$	$\gamma_q\left(\alpha_s(\mu);1\right)$	$\tilde{K}(\bar{b}_T;1/\bar{b}_T)$	$\tilde{C}_{j/j'}$	accuracy	accuracy (SCET)
—	—	—	—	0	QPM	
1	1	—	—	0	LO-LL	LL
2	2	1	1	0	LO-NLL	NLL
3	3	2	2	0	LO-NNLL	
2	2	1	1	1	NLO-NLL	NLL′
3	3	2	2	1	NLO-NNLL	NNLL
3	3	2	2	2	NNLO-NNLL	NNLL′
4	4	3	3	2	NNLO-N^{3}LL	N^{3}LL
4	4	3	3	3	N^{3}LO-N^{3}LL	N^{3}LL′

Table 4.2: Orders of accuracy needed for evolution of TMD PDFs and other ingredients entering the

Matching with collinear PDFs

$$\tilde{f}_{i/p}(x, \mathbf{b}_*, \mu_{b_*}, \mu_{b_*}^2) = \sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \tilde{C}_{i/j}(x/\hat{x}, b_T; \mu_{b_*}, \mu_{b_*}^2, \alpha_s(\mu_{b_*}))\, f_{j/p}(\hat{x}; \mu_{b_*}) + O((m\, b_*(b_T))^p)\,. \tag{4.30}$$

Matching with collinear PDFs

matching coefficients

$$\tilde{f}_{i/p}(x, \mathbf{b}_*, \mu_{b_*}, \mu_{b_*}^2) = \sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \, \tilde{C}_{i/j}(x/\hat{x}, b_T; \mu_{b_*}, \mu_{b_*}^2, \alpha_s(\mu_{b_*})) \, f_{j/p}(\hat{x}; \mu_{b_*}) + O((m \, b_*(b_T))^p) \, . \tag{4.30}$$

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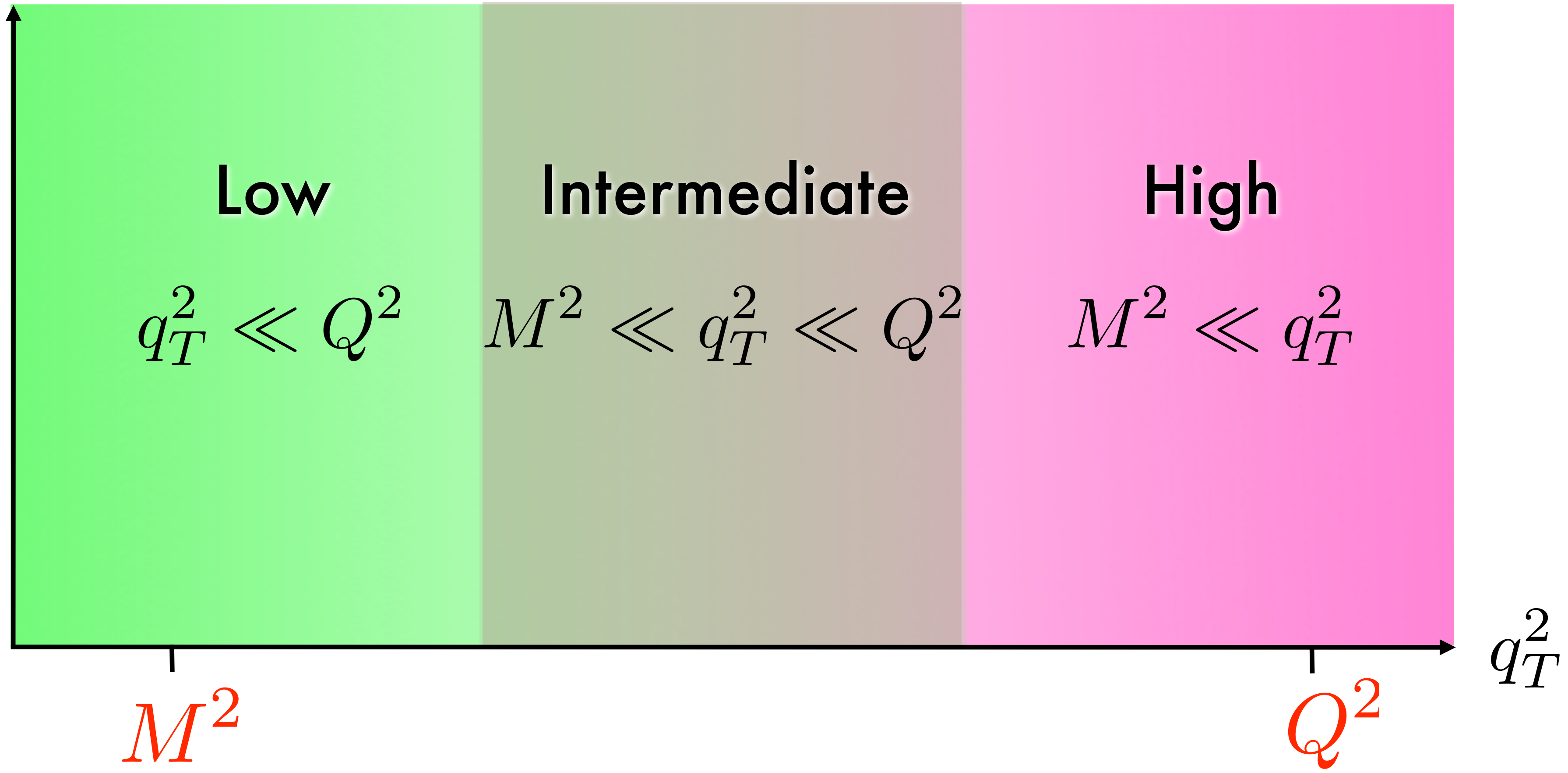

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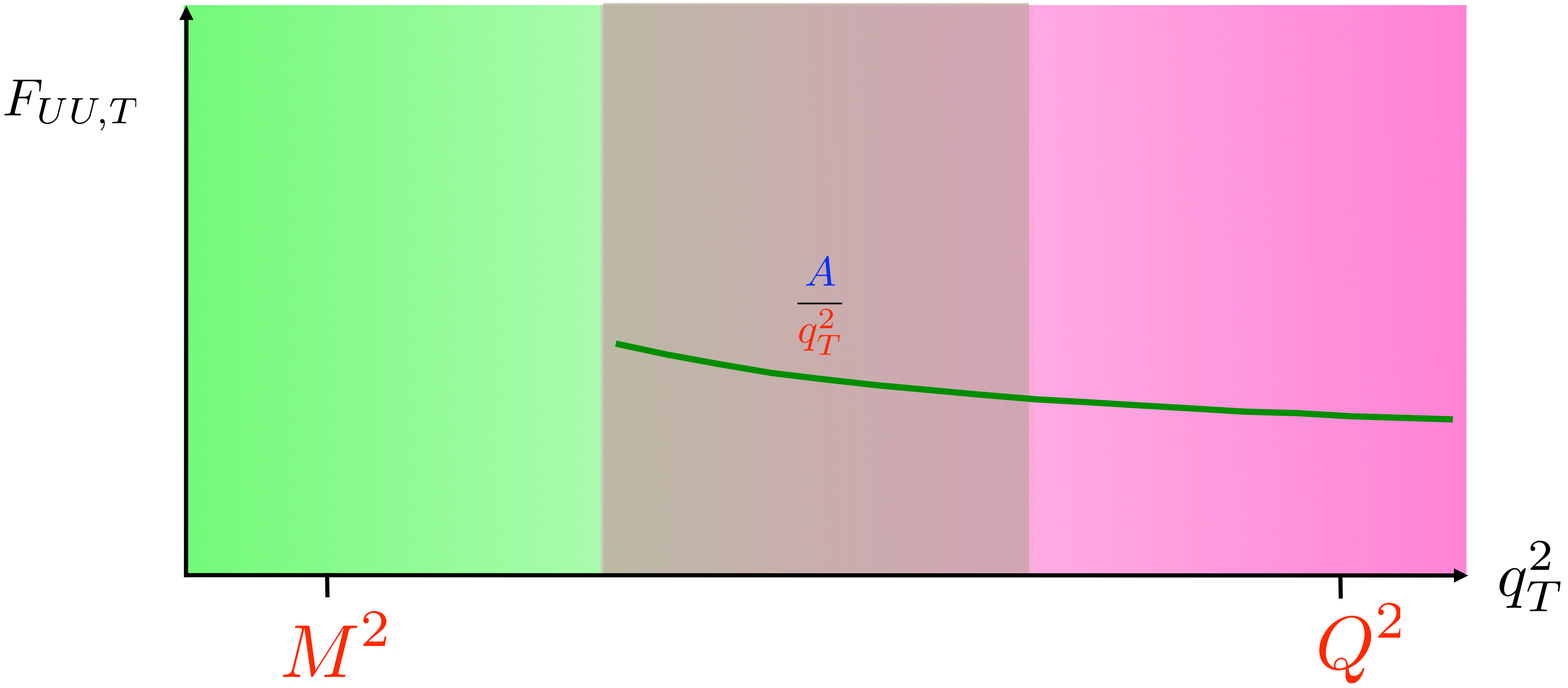

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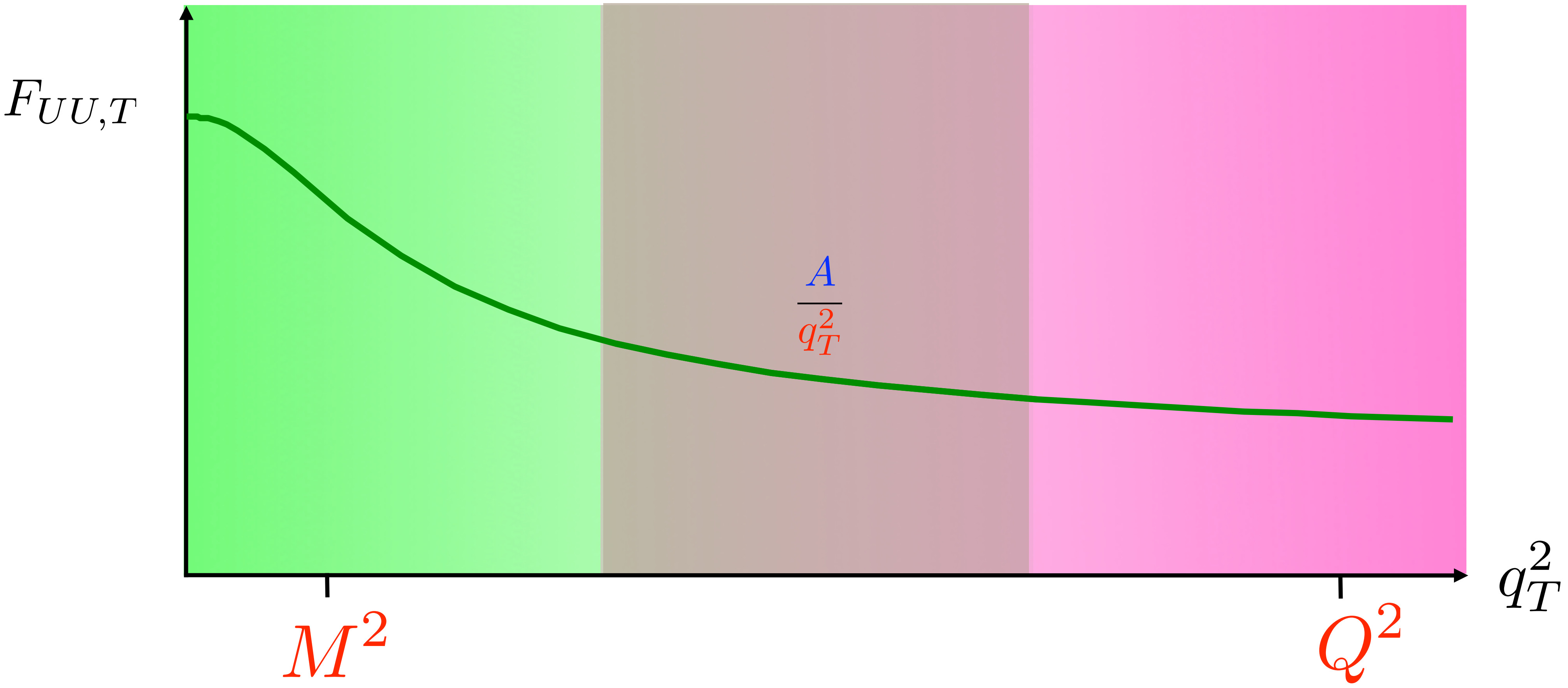

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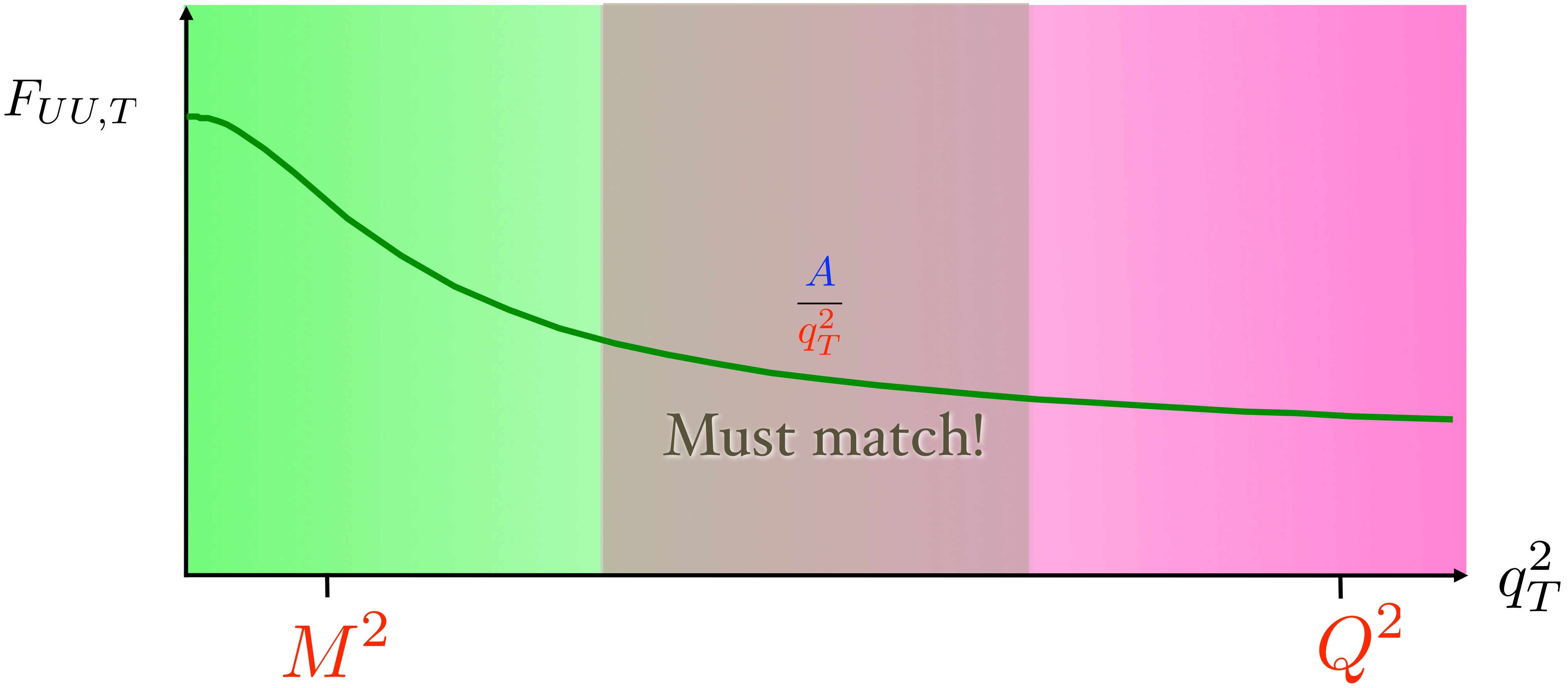

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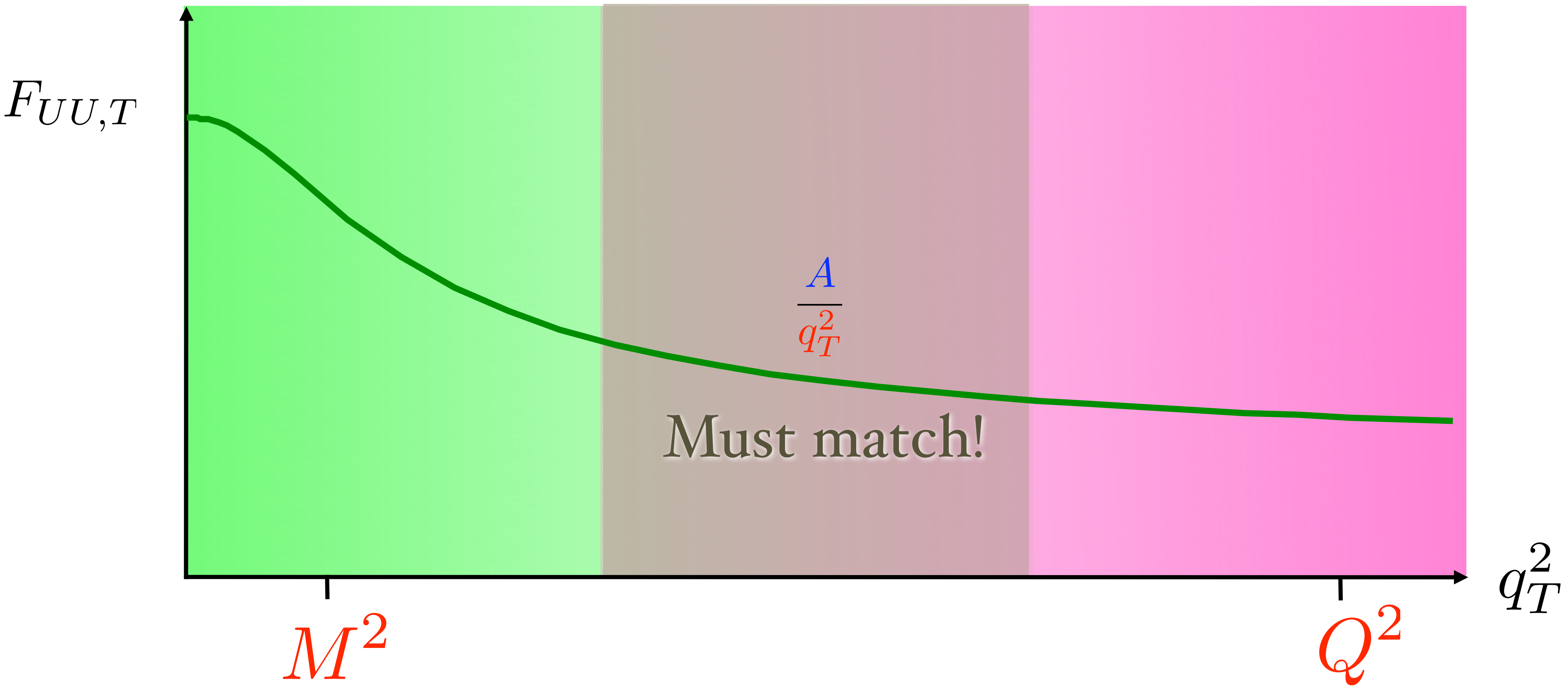

The leading high-transverse momentum part is just the “tail” of the leading low-transverse-momentum part

Ingredients and accuracy

Accuracy	H and C	K and γ_F	γ_K	PDF and α_s evolution
LL	0	-	1	-
NLL	0	1	2	LO
NLL$'$	1	1	2	NLO
NNLL	1	2	3	NLO
NNLL$'$	2	2	3	NNLO
N^{3}LL	2	3	4	NNLO