# Theory of TMDs

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# Plan of the lectures

✓ Review the idea of structure functions for DIS and introduce them for semi-inclusive DIS

- Introduce the idea of quark-quark correlation functions

- Parametrize correlation functions in terms of PDFs or Transverse Momentum Distributions (TMDs)

- Obtain the expression of structure functions for semi-inclusive DIS in terms of TMDs

- Discuss concept of TMD factorization and TMD evolution

- Discuss a bit of phenomenology

# Transverse Momentum Distributions

# About names and acronyms

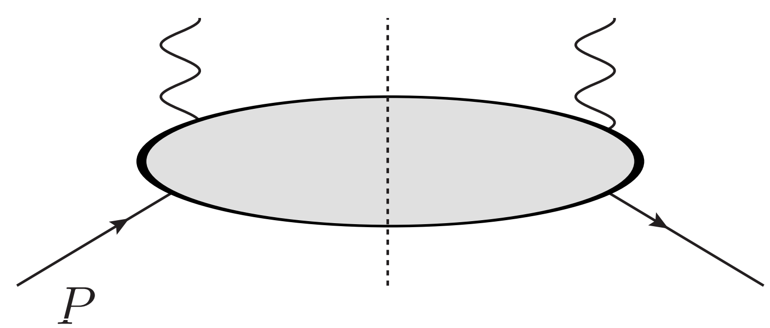
- TMD as an adjective stands for Transverse Momentum Dependent
- TMD as a noun stands for Transverse Momentum Distribution and it is usually meant to encompass both transverse-momentum-dependent PDFs and Fragmentation Functions (FFs)

# DIS cross section in terms of structure functions

$$\frac{d^3\sigma}{dx_B dy d\phi_S} = \frac{\alpha^2\, y}{2\, Q^4}\, L_{\mu\nu}(l, l', \lambda_e)\; 2MW^{\mu\nu}(q, P, S)$$

$F_{UU,T}(x, Q^2)$

$$2MW^{\mu\nu} = \frac{1}{x}\left[-g_{\perp}^{\mu\nu}\, F_{UU,T} + \hat{t}^{\mu}\hat{t}^{\nu}\, F_{UU,L}\right]$$



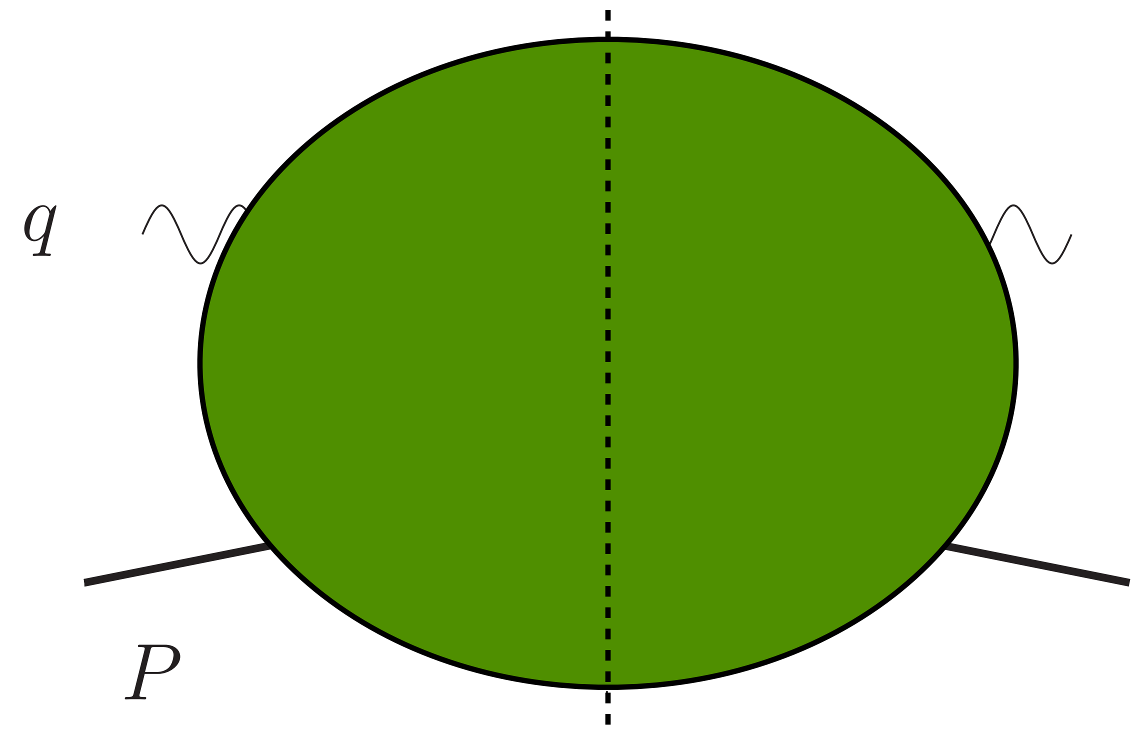

$$\frac{d\sigma}{dx_B\, dy\, d\phi_S} = \frac{2\alpha^2}{x_B y Q^2}\left\{\left(1 - y + \frac{y^2}{2}\right) F_{UU,T} + (1-y)\, F_{UU,L}\right\}$$

*see, e.g., A.B., Diehl, Goeke, Metz, Mulders, Schlegel, JHEP093 (07)*

# Hadronic tensor in terms of correlation function

This is the result, then we are going to motivate it

$$2MW^{\mu\nu}(q,P,S) \approx \sum_q e_q^2 \,\frac{1}{2}\mathrm{Tr}\left[\Phi(x_B,S)\,\gamma^{\mu}\gamma^{+}\gamma^{\nu}\right].$$



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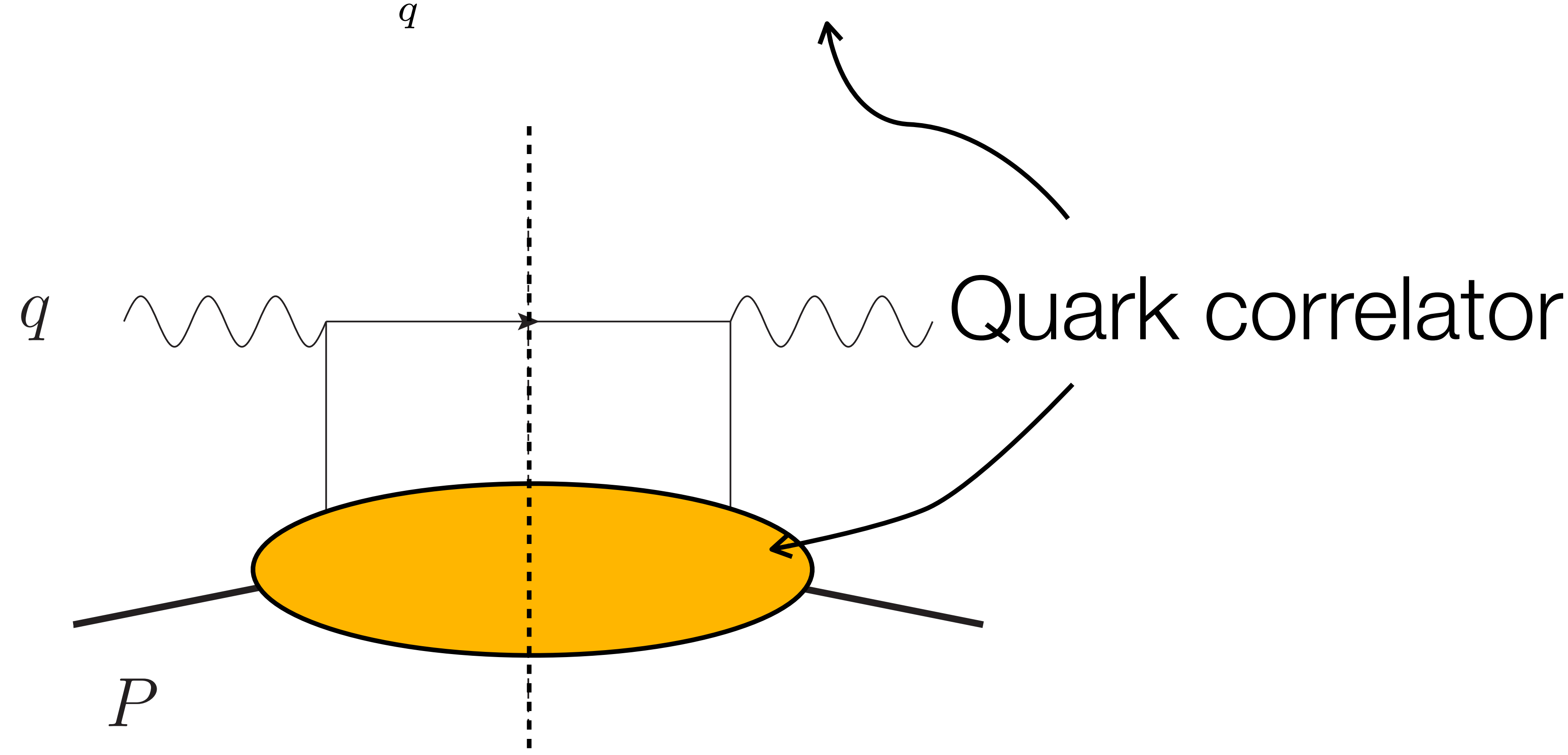



$$\Phi^q_{ji}(x,S) = \int \frac{d\xi^-}{2\pi}\, e^{-\mathrm{i}p\cdot\xi} \langle P,S|\, \bar{\psi}^q_i(\xi)\psi^q_j(0)\,|P,S\rangle\Big|_{\xi^+=\boldsymbol{\xi}_T=0,\ p^+=xP^+}$$

# Just a reminder: fermion fields

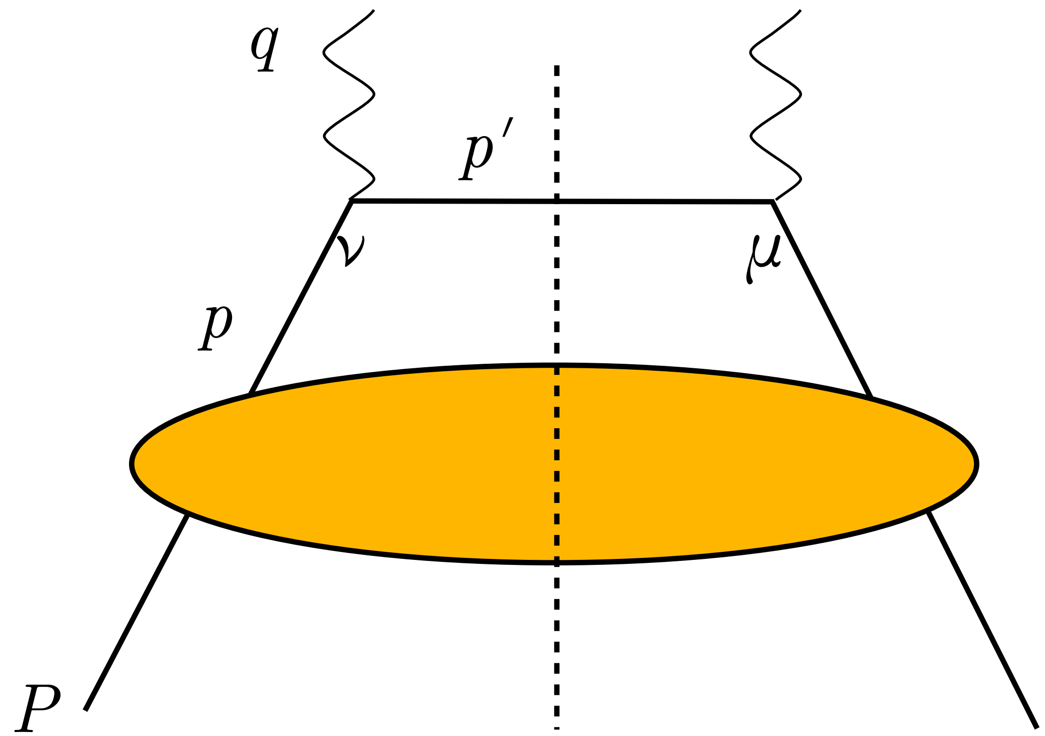
$$\overline{\psi}(x) = \sum_s \int \frac{d^3k}{(2\pi)^3 2\omega_k} \Big( d_s(\vec{k}) \overline{v}_s(\vec{k}) e^{-ikx} + c_s^\dagger(\vec{k}) \overline{u}_s(\vec{k}) e^{ikx} \Big)$$

positron absorption and electron creation

$$\psi(x) = \sum_s \int \frac{d^3k}{(2\pi)^3 2\omega_k} \Big( c_s(\vec{k}) u_s(\vec{k}) e^{-ikx} + d_s^\dagger(\vec{k}) v_s(\vec{k}) e^{ikx} \Big)$$

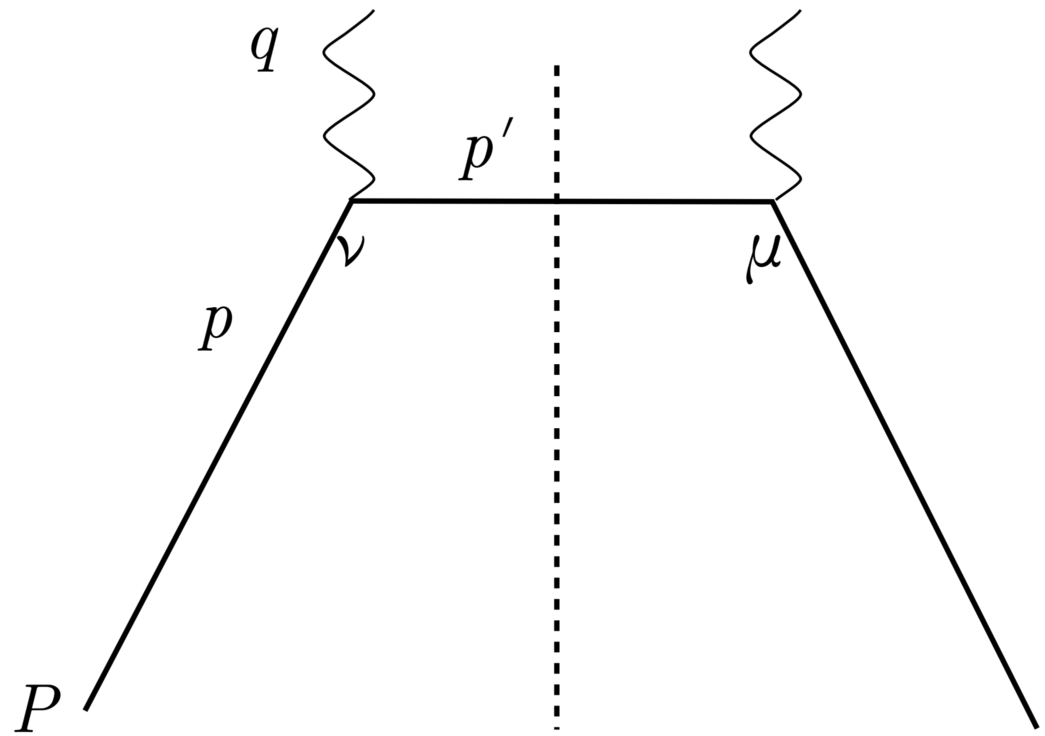
electron absorption and positron creation

# Quark-in-a-quark case




At order $\alpha^0$, the calculation should be simple and give us a result very similar to the leptonic tensor

# Quark-in-quark case




$$\Phi^q_{ji}(x,S) = \int \frac{d\xi^-}{2\pi}\, e^{-\mathrm{i}p\cdot\xi} \langle P,S|\, \overline{\psi}^q_i(\xi)\psi^q_j(0)\,|P,S\rangle \Big|_{\xi^+=\boldsymbol{\xi}_T=0,\ p^+=xP^+}$$

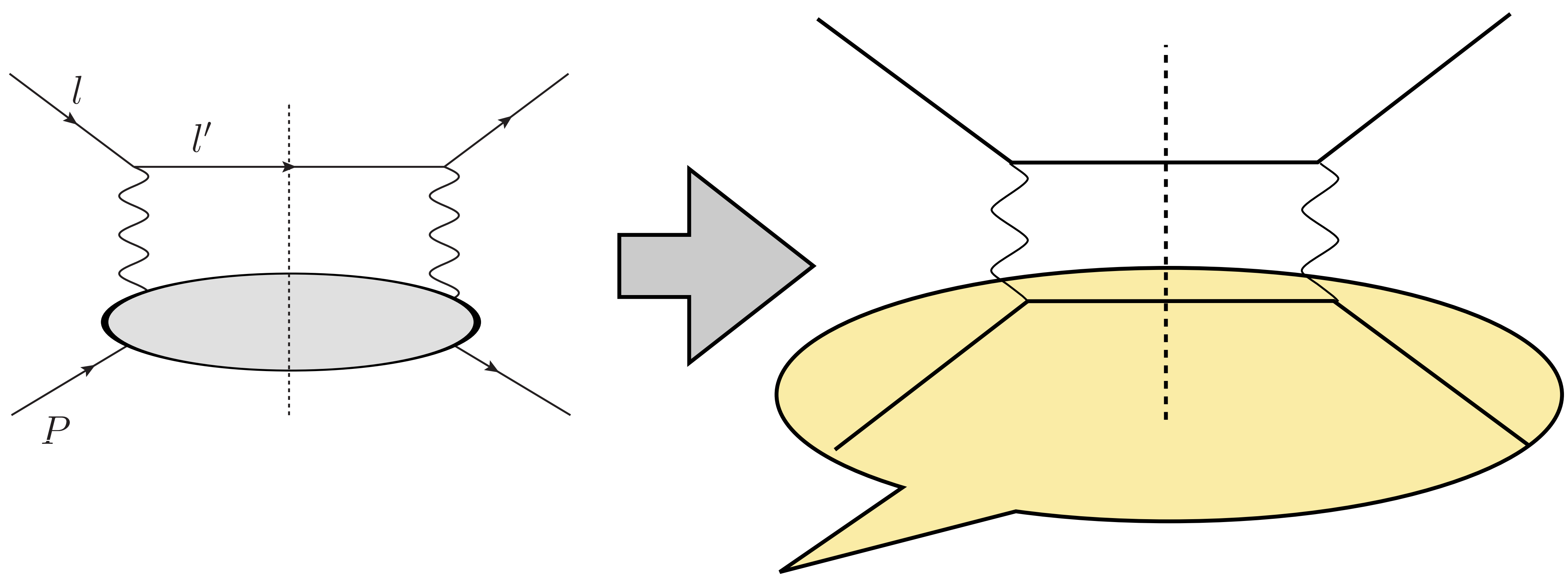
$$|P,S\rangle = c^\dagger(P)\,|0\rangle$$

unpolarized

$$\Phi^q_{ji}(x) = \int \frac{d\xi^-}{2\pi}\, e^{-\mathrm{i}(xP^+-P^+)\xi^-}\, \frac{1}{2}\sum_s u_j(P)\overline{u}_i(P) = \frac{1}{2}\, \not{P}_{ji}\ \delta(xP^+ - P^+)$$

# Are there more “relevant” and less “relevant” parts?

To address this problem, let’s first look at a full QED process like electron—muon scattering, where the cross section is proportional to the contraction of two leptonic tensors.




$$\mathrm{Tr}[(\not{P} + M)\gamma^\mu(\not{P} + \not{q} + M)\gamma^\nu] = 4P^\mu(P+q)^\nu + 4P^\nu(P+q)^\mu - 4g^{\mu\nu}P\cdot(P+q) + 4M^2 g^{\mu\nu}$$

# Are there more “relevant” and less “relevant” parts?

$$\mathrm{Tr}[(\not{P}+M)\gamma^\mu(\not{P}+\not{q}+M)\gamma^\nu] = 4P^\mu(P+q)^\nu + 4P^\nu(P+q)^\mu - 4g^{\mu\nu}P\cdot(P+q) + 4M^2g^{\mu\nu}$$

Introduce light-cone vectors $n_+$ and $n_-$ so that $(n_+)^2=(n_-)^2=0$, $n_+.n_-=1$.

Write P and q in terms of $n_+$ and $n_-$

$$P^\mu = P^+(n_+)^\mu + \frac{M^2}{2P^+}(n_-)^\mu$$

$$q^\mu = -x_B P^+(n_+)^\mu + \frac{Q^2}{2x_B P^+}(n_-)^\mu$$

Neglecting terms of order $M^4/Q^4$

$$(P+q)^\mu = (1-x_B)P^+(n_+)^\mu + \left(\frac{M^2}{2P^+} + \frac{Q^2}{2x_B P^+}\right)(n_-)^\mu$$

$$\Longrightarrow x_B = 1 - \frac{M^2}{Q^2}$$

# Are there more “relevant” and less “relevant” parts?

$$\mathrm{Tr}[(\not{P} + M)\gamma^\mu(\not{P} + \not{q} + M)\gamma^\nu] = 4P^\mu(P+q)^\nu + 4P^\nu(P+q)^\mu - 4g^{\mu\nu}P\cdot(P+q) + 4M^2 g^{\mu\nu}$$

Introduce light-cone vectors $n_+$ and $n_-$ so that $(n_+)^2=(n_-)^2=0$, $n_+ \cdot n_- = 1$.

Write P and q in terms of $n_+$ and $n_-$

It is sufficient to take into consideration the + component of P and the – component of P+q

$$P^\mu = P^+(n_+)^\mu$$

$$q^\mu = -x_B P^+(n_+)^\mu + \frac{Q^2}{2x_B P^+}(n_-)^\mu$$

$$(P+q)^\mu = \frac{Q^2}{2x_B P^+}(n_-)^\mu$$

Neglecting terms of order $M^2/Q^2$

$$\Longrightarrow x_B = 1$$

# Light cone coordinates

Light-cone vectors will be indicated as

$$a^\mu = \left[a^-,\ a^+,\ \boldsymbol{a}_T\right] = \left[\frac{a^0 - a^3}{\sqrt{2}},\ \frac{a^0 + a^3}{\sqrt{2}},\ a^1,\ a^2\right]. \tag{1}$$

The dot-product in light-cone components is

$$a \cdot b = a^+ b^- + a^- b^+ - \boldsymbol{a}_T \cdot \boldsymbol{b}_T \tag{2}$$

The light-cone decomposition of a vector can be written in a Lorentz covariant fashion using two light-like vectors $n_+$ and $n_-$ satisfying $n_\pm^2 = 0$ and $n_+ \cdot n_- = 1$ and promoting $\boldsymbol{a}_T$ to a four-vector $a_T^\mu = [0, 0, \boldsymbol{a}_T]$ so that

$$a^\mu = a^+ n_+^\mu + a^- n_-^\mu + a_T^\mu, \tag{3}$$

where

$$a^+ = a \cdot n_-, \qquad a^- = a \cdot n_+, \qquad a_T \cdot n_+ = a_T \cdot n_- = 0. \tag{4}$$

Note that

$$a_T \cdot b_T = -\boldsymbol{a}_T \cdot \boldsymbol{b}_T \tag{5}$$

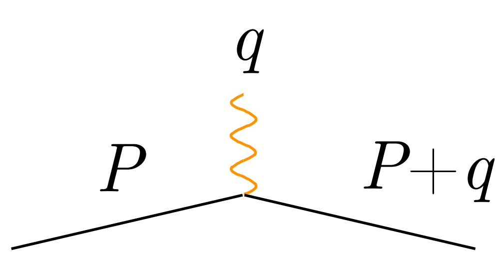
# Choice of frame (not necessary, but useful)

The most symmetric choice is (Breit frame)

$$P^+ = \frac{Q}{x_B\sqrt{2}}$$

but by changing $P^+$ we can select any other frame (e.g. proton rest frame)

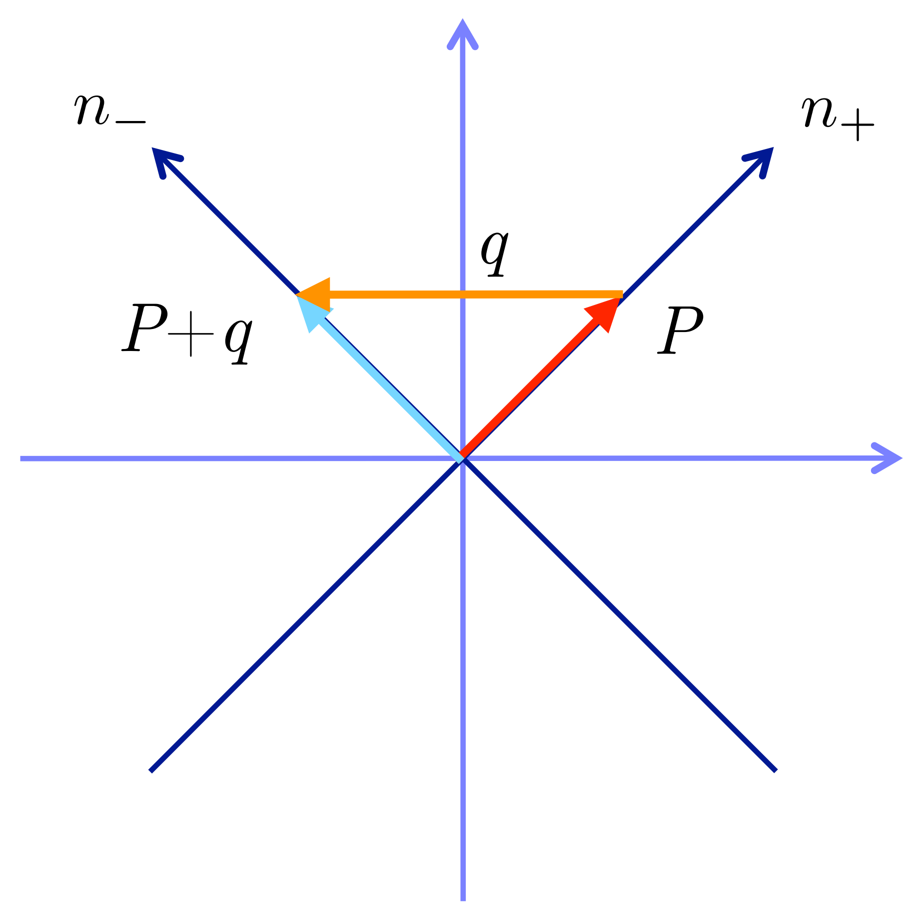
Neglecting masses and setting $x_B$=1 in this trivial case




$$P^\mu = \left[0,\ \frac{Q}{\sqrt{2}},\ 0,\ 0\right]$$

$$q^\mu = \left[\frac{Q}{\sqrt{2}},\ -\frac{Q}{\sqrt{2}},\ 0,\ 0\right]$$

$$(P+q)^\mu = \left[\frac{Q}{\sqrt{2}},\ 0,\ 0,\ 0\right]$$



# Key points

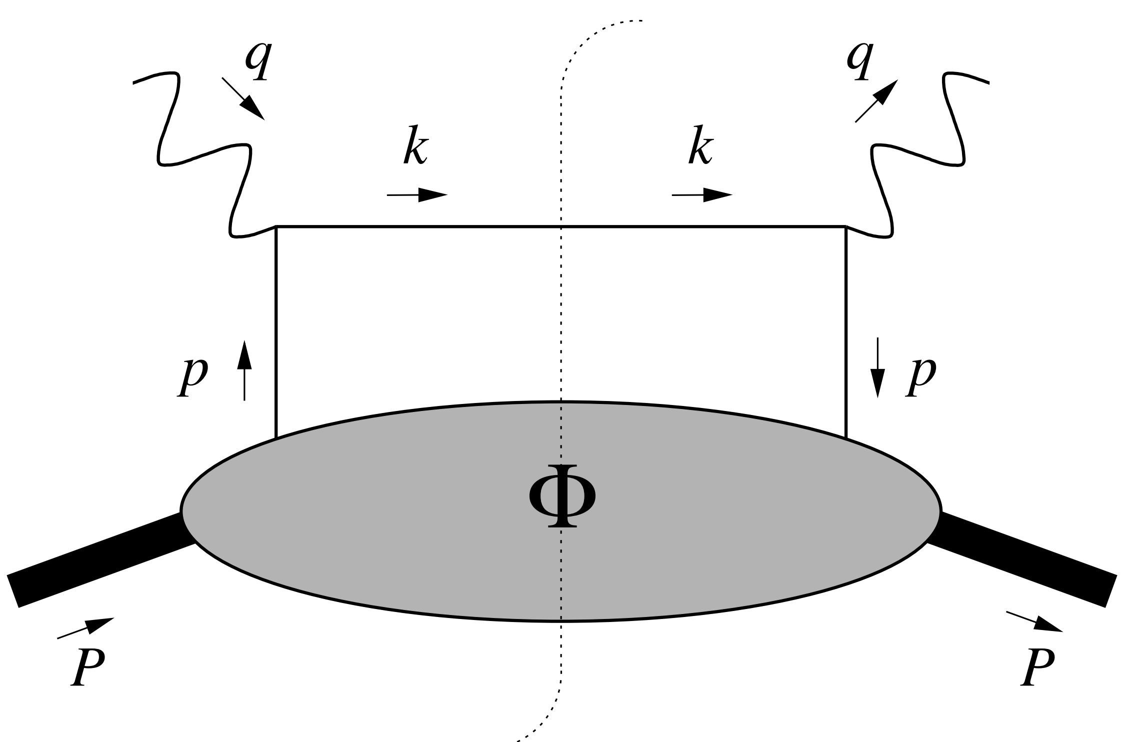
- The relevant components are the + components before the absorption of the photon and the – components after the absorption.
- The identification of the relevant components can be done in a frame-independent way, however it is convenient to work in certain frames to simplify the discussion

# Back to the hadron tensor

$$2MW^{\mu\nu}(q,P,S) = \sum_q e_q^2 \int \mathrm{d}^4p\, \delta\left(\left(p+q\right)^2 - m^2\right)\, \theta\left(p^0+q^0-m\right)$$
$$\times\, \mathrm{Tr}\left[\Phi(p,P,S)\gamma^\mu\left(\not{p}+\not{q}+m\right)\gamma^\nu\right]$$

compared to our lepton tensor analogy, there is an extra integration over the phase space of the final quark (assumed to go unobserved, but on shell)

$$\Phi_{ji}(p,P,S) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4\xi\, \mathrm{e}^{-\mathrm{i}p\cdot\xi}\langle P,S|\,\bar{\psi}_i(\xi)\,\psi_j(0)\,|P,S\rangle$$
$$= \sum_X \int \frac{\mathrm{d}^3\boldsymbol{P}_X}{(2\pi)^3\,2P_X^0}\,\langle P,S|\,\bar{\psi}_i(0)\,|X\rangle\langle X|\,\psi_j(0)\,|P,S\rangle\;\delta^{(4)}\Big(P-p-P_X\Big)$$



# Relevant parts

$$p^\mu = \left[\frac{p^2 + |\boldsymbol{p}_T|^2}{2xP^+},\ xP^+,\ \boldsymbol{p}_T\right]$$

Fraction of light-cone "+" momentum

Transverse momentum

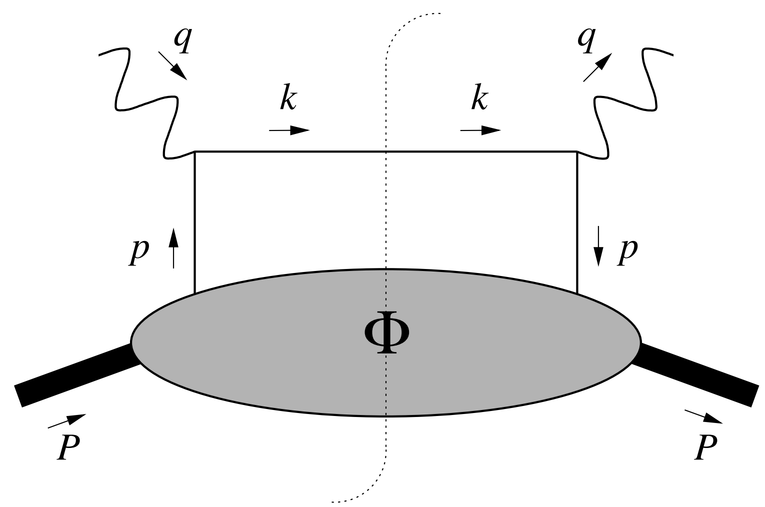
$$d^4p = d^2\boldsymbol{p}_T\, dp^-\, P^+\, dx$$

$$\delta\left(\left(p+q\right)^2 - m^2\right) \approx \delta(p^+ + q^+) \approx P^+\,\delta(x - x_B)$$

We assume that virtuality and transverse momentum of the quark are small compared to Q

$$\begin{aligned}
2MW^{\mu\nu}(q,P,S) &\approx \sum_q e_q^2 \int d^2\boldsymbol{p}_T dp^- dx\, \frac{P^+}{2P\cdot q}\,\delta\left(x - x_B\right) \\
&\quad \times \mathrm{Tr}\left[\Phi^q(p,P,S)\,\gamma^\mu\left(\not{p} + \not{q} + m\right)\gamma^\nu\right] \\
&= \sum_q e_q^2\,\frac{1}{2}\mathrm{Tr}\left[\Phi^q(x_B,S)\,\gamma^\mu\frac{P^+}{P\cdot q}\left(\not{p} + \not{q} + m\right)\gamma^\nu\right] \\
&= \sum_q e_q^2\,\frac{1}{2}\mathrm{Tr}\left[\Phi^q(x_B,S)\,\gamma^\mu\gamma^+\gamma^\nu\right]
\end{aligned}$$

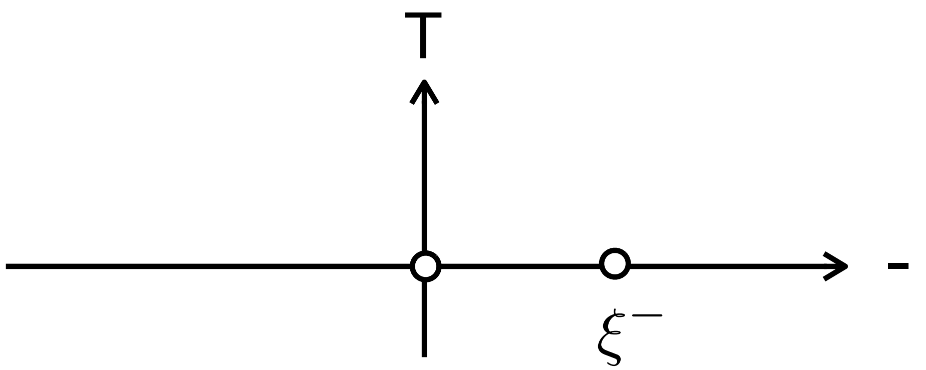
# Correlator




$$2MW^{\mu\nu}(q,P,S) \approx \sum_q e_q^2 \, \frac{1}{2} \mathrm{Tr}\left[\Phi(x_B,S)\,\gamma^\mu\gamma^+\gamma^\nu\right].$$

$$\Phi_{ij}(x,S) = \int d^2\boldsymbol{p}_T dp^- \; \Phi_{ij}(p,P,S)\Big|_{p^+=xP^+}$$

$$= \int \frac{d\xi^-}{2\pi}\, e^{ip\cdot\xi} \langle P,S|\,\bar{\psi}_j(0)\,\psi_i(\xi)\,|P,S\rangle\Big|_{\xi^+=\boldsymbol{\xi}_T=0}$$




$$\Phi_{ij}(p,P,S) = \frac{1}{(2\pi)^4}\int d^4\xi \; e^{ip\cdot\xi}\langle P,S|\,\bar{\psi}_j(0)\,\psi_i(\xi)\,|P,S\rangle$$

So far it was only INCLUSIVE DIS, what are the modifications in SEMI-INCLUSIVE DIS?

# Changes to be done

- We do not integrate over the final-state quark, but we detect a hadron in the final state and we integrate over all other fragments
- We have another external vector (final-state hadron momentum) that cannot be collinear to the other two (q and P).

$$q_T^\mu = q^\mu + (1-r)xP^\mu - P_h^\mu/z \qquad P_{h\perp}^\mu = -zq_T^\mu - 2rzxP^\mu$$

$$r = q_T^2/Q^2$$

If $q_T^2 << Q^2$, then $$P_{h\perp}^\mu = -zq_T^\mu$$

# Possible choices of light-cone vectors

Choice 1:
P and q have no transverse component.

$$P^\mu = P^+ n'^\mu_+ + \frac{M^2}{2P^+} n'^\mu_-,$$
$$q^\mu = -x_B P^+ n'^\mu_+ + \frac{Q^2}{2x_B P^+} n'^\mu_-.$$

$$P^\mu = \left[\frac{x_B M^2}{Q\sqrt{2}},\ \frac{Q}{x_B\sqrt{2}},\ \mathbf{0}\right]$$
$$q^\mu = \left[\frac{Q}{\sqrt{2}},\ -\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right]$$
$$P_h^\mu = \left[\frac{z_h Q}{\sqrt{2}},\ \frac{M_h^2 + |\boldsymbol{P}_{h\perp}|^2}{z_h Q\sqrt{2}},\ \boldsymbol{P}_{h\perp}\right]$$

$$\boldsymbol{q}_T = -\frac{\boldsymbol{P}_{h\perp}}{z_h}$$

Choice 2:
P and $P_h$ have no transverse component

$$P^\mu = P^+ n^\mu_+ + \frac{M^2}{2P^+} n^\mu_-,$$
$$P_h^\mu = \frac{M_h^2}{2P_h^-} n^\mu_+ + P_h^- n^\mu_-.$$

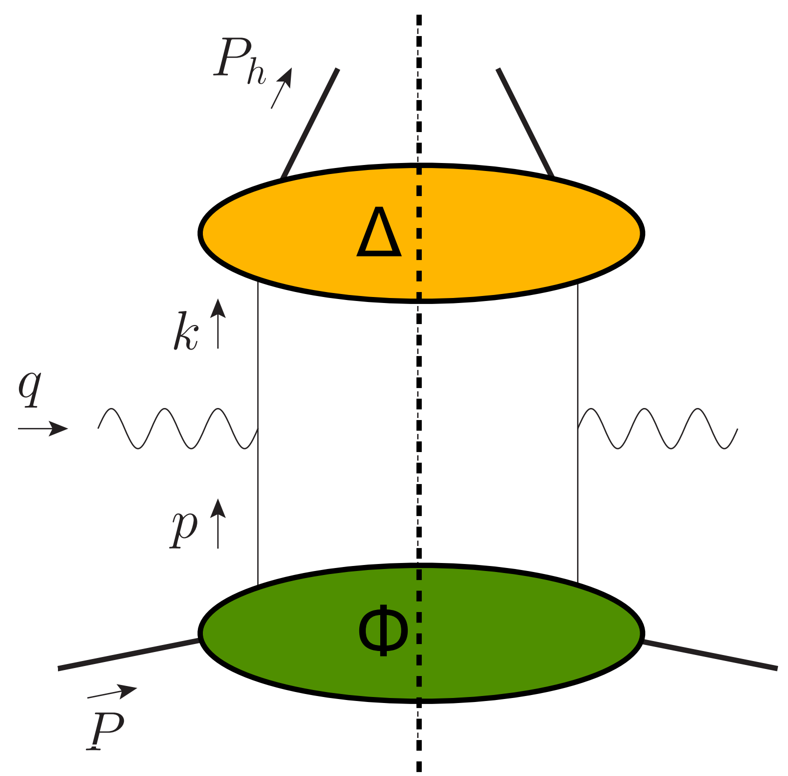
$$P^\mu = \left[\frac{x_B M^2}{Q\sqrt{2}},\ \frac{Q}{x_B\sqrt{2}},\ \mathbf{0}\right]$$
$$P_h^\mu = \left[\frac{z_h Q}{\sqrt{2}},\ \frac{M_h^2}{z_h Q\sqrt{2}},\ \mathbf{0}\right]$$
$$q^\mu = \left[\frac{Q}{\sqrt{2}},\ -\frac{(Q^2 - |\boldsymbol{q}_T|^2)}{Q\sqrt{2}},\ \boldsymbol{q}_T\right] \approx \left[\frac{Q}{\sqrt{2}},\ -\frac{Q}{\sqrt{2}},\ \boldsymbol{q}_T\right]$$

I work with this one. The results are independent of this choice

# Hadronic tensor for SIDIS

This is the result, then we are going to motivate it

$$2MW^{\mu\nu}(q,P,S,P_h) = \sum_q e_q^2 \int \mathrm{d}^4p\,\mathrm{d}^4k\;\delta^{(4)}\,(p+q-k)\;\mathrm{Tr}\,(\Phi(p,P,S)\,\gamma^\mu\,\Delta(k,P_h)\,\gamma^\nu)$$

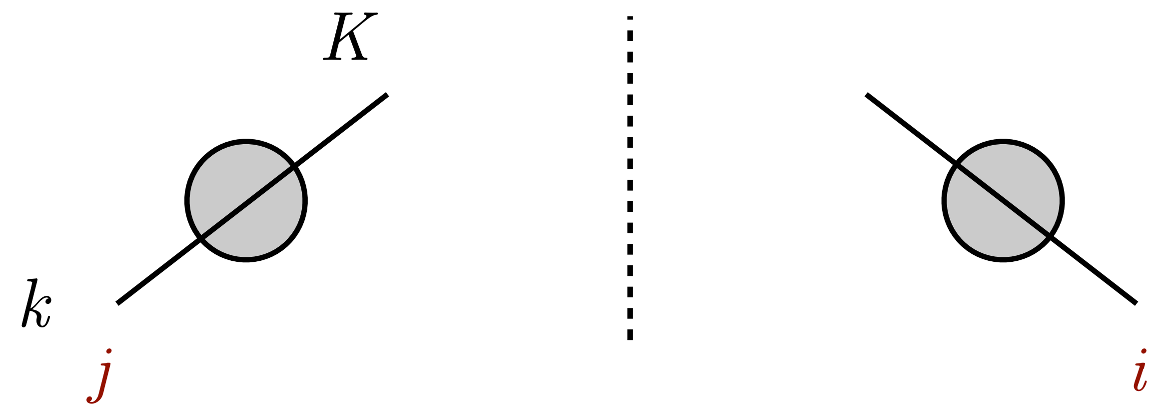


# Fragmentation correlator

$$\Delta_{ij}(k, P_h) = \frac{1}{(2\pi)^4} \int d^4\zeta \; e^{ik\cdot\zeta} \, \langle 0 | \, \psi_i(\zeta) \, | P_h \rangle \langle P_h | \, \bar{\psi}_j(0) \, | 0 \rangle$$

$$= \sum_Y \int \frac{d^3 \boldsymbol{P}_Y}{(2\pi)^3 \, 2P_Y^0} \, \langle 0 | \, \psi_i(0) \, | P_h, Y \rangle \langle P_h, Y | \, \bar{\psi}_j(0) \, | 0 \rangle \; \delta^{(4)}\Big( k - P_h - P_Y \Big).$$

# Electron case: zeroth order contribution




$$\Delta_{ij}(k,K) = \int \frac{d^4\zeta}{(2\pi)^4}\; e^{-i\zeta k}\; \langle 0|\psi_i(0)|e(K)\rangle\langle e(K)|\overline{\psi}_j(\zeta)|0\rangle$$
$$= \langle 0|\psi_i(0)|e(K)\rangle\langle e(K)|\overline{\psi}_j(\zeta)|0\rangle\delta^4(k-K)$$

$$\Delta(k,K) = u(K)\bar{u}(K)\delta^4(K-k) = (\not{K}+m)\delta^4(K-k)$$

# Hadronic tensor for SIDIS

$$2MW^{\mu\nu}(q,P,S,P_h) = \sum_a e_q^2 \int \mathrm{d}^4p\,\mathrm{d}^4k\;\delta^{(4)}\left(p+q-k\right)\,\mathrm{Tr}\left(\Phi(p,P,S)\,\gamma^\mu\,\Delta(k,P_h)\,\gamma^\nu\right)$$

$$p^\mu = \left[\frac{p^2+|\boldsymbol{p}_T|^2}{2xP^+},\; xP^+,\; \boldsymbol{p}_T\right],$$
$$k^\mu = \left[\frac{P_h^-}{z},\; \frac{z\,(k^2+|\boldsymbol{k}_T|^2)}{2P_h^-},\; \boldsymbol{k}_T\right]$$

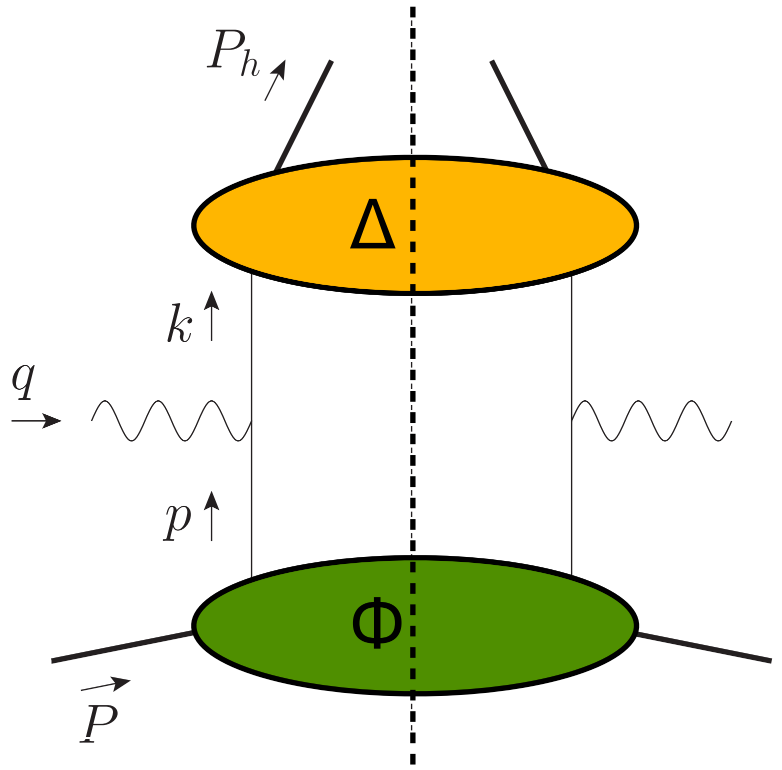
$$d^4p = d^2\boldsymbol{p}_T\,dp^-\;P^+\,dx$$

$$\mathrm{d}^4k = \mathrm{d}^2\boldsymbol{k}_T\;\mathrm{d}k^+\,P_h^-\,\frac{\mathrm{d}\,z}{z^2}$$

$$\begin{aligned}\delta^{(4)}\left(p+q-k\right) &\approx \delta(p^+ + q^+)\,\delta(q^- - k^-)\,\delta^{(2)}\Big(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T\Big)\\ &\approx \frac{1}{P^+\,P_h^-}\,\delta(x-x_B)\,\delta(1/z-1/z_h)\,\delta^{(2)}\Big(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T\Big)\end{aligned}$$

# Final formula for hadronic tensor

$$2MW^{\mu\nu}(q,P,S,P_h) = \frac{2z_h}{x_B}\,\mathcal{C}\Big[\mathrm{Tr}\big(\Phi(x_B,\boldsymbol{p}_T,S)\,\gamma^\mu\,\Delta(z_h,\boldsymbol{K}_T)\,\gamma^\nu\big)\Big]$$



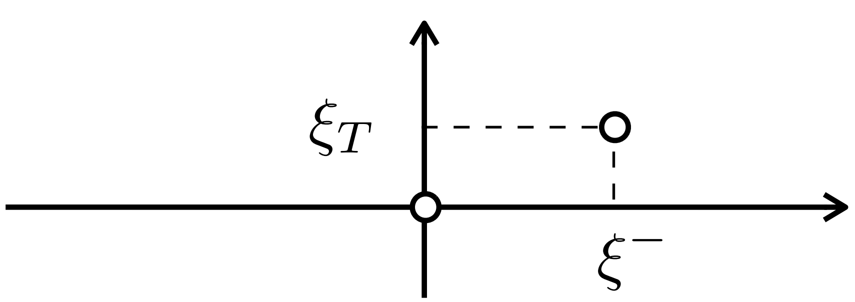

$$\mathcal{C}\big[wfD\big] = \sum_a x e_a^2 \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{K}_T\, \delta^{(2)}\big(z\boldsymbol{p}_T - \boldsymbol{K}_T - \boldsymbol{P}_{h\perp}\big)\, w(\boldsymbol{p}_T,\boldsymbol{K}_T)\, f^a(x,p_T^2)\, D^a(z,K_T^2),$$

*Only at low transverse momentum* $\boldsymbol{P}_{h\perp}^2 \ll Q^2$

# Quark-quark correlation functions

$$\Phi_{ij}(x,S) = \int d^2\boldsymbol{p}_T\ \Phi_{ij}(x,\boldsymbol{p}_T)$$
$$= \int \frac{d\xi^-}{2\pi}\ e^{ip\cdot\xi}\langle P,S|\,\bar{\psi}_j(0)\,\psi_i(\xi)\,|P,S\rangle\Big|_{\xi^+=\boldsymbol{\xi}_T=0}$$

$$\Phi_{ij}(x,\boldsymbol{p}_T,S) = \int dp^-\ \Phi(p,P,S)\Big|_{p^+=xP^+}$$
$$= \int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{(2\pi)^3}\ e^{ip\cdot\xi}\langle P,S|\,\bar{\psi}_j(0)\,\psi_i(\xi)\,|P,S\rangle\Big|_{\xi^+=0}$$



# Key points

- At “parton-model level,” the hadronic tensor can be written in terms of correlations functions

- For inclusive DIS, you need only the distribution function correlator, integrated over $p_T$

- For semi-inclusive DIS, you need distribution and fragmentation function correlators, not integrated over $p_T$