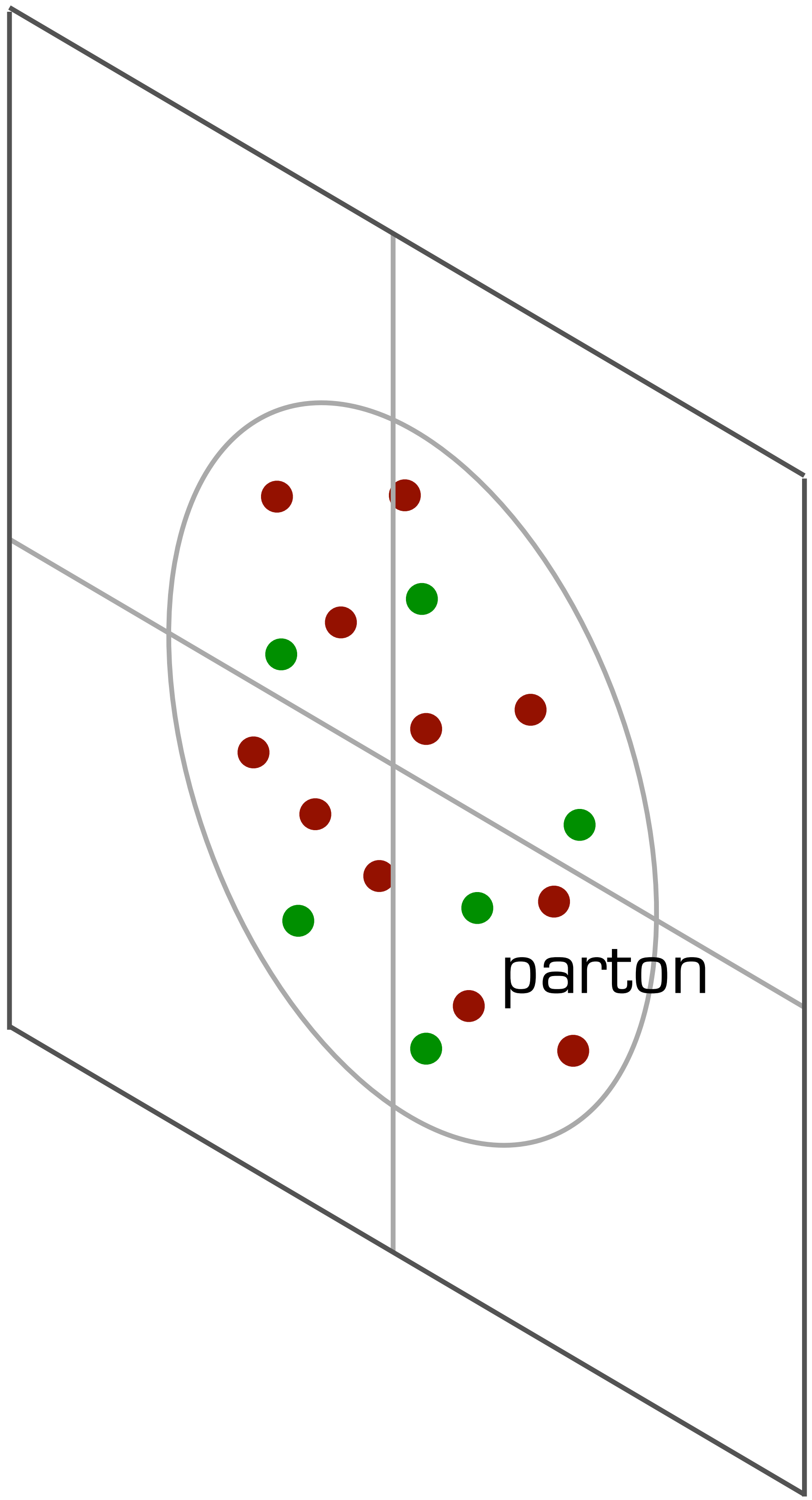
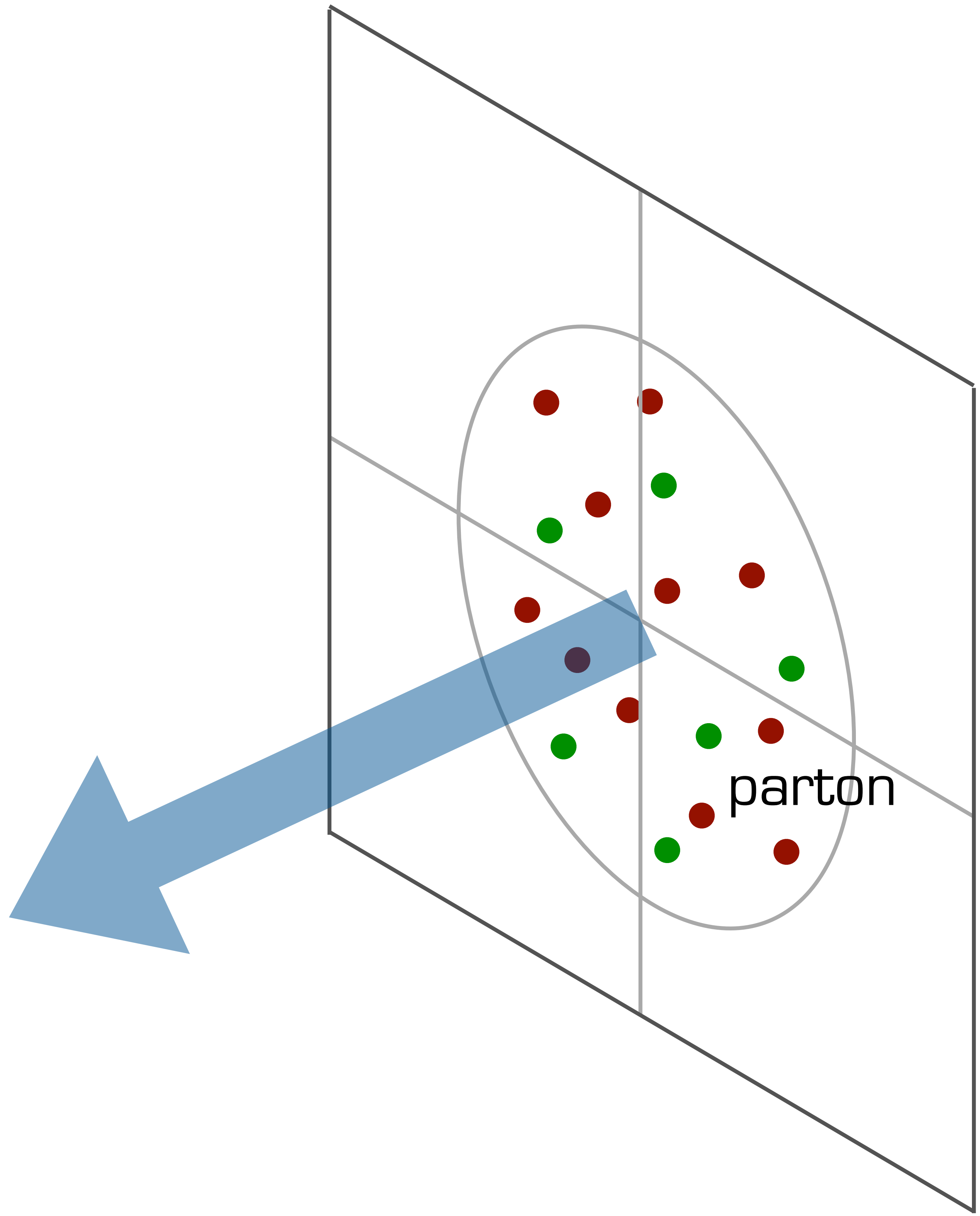
Theory of TMDs

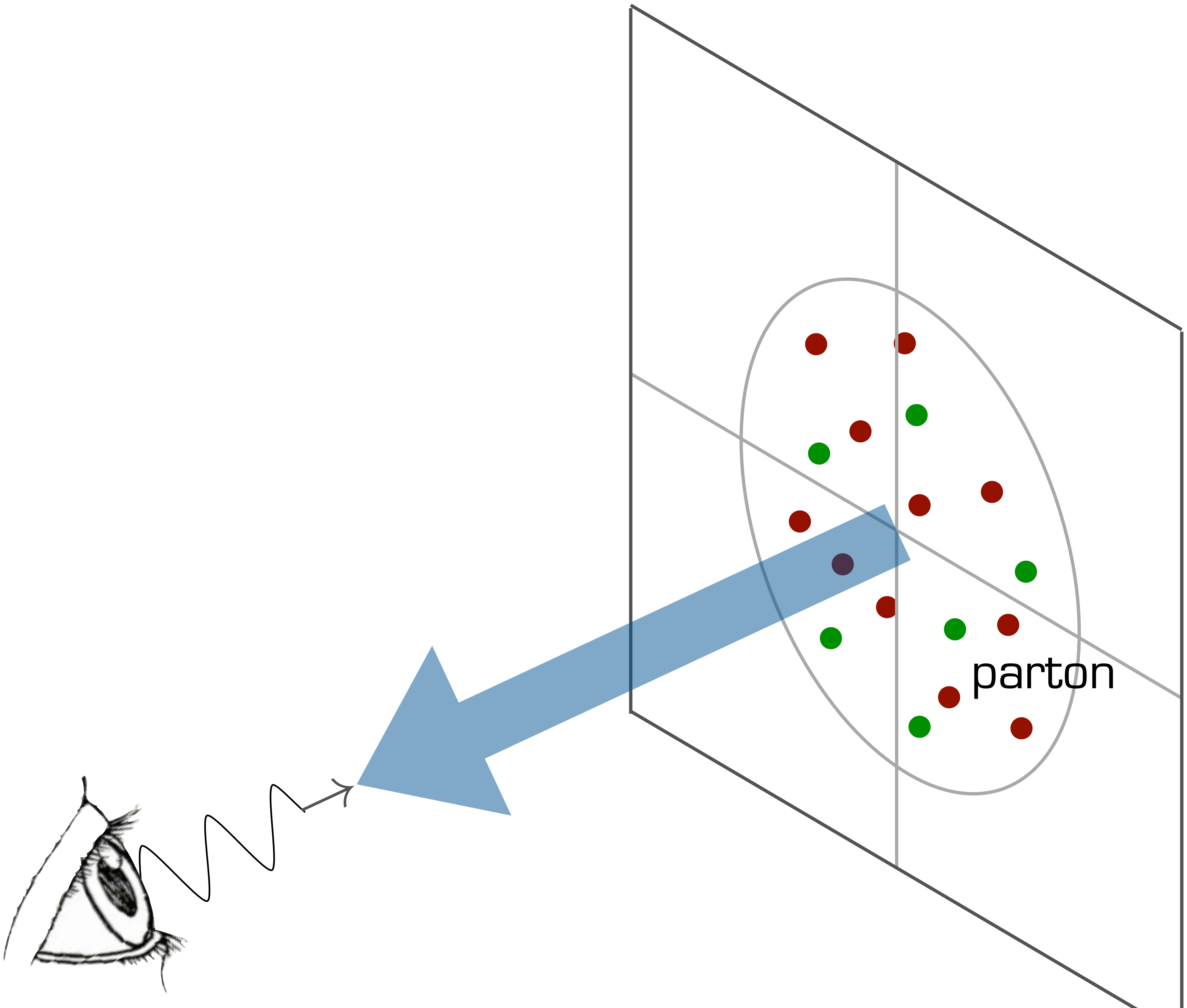
Alessandro Bacchetta (Pavia U. and INFN) alessandro.bacchetta@unipv.it

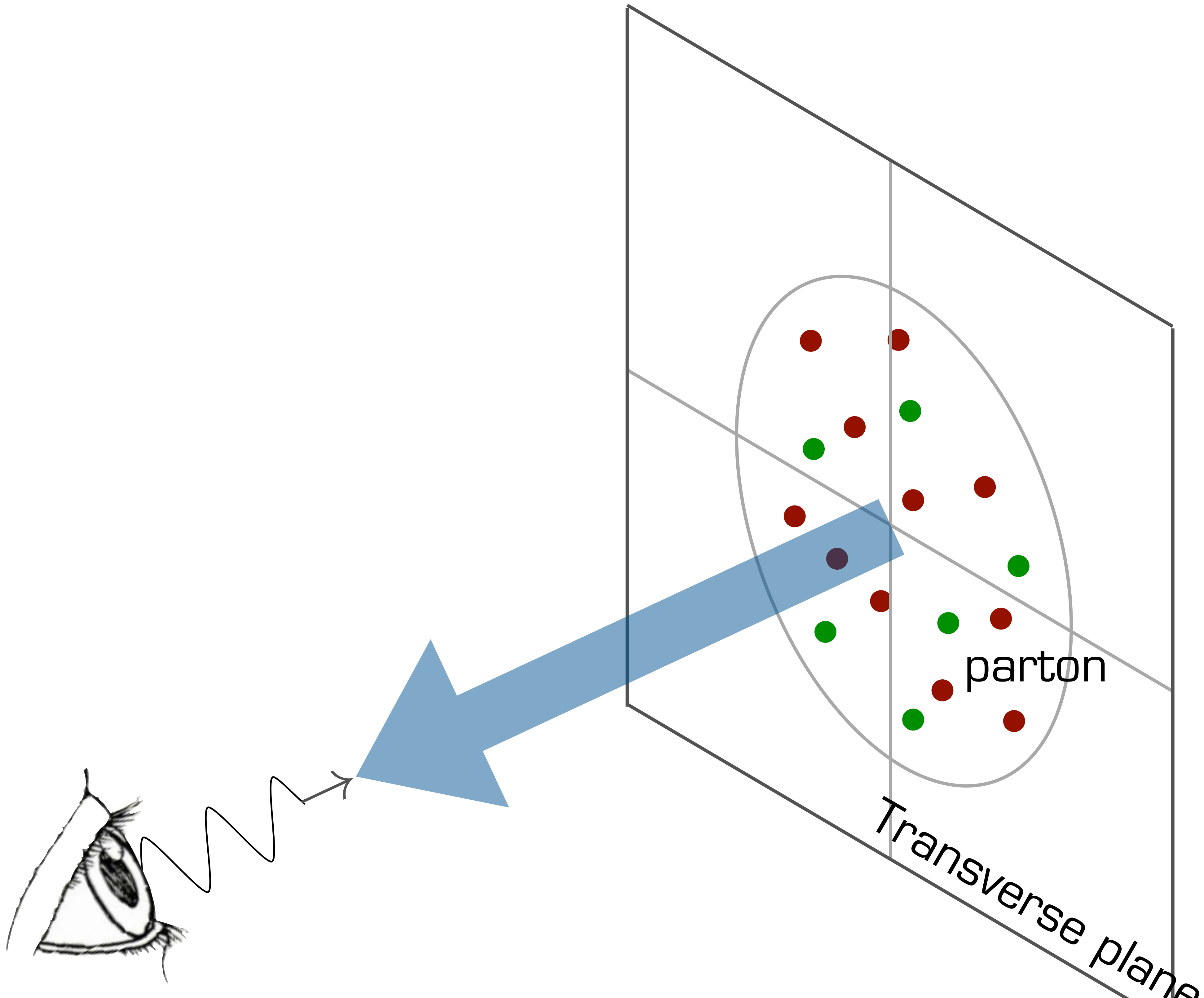
European Summer School on the Physics of the Electron-Ion Collider
June 18-22, 2023
Corigliano-Rossano
Italy

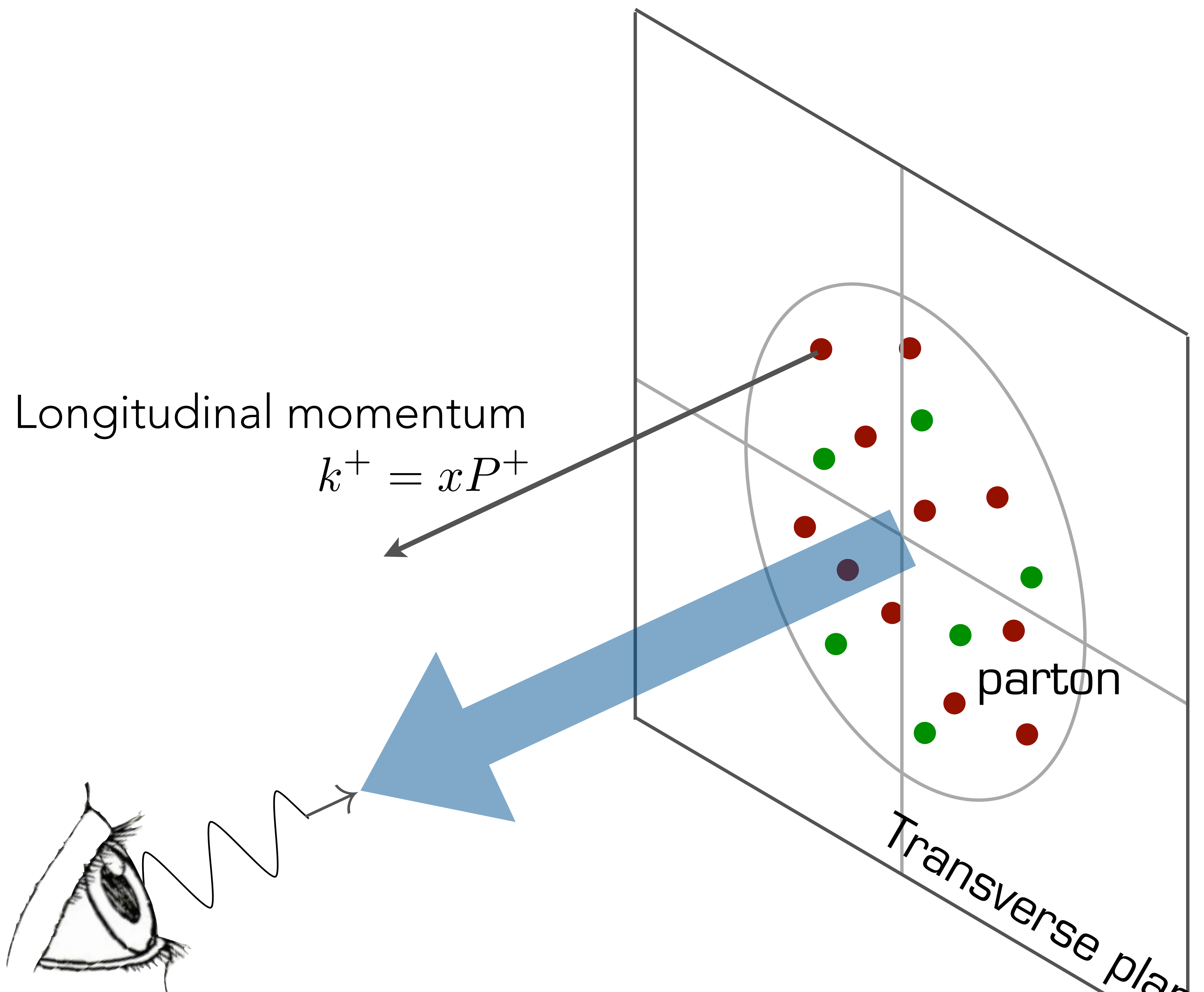
Some introduction


parton


parton


parton


parton
Transverse plane


Longitudinal momentum
$k^+ = xP^+$
parton
Transverse plane

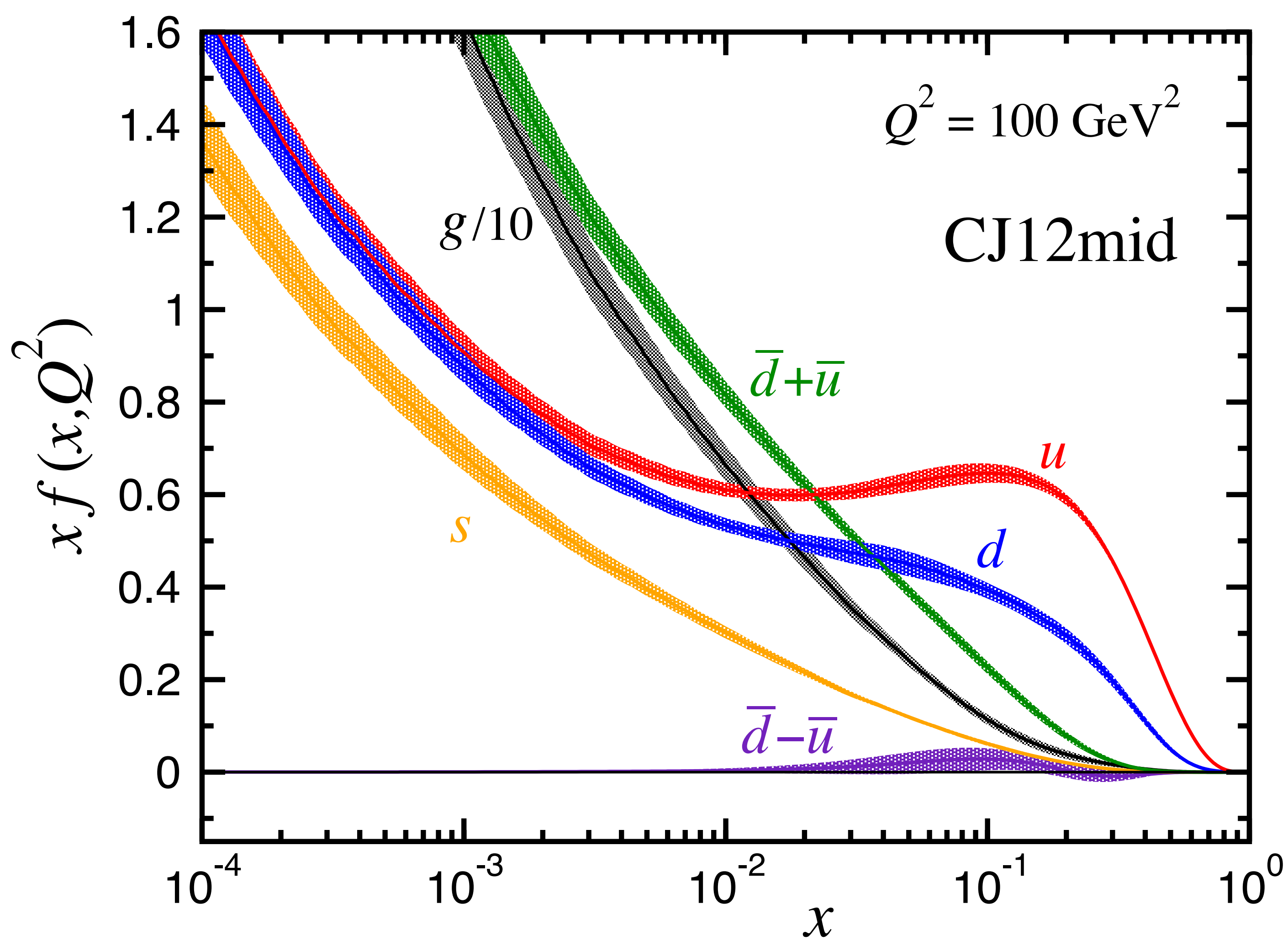
Standard parton distribution functions

Standard collinear PDFs describe the distribution of partons in one dimension in momentum space. They are extracted through global fits.

See lectures by E. Tassi

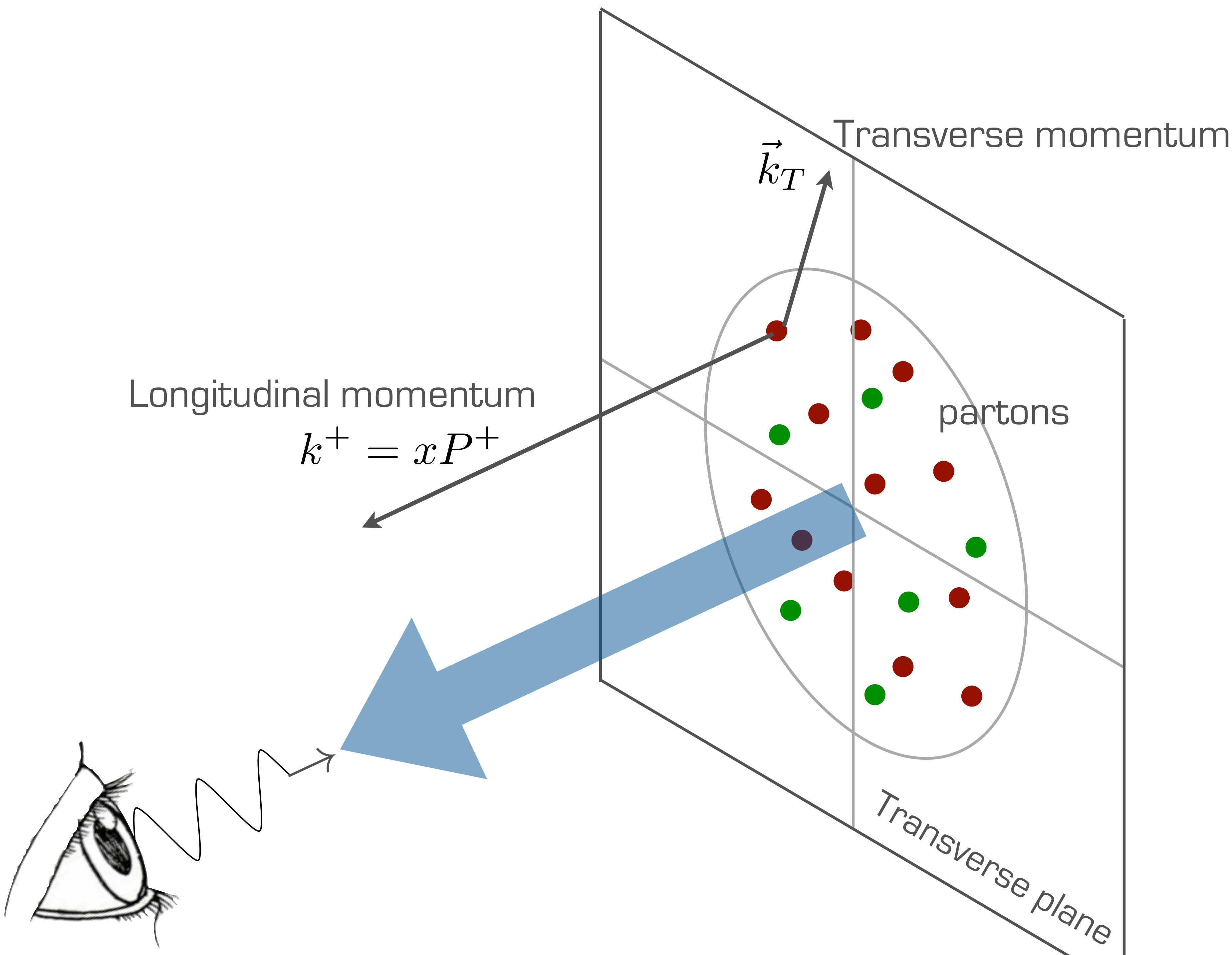
Standard parton distribution functions

Standard collinear PDFs describe the distribution of partons in one dimension in momentum space. They are extracted through global fits.
See lectures by E. Tassi



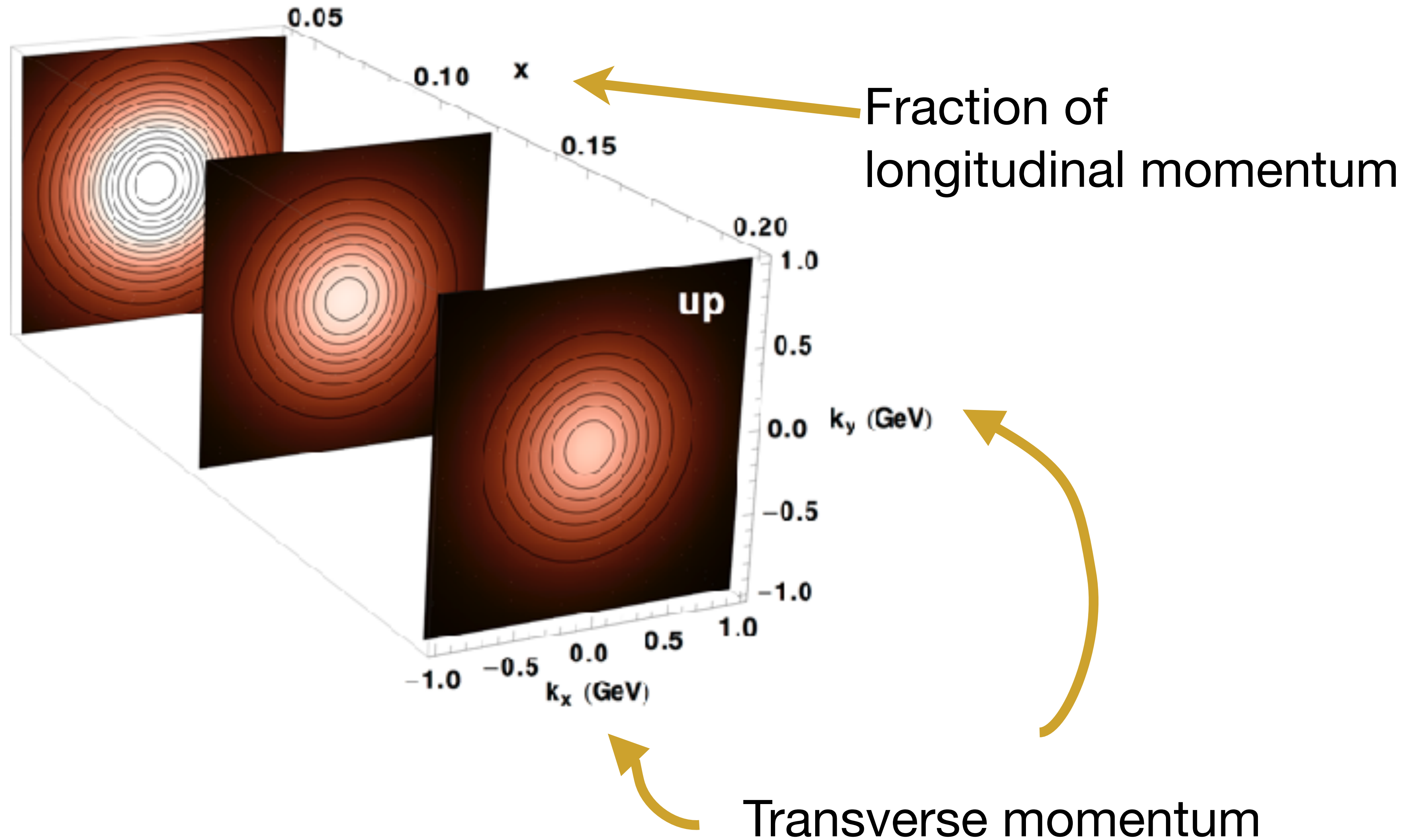
CTEQ-JLAB 12 set, Owens, Accardi, Melnitchouk, PRD87 (13)

Considering new dimensions

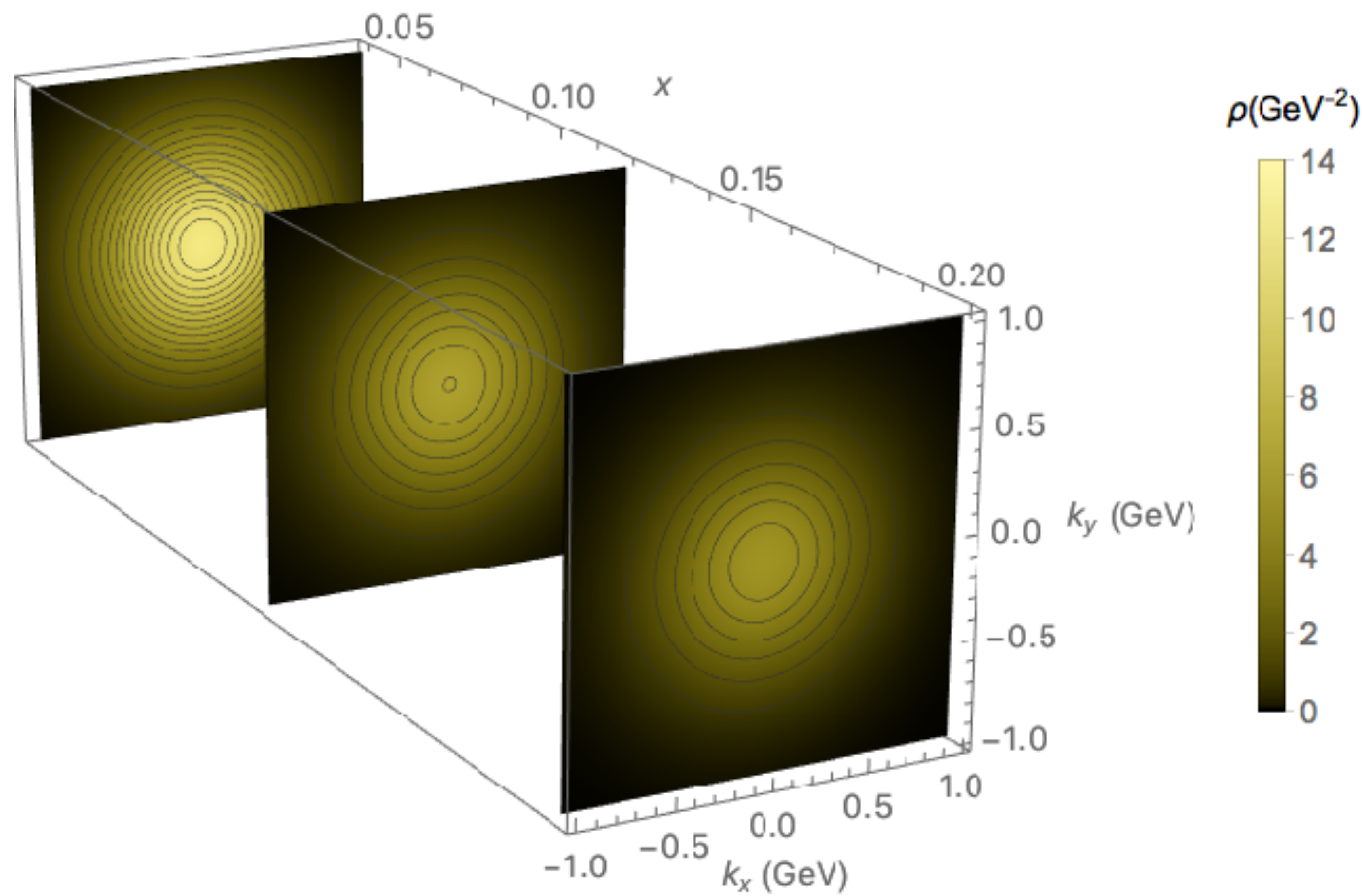


Transverse Momentum Distributions

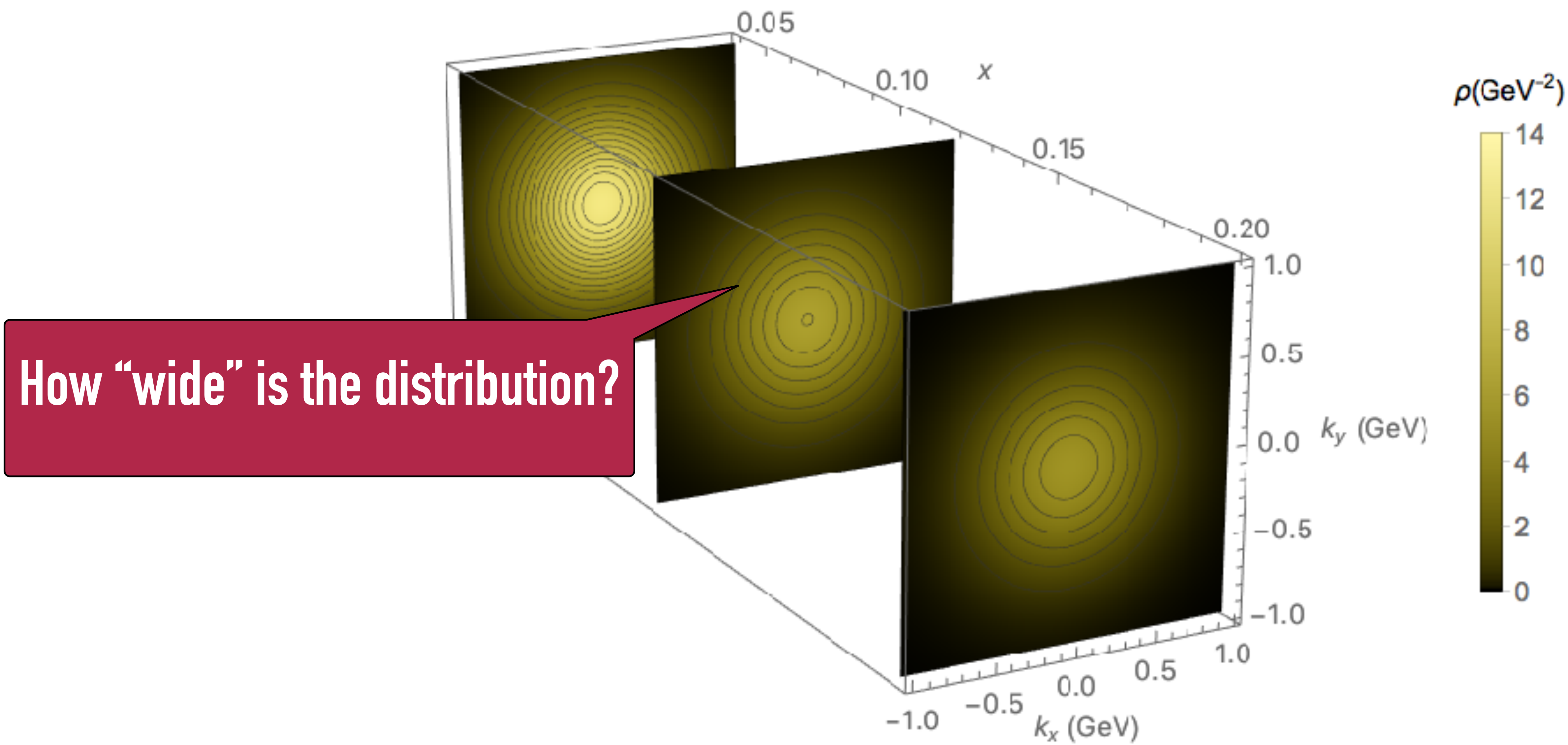
TMDs describe the distribution of partons in three dimensions in momentum space. They also have to be extracted through global fits.



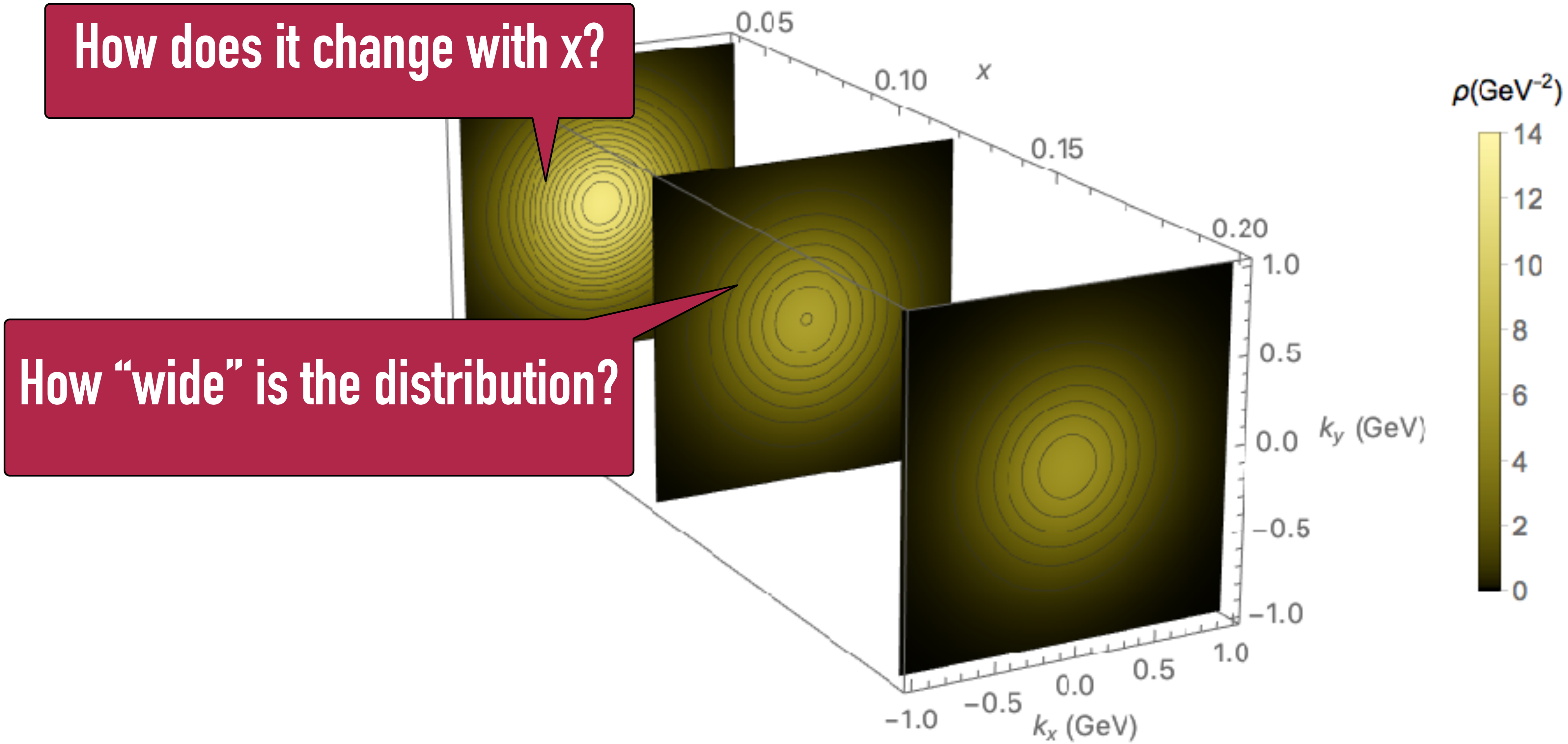
Some questions



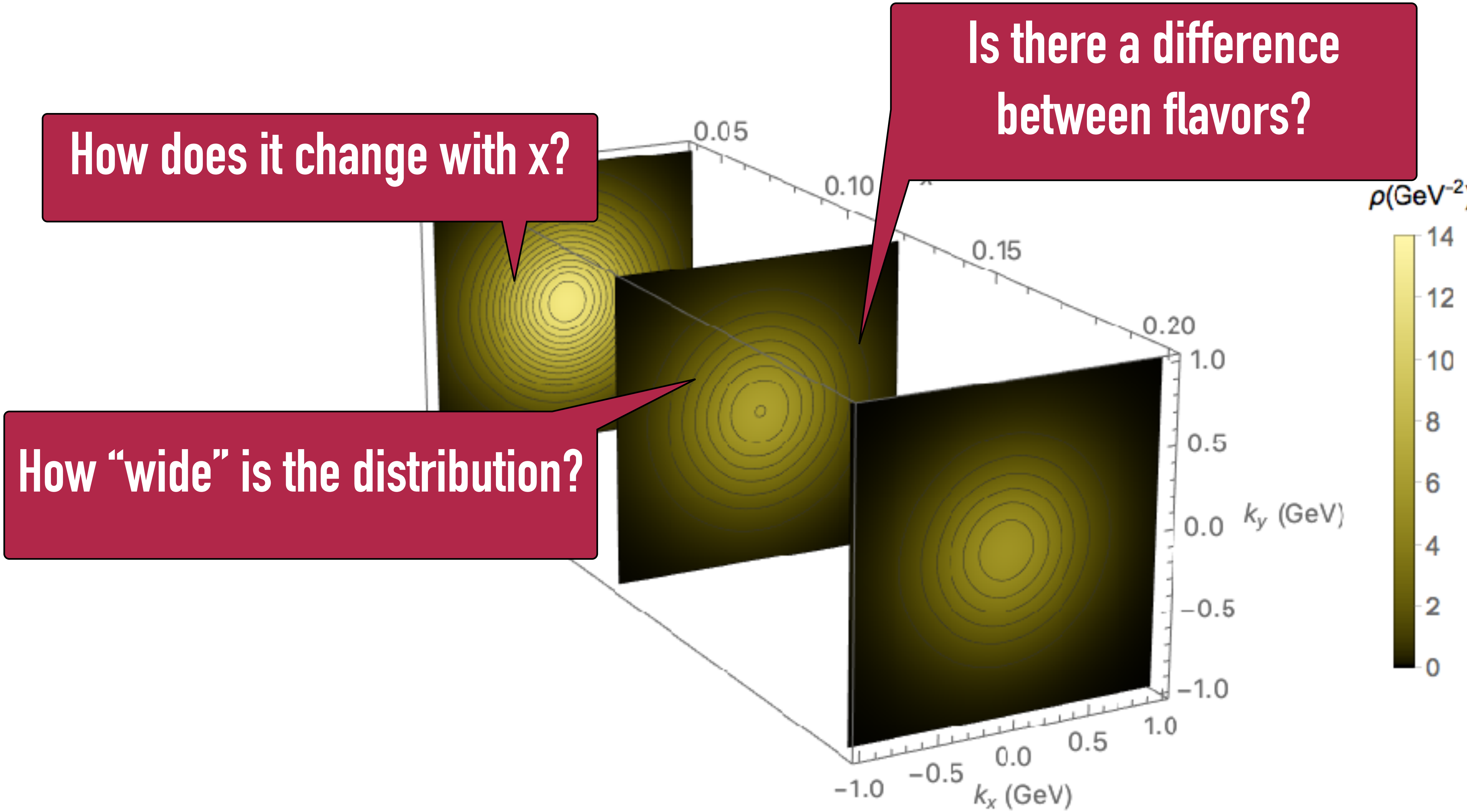
Some questions



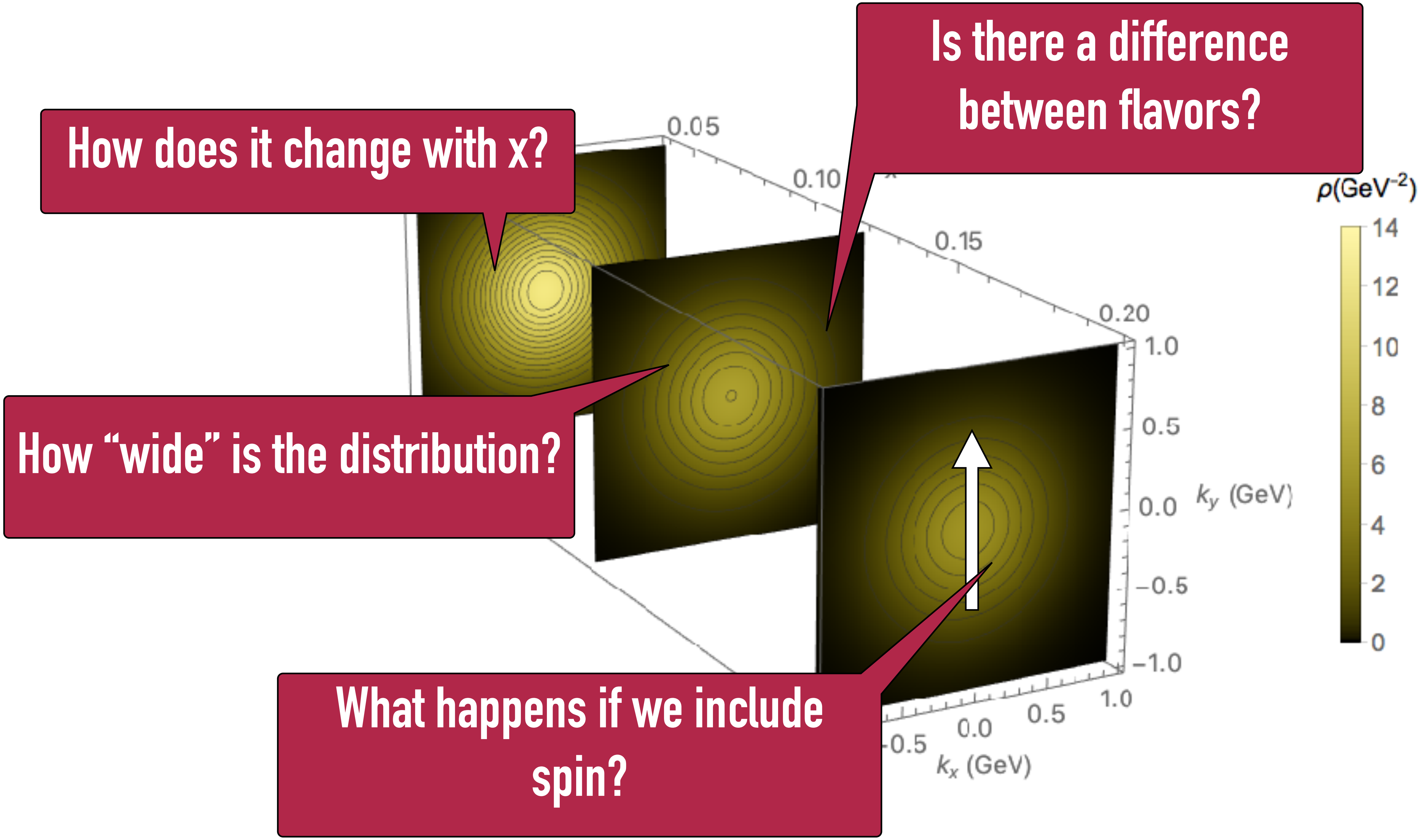
Some questions



Some questions

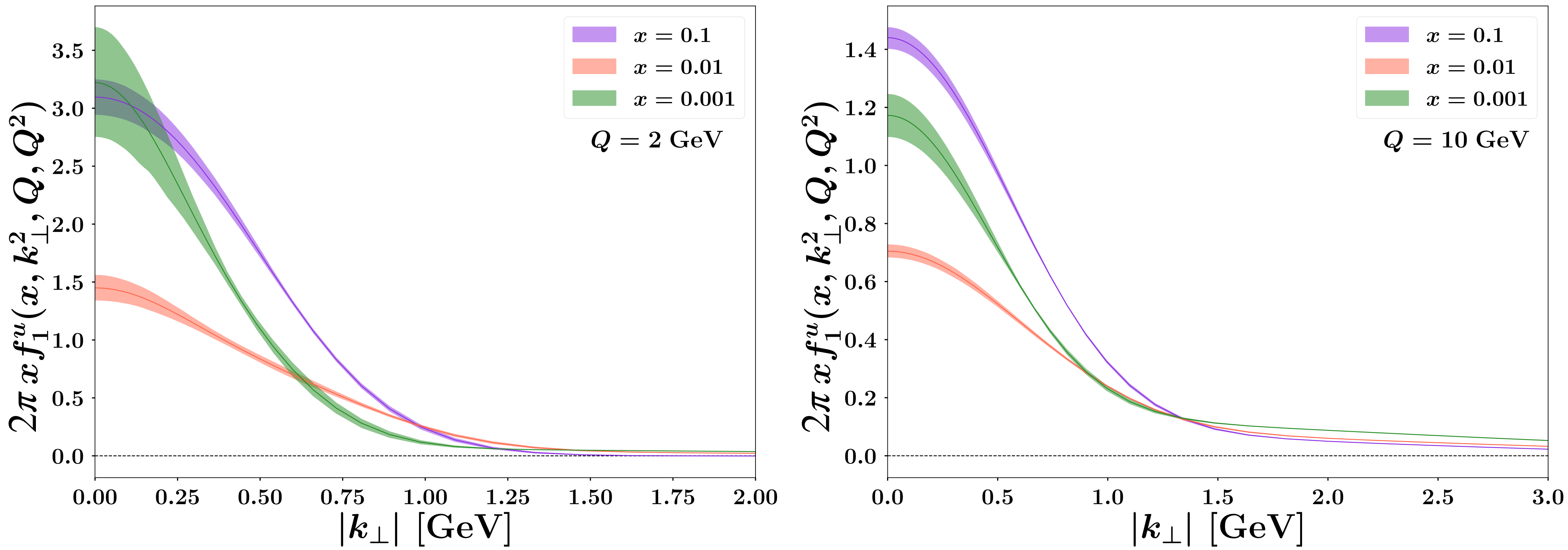


Some questions



Some answers

[MAP Collaboration](#)
[Bacchetta, Bertone, Bissolotti, Bozzi, Cerutti, Piacenza, Radici, Signori, arXiv:2206.07598](#)

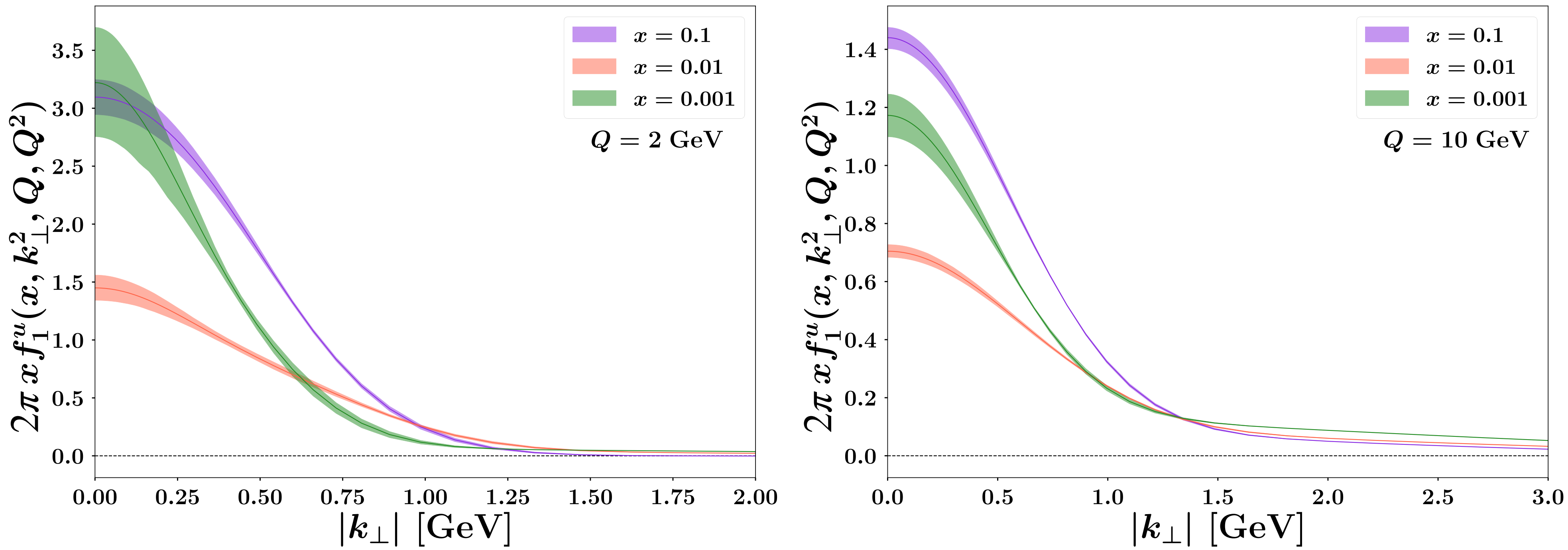


@ 2 GeV: the average transverse momentum squared is 0.4 GeV^2 (its square root is about 0.6 GeV)

@ 10 GeV: the average transverse momentum squared is 0.5 GeV^2 (its square root is about and 0.7 GeV)

Some answers

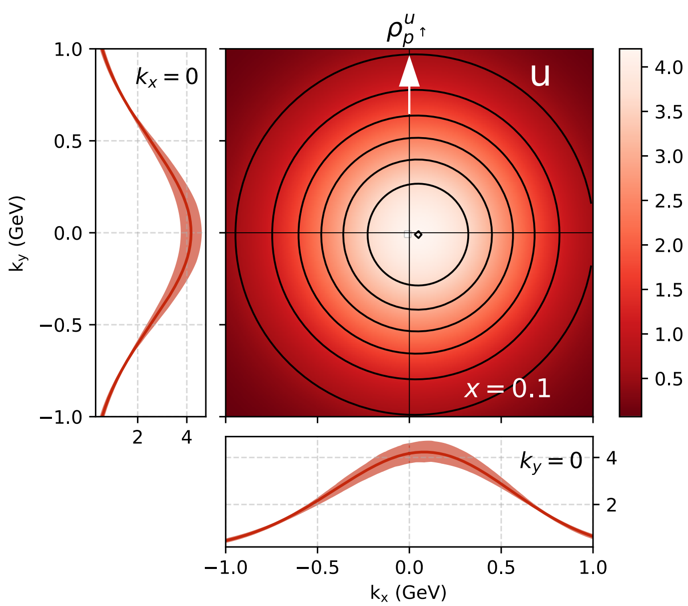
[MAP Collaboration](#)
[Bacchetta, Bertone, Bissolotti, Bozzi, Cerutti, Piacenza, Radici, Signori, arXiv:2206.07598](#)

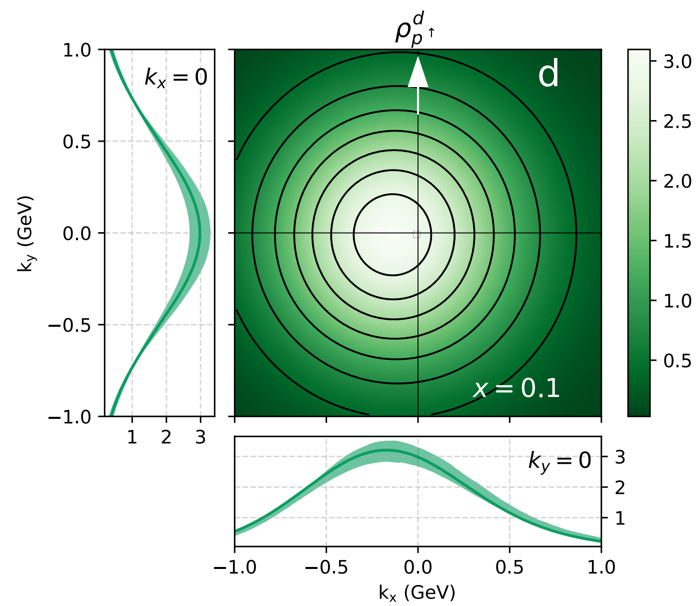


@ 2 GeV: the average transverse momentum at x=0.1 seems to be larger than at x=0.001

@ 10 GeV: the above behavior changes and the two become very similar, with a much higher tail at low x

Inclusion of spin

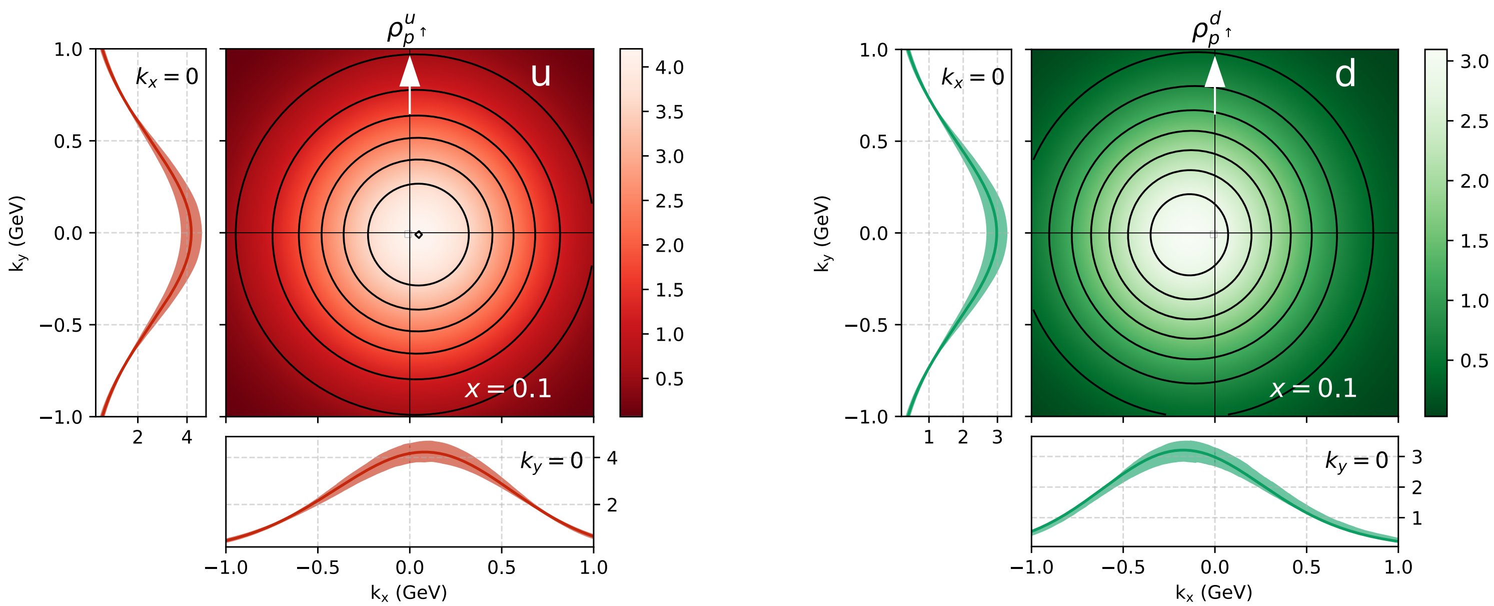




Q= 2GeV

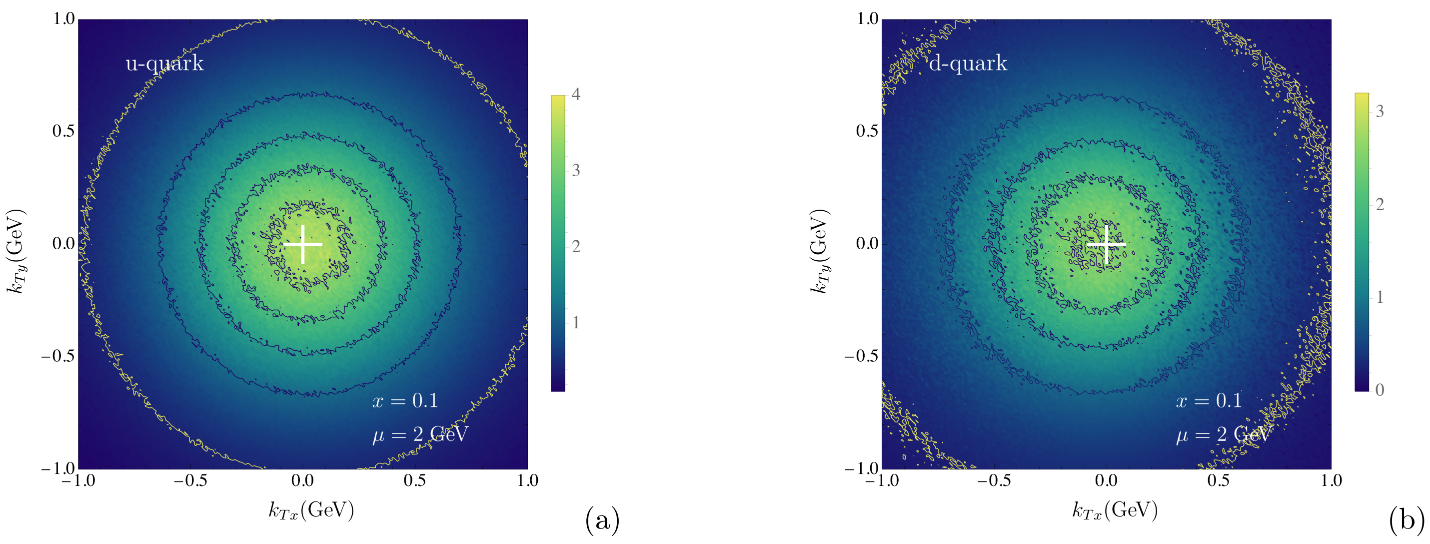
[Bacchetta, Delcarro, Pisano, Radici, arXiv:2004.14278]

Inclusion of spin



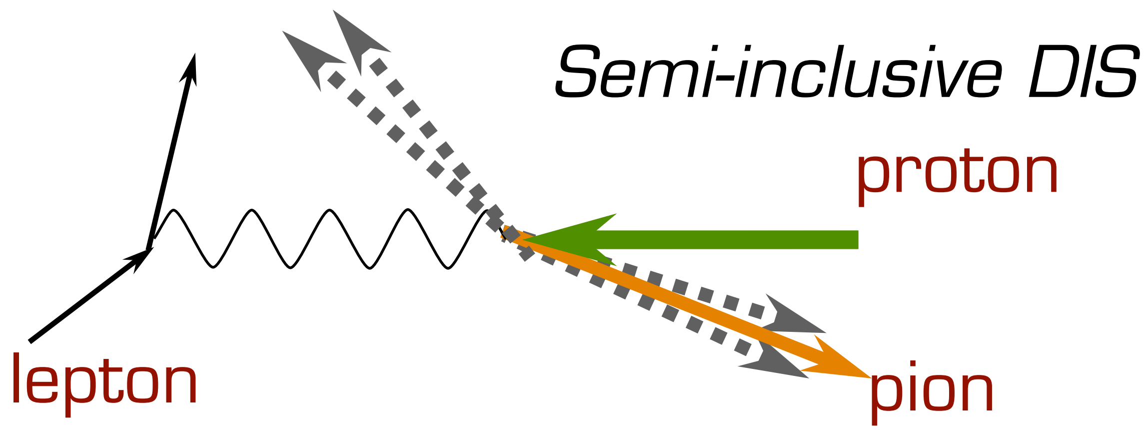
Q= 2GeV

Bacchetta, Delcarro, Pisano, Radici, arXiv:2004.14278

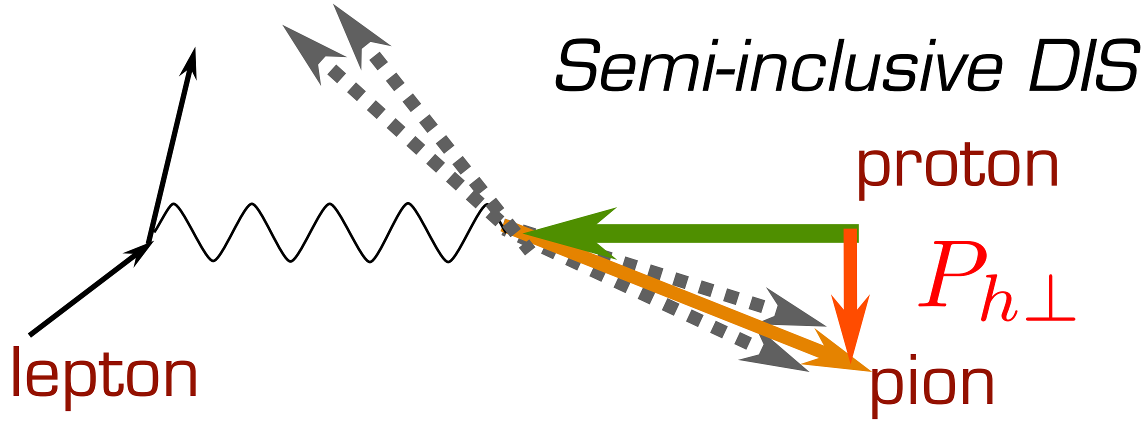


Bury, Prokudin, Vladimirov, arXiv:2103.03270

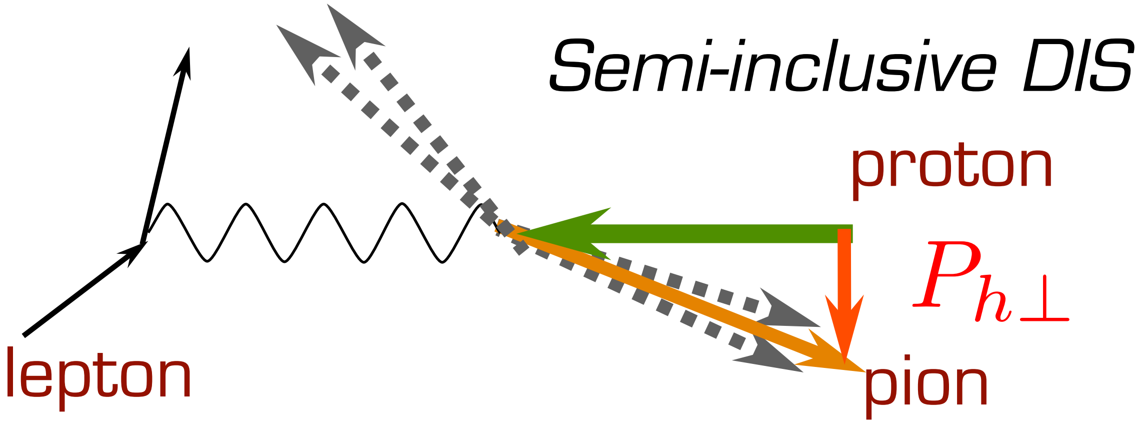
Where can we access TMDs?

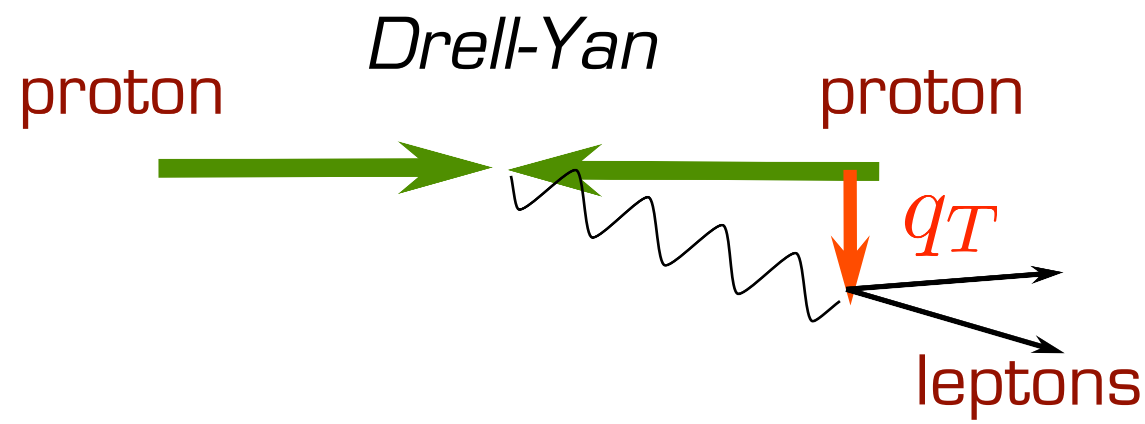


Where can we access TMDs?

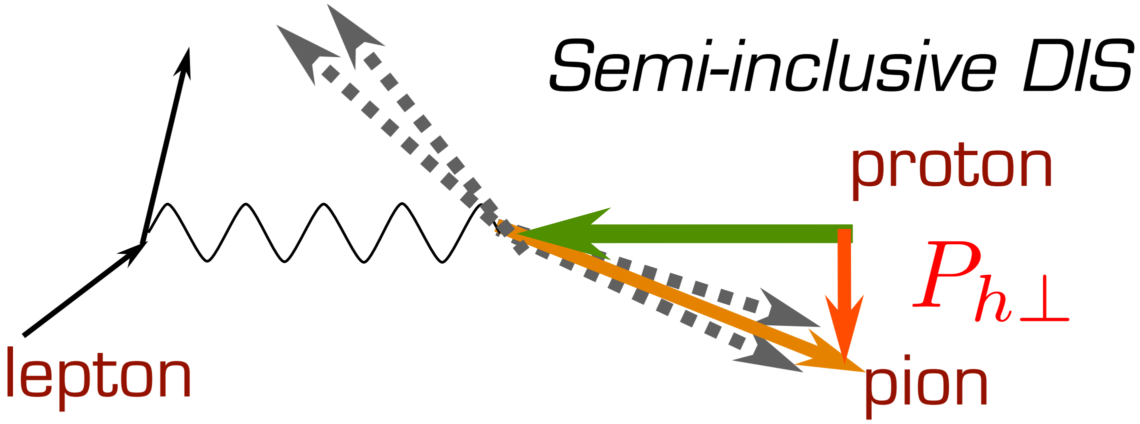


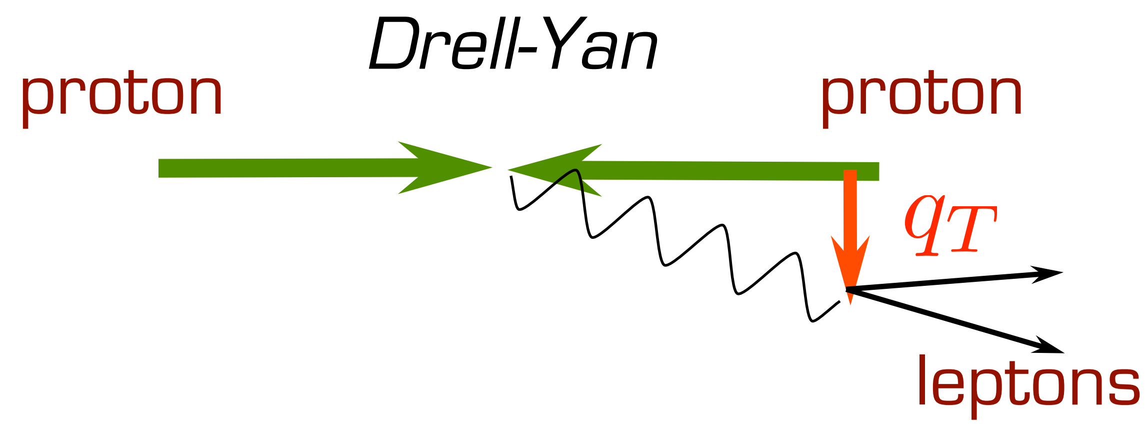
Where can we access TMDs?

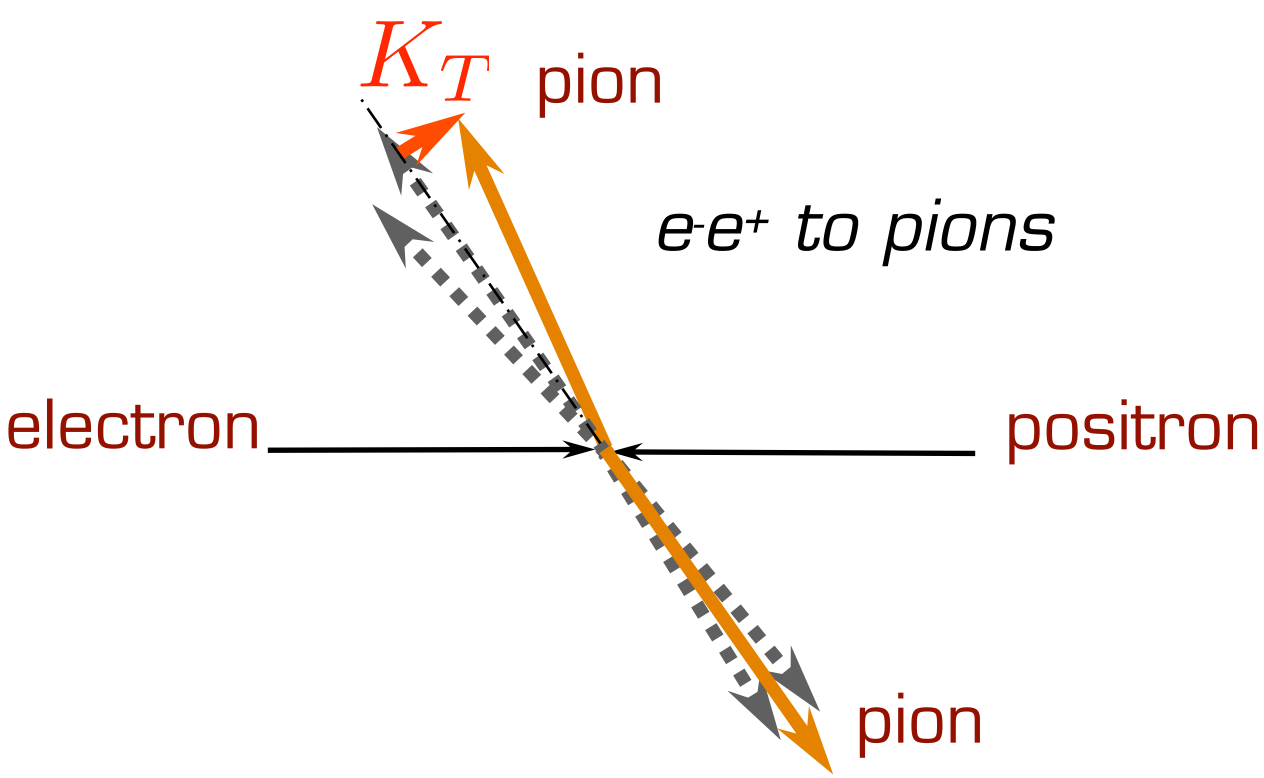




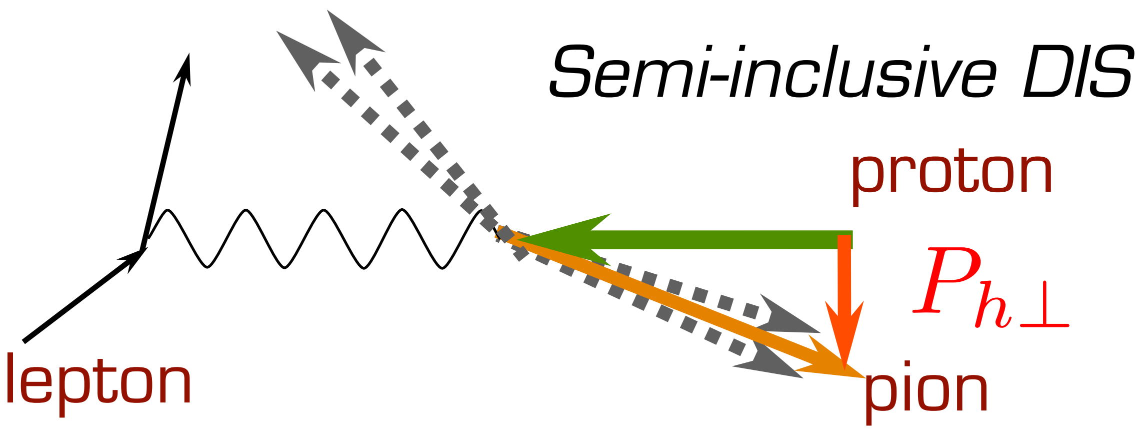
Where can we access TMDs?

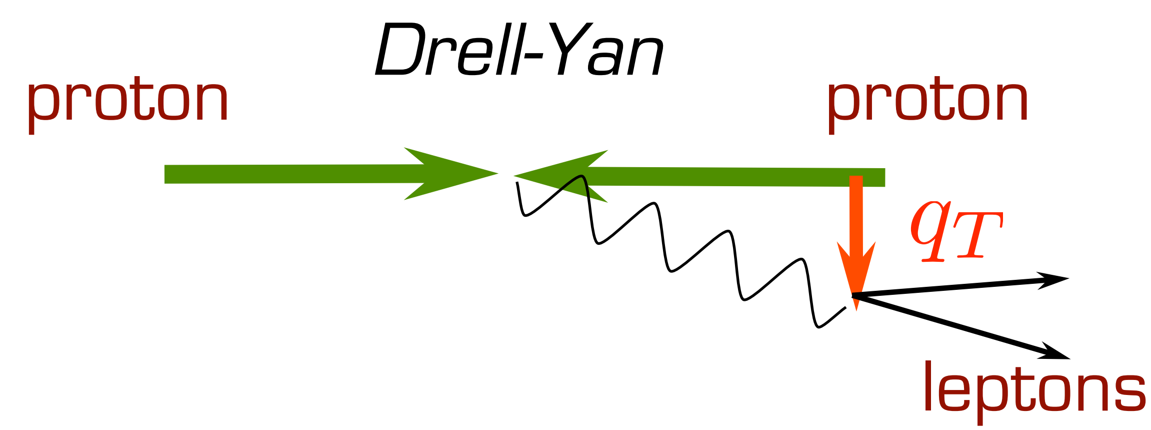


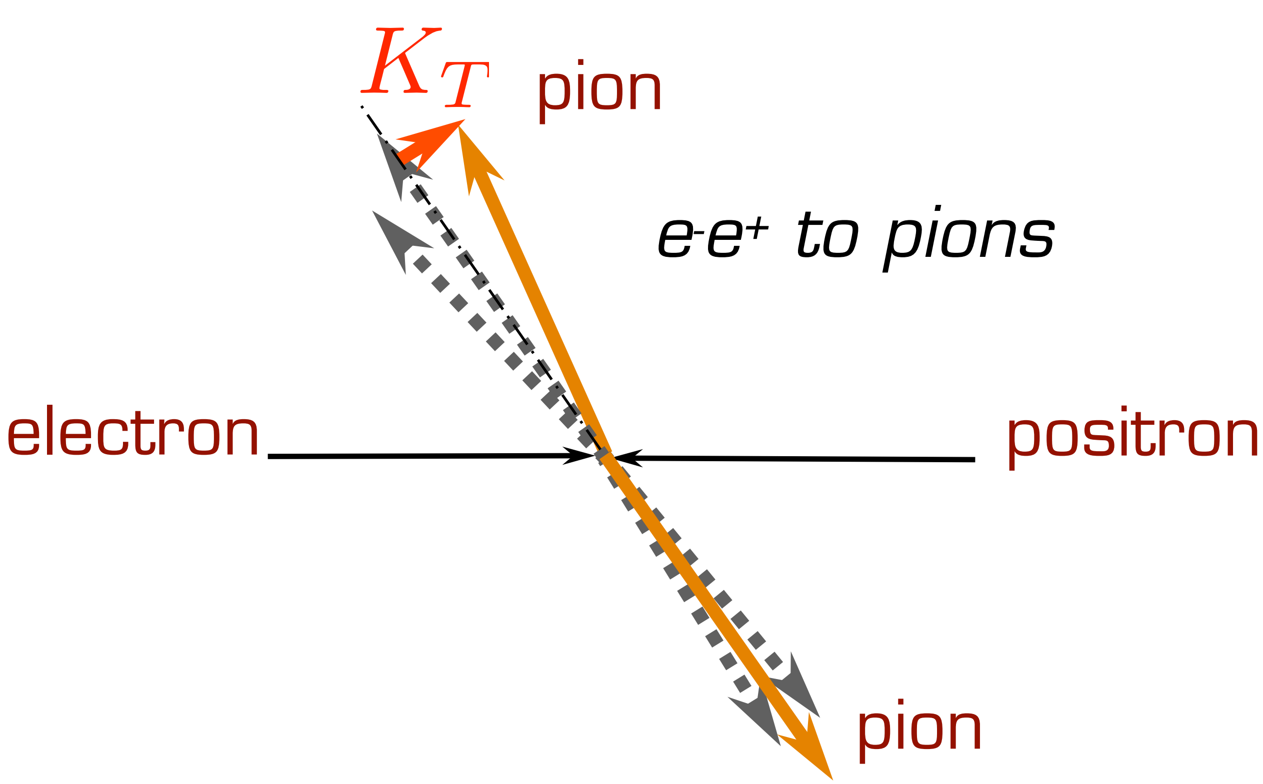


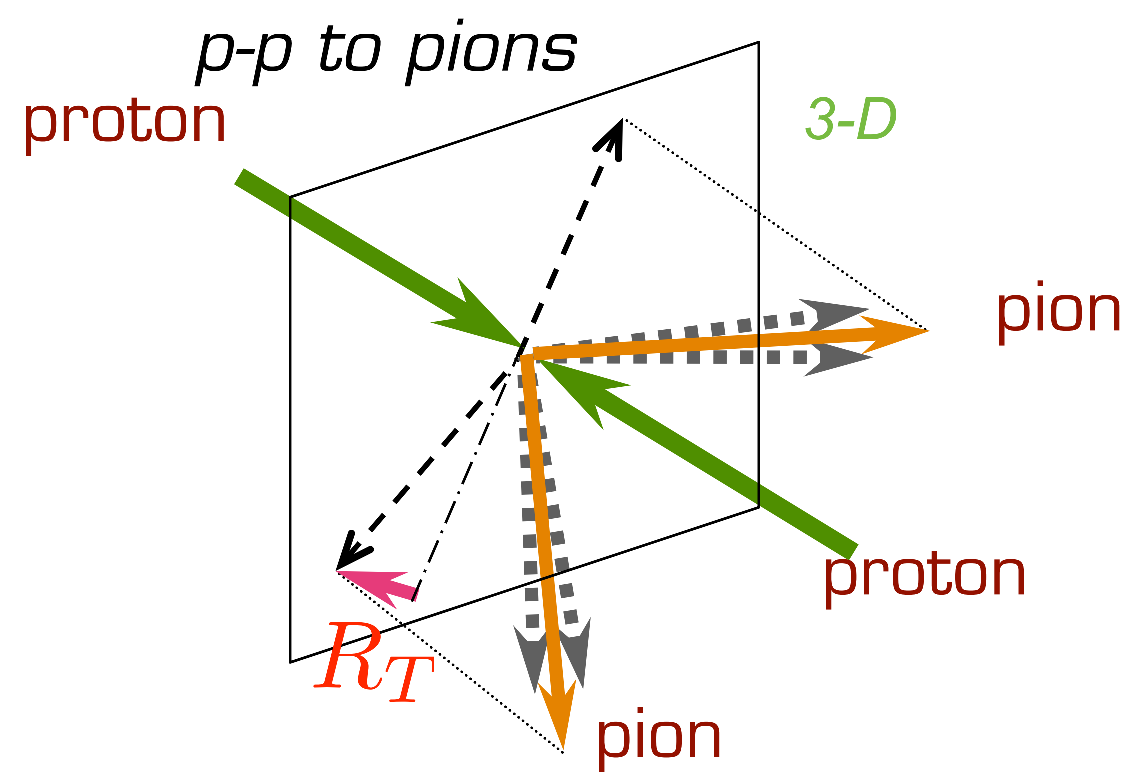


Where can we access TMDs?









Collinear PDFs

- Define structure functions

$$W^j_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) F^j_1(x, Q^2) + \frac{\hat{P}_\mu \hat{P}_\nu}{P \cdot q} F^j_2(x, Q^2)$$

- Write structure functions in the parton model

$$F_1 = \frac{1}{2} \sum_a e_a^2 f_1^a(x)$$

- Write structure functions based on QCD factorization

$$\mathcal{F}^k_a(x, Q^2) = \sum_i C^k_{a,i}(a_s, \frac{Q^2}{\mu_F^2}, m_c, m_b, \frac{x}{z}) \otimes f_i(a_s, z, \mu_F^2)$$

Coefficient functions   longitudinal convolution   scale dependence

We’ll try to do that also for semi-inclusive DIS (SIDIS)

Plan of the lectures

- Review the idea of structure functions for DIS and introduce them for semi-inclusive DIS
- Introduce the idea of quark-quark correlation functions
- Parametrize correlation functions in terms of PDFs or Transverse Momentum Distributions (TMDs)
- Obtain the expression of structure functions for semi-inclusive DIS in terms of TMDs
- Discuss concept of TMD factorization and TMD evolution
- Discuss a bit of phenomenology

Useful references

- Piet Mulders' lecture notes http://www.nikhef.nl/~pietm/COR-0.pdf
- The 3D structure of the nucleon https://doi.org/10.1140/epja/i2016-16164-4
- J. Collins, Foundations of Perturbative QCD (2011)
- TMD collaboration, "TMD Handbook," arXiv:2304.03302

Preprints: JLAB-THY-23-3780, LA-UR-21-20798, MIT-CTP/5386

TMD Handbook

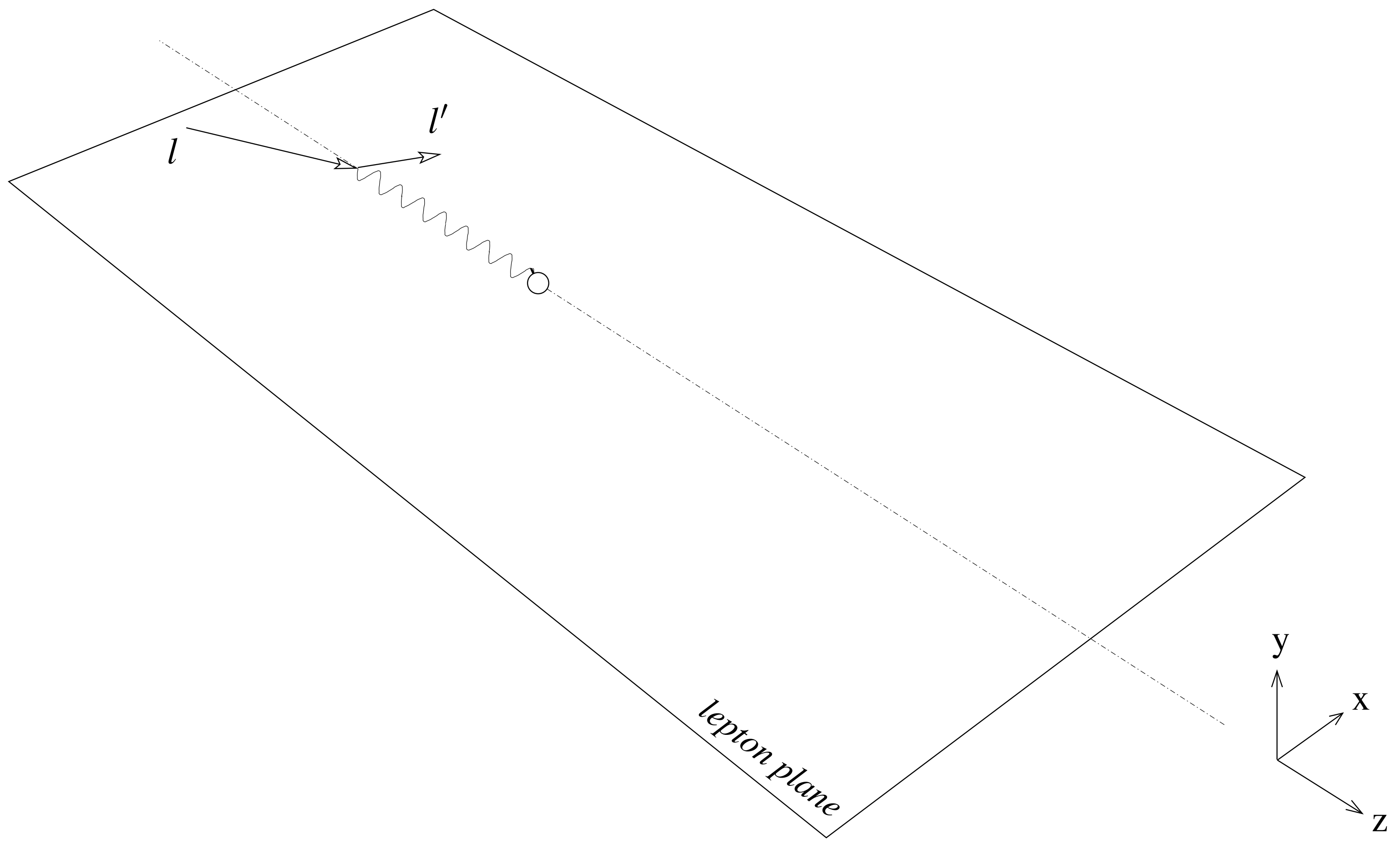
Renaud Boussarie[1], Matthias Burkardt[2], Martha Constantinou[3], William Detmold[4], Markus Ebert[4,5], Michael Engelhardt[2], Sean Fleming[6], Leonard Gamberg[7], Xiangdong Ji[8], Zhong-Bo Kang[9], Christopher Lee[10], Keh-Fei Liu[11], Simonetta Liuti[12], Thomas Mehen[13], Andreas Metz[3], John Negele[4], Daniel Pitonyak[14], Alexei Prokudin[7,16], Jian-Wei Qiu[16,17], Abha Rajan[12,18], Marc Schlegel[2,19], Phiala Shanahan[4], Peter Schweitzer[20], Iain W. Stewart[4], Andrey Tarasov[21,22], Raju Venugopalan[18], Ivan Vitev[10], Feng Yuan[23], Yong Zhao[24,4,18]

Structure functions

Inclusive DIS

$$\ell(l) + N(P) \to \ell(l') + X$$

$$x_B = \frac{Q^2}{2\,P \cdot q}, \qquad y = \frac{P \cdot q}{P \cdot l}$$



To discuss target mass corrections, also $\gamma = 2x_B M/Q$ is needed, but most of the time it will be neglected

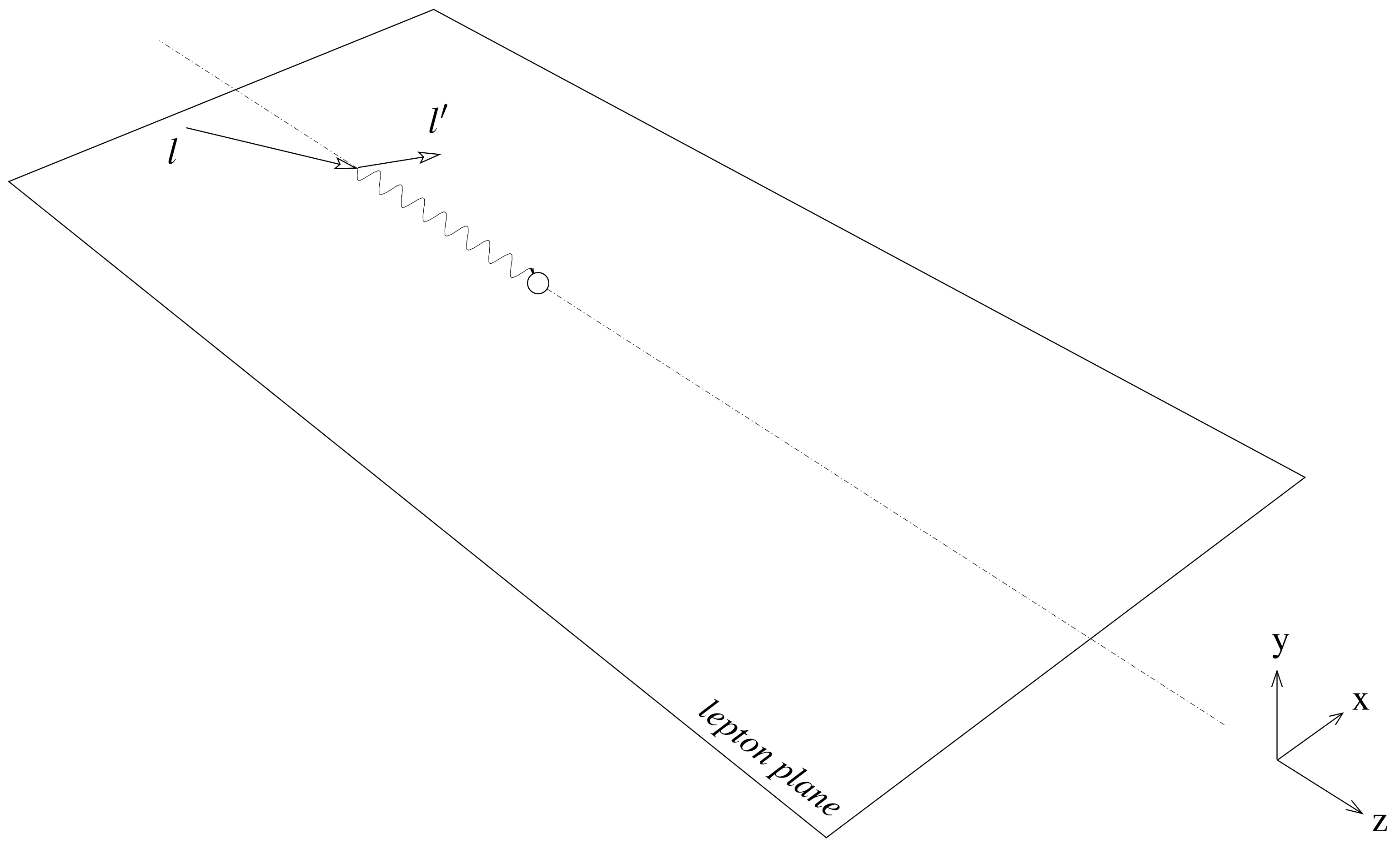
Inclusive DIS

$$\ell(l) + N(P) \to \ell(l') + X$$

$$x_B = \frac{Q^2}{2\,P\cdot q}, \qquad y = \frac{P\cdot q}{P\cdot l}$$

Bjorken limit

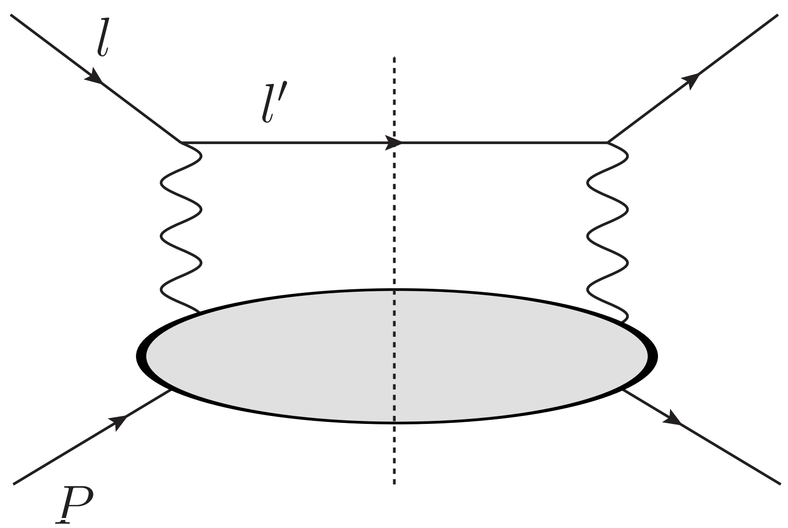
$$Q^2 \gg M^2, \quad x_B = \text{const.}$$



To discuss target mass corrections, also $\gamma = 2x_B M/Q$ is needed, but most of the time it will be neglected

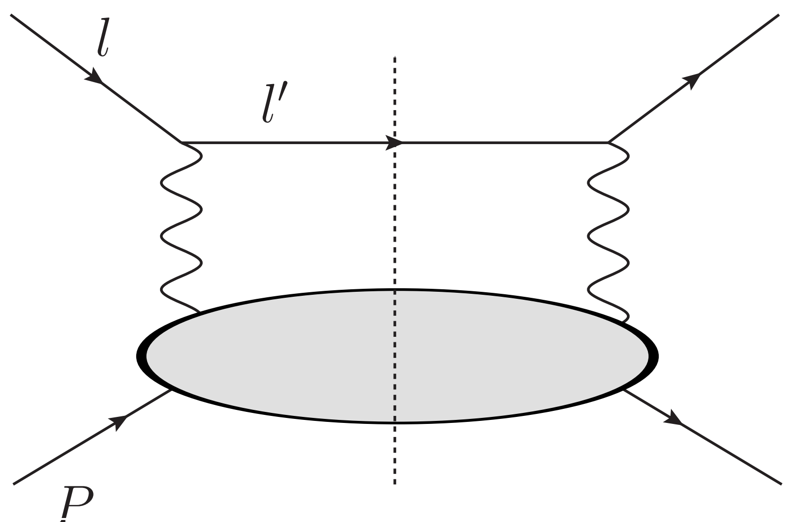
Cross section from leptonic and hadronic tensors

$$\frac{d^3\sigma}{dx_B\, dy\, d\phi_S} = \frac{\alpha^2\, y}{2\, Q^4}\, L_{\mu\nu}(l, l', \lambda_e)\; 2MW^{\mu\nu}(q, P, S)$$



Cross section from leptonic and hadronic tensors

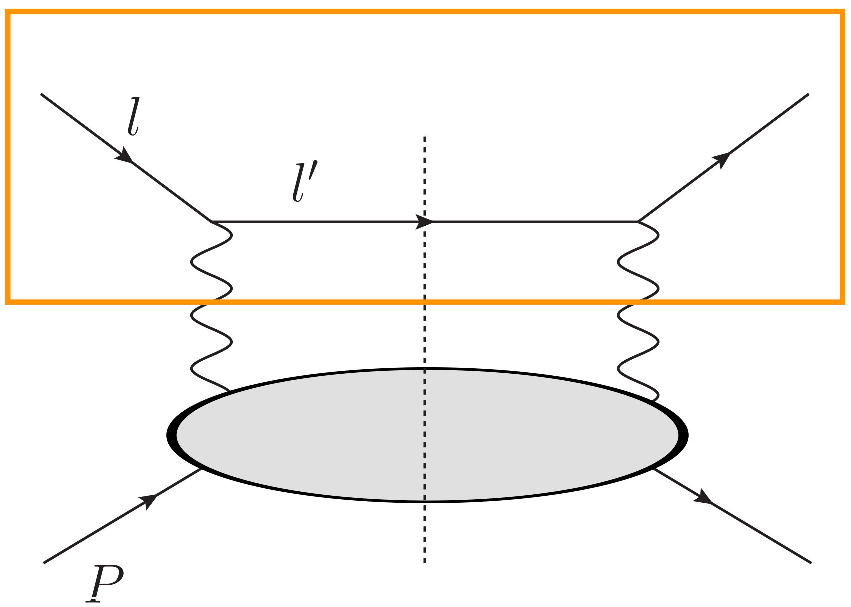
$$\frac{d^3\sigma}{dx_B\, dy\, d\phi_S} = \frac{\alpha^2\, y}{2\, Q^4}\, L_{\mu\nu}(l, l', \lambda_e)\; 2M W^{\mu\nu}(q, P, S)$$



Single-photon-exchange approximation

Cross section from leptonic and hadronic tensors

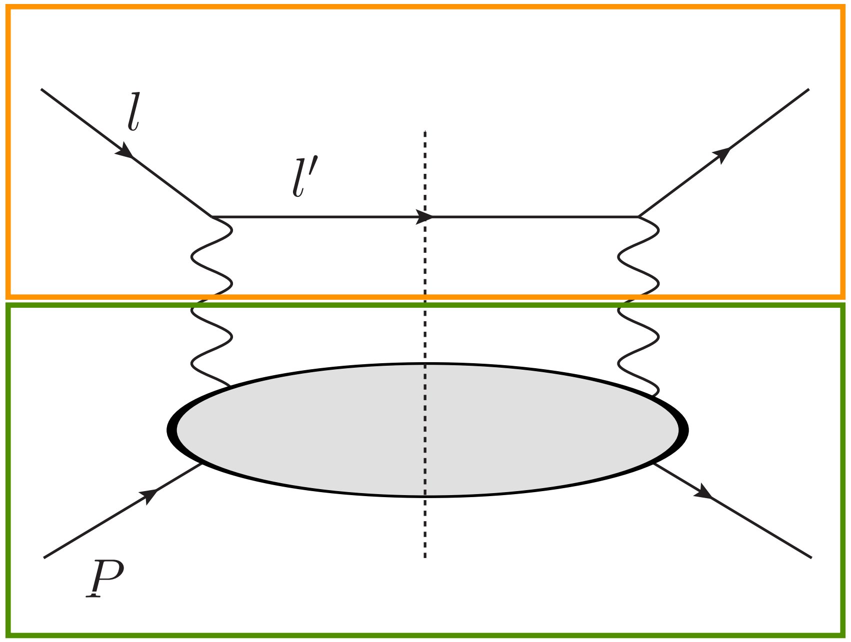
$$\frac{d^3\sigma}{dx_B\,dy\,d\phi_S} = \frac{\alpha^2\, y}{2\,Q^4}\, L_{\mu\nu}(l,l',\lambda_e)\; 2MW^{\mu\nu}(q,P,S)$$



Single-photon-exchange approximation

Cross section from leptonic and hadronic tensors

$$\frac{d^3\sigma}{dx_B\,dy\,d\phi_S} = \frac{\alpha^2\,y}{2\,Q^4}\,L_{\mu\nu}(l,l',\lambda_e)\;2MW^{\mu\nu}(q,P,S)$$



Single-photon-exchange approximation

Basis vectors

Orthogonal and normalized

$$\hat{q}^\mu = \frac{q^\mu}{Q},$$

$$\hat{t}^\mu = \frac{2x_B}{Q\sqrt{1+\gamma^2}}\left(P^\mu - \frac{P\cdot q}{q^2}q^\mu\right),$$

$$\hat{l}^\mu = -\frac{g_\perp^{\mu\nu} l_\nu}{|g_\perp^{\mu\nu} l_\nu|}$$

Basis vectors

Orthogonal and normalized

$$\hat{q}^\mu = \frac{q^\mu}{Q},$$

$$\hat{t}^\mu = \frac{2x_B}{Q\sqrt{1+\gamma^2}}\left(P^\mu - \frac{P\cdot q}{q^2}q^\mu\right),$$

$$\hat{l}^\mu = -\frac{g_\perp^{\mu\nu}l_\nu}{|g_\perp^{\mu\nu}l_\nu|}$$

"perp" projectors

$$g_\perp^{\mu\nu} = g^{\mu\nu} + \hat{q}^\mu\hat{q}^\nu - \hat{t}^\mu\hat{t}^\nu,$$

$$\epsilon_\perp^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\hat{t}_\rho\hat{q}_\sigma.$$

Basis vectors

Orthogonal and normalized

$$\hat{q}^\mu = \frac{q^\mu}{Q},$$
$$\hat{t}^\mu = \frac{2x_B}{Q\sqrt{1+\gamma^2}}\left(P^\mu - \frac{P\cdot q}{q^2}q^\mu\right),$$
$$\hat{l}^\mu = -\frac{g_\perp^{\mu\nu}l_\nu}{|g_\perp^{\mu\nu}l_\nu|}$$

"perp" projectors

$$g_\perp^{\mu\nu} = g^{\mu\nu} + \hat{q}^\mu\hat{q}^\nu - \hat{t}^\mu\hat{t}^\nu,$$
$$\epsilon_\perp^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\hat{t}_\rho\hat{q}_\sigma.$$

then

$$l^\mu = \frac{Q}{2}\hat{q}^\mu + \frac{(2-y)}{2y}\hat{t}^\mu + \frac{Q\sqrt{1-y}}{y}\hat{l}^\mu$$

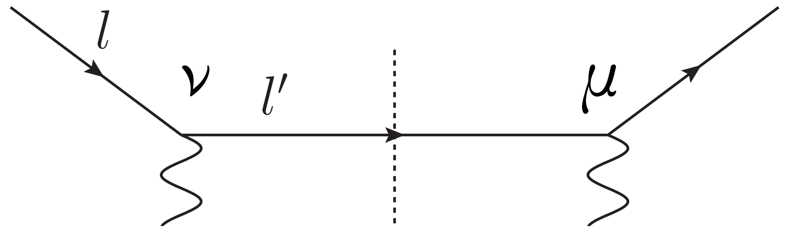
$$q^\mu = (0,\ 0,\ 0,\ Q)$$
$$l^\mu = \left(\frac{(2-y)Q}{2y},\ \frac{\sqrt{1-y}\,Q}{y},\ 0,\ \frac{Q}{2}\right)$$
$$l'^\mu = \left(\frac{(2-y)Q}{2y},\ \frac{\sqrt{1-y}\,Q}{y},\ 0,\ -\frac{Q}{2}\right)$$

Leptonic tensor (unpolarized)

ν μ

$$
\begin{aligned}
L_{\mu\nu} &= -Q^2 g_{\mu\nu} + 2\left(l_\mu l'_\nu + l'_\mu l_\nu\right) \\
&= \frac{2Q^2}{y^2}\Big[-\left(1 - y + \frac{y^2}{2}\right) g_{\perp\mu\nu} + 2(1-y)\,\hat{t}_\mu \hat{t}_\nu \\
&\quad + 2(1-y)\left(\hat{l}_{\perp\mu}\hat{l}_{\perp\nu} + \frac{1}{2} g_{\perp\mu\nu}\right) + \dots\Big]
\end{aligned}
$$

Leptonic tensor (unpolarized)



$$
\begin{aligned}
L_{\mu\nu} &= -Q^2 g_{\mu\nu} + 2\left(l_\mu l'_\nu + l'_\mu l_\nu\right) \\
&= \frac{2Q^2}{y^2}\left[-\left(1 - y + \frac{y^2}{2}\right) g_{\perp\mu\nu} + 2(1-y)\,\hat{t}_\mu \hat{t}_\nu \right. \\
&\quad \left. + 2(1-y)\left(\hat{l}_{\perp\mu}\hat{l}_{\perp\nu} + \frac{1}{2} g_{\perp\mu\nu}\right) + \ldots\right]
\end{aligned}
$$

Hadronic tensor (unpolarized)

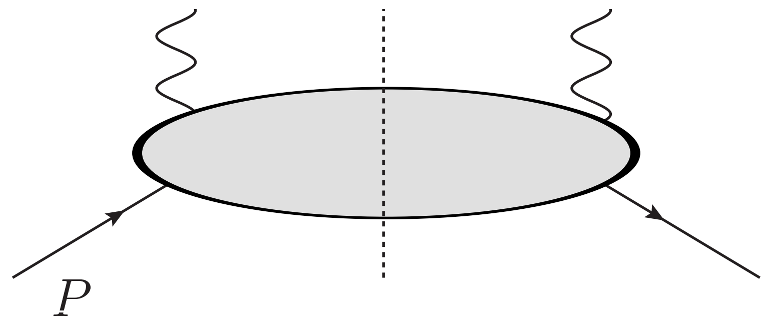
Conditions to be respected

Hermiticity: $W^*_{\mu\nu}(q,P,S) = W_{\nu\mu}(q,P,S)$,

parity: $L_\mu^\rho L_\nu^\sigma W_{\rho\sigma}(q,P,S) = W_{\mu\nu}(\tilde{q},\tilde{P},-\tilde{S})$,

time-reversal: $L_\mu^\rho L_\nu^\sigma W^*_{\rho\sigma}(q,P,S) = W_{\mu\nu}(\tilde{q},\tilde{P},\tilde{S})$

Hadronic tensor (unpolarized)



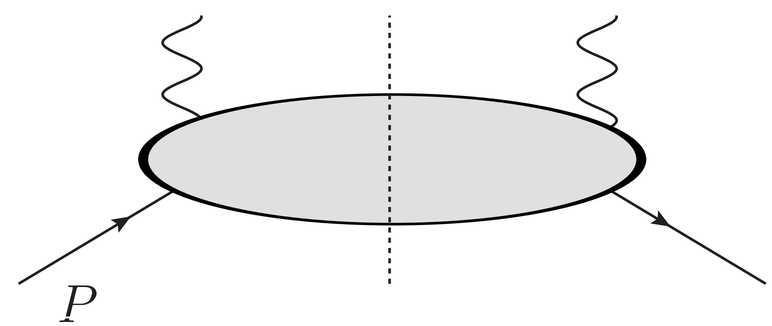
Conditions to be respected

Hermiticity: $W^*_{\mu\nu}(q, P, S) = W_{\nu\mu}(q, P, S)$,

parity: $L_\mu^\rho L_\nu^\sigma W_{\rho\sigma}(q, P, S) = W_{\mu\nu}(\tilde{q}, \tilde{P}, -\tilde{S})$,

time-reversal: $L_\mu^\rho L_\nu^\sigma W^*_{\rho\sigma}(q, P, S) = W_{\mu\nu}(\tilde{q}, \tilde{P}, \tilde{S})$

Hadronic tensor (unpolarized)

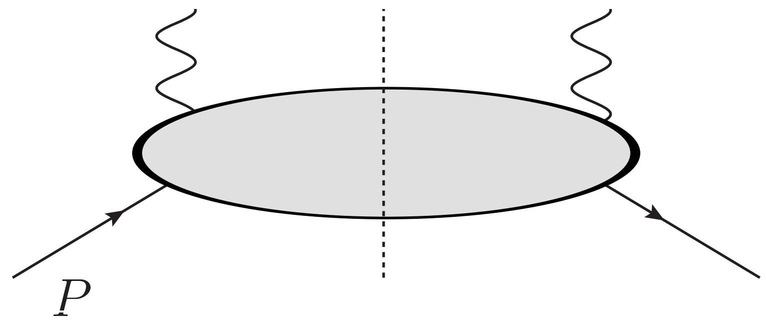


Conditions to be respected

Hermiticity:	$W^{*}_{\mu\nu}(q,P,S) = W_{\nu\mu}(q,P,S),$
parity:	$L^{\rho}_{\mu} L^{\sigma}_{\nu} W_{\rho\sigma}(q,P,S) = W_{\mu\nu}(\tilde{q},\tilde{P},-\tilde{S}),$
time-reversal:	$L^{\rho}_{\mu} L^{\sigma}_{\nu} W^{*}_{\rho\sigma}(q,P,S) = W_{\mu\nu}(\tilde{q},\tilde{P},\tilde{S})$

$$2MW^{\mu\nu} = 2M\left[A\, g^{\mu\nu} + B\, q^{\mu} q^{\nu} + C\, \frac{P^{\mu}P^{\nu}}{M^2} + D\, \frac{P^{\mu}q^{\nu} + q^{\mu}P^{\nu}}{M^2}\right]$$

Hadronic tensor (unpolarized)



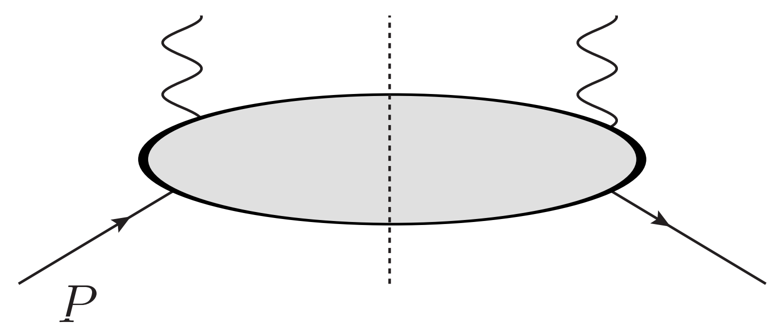
Conditions to be respected

Hermiticity:	$W^*_{\mu\nu}(q,P,S) = W_{\nu\mu}(q,P,S)$,
parity:	$L_\mu^\rho L_\nu^\sigma W_{\rho\sigma}(q,P,S) = W_{\mu\nu}(\tilde{q},\tilde{P},-\tilde{S})$,
time-reversal:	$L_\mu^\rho L_\nu^\sigma W^*_{\rho\sigma}(q,P,S) = W_{\mu\nu}(\tilde{q},\tilde{P},\tilde{S})$

$$2MW^{\mu\nu} = 2M\left[A\, g^{\mu\nu} + B\, q^\mu q^\nu + C\,\frac{P^\mu P^\nu}{M^2} + D\,\frac{P^\mu q^\nu + q^\mu P^\nu}{M^2}\right]$$

A structure like this $\mathrm{i}\epsilon^{\mu\nu\rho\sigma}P_\rho q_\sigma$ would be forbidden by?

Hadronic tensor (unpolarized)



Conditions to be respected

Hermiticity:	$W^*_{\mu\nu}(q, P, S) = W_{\nu\mu}(q, P, S)$,
parity:	$L_\mu^\rho L_\nu^\sigma W_{\rho\sigma}(q, P, S) = W_{\mu\nu}(\tilde{q}, \tilde{P}, -\tilde{S})$,
time-reversal:	$L_\mu^\rho L_\nu^\sigma W^*_{\rho\sigma}(q, P, S) = W_{\mu\nu}(\tilde{q}, \tilde{P}, \tilde{S})$

$$2MW^{\mu\nu} = 2M\left[A\, g^{\mu\nu} + B\, q^\mu q^\nu + C\, \frac{P^\mu P^\nu}{M^2} + D\, \frac{P^\mu q^\nu + q^\mu P^\nu}{M^2}\right]$$

A structure like this $\mathrm{i}\epsilon^{\mu\nu\rho\sigma} P_\rho q_\sigma$ would be forbidden by?

Further condition: gauge invariance $q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0$

Structure functions

$$2MW^{\mu\nu} = \frac{1}{x}\left[-g_{\perp}^{\mu\nu}F_{UU,T} + \hat{t}^{\mu}\hat{t}^{\nu}F_{UU,L}\right]$$

Structure functions

$$2MW^{\mu\nu} = \frac{1}{x}\left[-g_\perp^{\mu\nu} F_{UU,T} + \hat{t}^\mu \hat{t}^\nu F_{UU,L}\right]$$

Only two structure functions are left.
Note that this is true for any process that can be described with a hadronic tensor. Also electron-proton elastic scattering, exclusive processes, inelastic scattering at low Q…

Structure functions

$$2MW^{\mu\nu} = \frac{1}{x}\left[-g_\perp^{\mu\nu} F_{UU,T} + \hat{t}^\mu \hat{t}^\nu F_{UU,L}\right]$$

Only two structure functions are left.
Note that this is true for any process that can be described with a hadronic tensor. Also electron-proton elastic scattering, exclusive processes, inelastic scattering at low Q…

Connection with structure functions in Enrico's lectures

Structure functions

$$2MW^{\mu\nu} = \frac{1}{x}\left[-g_{\perp}^{\mu\nu} F_{UU,T} + \hat{t}^{\mu}\hat{t}^{\nu} F_{UU,L}\right]$$

Only two structure functions are left.
Note that this is true for any process that can be described with a hadronic tensor. Also electron-proton elastic scattering, exclusive processes, inelastic scattering at low Q…

Connection with structure functions in Enrico's lectures

$$F_{UU,T} = 2x_B F_1,$$

$$F_{UU,L} = (1+\gamma^2)F_2 - 2x_B F_1$$

Contraction with leptonic tensor

$$g_{\perp \mu\nu} g_{\perp}^{\mu\nu} = 2$$

$$\hat{t}_{\mu} \hat{t}_{\nu} \hat{t}^{\mu} \hat{t}^{\nu} = 1$$

Cross section in terms of structure functions

$$\frac{d^3\sigma}{dx_B\,dy\,d\phi_S} = \frac{\alpha^2\,y}{2\,Q^4}\,L_{\mu\nu}(l,l',\lambda_e)\;2MW^{\mu\nu}(q,P,S)$$

Cross section in terms of structure functions

$$\frac{d^3\sigma}{dx_B\,dy\,d\phi_S} = \frac{\alpha^2\,y}{2\,Q^4}\,L_{\mu\nu}(l,l',\lambda_e)\,2MW^{\mu\nu}(q,P,S)$$

$$2MW^{\mu\nu} = \frac{1}{x_B}\left[-g_\perp^{\mu\nu}\,F_{UU,T} + \hat{t}^\mu\hat{t}^\nu\,F_{UU,L}\right]$$

Cross section in terms of structure functions

$$\frac{d^3\sigma}{dx_B dy d\phi_S} = \frac{\alpha^2 \, y}{2\, Q^4} \, L_{\mu\nu}(l, l', \lambda_e) \, 2MW^{\mu\nu}(q, P, S)$$

$$F_{UU,T}(x_B, Q^2)$$

$$2MW^{\mu\nu} = \frac{1}{x_B}\left[-g_\perp^{\mu\nu} \, F_{UU,T} + \hat{t}^\mu \hat{t}^\nu \, F_{UU,L}\right]$$

Cross section in terms of structure functions

$$\frac{d^3\sigma}{dx_B dy d\phi_S} = \frac{\alpha^2 \, y}{2\, Q^4} \, L_{\mu\nu}(l, l', \lambda_e) \, 2MW^{\mu\nu}(q, P, S)$$

$F_{UU,T}(x_B, Q^2)$

$$2MW^{\mu\nu} = \frac{1}{x_B}\left[-g_\perp^{\mu\nu} F_{UU,T} + \hat{t}^\mu \hat{t}^\nu F_{UU,L} \right]$$

$$\frac{d\sigma}{dx_B \, dy \, d\phi_S} = \frac{2\alpha^2}{x_B y Q^2} \left\{ \left(1 - y + \frac{y^2}{2}\right) F_{UU,T} + (1-y) \, F_{UU,L} \right\}$$

Polarized cross section

$$\frac{d^3\sigma}{dx_B dy d\phi_S} = \frac{\alpha^2 \, y}{2 \, Q^4} \, L_{\mu\nu}(l, l', \lambda_e) \; 2MW^{\mu\nu}(q, P, S)$$

Polarized cross section

$$\frac{d^3\sigma}{dx_B dy d\phi_S} = \frac{\alpha^2\, y}{2\, Q^4}\, L_{\mu\nu}(l, l', \lambda_e)\; 2MW^{\mu\nu}(q, P, S)$$

$$2MW^{\mu\nu} = \frac{1}{x}\left[-g_\perp^{\mu\nu}\, F_{UU,T} + \hat{t}^\mu \hat{t}^\nu\, F_{UU,L} + iS_L \epsilon_\perp^{\mu\nu}\, F_{LL} - i\hat{t}^{[\mu}\epsilon_\perp^{\nu]\rho} S_\rho F_{LT}^{\cos\phi_S}\right]$$

Polarized cross section

$$\frac{d^3\sigma}{dx_B dy d\phi_S} = \frac{\alpha^2 \, y}{2\, Q^4} \, L_{\mu\nu}(l, l', \lambda_e) \; 2MW^{\mu\nu}(q, P, S)$$

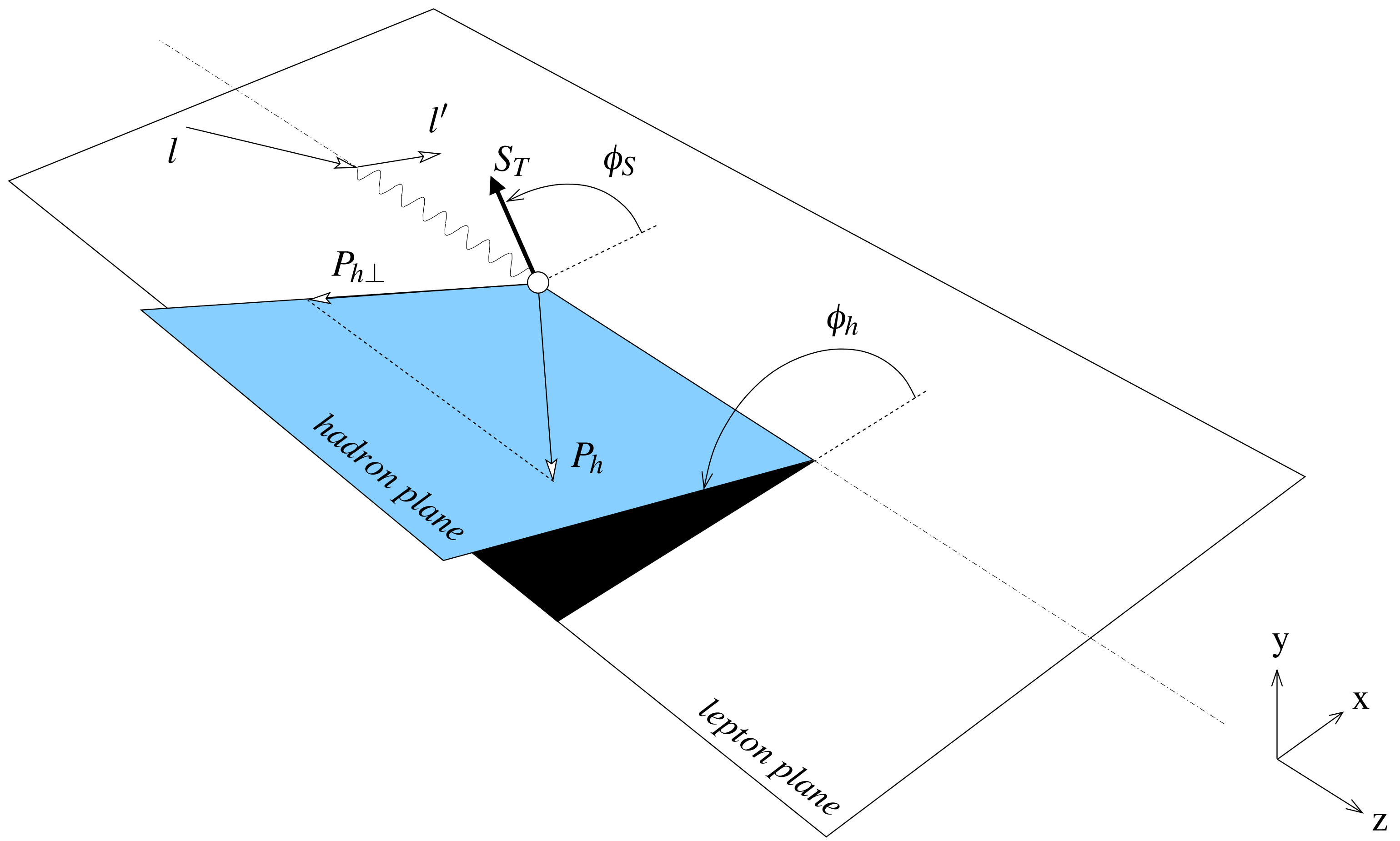
$$2MW^{\mu\nu} = \frac{1}{x}\Big[-g_\perp^{\mu\nu} F_{UU,T} + \hat{t}^\mu \hat{t}^\nu F_{UU,L} + iS_L \epsilon_\perp^{\mu\nu} F_{LL} - i\hat{t}^{[\mu}\epsilon_\perp^{\nu]\rho} S_\rho F_{LT}^{\cos\phi_S}\Big]$$

$$\frac{d\sigma}{dx_B\, dy\, d\phi_S} = \frac{2\alpha^2}{x_B y Q^2}\bigg\{\Big(1 - y + \frac{y^2}{2}\Big) F_{UU,T} + (1-y)\, F_{UU,L} + S_L \lambda_e\, y\Big(1 - \frac{y}{2}\Big)\, F_{LL}$$
$$+ |\boldsymbol{S}_T| \lambda_e\, y \sqrt{1-y}\, \cos\phi_S\, F_{LT}^{\cos\phi_s}\bigg\}$$

Semi-inclusive DIS (SIDIS)

$$\ell(l) + N(P) \to \ell(l') + h(P_h) + X,$$

$$x_B = \frac{Q^2}{2\,P\cdot q}, \qquad y = \frac{P\cdot q}{P\cdot l}, \qquad z_h = \frac{P\cdot P_h}{P\cdot q}.$$

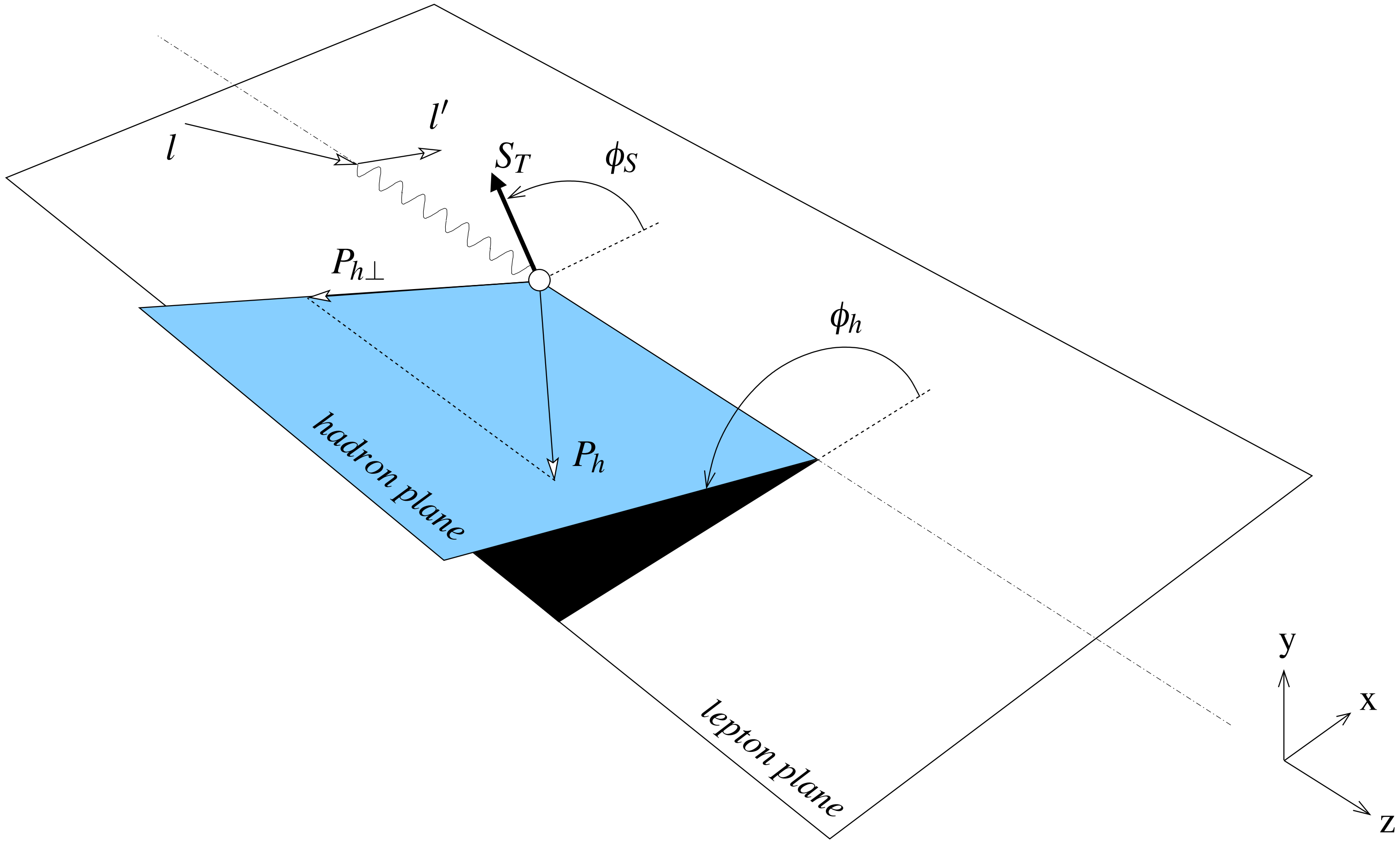


Semi-inclusive DIS (SIDIS)

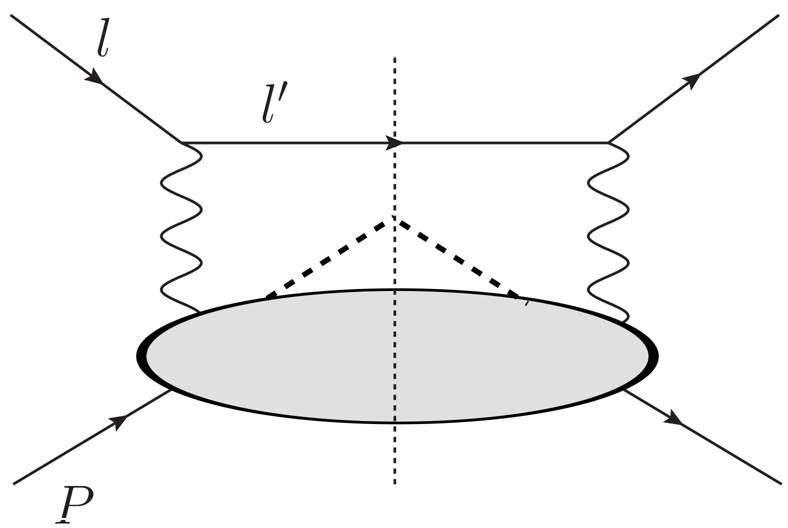
see, e.g., A.B., Diehl, Goeke, Metz, Mulders, Schlegel, JHEP093 (07)

$$\ell(l) + N(P) \to \ell(l') + h(P_h) + X,$$

$$x_B = \frac{Q^2}{2\,P\cdot q}, \qquad y = \frac{P\cdot q}{P\cdot l}, \qquad z_h = \frac{P\cdot P_h}{P\cdot q}.$$

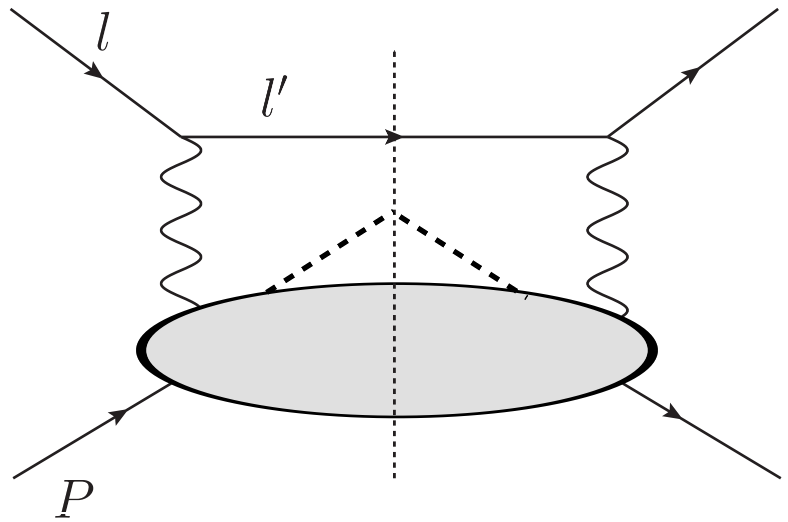


Cross section from leptonic and hadronic tensors

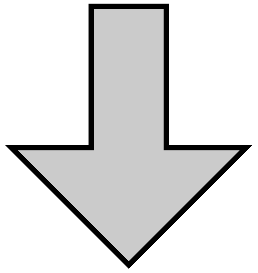


$$\frac{d^3\sigma}{dx_B dy d\phi_S} = \frac{\alpha^2\, y}{2\, Q^4}\, L_{\mu\nu}(l, l', \lambda_e)\; 2MW^{\mu\nu}(q, P, S)$$

Cross section from leptonic and hadronic tensors



$$\frac{d^3\sigma}{dx_B\,dy\,d\phi_S} = \frac{\alpha^2\,y}{2\,Q^4}\,L_{\mu\nu}(l,l',\lambda_e)\;2MW^{\mu\nu}(q,P,S)$$



$$\frac{d^6\sigma}{dx_B\,dy\,dz_h\,d\phi_S\,d\phi_h\,dP_{h\perp}^2} = \frac{\alpha^2\,y}{2\,z_h\,Q^4}\,L_{\mu\nu}(l,l',\lambda_e)\;2MW^{\mu\nu}(q,P,S,P_h)$$

Hadronic tensor (unpolarized)

Hadronic tensor (unpolarized)

$$
\begin{aligned}
2MW^{\mu\nu}(q,P,S) = \frac{2z_h}{x_B}\Big[& -g_\perp^{\mu\nu} F_{UU,T}(x_B,z_h,P_{h\perp}^2,Q^2) + \hat{t}^\mu \hat{t}^\nu F_{UU,L}(x_B,z_h,P_{h\perp}^2,Q^2) \\
& + \Big(\hat{t}^\mu \hat{h}^\nu + \hat{t}^\nu \hat{h}^\mu\Big) F_{UU}^{\cos\phi_h}(x_B,z_h,P_{h\perp}^2,Q^2) + \Big(\hat{h}^\mu \hat{h}^\nu + g_\perp^{\mu\nu}\Big) F_{UU}^{\cos 2\phi_h}(x_B,z_h,P_{h\perp}^2,Q^2) \\
& - \mathrm{i}\Big(\hat{t}^\mu \hat{h}^\nu - \hat{t}^\nu \hat{h}^\mu\Big) F_{LU}^{\sin\phi_h}(x_B,z_h,P_{h\perp}^2,Q^2)\Big],
\end{aligned}
$$

$$
\hat{h} = \frac{P_{h\perp}}{|P_{h\perp}|}
$$

Hadronic tensor (unpolarized)

$$
\begin{aligned}
2MW^{\mu\nu}(q,P,S) = \frac{2z_h}{x_B}\Big[& -g_\perp^{\mu\nu} F_{UU,T}(x_B,z_h,P_{h\perp}^2,Q^2) + \hat{t}^\mu \hat{t}^\nu F_{UU,L}(x_B,z_h,P_{h\perp}^2,Q^2) \\
& + \Big(\hat{t}^\mu \hat{h}^\nu + \hat{t}^\nu \hat{h}^\mu\Big) F_{UU}^{\cos\phi_h}(x_B,z_h,P_{h\perp}^2,Q^2) + \Big(\hat{h}^\mu \hat{h}^\nu + g_\perp^{\mu\nu}\Big) F_{UU}^{\cos 2\phi_h}(x_B,z_h,P_{h\perp}^2,Q^2) \\
& - \mathrm{i}\Big(\hat{t}^\mu \hat{h}^\nu - \hat{t}^\nu \hat{h}^\mu\Big) F_{LU}^{\sin\phi_h}(x_B,z_h,P_{h\perp}^2,Q^2)\Big],
\end{aligned}
$$

$$
\hat{h} = \frac{P_{h\perp}}{|P_{h\perp}|}
$$

There are in total five structure functions, instead of the two of (unpolarized) inclusive DIS. They depend on two more scalars.

Hadronic tensor (unpolarized)

$$
\begin{aligned}
2MW^{\mu\nu}(q,P,S) = \frac{2z_h}{x_B}\Big[& - g_\perp^{\mu\nu} F_{UU,T}(x_B, z_h, P_{h\perp}^2, Q^2) + \hat{t}^\mu \hat{t}^\nu F_{UU,L}(x_B, z_h, P_{h\perp}^2, Q^2) \\
& + \left(\hat{t}^\mu \hat{h}^\nu + \hat{t}^\nu \hat{h}^\mu\right) F_{UU}^{\cos\phi_h}(x_B, z_h, P_{h\perp}^2, Q^2) + \left(\hat{h}^\mu \hat{h}^\nu + g_\perp^{\mu\nu}\right) F_{UU}^{\cos 2\phi_h}(x_B, z_h, P_{h\perp}^2, Q^2) \\
& - \mathrm{i}\left(\hat{t}^\mu \hat{h}^\nu - \hat{t}^\nu \hat{h}^\mu\right) F_{LU}^{\sin\phi_h}(x_B, z_h, P_{h\perp}^2, Q^2)\Big],
\end{aligned}
$$

$$
\hat{h} = \frac{P_{h\perp}}{|P_{h\perp}|}
$$

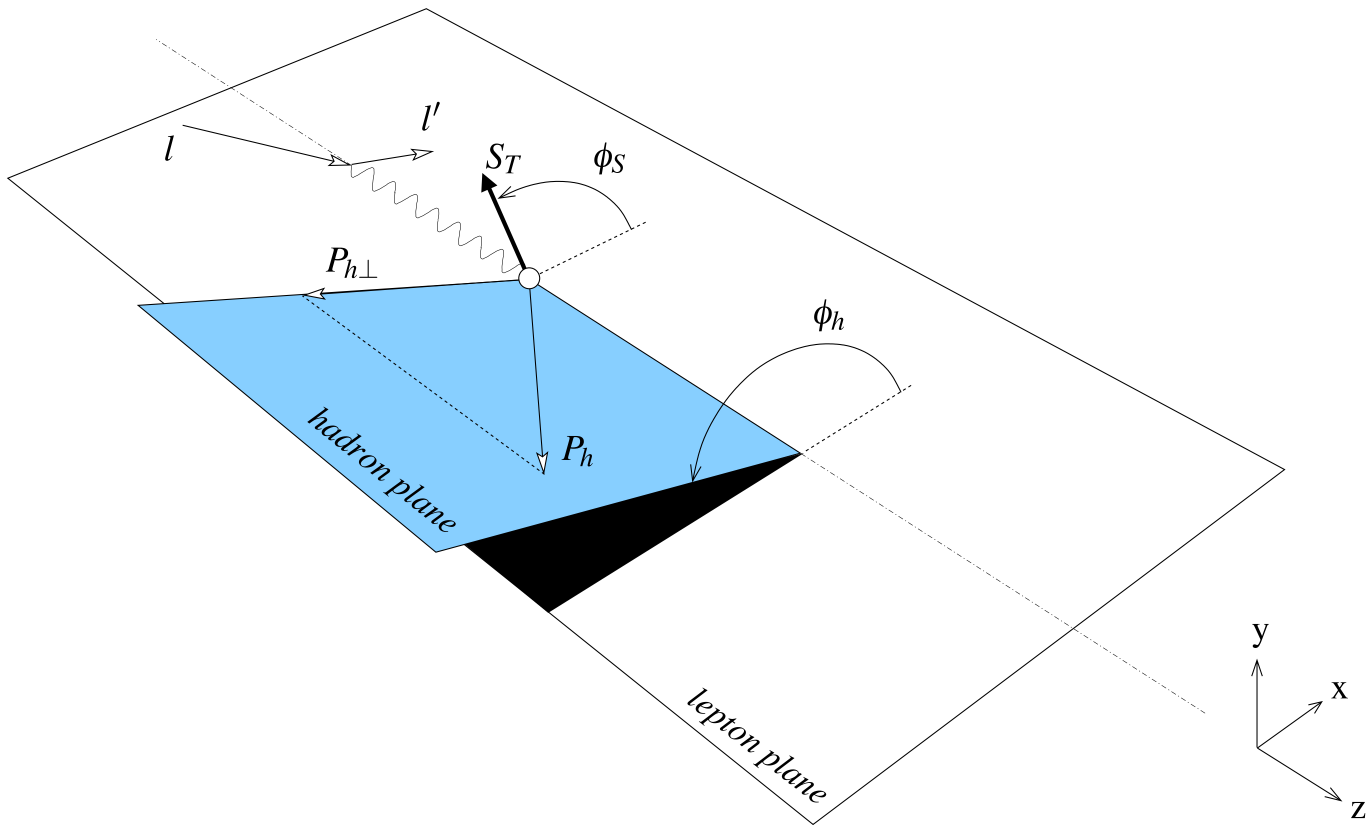
There are in total five structure functions, instead of the two of (unpolarized) inclusive DIS. They depend on two more scalars.

What is the origin of their names? The second subscript indicate the polarization of the target. The first subscript indicate whether polarization of the lepton beam is required or not. The cosines and sines indicate what kind of azimuthal modulation will be generated in the cross section.

Contraction with leptonic tensor

$$\left(\hat{t}_\mu \hat{l}_\nu + \hat{t}_\nu \hat{l}_\mu\right)\left(\hat{t}^\mu \hat{h}^\nu + \hat{t}^\nu \hat{h}^\mu\right) = -g_\perp^{\mu\nu} \hat{l}_\mu \hat{h}_\nu \equiv \cos\phi_h$$

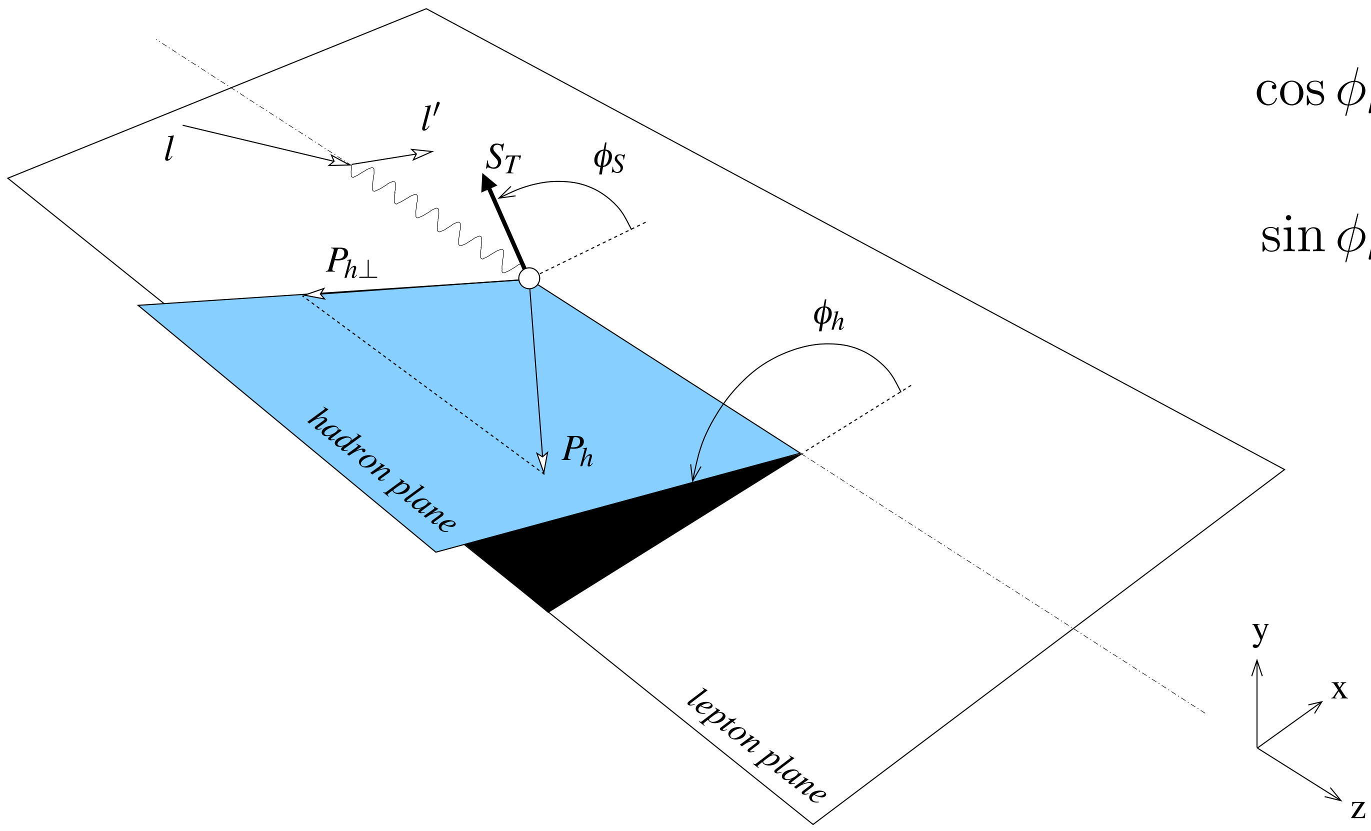
$$\left(\hat{t}_\mu \epsilon_{\perp\nu\rho} \hat{l}^\rho + \hat{t}_\nu \epsilon_{\perp\mu\rho} \hat{l}^\rho\right)\left(\hat{t}^\mu \hat{h}^\nu + \hat{t}^\nu \hat{h}^\mu\right) = -\epsilon_\perp^{\mu\nu} \hat{l}_\mu \hat{h}_\nu \equiv \sin\phi_h$$



Contraction with leptonic tensor

$$\left(\hat{t}_\mu \hat{l}_\nu + \hat{t}_\nu \hat{l}_\mu\right)\left(\hat{t}^\mu \hat{h}^\nu + \hat{t}^\nu \hat{h}^\mu\right) = -g_\perp^{\mu\nu} \hat{l}_\mu \hat{h}_\nu \equiv \cos\phi_h$$

$$\left(\hat{t}_\mu \epsilon_{\perp\nu\rho} \hat{l}^\rho + \hat{t}_\nu \epsilon_{\perp\mu\rho} \hat{l}^\rho\right)\left(\hat{t}^\mu \hat{h}^\nu + \hat{t}^\nu \hat{h}^\mu\right) = -\epsilon_\perp^{\mu\nu} \hat{l}_\mu \hat{h}_\nu \equiv \sin\phi_h$$



$$\cos\phi_h = \frac{(\hat{\boldsymbol{q}} \times \boldsymbol{l})}{|\hat{\boldsymbol{q}} \times \boldsymbol{l}|} \cdot \frac{(\hat{\boldsymbol{q}} \times \boldsymbol{P}_h)}{|\hat{\boldsymbol{q}} \times \boldsymbol{P}_h|},$$

$$\sin\phi_h = \frac{(\boldsymbol{l} \times \boldsymbol{P}_h) \cdot \hat{\boldsymbol{q}}}{|\hat{\boldsymbol{q}} \times \boldsymbol{l}|\,|\hat{\boldsymbol{q}} \times \boldsymbol{P}_h|}.$$

see, e.g., A.B., D'Alesio, Diehl, Miller, hep-ph/0410050

Key points

- Structure functions arise from the combinations of available four-vectors
- They are frame independent
- They are gauge invariant

SIDIS cross section in terms of structure functions

$$\frac{d\sigma}{dx\,dy\,d\phi_S\,dz\,d\phi_h\,dP_{h\perp}^2} = \frac{\alpha^2}{x\,y\,Q^2}\,\frac{y^2}{2\,(1-\varepsilon)}\left\{F_{UU,T} + \varepsilon\,F_{UU,L} + \sqrt{2\,\varepsilon(1+\varepsilon)}\,\cos\phi_h\,F_{UU}^{\cos\phi_h} + \varepsilon\cos(2\phi_h)\,F_{UU}^{\cos 2\phi_h} + \lambda_e\,\sqrt{2\,\varepsilon(1-\varepsilon)}\,\sin\phi_h\,F_{LU}^{\sin\phi_h}\right\}$$

SIDIS cross section in terms of structure functions

$$F_{UU,T}(x, z, P_{h\perp}^2, Q^2)$$

$$\frac{d\sigma}{dx\, dy\, d\phi_S\, dz\, d\phi_h\, dP_{h\perp}^2} = \frac{\alpha^2}{x\, y\, Q^2}\, \frac{y^2}{2\,(1-\varepsilon)} \Big\{ F_{UU,T} + \varepsilon\, F_{UU,L} + \sqrt{2\,\varepsilon(1+\varepsilon)}\, \cos\phi_h\, F_{UU}^{\cos\phi_h} + \varepsilon \cos(2\phi_h)\, F_{UU}^{\cos 2\phi_h}$$
$$+ \lambda_e\, \sqrt{2\,\varepsilon(1-\varepsilon)}\, \sin\phi_h\, F_{LU}^{\sin\phi_h} \Big\}$$

Polarized SIDIS cross section

$$
\begin{aligned}
&\frac{d\sigma}{dx\,dy\,d\phi_S\,dz\,d\phi_h\,dP_{h\perp}^2} \\
&= \frac{\alpha^2}{x\,y\,Q^2}\,\frac{y^2}{2\,(1-\varepsilon)}\Bigg\{F_{UU,T}+\varepsilon\,F_{UU,L}+\sqrt{2\,\varepsilon(1+\varepsilon)}\,\cos\phi_h\,F_{UU}^{\cos\phi_h}+\varepsilon\cos(2\phi_h)\,F_{UU}^{\cos 2\phi_h} \\
&\quad+\lambda_e\,\sqrt{2\,\varepsilon(1-\varepsilon)}\,\sin\phi_h\,F_{LU}^{\sin\phi_h}+S_L\left[\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_h\,F_{UL}^{\sin\phi_h}+\varepsilon\sin(2\phi_h)\,F_{UL}^{\sin 2\phi_h}\right] \\
&\quad+S_L\,\lambda_e\left[\sqrt{1-\varepsilon^2}\,F_{LL}+\sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_h\,F_{LL}^{\cos\phi_h}\right] \\
&\quad+S_T\left[\sin(\phi_h-\phi_S)\left(F_{UT,T}^{\sin(\phi_h-\phi_S)}+\varepsilon\,F_{UT,L}^{\sin(\phi_h-\phi_S)}\right)+\varepsilon\,\sin(\phi_h+\phi_S)\,F_{UT}^{\sin(\phi_h+\phi_S)}\right. \\
&\qquad+\varepsilon\,\sin(3\phi_h-\phi_S)\,F_{UT}^{\sin(3\phi_h-\phi_S)}+\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_S\,F_{UT}^{\sin\phi_S} \\
&\qquad\left.+\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin(2\phi_h-\phi_S)\,F_{UT}^{\sin(2\phi_h-\phi_S)}\right]+S_T\lambda_e\left[\sqrt{1-\varepsilon^2}\,\cos(\phi_h-\phi_S)\,F_{LT}^{\cos(\phi_h-\phi_S)}\right. \\
&\qquad\left.+\sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_S\,F_{LT}^{\cos\phi_S}+\sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos(2\phi_h-\phi_S)\,F_{LT}^{\cos(2\phi_h-\phi_S)}\right]\Bigg\}
\end{aligned}
$$

Polarized SIDIS cross section

$$
\begin{aligned}
&\frac{d\sigma}{dx\,dy\,d\phi_S\,dz\,d\phi_h\,dP_{h\perp}^2} \\
&= \frac{\alpha^2}{x\,y\,Q^2}\,\frac{y^2}{2\,(1-\varepsilon)}\Bigg\{ F_{UU,T} + \varepsilon\,F_{UU,L} + \sqrt{2\,\varepsilon(1+\varepsilon)}\,\cos\phi_h\,F_{UU}^{\cos\phi_h} + \varepsilon\cos(2\phi_h)\,F_{UU}^{\cos 2\phi_h} \\
&\quad + \lambda_e\,\sqrt{2\,\varepsilon(1-\varepsilon)}\,\sin\phi_h\,F_{LU}^{\sin\phi_h} + S_L\left[\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_h\,F_{UL}^{\sin\phi_h} + \varepsilon\sin(2\phi_h)\,F_{UL}^{\sin 2\phi_h}\right] \\
&\quad + S_L\,\lambda_e\left[\sqrt{1-\varepsilon^2}\,F_{LL} + \sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_h\,F_{LL}^{\cos\phi_h}\right] \\
&\quad + S_T\left[\sin(\phi_h-\phi_S)\left(F_{UT,T}^{\sin(\phi_h-\phi_S)} + \varepsilon\,F_{UT,L}^{\sin(\phi_h-\phi_S)}\right) + \varepsilon\,\sin(\phi_h+\phi_S)\,F_{UT}^{\sin(\phi_h+\phi_S)}\right. \\
&\qquad + \varepsilon\,\sin(3\phi_h-\phi_S)\,F_{UT}^{\sin(3\phi_h-\phi_S)} + \sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_S\,F_{UT}^{\sin\phi_S} \\
&\qquad \left. + \sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin(2\phi_h-\phi_S)\,F_{UT}^{\sin(2\phi_h-\phi_S)}\right] + S_T\lambda_e\left[\sqrt{1-\varepsilon^2}\,\cos(\phi_h-\phi_S)\,F_{LT}^{\cos(\phi_h-\phi_S)}\right. \\
&\qquad \left. + \sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_S\,F_{LT}^{\cos\phi_S} + \sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos(2\phi_h-\phi_S)\,F_{LT}^{\cos(2\phi_h-\phi_S)}\right]\Bigg\}
\end{aligned}
$$

A.B., Diehl, Goeke, Metz, Mulders, Schlegel, JHEP093 (07)

Drell—Yan processes

$$H_a + H_b \to \gamma^*(q) + X \to l^- + l^+ + X$$

$$\frac{d\sigma}{d^4q\, d\Omega} = \frac{\alpha_{em}^2}{2\, F\, q^4}\, L_{\mu\nu} W^{\mu\nu}$$

Cross section in terms of structure functions

$$\frac{d\sigma}{d^4q\,d\Omega} = \frac{\alpha_{em}^2}{F\,q^2} \times$$

$$\Big\{ \Big((1+\cos^2\theta)\,F_{UU}^1 + (1-\cos^2\theta)\,F_{UU}^2 + \sin 2\theta \cos\phi\, F_{UU}^{\cos\phi} + \sin^2\theta \cos 2\phi\, F_{UU}^{\cos 2\phi} \Big)$$

$$+ \; S_{aL} \Big(\sin 2\theta \sin\phi\, F_{LU}^{\sin\phi} + \sin^2\theta \sin 2\phi\, F_{LU}^{\sin 2\phi} \Big)$$

$$+ \; |\vec{S}_{aT}| \Big[\sin\phi_a \Big((1+\cos^2\theta)\,F_{TU}^1 + (1-\cos^2\theta)\,F_{TU}^2 + \sin 2\theta \cos\phi\, F_{TU}^{\cos\phi} + \sin^2\theta \cos 2\phi\, F_{TU}^{\cos 2\phi} \Big)$$

$$+ \cos\phi_a \Big(\sin 2\theta \sin\phi\, F_{TU}^{\sin\phi} + \sin^2\theta \sin 2\phi\, F_{TU}^{\sin 2\phi} \Big) \Big]$$

Only unpolarized and single-polarized part.
Different frames (and different definitions of the angles) are in use (Collins—Soper, Gottfried—Jackson).
Structure functions are different if defined in the different frames.

Cross section in terms of structure functions

$$\frac{d\sigma}{d^4q\, d\Omega} = \frac{\alpha_{em}^2}{F\, q^2} \times$$

$$\Big\{ \Big((1+\cos^2\theta)\, F_{UU}^1 + (1-\cos^2\theta)\, F_{UU}^2 + \sin 2\theta \cos\phi\, F_{UU}^{\cos\phi} + \sin^2\theta \cos 2\phi\, F_{UU}^{\cos 2\phi} \Big)$$

$$+ \; S_{aL} \Big(\sin 2\theta \sin\phi\, F_{LU}^{\sin\phi} + \sin^2\theta \sin 2\phi\, F_{LU}^{\sin 2\phi} \Big)$$

$$+ \; |\vec{S}_{aT}| \Big[\sin\phi_a \Big((1+\cos^2\theta)\, F_{TU}^1 + (1-\cos^2\theta)\, F_{TU}^2 + \sin 2\theta \cos\phi\, F_{TU}^{\cos\phi} + \sin^2\theta \cos 2\phi\, F_{TU}^{\cos 2\phi} \Big)$$

$$+ \cos\phi_a \Big(\sin 2\theta \sin\phi\, F_{TU}^{\sin\phi} + \sin^2\theta \sin 2\phi\, F_{TU}^{\sin 2\phi} \Big) \Big]$$

Only unpolarized and single-polarized part.
Different frames (and different definitions of the angles) are in use (Collins—Soper, Gottfried—Jackson).
Structure functions are different if defined in the different frames.

S. Arnold, Metz, Schlegel, arXiv:0809.2262