# Unimodular gravity

Roberto Percacci[1]

[1]SISSA, Trieste, Italy

Dark Energy from Fundamental Theories to Observations
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## The cosmological constant problem(s)

1. naive estimates of the vacuum energy in QFT, inserted in the Einstein equations, give values of the curvature that far exceed observation
2. in order to have small vacuum energy now, the vacuum energy before the electroweak/strong phase transition had to be highly fine tuned

# Outline

1. (Gauge) equivalent formulations of gravity
2. Classical EG vs UG
3. Quantum EG vs UG

## Reformulations of GR

What is the gauge group of gravity?

not a physically meaningful question.

Can recast the theory in various forms with smaller or larger gauge group.

# Five formulations of GR: EG

$$S_{\mathrm{EG}}(g) = Z_N \int d^4x \sqrt{|g|}\, R \qquad \text{where} \qquad Z_N = \frac{1}{16\pi G} \, .$$

Invariant under *Diff M*

# Five formulations of GR: DG

Apply Stückelberg trick to EG.

$$S_{\mathrm{DG}}(g,\chi) = S_{\mathrm{EG}}\left(\frac{1}{Z_N}\chi^2 g_{\mu\nu}\right) = \int d^4x\,\sqrt{|g|}\left[\chi^2 R - 6\chi\nabla^2\chi\right]$$

Invariant under *Diff* $\ltimes$ *Weyl*

$$g_{\mu\nu}\to\Omega^2 g_{\mu\nu}\,,\qquad \chi\to\Omega^{-1}\chi\,.$$

## Five formulations of GR: unimodular

Under a diffeomorphism,

$$\sqrt{g'} = \det\left|\frac{\partial x}{\partial x'}\right|\sqrt{g}$$

thus we can find a diffeomorphism such that

$$\sqrt{|g|} = \omega$$

(in particular $\omega = 1$).

$$S_{\rm UG}(g) = Z_N \int d^4x\, \omega\, R\,.$$

Invariant under *SDiff M*

# Five formulations of GR: UD

$$S_{\mathrm{UD}}(g,\chi) = \int d^4x\, \omega \left[\chi^2 R - 6\chi\nabla^2\chi\right] .$$

Invariant under $S(Diff \ltimes Weyl) = Diff^*$.

## Five formulations of GR: *WTDiff*

$$S_X(g_{\mu\nu}) = S_{EG}\left(\left(\frac{|g|}{\omega^2}\right)^{-1/4} g_{\mu\nu}\right)$$
$$= Z_N \int d^4x\, |g|^{\frac{1}{4}}\, \omega^{\frac{1}{2}} \left[R + \frac{3}{32}\left(|g|^{-1}\nabla|g| - 2\omega^{-1}\nabla\omega\right)^2\right]$$

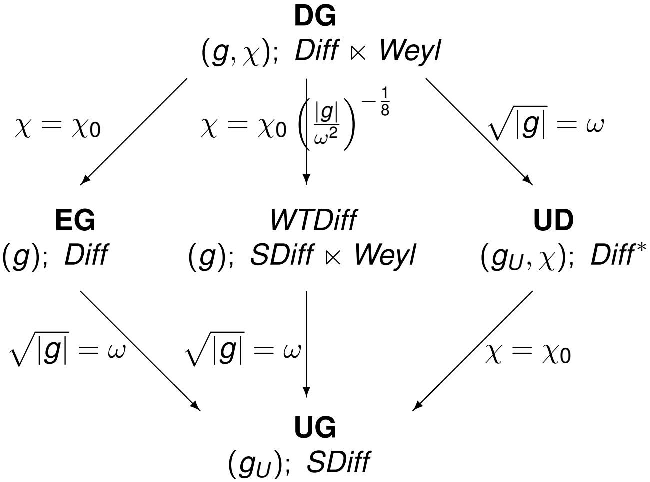
Invariant under *SDiff M* $\ltimes$ *Weyl*

Linearized form

$$\frac{1}{4}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \frac{1}{2}\partial_\mu h^{\mu\alpha}\partial_\nu h^\nu{}_\alpha + \frac{1}{2}\partial_\mu h^{\mu\alpha}\partial_\alpha h - \frac{3}{32}\partial_\mu h \partial^\mu h$$

E. Alvarez, D. Blas, J. Garriga and E. Verdaguer, Transverse Fierz-Pauli symmetry, Nucl. Phys. B 756 (2006) 148-170, arXiv: hep-th/0606019 [hep-th]

# Five formulations of GR: summary



# Hamiltonian formulation

| | **DG** | **EG** | **UG** |
|---|---|---|---|
| fields | $q_{ij}$,$N_i$,$N$,$\chi$ | $q_{ij}$,$N_i$,$N$ | $q_{ij}$,$N_i$ |
| momenta | $p^{ij}$,$P^i$,$P$,$\pi$ | $p^{ij}$,$P^i$,$P$ | $p^{ij}$,$P^i$ |
| # can. variables | 22 | 20 | 18 |
| primary constr. | $P^i$, $P$, $C$ | $P^i$, $P$ | $P^i$ |
| secondary constr. | $\mathcal{H}_i$,$\mathcal{H}$ | $\mathcal{H}_i$,$\mathcal{H}$ | $\mathcal{H}_i$,$\mathcal{H}_\Lambda$ |
| # 1st cl. constr. | 9 | 8 | 7 |
| # canonical d.o.f. | 4 | 4 | 4 |

R. de Leon Ardon, S. Gielen, R.P, Gravity with more or less gauging, C.Q.G. 35 (2018) 195009, arXiv:1805.11626 [gr-qc]

## Which is to be preferred?

1. extending the gauge group is useful to recognize equivalences between different formulations
2. in DG certain singular configurations can be interpreted as gauge artifacts, e.g. I. Bars, S.H. Chen, P. Steinhardt and N. Turok, Antigravity and the big bang/big crunch transition, Phys. Lett. B715 (2012) 278
3. suggest routes to unification

## Which is to be preferred?

Gauge invariances needed in order to deal with local d.o.f. UG has the smallest gauge group compatible with locality.

Traceless fluctuation $\rightarrow$ simpler Feynman rules

The metric is a nonlinear field with values in $GL(4)/O(1,3)$ and GR can be treated by methods of EFT. Analogy with chiral models with $U \in \frac{SU(2)_L \times SU(2)_R}{SU(2)_V}$

# EFT of gravity

Chiral action

$$S = \int dx \left[ \frac{f_\pi^2}{4} tr(U^{-1}\partial U)^2 + \ell_1 tr((U^{-1}\partial U)^2)^2 + \ell_2 tr((U^{-1}\partial U)^2)^2 + O(\partial^6) \right]$$

Gravitational action

$$S = \int dx \sqrt{g} \left[ 2m_P^2 \Lambda + m_P^2 R + \ell_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \ell_2 R_{\mu\nu} R^{\mu\nu} + \ell_3 R^2 + O(\partial^6) \right]$$

$$R \sim \Gamma\Gamma \sim (g^{-1}\partial g)^2$$

Differences disappear for unimodular gravity.

# Unimodularity as gauge fixing

$$\sqrt{|g|} = \omega$$

is a gauge condition breaking *Diff* to *SDiff*.

It removes one d.o.f. per spacetime point
Almost all of these are unphysical

However

$$\int d^4x \sqrt{|g|} = \int d^4x \omega$$

is a physical statement.

UG is EG with fixed total volume.

EG and UG differ by ONE physical d.o.f.

## General action

$S_{EG} = S^g_{EG} + S^m_{EG}$
an arbitrary *Diff*-invariant action for gravity and matter, where

$$S^g_{EG}(g) = \int d^dx\,\sqrt{|g|}\,\mathcal{L}_g(g) \text{ and } S^m_{EG}(\psi, g) = \int d^dx\,\sqrt{|g|}\,\mathcal{L}_m(\psi, g)$$

$$S_{UG} = S^g_{UG} + S^m_{UG}$$

$$S^g_{UG}(g) = \int d^dx\,\omega\,\mathcal{L}_g(g)\ ; \qquad S^m_{UG}(\psi, g) = \int d^dx\,\omega\,\mathcal{L}_m(\psi, g)\ .$$

## EOM

In varying $S_{UG}$ one must impose

$$0 = \delta\sqrt{|g|} \equiv \frac{1}{2} g^{\mu\nu} \delta g_{\mu\nu} \ .$$

This leads to the tracefree equations

$$-\tilde{E}^{\mu\nu} + \frac{1}{d} g^{\mu\nu} \tilde{E} = \frac{1}{2}\left( \tilde{T}^{\mu\nu} - \frac{1}{d} g^{\mu\nu} \tilde{T} \right) \ ,$$

where

$$\tilde{E}^{\alpha\beta} = \frac{1}{\omega} \frac{\delta S^{g}_{UG}}{\delta g_{\alpha\beta}} \qquad \text{and} \qquad \tilde{T}^{\alpha\beta} = \frac{2}{\omega} \frac{\delta S^{m}_{UG}}{\delta g_{\alpha\beta}}$$

is a symmetric but not conserved energy-momentum tensor.

# The rhs of the EOM

Under and infinitesimal diffeomorphism $\epsilon^\mu$

$$\delta\sqrt{g} = \sqrt{g}\nabla_\mu\epsilon^\mu$$

So, the group *SDiff* is generated by transverse vectorfields:

$$\nabla_\mu v^\mu = 0 \; .$$

From the *SDiff*-invariance of $S^m_{UG}$,

$$0 = \int d^4x \frac{\delta S^m_{UG}}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = \int d^4x \tilde{T}^{\mu\nu}\nabla_\mu\epsilon_\nu = -\int d^4x \nabla_\mu \tilde{T}^{\mu\nu}\epsilon_\nu$$

there follows

$$\nabla_\mu \tilde{T}^{\mu\nu} = \nabla^\nu\Sigma \; ,$$

where $\Sigma$ is some scalar field.

# The rhs of the EOM

What is $\Sigma$? For a scalar

$$\tilde{T}^{\mu\nu} = \nabla^{\mu}\phi\nabla^{\nu}\phi$$

Using the eom (and the Bianchi identity) one finds

$$\nabla_{\mu}\tilde{T}^{\mu\nu} = -\nabla^{\nu}\mathcal{L}_m \, .$$

The term coming from the variation of $\sqrt{|g|}$ is missing, and this term is proportional to the Lagrangian density. We will therefore define an "improved" symmetric, conserved energy-momentum tensor

$$T_{\mu\nu} = \tilde{T}_{\mu\nu} + g_{\mu\nu}\mathcal{L}_m \, .$$

This is just the usual energy-momentum tensor used in GR:

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\delta S^m_{EG}}{\delta g_{\mu\nu}} \, .$$

# The rhs of the EOM

Since $\tilde{T}_{\mu\nu}$ and $T_{\mu\nu}$ have the same tracefree part, we can replace $\tilde{T}_{\mu\nu}$ by $T_{\mu\nu}$ in the tracefree eom:

$$-\tilde{E}^{\mu\nu} + \frac{1}{d} g^{\mu\nu} \tilde{E} = \frac{1}{2} \left( T^{\mu\nu} - \frac{1}{d} g^{\mu\nu} T \right) \ .$$

## The lhs of the EOM

*Diff*-invariance of $S^g_{EG}$ implies the differential identity

$$\nabla_\mu \tilde{E}^{\mu\nu} + \frac{1}{2}\nabla^\nu \mathcal{L}_g = 0 \ .$$

Acting with $\nabla_\mu$ on the EOM and using this we obtain

$$\nabla^\nu \left( \frac{1}{2}\mathcal{L}_g + \frac{1}{d}\tilde{E} + \frac{1}{2d} T \right) = 0 \ .$$

$$\frac{1}{2}\mathcal{L}_g + \frac{1}{d}\tilde{E} + \frac{1}{2d} T = Z_N \Lambda$$

where $\Lambda$ is an arbitrary constant of integration.

$$-\tilde{E}^{\mu\nu} - \frac{1}{2} g^{\mu\nu}\mathcal{L}_g + Z_N \Lambda g^{\mu\nu} = \frac{1}{2} T^{\mu\nu} \ .$$

This is the EOM of $S_{EG}$, except for the $\Lambda$ term.

## Vacuum energy does not gravitate

A cosmological term in the UG action is metric-independent and does not affect the EOM.

The cosmological term in the EOM is an integration constant, unrelated to the vacuum energy.

First cosmological constant problem is solved.

## Inflation

Apparent issue for inflation.

Whereas a constant vacuum energy does not affect the EOM, a change in the vacuum energy will.

G.F.R. Ellis,The tracefree EInstein equations and inflation, arXiv: 1306.3021 [gr-qc]

## Phase transitions

Likewise if the vacuum energy changes due to a phase transition, this is felt by the gravitational field.

Thus if we fix the integration constant to fit the cc seen today, its value before the phase transition had to be highly fine tuned.

The second cosmological constant problem is unsolved.

## Quantum questions

Can we maintain these equivalences in the quantum theory?

Conflicting statements in the literature.

R. de León Ardón, N. Ohta and R. P., “Path integral of unimodular gravity,” Phys. Rev. D **97** (2018) no.2, 026007 arXiv:1710.02457 [gr-qc] ,

R. P., “Unimodular quantum gravity and the cosmological constant,” Found. Phys. **48** (2018) no.10, 1364-1379, arXiv:1712.09903 [gr-qc] ,

G.P. de Brito, O. Melichev, R.P. and A.D. Pereira, “Can quantum fluctuations differentiate between standard and unimodular gravity?,” JHEP **12** (2021), 090, arXiv:2105.13886 [gr-qc] .

## York variables

$$h_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu + \left( \bar{\nabla}_\mu \bar{\nabla}_\nu - \frac{1}{d} \bar{g}_{\mu\nu} \bar{\nabla}^2 \right) \sigma + \frac{1}{d} \bar{g}_{\mu\nu} h \,,$$

$$\bar{\nabla}^\mu h^{\mathrm{TT}}_{\mu\nu} = 0 \,, \quad \bar{g}^{\mu\nu} h^{\mathrm{TT}}_{\mu\nu} = 0 \,, \quad \bar{\nabla}^\mu \xi_\mu = 0 \,, \quad h = \bar{g}^{\mu\nu} h_{\mu\nu} \,.$$

$$J_1 = \det \left( \Delta_{L1} - \frac{2\bar{R}}{d} \right)^{1/2} \det \left( \Delta_{L0} \right)^{1/2} \det \left( \Delta_{L0} - \frac{\bar{R}}{d-1} \right)^{1/2}$$

$$\begin{aligned} \Delta_{L0} \chi &= -\bar{\nabla}^2 \chi, \\ \Delta_{L1} A_\mu &= -\bar{\nabla}^2 A_\mu + \bar{R}_\mu{}^\rho A_\rho, \\ \Delta_{L2} h_{\mu\nu} &= -\bar{\nabla}^2 h_{\mu\nu} + \bar{R}_\mu{}^\rho h_{\rho\nu} + \bar{R}_\nu{}^\rho h_{\mu\rho} - \bar{R}_{\mu\rho\nu\sigma} h^{\rho\sigma} - \bar{R}_{\mu\rho\nu\sigma} h^{\sigma\rho} \,. \end{aligned}$$

## Transformations under *Diff*

$$\delta h_{\mu\nu} = \bar{\nabla}_\mu \epsilon_\nu + \bar{\nabla}_\nu \epsilon_\mu$$

$$\epsilon^\mu = \epsilon^{T\mu} + \bar{\nabla}_\mu \phi \; ; \qquad \bar{\nabla}_\mu \epsilon^{T\mu} = 0 \; .$$

$$\delta_{\epsilon^T} \xi^\mu = \epsilon^{T\mu} \; ; \qquad \delta_\phi h = -2\Delta_{L0}\phi \; ; \qquad \delta_\phi \sigma = 2\phi \; ,$$

$h^{TT}_{\mu\nu}$ and $s = h + \Delta_{L0}\sigma$ are invariant.

$\xi_\mu$ and $\psi = \frac{((d-1)\Delta_{L0} - \bar{R})\sigma + \beta h}{(d-1-\beta)\Delta_{L0} - \bar{R}}$ are gauge d.o.f.

# One-loop EG

$$S = \frac{Z_N}{2}\int d^dx\,\sqrt{\bar{g}}\Bigg\{\frac{1}{2}h^{\mathrm{TT}}_{\mu\nu}\left(\Delta_{L2}-\frac{2\bar{R}}{d}\right)h^{\mathrm{TT}\mu\nu} - \frac{(d-1)(d-2)}{2d^2}s\left(\Delta_{L0}-\frac{\bar{R}}{d-1}\right)s-\frac{d-2}{4d}Eh^2\Bigg\}.$$

where EOM implies

$$0 = E \equiv \bar{R}-\frac{2d\Lambda}{d-2}$$

## Gauge fixing

$$
\begin{aligned}
F_\mu &= \bar{\nabla}_\rho h^\rho{}_\mu - \frac{\beta+1}{d}\bar{\nabla}_\mu h \\
&= -\left(\Delta_{L1} - \frac{2\bar{R}}{d}\right)\xi_\mu - \frac{d-1-\beta}{d}\nabla_\mu\left(\Delta_{L0} - \frac{\bar{R}}{d-1-\beta}\right)\psi
\end{aligned}
$$

$$
\begin{aligned}
S_{GF} &= = \frac{Z_N}{2\alpha}\int d^dx\sqrt{\bar{g}}\,\bar{g}^{\mu\nu}F_\mu F_\nu \\
&= \frac{Z_N}{2\alpha}\int d^dx\sqrt{\bar{g}}\Bigg[\xi_\mu\left(\Delta_{L1} - \frac{2\bar{R}}{d}\right)^2\xi^\mu \\
&\qquad + \frac{(d-1-\beta)^2}{d^2}\psi\Delta_{L0}\left(\Delta_{L0} - \frac{\bar{R}}{d-1-\beta}\right)^2\psi\Bigg]
\end{aligned}
$$

## Ghosts

$$C_\nu = C_\nu^T + \nabla_\nu \frac{1}{\sqrt{-\bar{\nabla}^2}} C^L$$

$$S_{gh} = \int d^d x \sqrt{\bar{g}} \Big[ \bar{C}^{T\mu} \left( \Delta_{L1} - \frac{2\bar{R}}{d} \right) C_\mu^T + 2 \frac{d-1-\beta}{d} \bar{C}^L \left( \Delta_{L0} - \frac{\bar{R}}{d-1-\beta} \right) C^L \Big]$$

# 1-loop EG with cosmological term

$$Z_{EG}^{(1)}(\bar{g}) = e^{-S(\bar{g})} \int (d\epsilon) \frac{\mathrm{Det}_1\left(\Delta_{L1} - \frac{2\bar{R}}{d}\right)^{1/2}}{\mathrm{Det}_2\left(\Delta_{L2} - \frac{2\bar{R}}{d}\right)^{1/2}}$$

$$V_{Diff} = \int (d\epsilon)$$

on shell $\bar{R} = \frac{2d\Lambda}{d-2}$

# 1-loop EG with cosmological term

$$\Gamma^{(1)}(\bar{g}) = S(\bar{g}) + \frac{1}{2}\mathrm{Tr}\log\left(\Delta_{L2} - \frac{2\bar{R}}{d}\right) - \frac{1}{2}\mathrm{Tr}\log\left(\Delta_{L1} - \frac{2\bar{R}}{d}\right)$$

$$\Gamma_{log}(\bar{g}) \;=\; -\frac{1}{2(4\pi)^2}\int d^4x\,\sqrt{\bar{g}}\,\log\left(\frac{\Lambda^2}{\mu^2}\right)\left(\frac{53}{45}\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} - \frac{29}{40}\bar{R}^2\right)$$

S.M. Christensen, M.J. Duff Nucl. Phys. B170 (1980) 480-506

## Quantum UG

Usual linear splitting $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ does not fit well with unimodularity condition.
Use instead exponential splitting:

$$g_{\mu\nu} = \bar{g}_{\mu\rho} \left( e^X \right)^{\rho}{}_{\nu}$$

where $\det \bar{g} = \omega$ and $X^{\rho}{}_{\nu} = \bar{g}^{\rho\sigma} h_{\sigma\nu}$.
Symmetric in spite of appearance.
Respects nonlinear structure.
Can use York decomposition on $h_{\mu\nu}$ as before.
On shell results same for EG.

## Gauge fixing for *SDiff*

*SDiff* is generated by transverse vector fields

$$\bar{\nabla}_\mu \epsilon^\mu = 0$$

$$F_\mu = T_{\mu\nu} \bar{\nabla}_\rho h^{\rho\nu} = -\left(\Delta_{L1} - \frac{2\bar{R}}{d}\right) \xi_\mu$$

where

$$T^\mu{}_\nu = \delta^\mu_\nu - \bar{\nabla}^\mu \frac{1}{\bar{\nabla}^2} \bar{\nabla}_\nu$$

$$S_{GF} = \frac{Z_N}{2\alpha} \int d^d x \, \omega F_\mu T^{\mu\nu} F_\nu = \frac{Z_N}{2\alpha} \int d^d x \, \omega \, \xi_\mu \left(-\bar{\nabla}^2 - \frac{\bar{R}}{d}\right)^2 \xi^\mu$$

$$S_{gh} = \int d^d x \, \omega \, \bar{C}^T_\mu \left(-\bar{\nabla}^2 - \frac{\bar{R}}{d}\right) C^{\mu\,T}$$

The one-loop partition function of unimodular gravity is

$$Z_{UG}^{(1)} = e^{-S(\bar{g})}\left(\int (d\epsilon^T)\right)\frac{\text{Det}_1\left(\Delta_{L1}-\frac{2\bar{R}}{d}\right)^{1/2}}{\text{Det}_2\left(\Delta_{L2}-\frac{2\bar{R}}{d}\right)^{1/2}\text{Det}\Delta_{L_0}^{1/2}}$$

## Volume of *SDiff*

$$(d\epsilon) = (d\epsilon^T)(d\phi)\mathrm{Det}\Delta_{L_0}^{1/2}$$

Since $\delta_\phi h = -2\Delta_{L0}\phi$, the measure on $Q = Diff/SDiff$ is

$$(dh) = (d\phi)\,\mathrm{Det}\Delta_{L0}\ .$$

Thus

$$V_{Diff} = \int (d\epsilon) = \int (d\epsilon^T)\,\det\Delta_{L_0}^{-1/2} \int (d\phi)\,\det\Delta_{L0} = V_{SDiff}\,V_Q\ ,$$

where

$$V_{SDiff} = \int (d\epsilon^T)\mathrm{Det}\Delta_{L_0}^{-1/2}$$

# Alternatively

$$
\begin{aligned}
V_{SDiff} &= \int (d\epsilon)\delta(\bar{\nabla}^{\mu}\epsilon_{\mu}) \\
&= \int (d\epsilon^{T})(d\phi)\mathrm{Det}\Delta_{L_0}^{1/2}\delta(\Delta_{L0}\phi) \\
&= \int (d\epsilon^{T})\mathrm{Det}\Delta_{L_0}^{-1/2}
\end{aligned}
$$

# In conclusion

$$Z_{UG}^{(1)} = e^{-S(\bar{g})} V_{SDiff} \frac{\mathrm{Det}_1 \left( \Delta_{L1} - \frac{2\bar{R}}{d} \right)^{1/2}}{\mathrm{Det}_2 \left( \Delta_{L2} - \frac{2\bar{R}}{d} \right)^{1/2}}$$

and

$$\Gamma_{UG}^{(1)}(g) = \Gamma_{EG}^{(1)}(g)\Big|_{\det g = \omega} + \text{constant}$$

## General path integral

Two-step gauge fixing in EG leads to

$$Z_{EG} = Z_{UG} \times \text{constant}$$

equivalence extends (formally) to all orders in perturbation theory

## Conclusions

- Equivalence between various formulations of GR can be maintained at one loop.
- UG interesting because of different role of vacuum energy. This extends to quantum UG.