

A novel method for calculating Bose-Einstein correlation functions with Coulomb final-state interaction

Márton Nagy, Máté Csanád, Aletta Purzsa, Dániel Kincses
(Eötvös University, Budapest)

Based on: arXiv:2308.10745; just about to be published in EPJ C.

WPCF – Resonance Workshop 2023
November 6, 2023



ELTE
EÖTVÖS LORÁND
UNIVERSITY

Outline

- Introduction
 - HBT correlations, Coulomb effect, basic formulas
 - Need for refinement: non-Gaussian sources, precision measurements
 - Lévy sources in heavy ion collisions
- New method for treatment of Coulomb interaction
 - Numerical & methodological motivation
 - Calculation of the Coulomb integral kernel
 - Rigorous mathematics needed
 - Spherically symmetric case: limiting functional expressed
 - Implementation; esp. for Lévy-type sources
- Outlook
 - Ready to use in experimental analyses
 - Generalizations: non-spherically symmetric case, strong interaction

Introduction

- Bose-Einstein-correlations of like particles ($\pi^+\pi^+$, $\pi^-\pi^-$, K^+K^+ ...): measure fm-scale space-time extent of particle emitting source
- Some definitions:

source function: $S(x, \mathbf{p})$

single part. distr.: $N_1(\mathbf{p}) = \int dx S(x, \mathbf{p})$

pair wave function: $\psi^{(2)}(x_1, x_2)$

pair mom. distr.: $N_2(\mathbf{p}_1, \mathbf{p}_2) = \int dx_1 dx_2 S(x_1, \mathbf{p}_1) S(x_2, \mathbf{p}_2) |\psi^{(2)}(x_1, x_2)|^2$

corr. function: $C(\mathbf{p}_1, \mathbf{p}_2) = \frac{N_2(\mathbf{p}_1, \mathbf{p}_2)}{N_1(\mathbf{p}_1)N_1(\mathbf{p}_2)}$

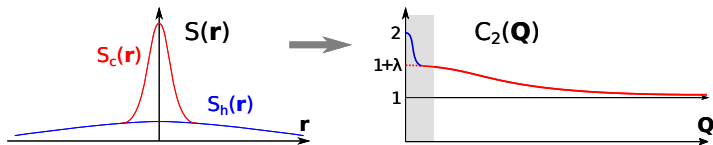
pair source: $D(\mathbf{r}, \mathbf{K}) = \int d^4\rho S(\rho + \frac{r}{2}, \mathbf{K}) S(\rho - \frac{r}{2}, \mathbf{K})$

- Approximately thus

$$C(\mathbf{k}, \mathbf{K}) = \frac{\int D(\mathbf{r}, \mathbf{K}) |\psi_{\mathbf{k}}(\mathbf{r})|^2 d\mathbf{r}}{\int D(\mathbf{r}, \mathbf{K}) d\mathbf{r}}, \quad \mathbf{K} := \frac{\mathbf{p}_1 + \mathbf{p}_2}{2}, \quad \mathbf{k} := \frac{\mathbf{p}_1 - \mathbf{p}_2}{2}.$$

Introduction

- Core-halo model for intercept parameter λ (Csörgő, Lörstad, Zimányi, *Z. Phys. C* 71, 491 (1996))



λ measures core fraction: $S = \sqrt{\lambda}S_c + (1 - \sqrt{\lambda})S_h \Rightarrow$ Bowler-Sinyukov formula:

$$\text{for „large” } S_h, C(\mathbf{k}, \mathbf{K}) = 1 - \lambda + \lambda \frac{\int D_c(\mathbf{r}, \mathbf{K}) |\psi_{\mathbf{k}}(\mathbf{r})|^2 d\mathbf{r}}{\int D_c(\mathbf{r}, \mathbf{K}) d\mathbf{r}}.$$

- No final state interactions: $C(\mathbf{k}) \equiv C^{(0)}(\mathbf{k})$, Fourier transform of source

$$|\psi_{\mathbf{k}}^{(0)}(\mathbf{r})|^2 = 1 + \cos(2\mathbf{k}\mathbf{r}) \Rightarrow C^{(0)}(\mathbf{k}) = 1 + \lambda \frac{\int D_c(\mathbf{r}, \mathbf{K}) \cos(2\mathbf{k}\mathbf{r}) d\mathbf{r}}{\int D_c(\mathbf{r}, \mathbf{K}) d\mathbf{r}}.$$

- Final state Coulomb interaction: $\psi^{(0)}$ replaced by solution of two-body Coulomb Schr. eq.; NR case: well known formulas (see below)

$$C^{(0)}(\mathbf{k}) = \frac{C(\mathbf{k})}{K(\mathbf{k})}, \quad K(\mathbf{k}) \equiv \frac{\int D_c(\mathbf{r}) |\psi_{\mathbf{k}}(\mathbf{r})|^2 d\mathbf{r}}{\int D_c(\mathbf{r}) |\psi_{\mathbf{k}}^{(0)}(\mathbf{r})|^2 d\mathbf{r}} \quad \text{Coulomb correction}$$

Source types; Lévy source functions

- Gaussian: usual choice; $D_{cc}(\mathbf{r}) \propto \exp(-r_{kl}(\mathbf{R}^2)_{kl}^{-1})$.
 - Fit parameters: $R_{kl}(\mathbf{K})$ and $\lambda(\mathbf{K})$
 - A generalization: Edgeworth expansion of $C(k)$; in this source: FT of $C^{(0)}(k)$
see eg. Csörgő, Hegyi, *PLB* 489, 15 (2000)
- Cauchy sources \Leftrightarrow exponential $C(k)$; employed at CMS in pp collisions
- Lévy-type sources (Csörgő, Hegyi, Zajc, *EPJ C* 36, 67 (2004)): generalized Gaussian
new parameter: $\alpha \in \mathbb{R}^+$ *stability index*; $\alpha \leq 2$. Expression with a Fourier transform:

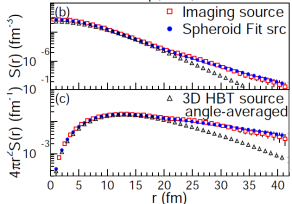
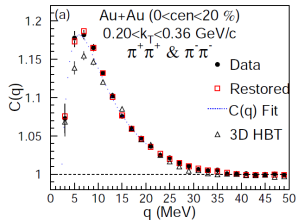
$$D_{cc}(\mathbf{r}) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\mathbf{r}} \exp(-|\mathbf{q}\mathbf{R}|^\alpha) \Leftrightarrow C_2^{(0)}(\mathbf{Q}) = 1 + \lambda \exp(-|Q\mathbf{R}|^\alpha).$$

- Generalization: Levy polynomials (same as Edgeworth for Gaussians)
Novák et al., *Acta Phys. Polon. Supp.* 9, 289 (2016)
- $\alpha=2$: Gaussian; $\alpha=1$: Cauchy distribution; $\alpha \neq 1, 2$: $D_{cc}(\mathbf{r})$ itself only numerically
- Possible reasons for Lévy sources: all rest on *stability* (just as for Gaussian)
 - Fractal structure of jet fragmentation (Csörgő, Hegyi, Novák, Zajc, *Acta Phys. Polon. B* 36, 329 (2005))
 - Anomalous diffusion (Csanád, Csörgő, MN, *Braz. J. Phys.* 37, 1002 (2007))
 - Critical phenomena, closeness of CEP (Csörgő, Hegyi, Novák, Zajc, *AIP Conf. Proc.* 828, 525 (2006))
 - Event & directional averaging (Cimerman, Plumberg, Tomasik, *Phys. Part. Nucl.* 51, 282 (2020))
However: event-by-event Lévy shape in EPOS simulation (Kincses, Stefaniak, Csanád, *Entropy* 24, 308 (2022))

Lévy sources in heavy ion collisions

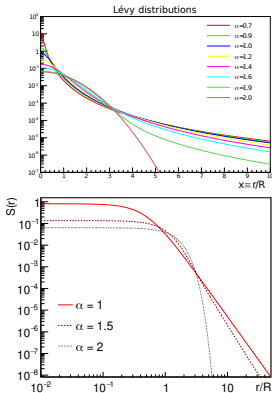
- Non-Gaussian behavior:

- Source extraction („imaging”)
 - Brown, Danielewicz, *PLB* 398, 252 (1997)
 - PHENIX, PRL 98 (2007) 132301



- \Rightarrow experimental motivation for Lévy sources

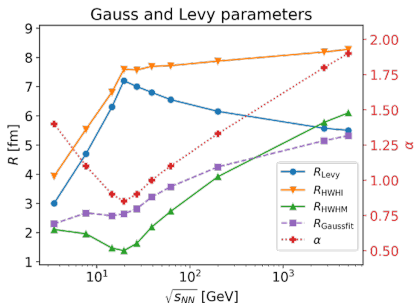
- For $\alpha \neq 2$, power law like $r \rightarrow \infty$ decrease ($\sim r^{-3-\alpha}$); no finite variance



- Meaning of Lévy R : through FWHM, FWHI (full width at half maximum/integral)

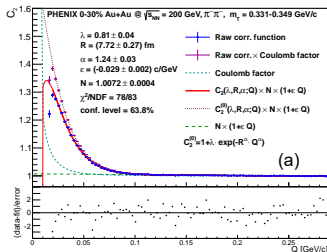
Lévy sources in heavy ion collisions

- Need for precision investigations. . .
- Illustration (courtesy of M. Csanád, see talk at ISMD2023): „simulating” Lévy $C(Q)$ with toy R and α , fit with Gaussian



- Interplay of α and R can hide interesting details, eg. non-monotonicity w.r.t. $\sqrt{s_{NN}}$, when fitting w/ Gaussians

- First Lévy HBT in heavy-ion collisions (PHENIX, *PRC* 97, 064911 (2018))



- $\alpha \neq 2$ confirmed m_t -independently
- Since then, done at several experiments, *some to be shown at this conference*:
 - STAR: see D. Kincses's talk (today)
 - CMS: see M. Csanád's talk (Thursday)
 - NA61: see B. Pórfy's talk (Thursday)

Coulomb interaction

- An essential ingredient for precision HBT measurements
- Non-relativistic treatment: valid in Pair Co-Moving System (PCMS).
- $\mathbf{p} = \hbar \mathbf{k}$: relative momentum, $E = \frac{p^2}{2m}$, m : reduced mass
- Sommerfeld parameter (Coulomb parameter) η : ratio of classical closest distance $r_0 \equiv \frac{q_e^2}{4\pi\epsilon_0} \frac{1}{E}$ to wavelength $\lambda \equiv \frac{2\pi\hbar}{p}$:

$$\eta \equiv \alpha_{\text{em}} \frac{mc}{\hbar k} = \frac{\pi r_0}{\lambda}, \quad \text{with} \quad \alpha_{\text{em}} \equiv \frac{q_e^2}{4\pi\epsilon_0} \frac{1}{\hbar c} \approx \frac{1}{137}.$$

- Two-particle wave function: symmetrized scattering „out” state
 - „out” states asymptotically plane wave + *incoming* spherical wave
 - alternate „in” state (plane wave + outgoing spherical wave) yields same results

$$\psi^{(C)} = e^{i\mathbf{K}\mathbf{R}} \times \frac{\mathcal{N}^*}{\sqrt{2}} e^{-ikr} \left\{ M(1-i\eta, 1, i(kr + \mathbf{k}\mathbf{r})) + (\mathbf{k} \leftrightarrow -\mathbf{k}) \right\}.$$

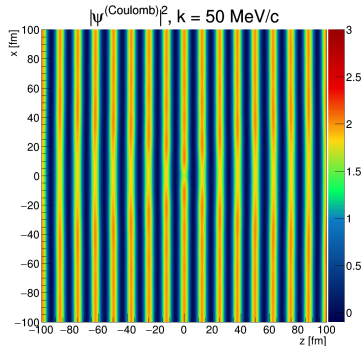
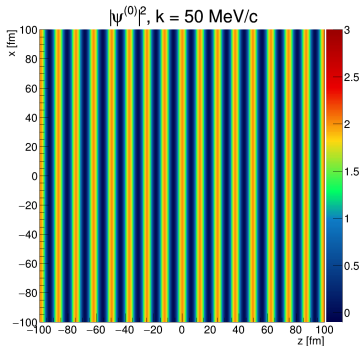
Making use of the $M(a, b, z)$ confluent hypergeometric function

- Normalization (\mathcal{N}) and Gamow factor ($|\mathcal{N}|^2$):

$$\mathcal{N} = e^{-\pi\eta/2} \Gamma(1+i\eta), \quad |\mathcal{N}|^2 = e^{-\pi\eta} |\Gamma(1+i\eta)|^2 = \frac{2\pi\eta}{e^{2\pi\eta} - 1}.$$

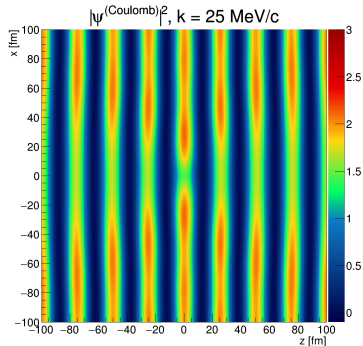
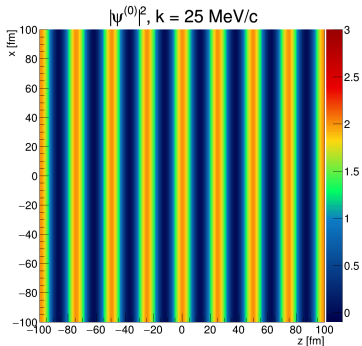
Coulomb interaction

- Coulomb wave function: distorted plane wave, asymptotically logarithmic corrections



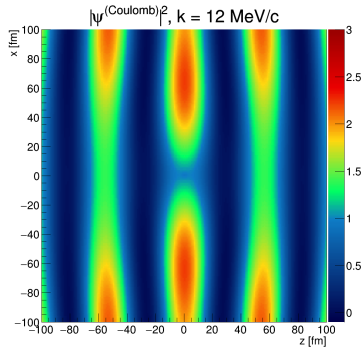
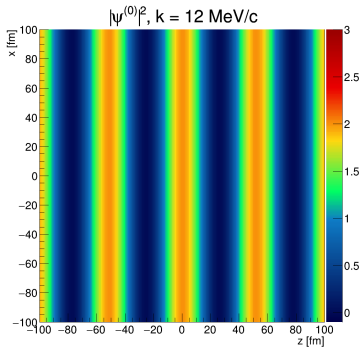
Coulomb interaction

- Coulomb wave function: distorted plane wave, asymptotically logarithmic corrections



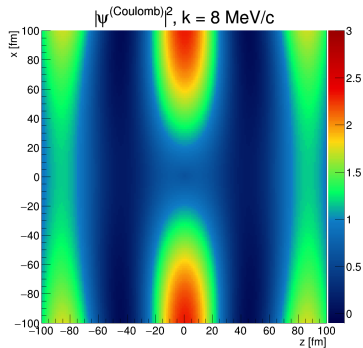
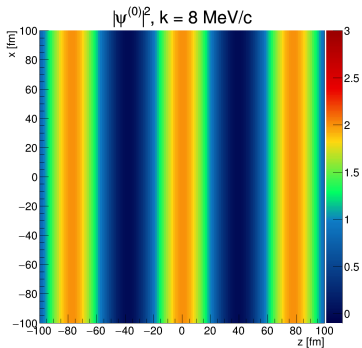
Coulomb interaction

- Coulomb wave function: distorted plane wave, asymptotically logarithmic corrections



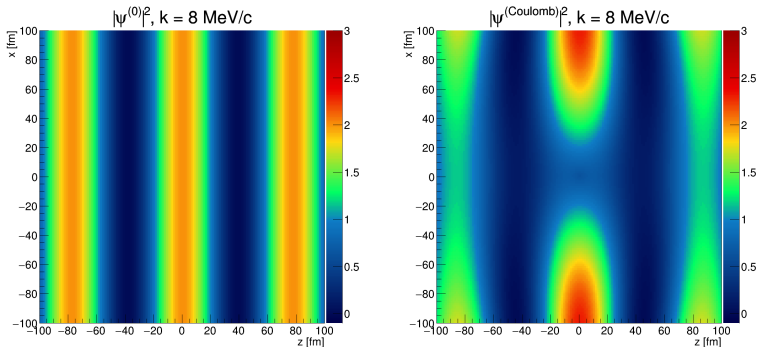
Coulomb interaction

- Coulomb wave function: distorted plane wave, asymptotically logarithmic corrections



Coulomb interaction

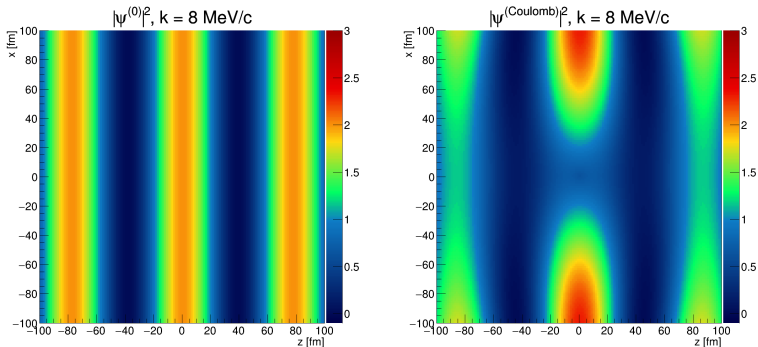
- Coulomb wave function: distorted plane wave, asymptotically logarithmic corrections



- Gamow correction captures only the value at the origin

Coulomb interaction

- Coulomb wave function: distorted plane wave, asymptotically logarithmic corrections



- Gamow correction captures only the value at the origin
- Computational methods:
 - Direct integrating $D(\mathbf{r})|\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2$ during fit: time-consuming, even nowadays
 - Pre-calculate a „Coulomb correction” with fix parameters (say, $R = 5 \text{ fm}$ Gaussian): fast but inconsistent
 - Use iterative method, use memory lookup table. . .

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int d^3\mathbf{r} D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
 - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r}) \dots$
 - Awkward: Fourier transform, then „almost inverse” Fourier transform, *numerically*...
- Natural idea: „interchange order of „integrals”

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int d^3\mathbf{r} D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
 - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r}) \dots$
 - Awkward: Fourier transform, then „almost inverse” Fourier transform, *numerically*...
- Natural idea: „interchange order of „integrals”

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int d^3\mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}}$$

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int d^3\mathbf{r} D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
 - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r}) \dots$
 - Awkward: Fourier transform, then „almost inverse” Fourier transform, *numerically*...
- Natural idea: „interchange order of „integrals”

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int d^3\mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \frac{1}{(2\pi)^3} \int d^3\mathbf{r} \int d^3\mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2$$

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int d^3\mathbf{r} D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
 - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r}) \dots$
 - Awkward: Fourier transform, then „almost inverse” Fourier transform, *numerically*...
- Natural idea: „interchange order of „integrals”

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int d^3\mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \frac{1}{(2\pi)^3} \int d^3\mathbf{r} \int d^3\mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2$$

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \Leftrightarrow f(\mathbf{q}) = \int d^3\mathbf{r} D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
 - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r}) \dots$
 - Awkward: Fourier transform, then „almost inverse” Fourier transform, *numerically*...
- Natural idea: „interchange order of „integrals”

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int d^3\mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \frac{1}{(2\pi)^3} \int d^3\mathbf{r} \int d^3\mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \stackrel{??}{=}$$

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int d^3\mathbf{r} D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
 - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r}) \dots$
 - Awkward: Fourier transform, then „almost inverse” Fourier transform, *numerically*...
- Natural idea: „interchange order of „integrals”

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int d^3\mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \frac{1}{(2\pi)^3} \int d^3\mathbf{r} \int d^3\mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \stackrel{??}{=} \stackrel{??}{=} \frac{1}{(2\pi)^3} \int d^3\mathbf{q} \int d^3\mathbf{r} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2$$

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int d^3\mathbf{r} D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
 - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r}) \dots$
 - Awkward: Fourier transform, then „almost inverse” Fourier transform, *numerically*...
- Natural idea: „interchange order of „integrals”

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int d^3\mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \frac{1}{(2\pi)^3} \int d^3\mathbf{r} \int d^3\mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \stackrel{??}{=} \\ \stackrel{??}{=} \frac{1}{(2\pi)^3} \int d^3\mathbf{q} \int d^3\mathbf{r} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 =$$

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{d^3 \mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int d^3 \mathbf{r} D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
 - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r}) \dots$
 - Awkward: Fourier transform, then „almost inverse” Fourier transform, *numerically*...
- Natural idea: „interchange order of „integrals”

$$\begin{aligned} C_2(\mathbf{Q}) &= \frac{1}{(2\pi)^3} \int d^3 \mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int d^3 \mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \frac{1}{(2\pi)^3} \int d^3 \mathbf{r} \int d^3 \mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \stackrel{??}{=} \\ &\stackrel{??}{=} \frac{1}{(2\pi)^3} \int d^3 \mathbf{q} \int d^3 \mathbf{r} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 = \\ &= \frac{1}{(2\pi)^3} \int d^3 \mathbf{q} f(\mathbf{q}) \int d^3 \mathbf{r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \end{aligned}$$

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{d^3 \mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int d^3 \mathbf{r} D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
 - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r}) \dots$
 - Awkward: Fourier transform, then „almost inverse” Fourier transform, *numerically*...
- Natural idea: „interchange order of „integrals”

$$\begin{aligned} C_2(\mathbf{Q}) &= \frac{1}{(2\pi)^3} \int d^3 \mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int d^3 \mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \frac{1}{(2\pi)^3} \int d^3 \mathbf{r} \int d^3 \mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \stackrel{??}{=} \\ &\stackrel{??}{=} \frac{1}{(2\pi)^3} \int d^3 \mathbf{q} \int d^3 \mathbf{r} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 = \\ &= \frac{1}{(2\pi)^3} \int d^3 \mathbf{q} f(\mathbf{q}) \int d^3 \mathbf{r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \quad \color{red}{\leftarrow \leftarrow \leftarrow} \end{aligned}$$

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \Leftrightarrow f(\mathbf{q}) = \int d^3\mathbf{r} D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
 - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r}) \dots$
 - Awkward: Fourier transform, then „almost inverse” Fourier transform, *numerically*...
- Natural idea: „interchange order of „integrals”

$$\begin{aligned} C_2(\mathbf{Q}) &= \frac{1}{(2\pi)^3} \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int d^3\mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \frac{1}{(2\pi)^3} \int d^3\mathbf{r} \int d^3\mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \stackrel{??}{=} \\ &\stackrel{??}{=} \frac{1}{(2\pi)^3} \int d^3\mathbf{q} \int d^3\mathbf{r} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 = \\ &= \frac{1}{(2\pi)^3} \int d^3\mathbf{q} f(\mathbf{q}) \int d^3\mathbf{r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \quad \color{red}{\leftarrow\leftarrow\leftarrow} \end{aligned}$$

- Not working in this form

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{d^3 \mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \Leftrightarrow f(\mathbf{q}) = \int d^3 \mathbf{r} D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
 - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r}) \dots$
 - Awkward: Fourier transform, then „almost inverse” Fourier transform, *numerically*...
- Natural idea: „interchange order of „integrals”

$$\begin{aligned} C_2(\mathbf{Q}) &= \frac{1}{(2\pi)^3} \int d^3 \mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int d^3 \mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \frac{1}{(2\pi)^3} \int d^3 \mathbf{r} \int d^3 \mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \stackrel{??}{=} \\ &\stackrel{??}{=} \frac{1}{(2\pi)^3} \int d^3 \mathbf{q} \int d^3 \mathbf{r} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 = \\ &= \frac{1}{(2\pi)^3} \int d^3 \mathbf{q} f(\mathbf{q}) \int d^3 \mathbf{r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \quad \color{red}{\leftarrow \leftarrow \leftarrow} \end{aligned}$$

- Not working in this form: Fourier transform \neq integral (Lebesgue)

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{d^3 \mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \Leftrightarrow f(\mathbf{q}) = \int d^3 \mathbf{r} D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
 - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r}) \dots$
 - Awkward: Fourier transform, then „almost inverse” Fourier transform, *numerically*...
- Natural idea: „interchange order of „integrals”

$$\begin{aligned} C_2(\mathbf{Q}) &= \frac{1}{(2\pi)^3} \int d^3 \mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int d^3 \mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \frac{1}{(2\pi)^3} \int d^3 \mathbf{r} \int d^3 \mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \stackrel{??}{=} \\ &\stackrel{??}{=} \frac{1}{(2\pi)^3} \int d^3 \mathbf{q} \int d^3 \mathbf{r} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 = \\ &= \frac{1}{(2\pi)^3} \int d^3 \mathbf{q} f(\mathbf{q}) \int d^3 \mathbf{r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \quad \color{red}{\leftarrow \leftarrow \leftarrow} \end{aligned}$$

- Not working in this form: Fourier transform \neq integral (Lebesgue)
 - Workaround: regularization, $\lambda \in \mathbb{R}^+$, then $\lambda \rightarrow 0$.

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{d^3 \mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \Leftrightarrow f(\mathbf{q}) = \int d^3 \mathbf{r} D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
 - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r}) \dots$
 - Awkward: Fourier transform, then „almost inverse” Fourier transform, *numerically*...
- Natural idea: „interchange order of „integrals”

$$\begin{aligned} C_2(\mathbf{Q}) &= \frac{1}{(2\pi)^3} \int d^3 \mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int d^3 \mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \frac{1}{(2\pi)^3} \int d^3 \mathbf{r} \int d^3 \mathbf{q} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \stackrel{??}{=} \\ &\stackrel{??}{=} \frac{1}{(2\pi)^3} \int d^3 \mathbf{q} \int d^3 \mathbf{r} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 = \\ &= \frac{1}{(2\pi)^3} \int d^3 \mathbf{q} f(\mathbf{q}) \int d^3 \mathbf{r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \quad \color{red}{\leftarrow \leftarrow \leftarrow} \end{aligned}$$

- Not working in this form: Fourier transform \neq integral (Lebesgue)
 - Workaround: regularization, $\lambda \in \mathbb{R}^+$, then $\lambda \rightarrow 0$.
 - Careful math needed (once in a physicist's lifetime...)

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q})$$

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q}) = \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r})$$

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q}) = \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r})$$

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q}) = \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r})$$

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q}) = \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r})$$

Details of derivation

- Interchanging our integrals in a careful way:

$$\begin{aligned}
 C_2(\mathbf{Q}) &= \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r})
 \end{aligned}$$

Details of derivation

- Interchanging our integrals in a careful way:

$$\begin{aligned}
 C_2(\mathbf{Q}) &= \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r})
 \end{aligned}$$

Details of derivation

- Interchanging our integrals in a careful way:

$$\begin{aligned}
 C_2(\mathbf{Q}) &= \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r})
 \end{aligned}$$

Details of derivation

- Interchanging our integrals in a careful way:

$$\begin{aligned}
 C_2(\mathbf{Q}) &= \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 e^{i\mathbf{q}\mathbf{r}}
 \end{aligned}$$

Details of derivation

- Interchanging our integrals in a careful way:

$$\begin{aligned}
 C_2(\mathbf{Q}) &= \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 e^{i\mathbf{q}\mathbf{r}}
 \end{aligned}$$

Details of derivation

- Interchanging our integrals in a careful way:

$$\begin{aligned}
 C_2(\mathbf{Q}) &= \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 e^{i\mathbf{q}\mathbf{r}}
 \end{aligned}$$

Details of derivation

- Interchanging our integrals in a careful way:

$$\begin{aligned}
 C_2(\mathbf{Q}) &= \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 e^{i\mathbf{q}\mathbf{r}} = \\
 &= \lim_{\lambda \rightarrow 0} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) \int d^3\mathbf{r} e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2.
 \end{aligned}$$

Details of derivation

- Interchanging our integrals in a careful way:

$$\begin{aligned}
 C_2(\mathbf{Q}) &= \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 e^{i\mathbf{q}\mathbf{r}} = \\
 &= \lim_{\lambda \rightarrow 0} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) \int d^3\mathbf{r} e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2.
 \end{aligned}$$

Details of derivation

- Interchanging our integrals in a careful way:

$$\begin{aligned}
 \underline{\underline{C_2(\mathbf{Q})}} &= \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 e^{i\mathbf{q}\mathbf{r}} = \\
 &= \underline{\underline{\lim_{\lambda \rightarrow 0} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) \int d^3\mathbf{r} e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2}}.
 \end{aligned}$$

Details of derivation

- Interchanging our integrals in a careful way:

$$\begin{aligned}
 \underline{\underline{C_2(\mathbf{Q})}} &= \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 e^{i\mathbf{q}\mathbf{r}} = \\
 &= \underline{\underline{\lim_{\lambda \rightarrow 0} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) \int d^3\mathbf{r} e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2}}.
 \end{aligned}$$

Details of derivation

- Interchanging our integrals in a careful way:

$$\begin{aligned}
 \underline{\underline{C_2(\mathbf{Q})}} &= \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 e^{i\mathbf{q}\mathbf{r}} = \\
 &= \lim_{\lambda \rightarrow 0} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) \underline{\underline{\int d^3\mathbf{r} e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2}}.
 \end{aligned}$$

- „Ingredients“:

Details of derivation

- Interchanging our integrals in a careful way:

$$\begin{aligned}
 \underline{\underline{C_2(\mathbf{Q})}} &= \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 e^{i\mathbf{q}\mathbf{r}} = \\
 &= \lim_{\lambda \rightarrow 0} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) \underline{\underline{\int d^3\mathbf{r} e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2}}.
 \end{aligned}$$

- „Ingredients“:
 - Conditions of (Lebesgue) integrability

Details of derivation

- Interchanging our integrals in a careful way:

$$\begin{aligned}
 \underline{\underline{C_2(\mathbf{Q})}} &= \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 e^{i\mathbf{q}\mathbf{r}} = \\
 &= \lim_{\lambda \rightarrow 0} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) \underline{\underline{\int d^3\mathbf{r} e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2}}.
 \end{aligned}$$

- „Ingredients“:
 - Conditions of (Lebesgue) integrability
 - *Lebesgue theorem* (for interchanging integrals and limits)

Details of derivation

- Interchanging our integrals in a careful way:

$$\begin{aligned}
 \underline{\underline{C_2(\mathbf{Q})}} &= \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 e^{i\mathbf{q}\mathbf{r}} = \\
 &= \lim_{\lambda \rightarrow 0} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) \underline{\underline{\int d^3\mathbf{r} e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2}}.
 \end{aligned}$$

- „Ingredients“:
 - Conditions of (Lebesgue) integrability
 - *Lebesgue theorem* (for interchanging integrals and limits)
 - *Fubini's theorem* (for interchanging repeated integrals)

Details of derivation

- Interchanging our integrals in a careful way:

$$\begin{aligned}
 \underline{\underline{C_2(\mathbf{Q})}} &= \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 e^{i\mathbf{q}\mathbf{r}} = \\
 &= \lim_{\lambda \rightarrow 0} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) \underline{\underline{\int d^3\mathbf{r} e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2}}.
 \end{aligned}$$

- „Ingredients“:
 - Conditions of (Lebesgue) integrability
 - *Lebesgue theorem* (for interchanging integrals and limits)
 - *Fubini's theorem* (for interchanging repeated integrals)
- In last step, *cannot* interchange $\int d^3\mathbf{q}$ and $\lim_{\lambda \rightarrow 0}$.

Details of derivation

- Interchanging our integrals in a careful way:

$$\begin{aligned}
 \underline{\underline{C_2(\mathbf{Q})}} &= \int d^3\mathbf{r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int d^3\mathbf{r} \lim_{\lambda \rightarrow 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \\
 &= \lim_{\lambda \rightarrow 0} \int d^3\mathbf{r} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 e^{i\mathbf{q}\mathbf{r}} = \\
 &= \lim_{\lambda \rightarrow 0} \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) \int d^3\mathbf{r} e^{-\lambda r} e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2.
 \end{aligned}$$

- „Ingredients“:
 - Conditions of (Lebesgue) integrability
 - *Lebesgue theorem* (for interchanging integrals and limits)
 - *Fubini's theorem* (for interchanging repeated integrals)
- In last step, *cannot* interchange $\int d^3\mathbf{q}$ and $\lim_{\lambda \rightarrow 0}$.
- As of now, continuing only in the spherically symmetric case:
 $f(\mathbf{q}) \equiv f_s(q)$, $D_{cc}(r) = 2\pi \int_0^\infty dq q^2 \sin(qr) f_s(q)$.

Details of derivation (cont'd)

- After substituting $\psi_{\mathbf{k}}^{(2)}(\mathbf{r})$, „master” formula thus reads as

$$C_2(Q) = \frac{|\mathcal{N}|^2}{2\pi^2} \lim_{\lambda \rightarrow 0} \int_0^\infty dq q^2 f_s(q) \left[\mathcal{D}_{1\lambda s}(q) + \mathcal{D}_{2\lambda s}(q) \right], \quad \text{where}$$

$$\mathcal{D}_{1\lambda s}(q) = \int d^3\mathbf{r} \frac{\sin(qr)}{qr} e^{-\lambda r} M(1+i\eta, 1, -i(kr+\mathbf{k}\mathbf{r})) M(1-i\eta, 1, i(kr+\mathbf{k}\mathbf{r})),$$

$$\mathcal{D}_{2\lambda s}(q) = \int d^3\mathbf{r} \frac{\sin(qr)}{qr} e^{-\lambda r} M(1+i\eta, 1, -i(kr-\mathbf{k}\mathbf{r})) M(1-i\eta, 1, i(kr+\mathbf{k}\mathbf{r})).$$

- These can be calculated (using complex analysis; method pioneered by Nordsieck in the theory of bremsstrahlung & pair creation)

A. Nordsieck, *Phys. Rev.* 93, 785 (1954).

$$\mathcal{D}_{1\lambda s}(q) = \frac{4\pi}{q} \text{Im} \left[\frac{1}{(\lambda-iq)^2} \left(1 + \frac{2k}{q+i\lambda} \right)^{2i\eta} \mathcal{F}_+ \left(\frac{4k^2}{(q+i\lambda)^2} \right) \right],$$

$$\mathcal{D}_{2\lambda s}(q) = \frac{4\pi}{q} \text{Im} \left[\frac{(\lambda-iq-2ik)^{i\eta} (\lambda-iq+2ik)^{-i\eta}}{(\lambda-iq)^2 + 4k^2} \right],$$

Here $\mathcal{F}_+(x) \equiv {}_2F_1(i\eta, 1+i\eta, 1, x)$ is the hypergeometric function

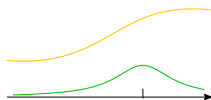
- For $\lim_{\lambda \rightarrow 0}$, function forms of $\mathcal{D}_{1\lambda s}$, $\mathcal{D}_{2\lambda s}$ become „ill-behaved”
- Need to calculate & simplify $\lambda \rightarrow 0$ limit (numerical limit-taking... $\Rightarrow \zeta$)

The main result

- *A remark:* approximation of $\delta(x)$ Dirac delta: a well known simpler similar case

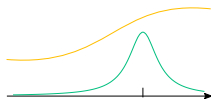
The main result

- *A remark:* approximation of $\delta(x)$ Dirac delta: a well known simpler similar case



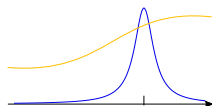
The main result

- *A remark:* approximation of $\delta(x)$ Dirac delta: a well known simpler similar case



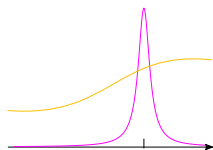
The main result

- *A remark:* approximation of $\delta(x)$ Dirac delta: a well known simpler similar case



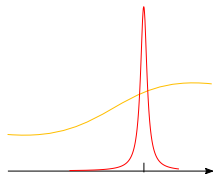
The main result

- *A remark:* approximation of $\delta(x)$ Dirac delta: a well known simpler similar case



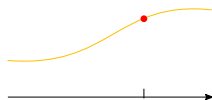
The main result

- *A remark:* approximation of $\delta(x)$ Dirac delta: a well known simpler similar case



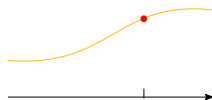
The main result

- *A remark:* approximation of $\delta(x)$ Dirac delta: a well known simpler similar case



The main result

- A remark: approximation of $\delta(x)$ Dirac delta: a well known simpler similar case



- In our case, result of the calculation is

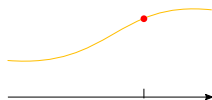
$$C_2(Q) = |\mathcal{N}|^2 \left(1 + f_s(2k) + \frac{\eta}{\pi} [\mathcal{A}_{1s} + \mathcal{A}_{2s}] \right), \quad \text{where}$$

$$\mathcal{A}_{1s} = -\frac{2}{\eta} \int_0^{\infty} dq \frac{f_s(q) - f_s(0)}{q} \operatorname{Im} \left[\left(1 + \frac{2k}{q} \right)^{2i\eta} \mathcal{F}_+ \left(\frac{4k^2}{q^2} - i0 \right) \right],$$

$$\mathcal{A}_{2s} = -\frac{2}{\eta} \int_0^{\infty} dq \frac{f_s(q) - f_s(2k)}{q-2k} \frac{q}{q+2k} \operatorname{Im} \frac{(q+2k)^{i\eta}}{(q-2k+i0)^{i\eta}}.$$

The main result

- A remark: approximation of $\delta(x)$ Dirac delta: a well known simpler similar case



- In our case, result of the calculation is

$$C_2(Q) = |\mathcal{N}|^2 \left(1 + f_s(2k) + \frac{\eta}{\pi} [\mathcal{A}_{1s} + \mathcal{A}_{2s}] \right), \quad \text{where}$$

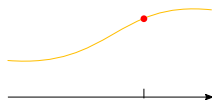
$$\mathcal{A}_{1s} = -\frac{2}{\eta} \int_0^{\infty} dq \frac{f_s(q) - f_s(0)}{q} \operatorname{Im} \left[\left(1 + \frac{2k}{q} \right)^{2i\eta} \mathcal{F}_+ \left(\frac{4k^2}{q^2} - i0 \right) \right],$$

$$\mathcal{A}_{2s} = -\frac{2}{\eta} \int_0^{\infty} dq \frac{f_s(q) - f_s(2k)}{q-2k} \frac{q}{q+2k} \operatorname{Im} \frac{(q+2k)^{i\eta}}{(q-2k+i0)^{i\eta}}.$$

- $\eta \rightarrow 0$: free $C_2^{(0)}(Q) = 1 + f_s(Q)$ recovered (NB: $Q = 2k$)

The main result

- A remark: approximation of $\delta(x)$ Dirac delta: a well known simpler similar case



- In our case, result of the calculation is

$$C_2(Q) = |\mathcal{N}|^2 \left(1 + f_s(2k) + \frac{\eta}{\pi} [\mathcal{A}_{1s} + \mathcal{A}_{2s}] \right), \quad \text{where}$$

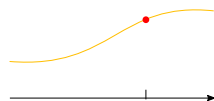
$$\mathcal{A}_{1s} = -\frac{2}{\eta} \int_0^{\infty} dq \frac{f_s(q) - f_s(0)}{q} \operatorname{Im} \left[\left(1 + \frac{2k}{q} \right)^{2i\eta} \mathcal{F}_+ \left(\frac{4k^2}{q^2} - i0 \right) \right],$$

$$\mathcal{A}_{2s} = -\frac{2}{\eta} \int_0^{\infty} dq \frac{f_s(q) - f_s(2k)}{q-2k} \frac{q}{q+2k} \operatorname{Im} \frac{(q+2k)^{i\eta}}{(q-2k+i0)^{i\eta}}.$$

- $\eta \rightarrow 0$: free $C_2^{(0)}(Q) = 1 + f_s(Q)$ recovered (NB: $Q = 2k$)
- $|\mathcal{N}|^2$ factor only: Gamow correction

The main result

- A remark: approximation of $\delta(x)$ Dirac delta: a well known simpler similar case



- In our case, result of the calculation is

$$C_2(Q) = |\mathcal{N}|^2 \left(1 + f_s(2k) + \frac{\eta}{\pi} [\mathcal{A}_{1s} + \mathcal{A}_{2s}] \right), \quad \text{where}$$

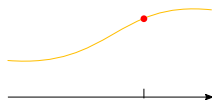
$$\mathcal{A}_{1s} = -\frac{2}{\eta} \int_0^{\infty} dq \frac{f_s(q) - f_s(0)}{q} \operatorname{Im} \left[\left(1 + \frac{2k}{q} \right)^{2i\eta} \mathcal{F}_+ \left(\frac{4k^2}{q^2} - i0 \right) \right],$$

$$\mathcal{A}_{2s} = -\frac{2}{\eta} \int_0^{\infty} dq \frac{f_s(q) - f_s(2k)}{q-2k} \frac{q}{q+2k} \operatorname{Im} \frac{(q+2k)^{i\eta}}{(q-2k+i0)^{i\eta}}.$$

- $\eta \rightarrow 0$: free $C_2^{(0)}(Q) = 1 + f_s(Q)$ recovered (NB: $Q = 2k$)
- $|\mathcal{N}|^2$ factor only: Gamow correction $\Rightarrow \mathcal{A}_{1s}, \mathcal{A}_{2s}$ „correct the Gamow correction”

The main result

- A remark: approximation of $\delta(x)$ Dirac delta: a well known simpler similar case



- In our case, result of the calculation is

$$C_2(Q) = |\mathcal{N}|^2 \left(1 + f_s(2k) + \frac{\eta}{\pi} [\mathcal{A}_{1s} + \mathcal{A}_{2s}] \right), \quad \text{where}$$

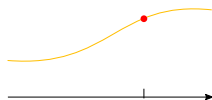
$$\mathcal{A}_{1s} = -\frac{2}{\eta} \int_0^{\infty} dq \frac{f_s(q) - f_s(0)}{q} \operatorname{Im} \left[\left(1 + \frac{2k}{q} \right)^{2i\eta} \mathcal{F}_+ \left(\frac{4k^2}{q^2} - i0 \right) \right],$$

$$\mathcal{A}_{2s} = -\frac{2}{\eta} \int_0^{\infty} dq \frac{f_s(q) - f_s(2k)}{q - 2k} \frac{q}{q + 2k} \operatorname{Im} \frac{(q + 2k)^{i\eta}}{(q - 2k + i0)^{i\eta}}.$$

- $\eta \rightarrow 0$: free $C_2^{(0)}(Q) = 1 + f_s(Q)$ recovered (NB: $Q = 2k$)
- $|\mathcal{N}|^2$ factor only: Gamow correction $\Rightarrow \mathcal{A}_{1s}, \mathcal{A}_{2s}$ „correct the Gamow correction”
- \mathcal{A}_{1s} and \mathcal{A}_{2s} : not proper integral transforms but well-defined *functionals* of $f_s(q)$

The main result

- A remark: approximation of $\delta(x)$ Dirac delta: a well known simpler similar case



- In our case, result of the calculation is

$$C_2(Q) = |\mathcal{N}|^2 \left(1 + f_s(2k) + \frac{\eta}{\pi} [\mathcal{A}_{1s} + \mathcal{A}_{2s}] \right), \quad \text{where}$$

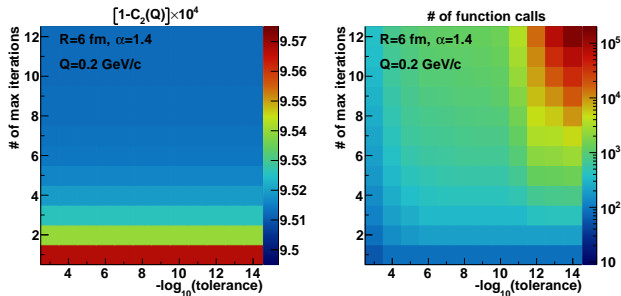
$$\mathcal{A}_{1s} = -\frac{2}{\eta} \int_0^{\infty} dq \frac{f_s(q) - f_s(0)}{q} \operatorname{Im} \left[\left(1 + \frac{2k}{q} \right)^{2i\eta} \mathcal{F}_+ \left(\frac{4k^2}{q^2} - i0 \right) \right],$$

$$\mathcal{A}_{2s} = -\frac{2}{\eta} \int_0^{\infty} dq \frac{f_s(q) - f_s(2k)}{q-2k} \frac{q}{q+2k} \operatorname{Im} \frac{(q+2k)^{i\eta}}{(q-2k+i0)^{i\eta}}.$$

- $\eta \rightarrow 0$: free $C_2^{(0)}(Q) = 1 + f_s(Q)$ recovered (NB: $Q = 2k$)
- $|\mathcal{N}|^2$ factor only: Gamow correction $\Rightarrow \mathcal{A}_{1s}, \mathcal{A}_{2s}$ „correct the Gamow correction”
- \mathcal{A}_{1s} and \mathcal{A}_{2s} : not proper integral transforms but well-defined *functionals* of $f_s(q)$
 - Care needed about branch cuts ($\pm i0$ terms) of $\mathcal{F}_+(x)$ and complex powers

Numerical implementation

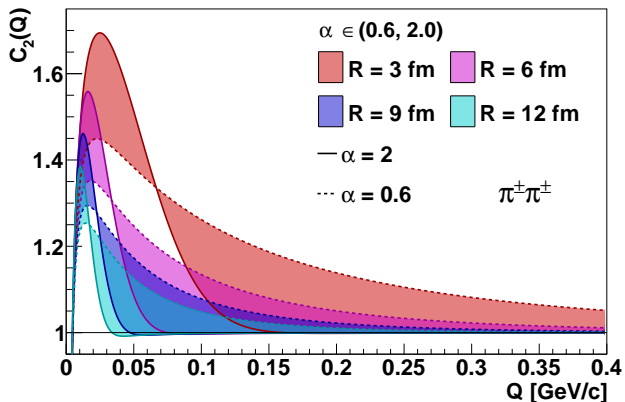
- Final numerical integrals needed: in $\mathcal{A}_{1\lambda_s}$ and $\mathcal{A}_{2\lambda_s}$
- Transform integral to $x \in [0, 1] \Rightarrow$ smooth, „beautiful” integrands
- Gauss-Kronrod quadrature (from C++ boost library) used:
 - Main parameters: # of max iterations & tolerance
 - Investigated; optimal value found:
few hundred integrand evaluations (instead of many 10000-s)



- Real-time calculation (during fit procedure) possible!
- Codes archived at: github.com/csanadm/CoulCorrLevyIntegral

Example calculations: illustrations

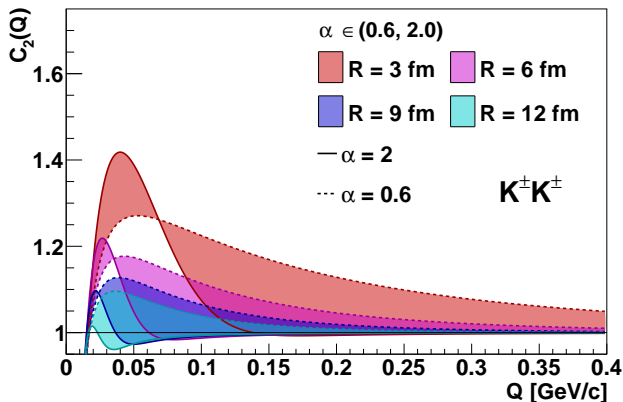
- For Lévy sources, for pion ($\pi^+\pi^+$, $\pi^-\pi^-$) pairs:



- most frequent target of HBT measurements
- Shaded region „swept” over by $C_2(Q)$ as α changes
- Apparent „nodes” disappear with increased zooming in

Example calculations: illustrations

- For Lévy sources, for kaon (K^+K^+ , K^-K^-) pairs:

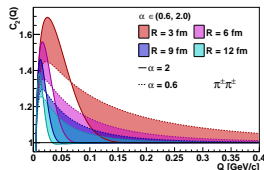


- Similarly to the case of pions; „nodes” are only apparent
- Coulomb effect stronger ($m_K > m_\pi$; η increases)
- Considerable interplay of experimentally measurable λ , R , α

Summary and outlook

- Efficient new method for Coulomb interacting HBT correlation function calculation
 - Calculations directly in momentum (Fourier) space
 - Distribution theory motivated careful mathematical methods invoked
 - Cross-checked with previous direct calculations
 - Numerical implementation done, ready for use in data analysis
- As of now, implementation only for spherically symmetric sources
 - Prospective generalization I: go beyond spherical symmetry
This is where efficiency becomes crucial. . .
 - Prospective generalization II: short-range final state strong interactions
Usual treatment: only s -wave (1 parameter: strong scattering length f_0)
For a direct calculation for Lévy sources, see: Kincses, MN, Csanád, *PRC* 102, 064912 (2020)
 - Prospective generalization (in fact, simplification) for non-identical particle correlations: only $\mathcal{D}_{1\lambda s}$ (ie. \mathcal{A}_{1s}) term needed

New exact analytic formulas for QM Coulomb problem! 😊



*Thank you for
your attention!*

