THE ROYAL SOCIETY

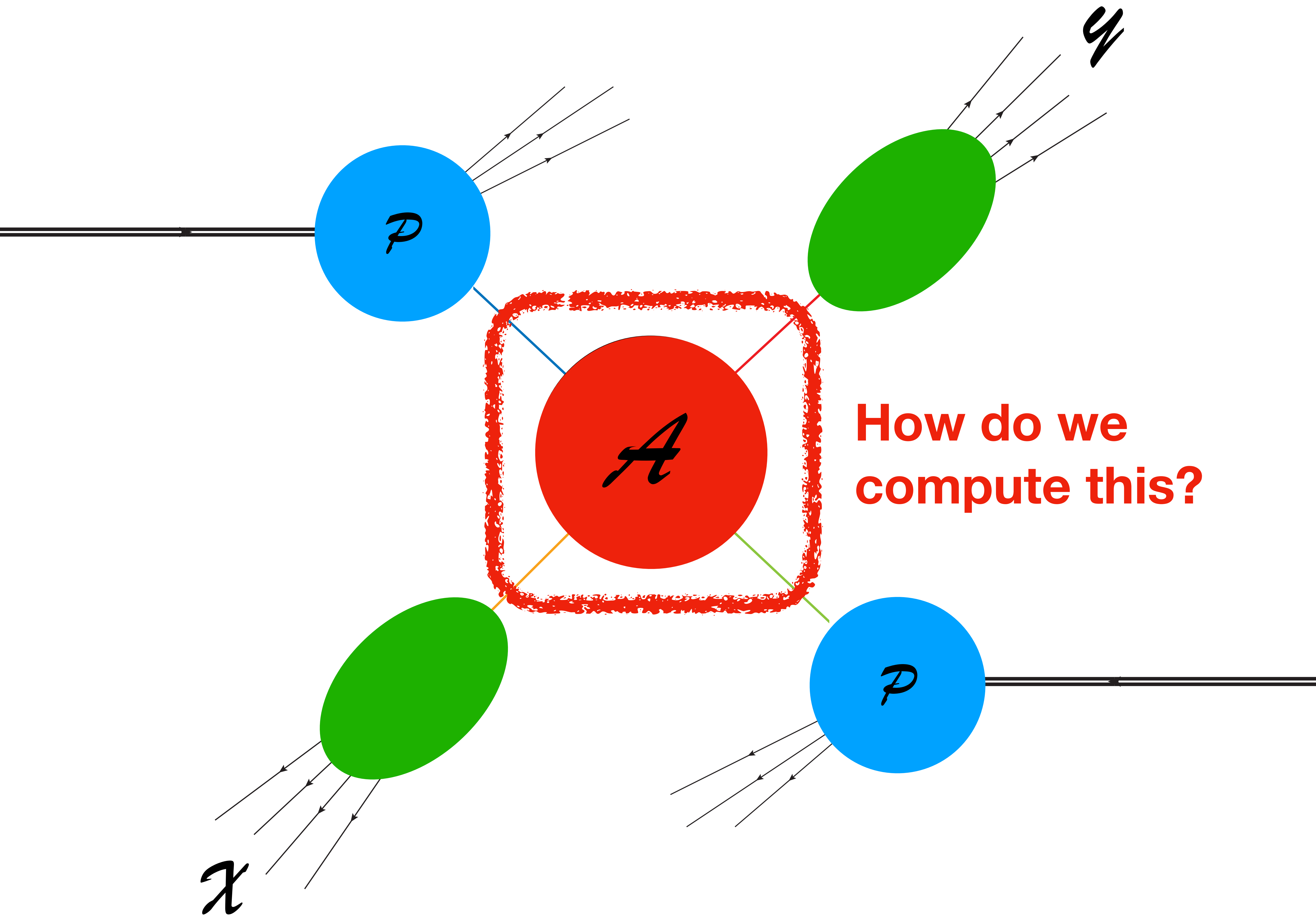
UNIVERSITY OF OXFORD

Scattering Amplitudes

Mathematical Methods and Novel Results

Giulio Gambuti - 16/12/2022 - Università di Genova

Based on: 2108.00055, 2112.11097, 2207.03503, 2212.?????



𝒫
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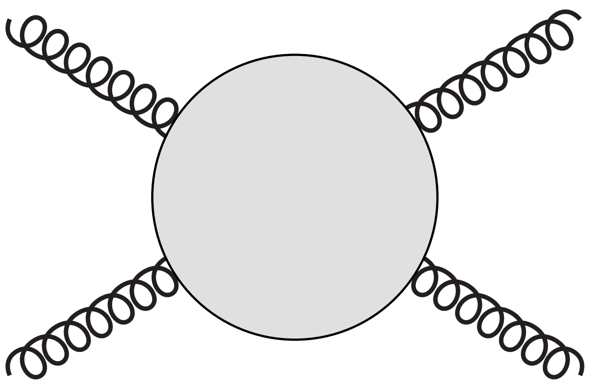
How do we compute this?
𝒫
𝒳

Outline

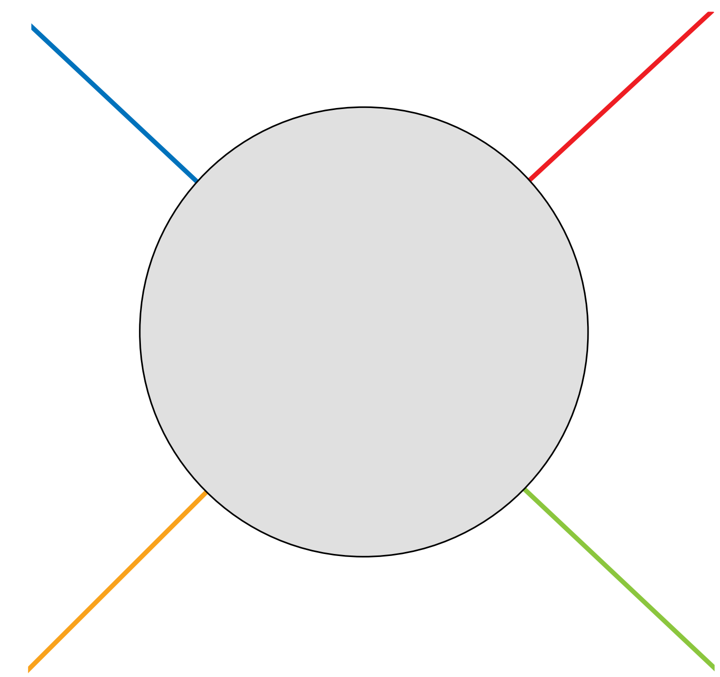


- **Why Amplitudes?**
- **How to compute them..**
- **What can we learn from them?**
  - **Infrared physics**
  - **Regge limit**

Small parameter(s)

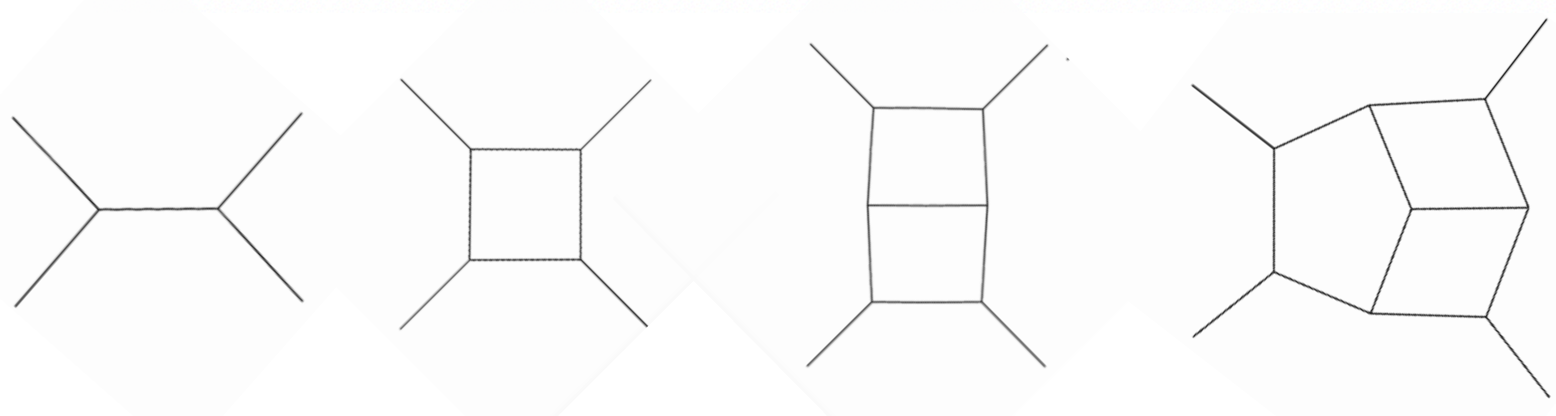
$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\text{int}}$$

Feynman Rules

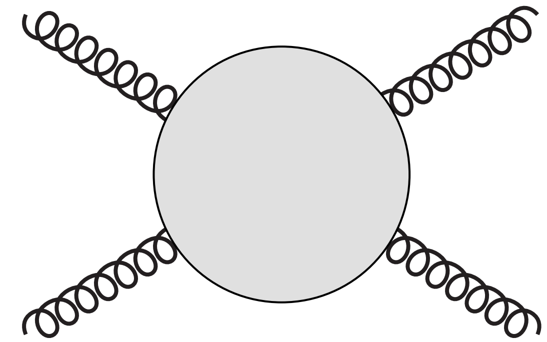


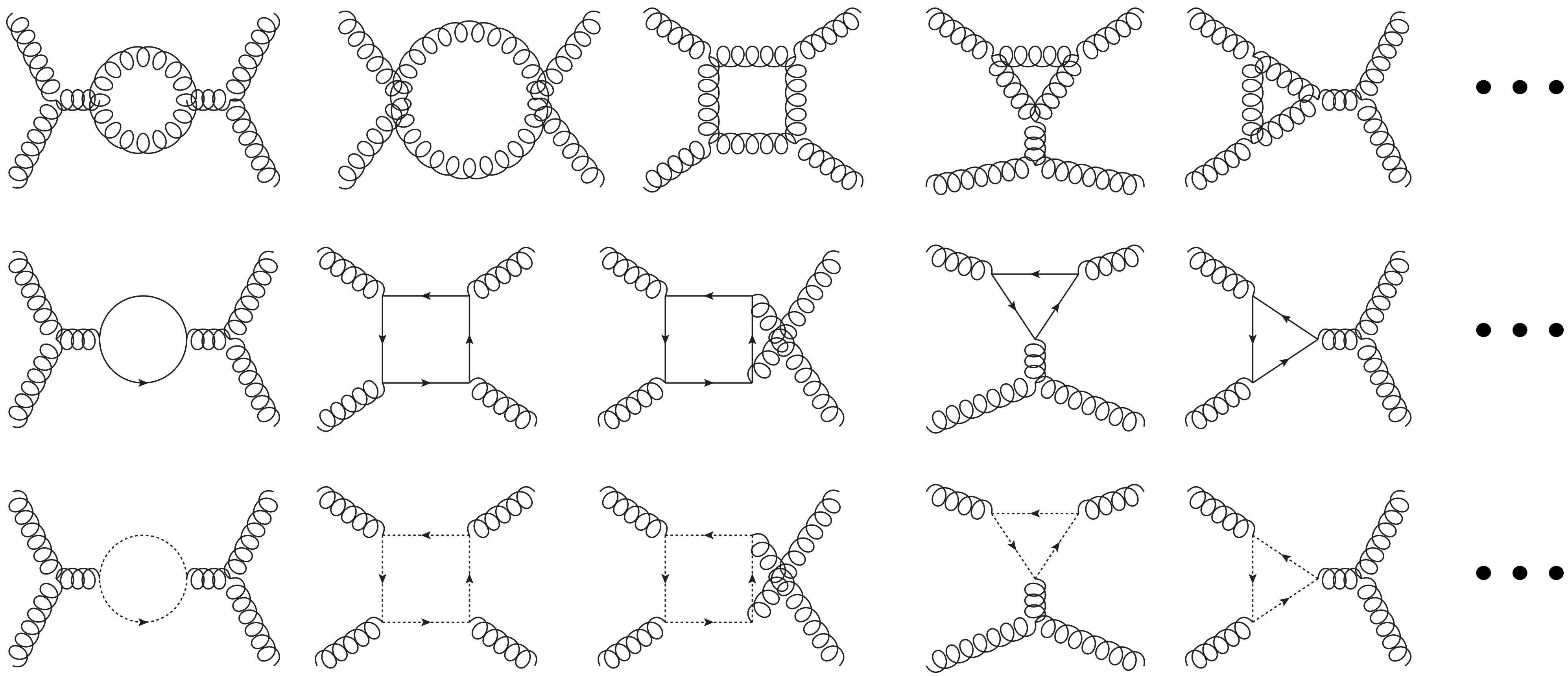
$$\mathbf{A}(|i\rangle \to |f\rangle) =$$

$$\mathbf{A}^{(0)} + \alpha \mathbf{A}^{(1)} + \alpha^2 \mathbf{A}^{(2)} + \alpha^3 \mathbf{A}^{(3)} + \ldots$$



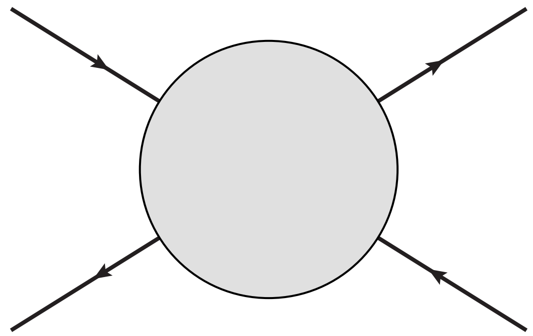
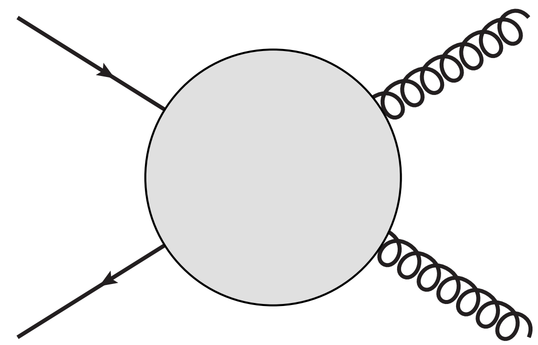
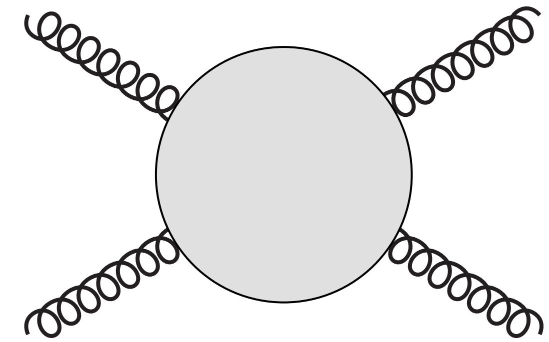
$$A_{gg \to gg}$$





81 diagrams!

Already in trouble

# of **QCD** Feynman diagrams			
tree level	1	3	4
1-loop	9	30	81
2-loop	158	595	1771
3-loop	3584	14971	48723 !?!

Dealing with the complexity:

Numerically

- faster
- manageable complexity

- subtraction of infinities
- has to be repeated

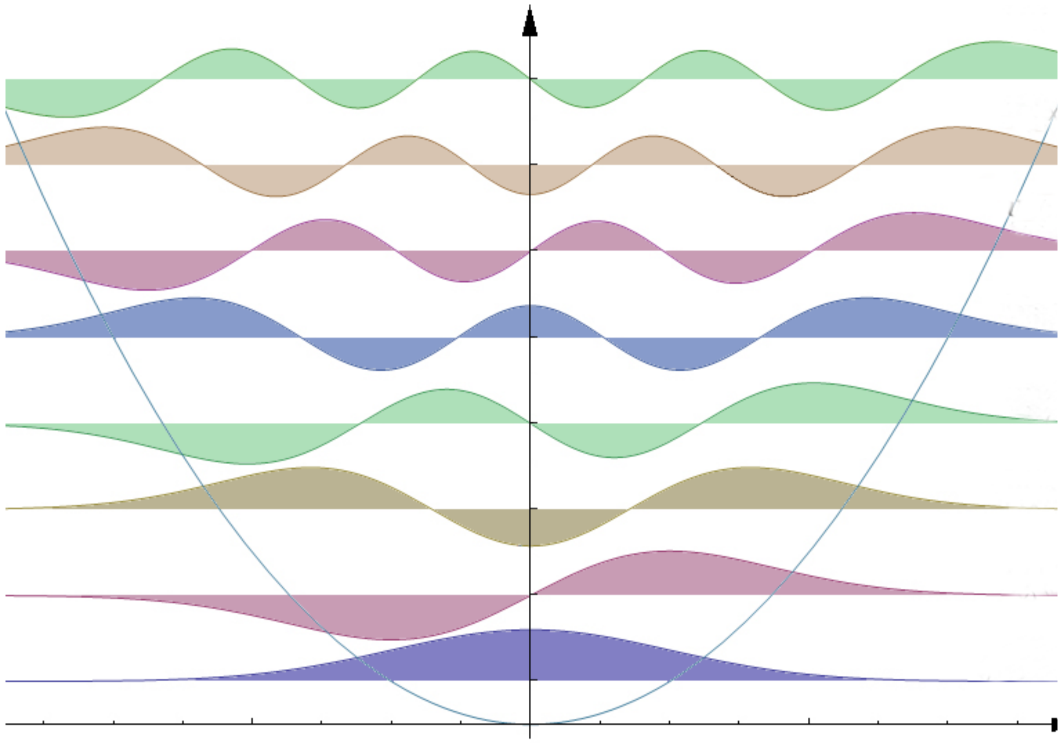
Analytically

- once and for all
- full information
- infinities are regulated

- more complex structures

Harmonic Oscillator

Quantum Mechanics



N=4 super Yang-Mills

Scattering Amplitudes

N=4 super Yang-Mills

Non-Abelian Gauge theory	gluon	4 fermions	3 scalars

UV finite

Conformal Invariant

Dual Conformal Invariant

Supersymmetric Ward Identities

Generalised Unitarity at all orders

Real World (QCD)

Non-Abelian Gauge theory	gluon	5 fermions	NO scalars

?????????

“Easy” → “Hard”

N=4 super Yang-Mills

↓

N=2 super Yang-Mills

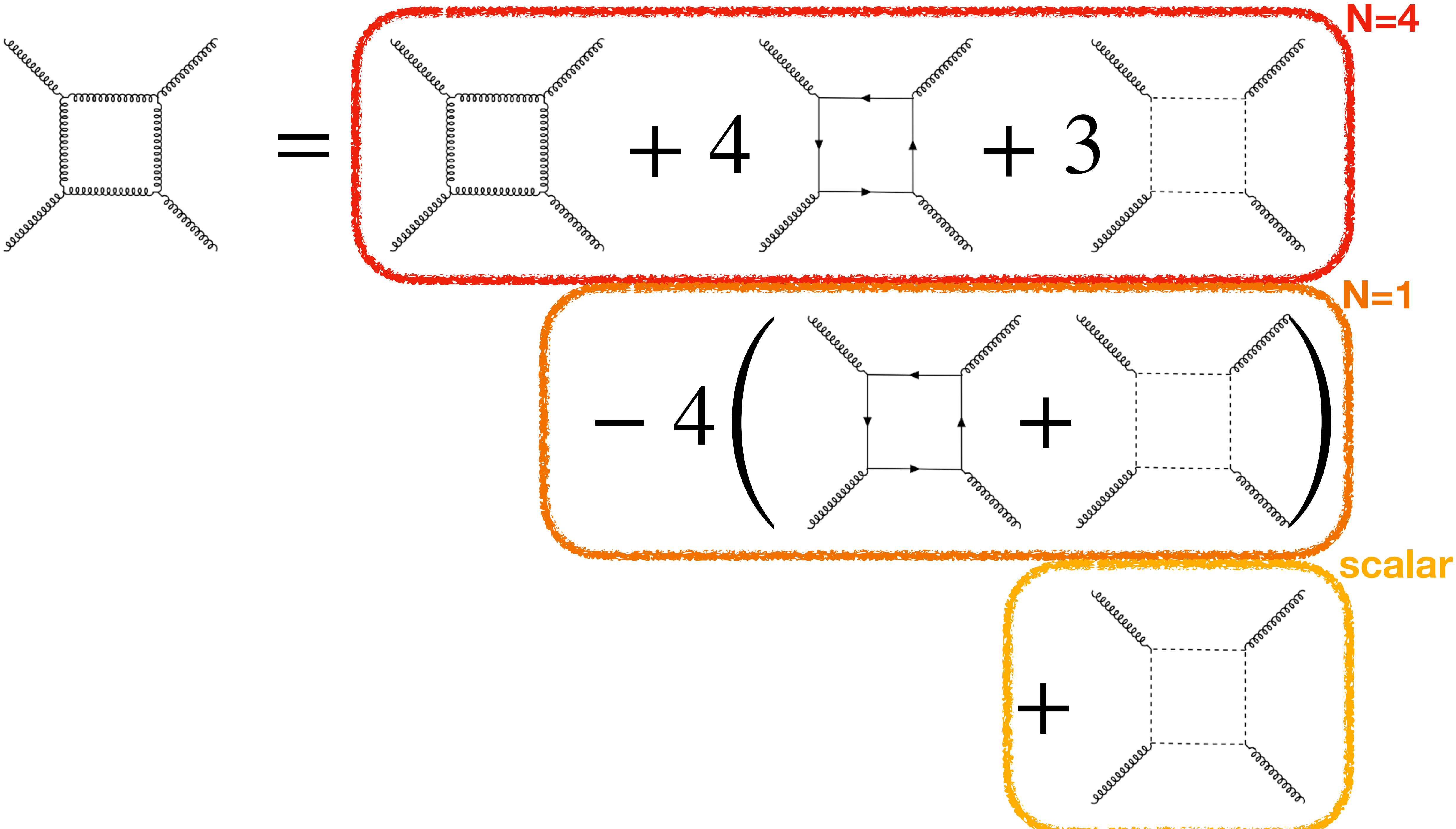
↓

N=1 super Yang-Mills

↓

Real World (QCD)

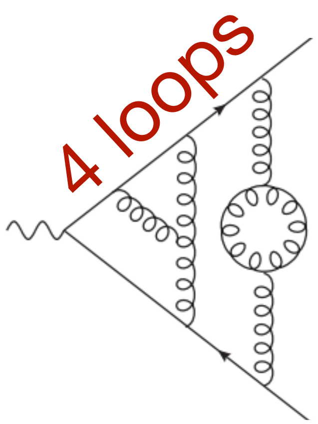
$$A_g = A_{N=4} - 4\,A_{N=1} + A_\phi$$



- Amplitudes ➜ Phenomenology and Formal QFT
- There is a “simple” model: N=4 sYM
- Can it help in QCD?
- We need analytic data !!

Massless State of the Art

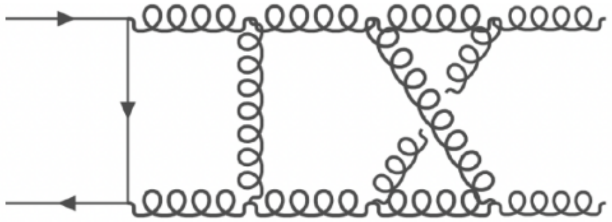
3-point function



Lee, von Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser: 2202.04660(PRL)

4-point function

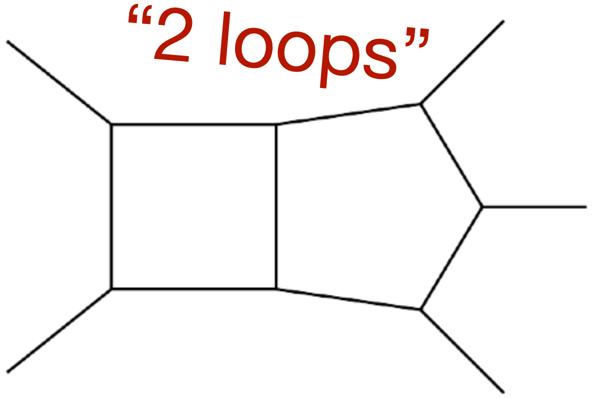
3 loops



- Caola, von Manteuffel, Tancredi: 2011.13946(PRL)
- Bargiela, Caola, von Manteuffel, Tancredi: 2111.13595(JHEP)
- Chakraborty, Caola, **GG**, Tancredi, von Manteuffel: 2108.00055(JHEP), 2207.03503(JHEP), 2112.11097(PRL)
- Bargiela, Chakraborty, **GG**: (2212.?????)

5-point function

“2 loops”



- Agrawal, Buccioni, von Manteuffel, Tancredi: 2105.04585(PRL)
- Badger, Gehrmann, Heinrich, Henn: 1905.03733(PRL)
- Abreu, Dormans, Cordero, Ita, Page: 1812.04586(PRL)

6-point function

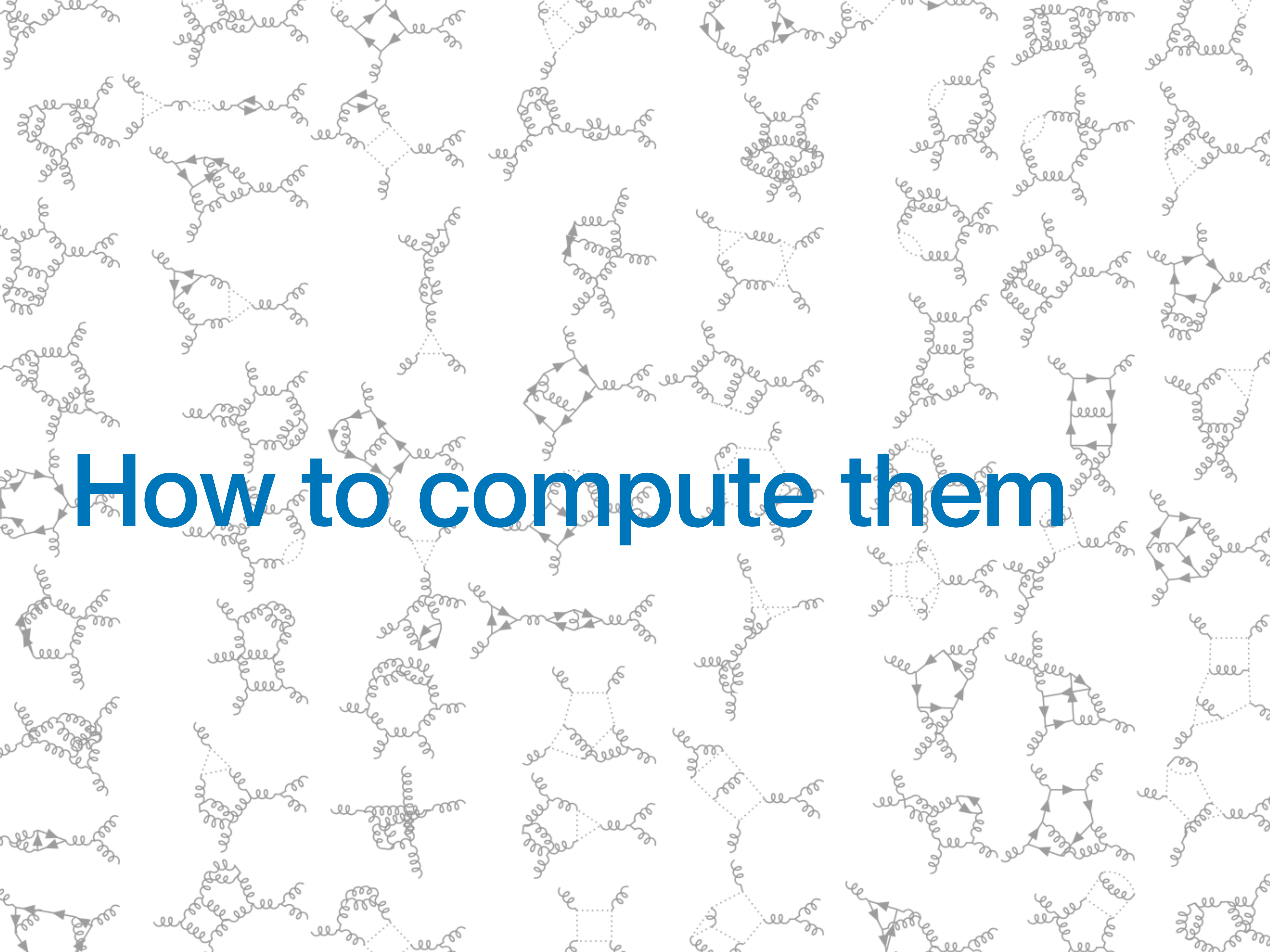
1 loop …

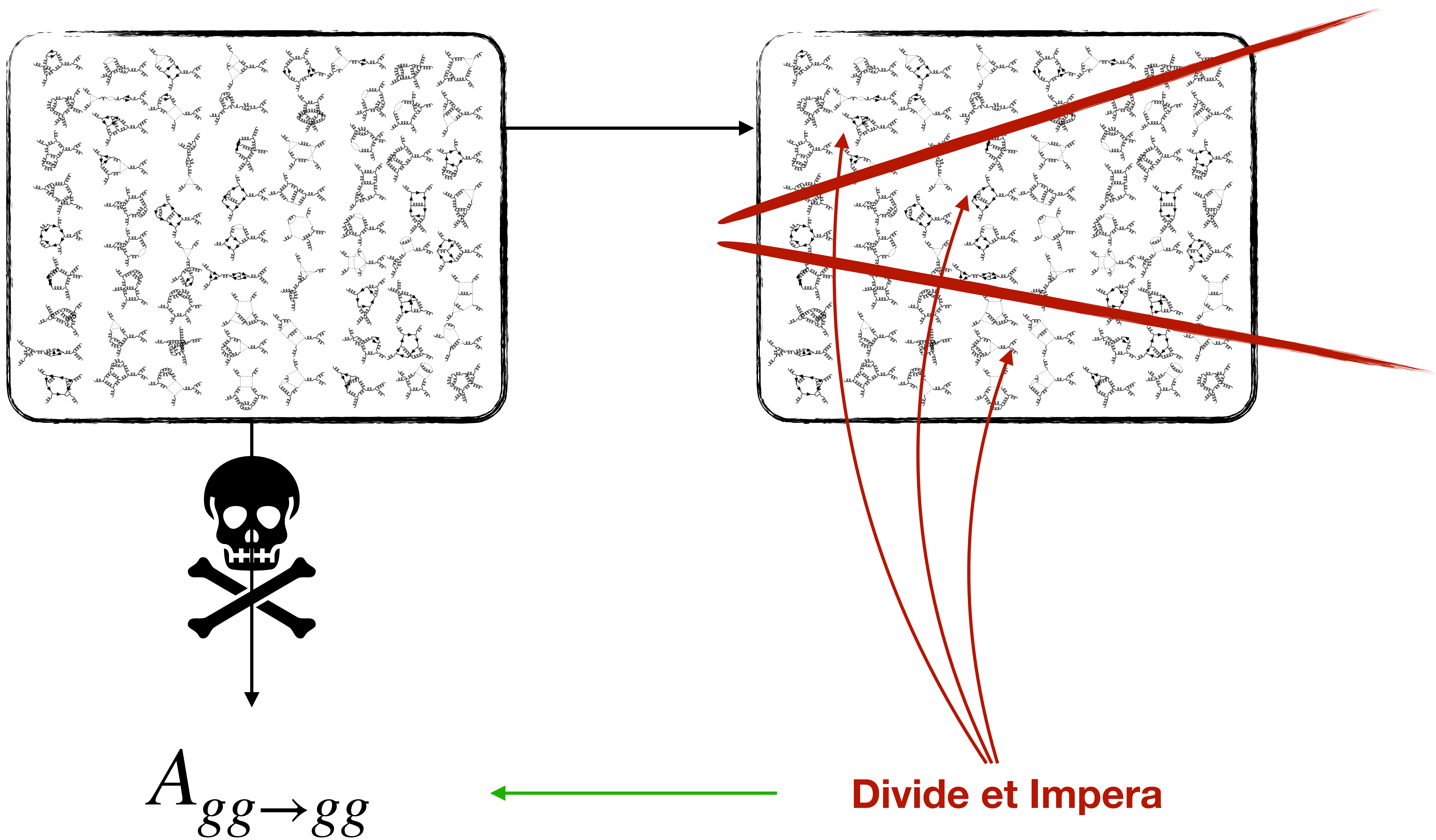
Openloops 2: 1907.13071(EPJC)

…

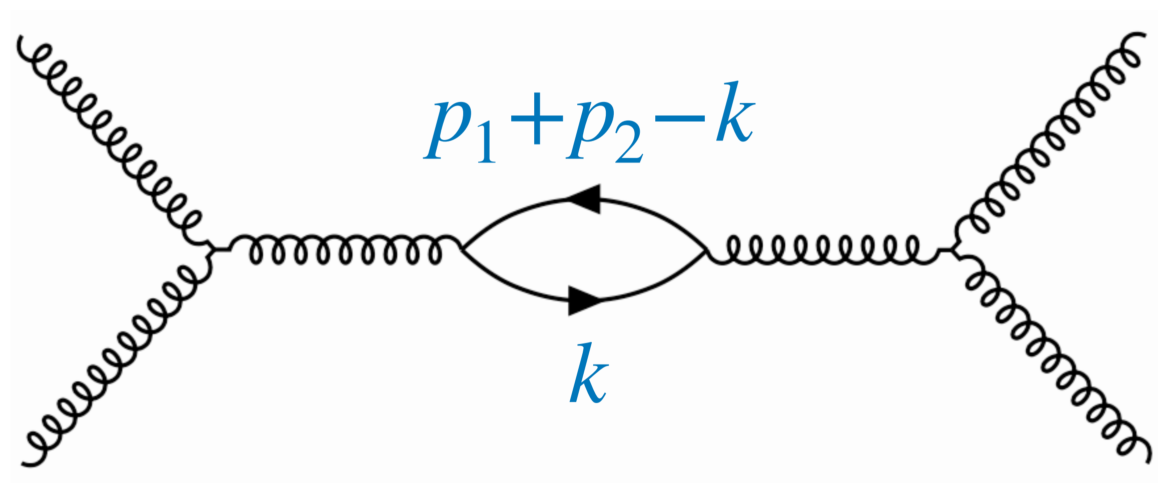
Ellis, Giele, Zanderighi: 0602185(JHEP)

How to compute them

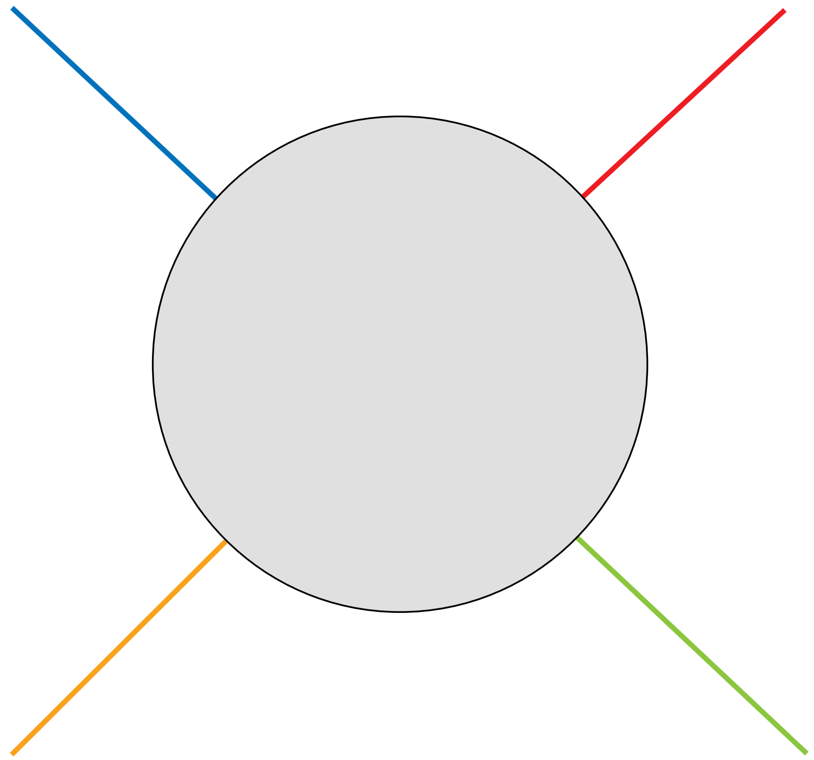




$A_{gg \to gg}$
Divide et Impera

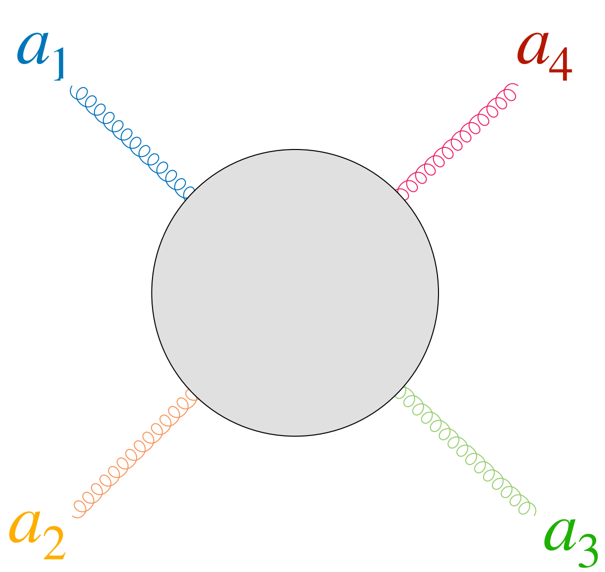


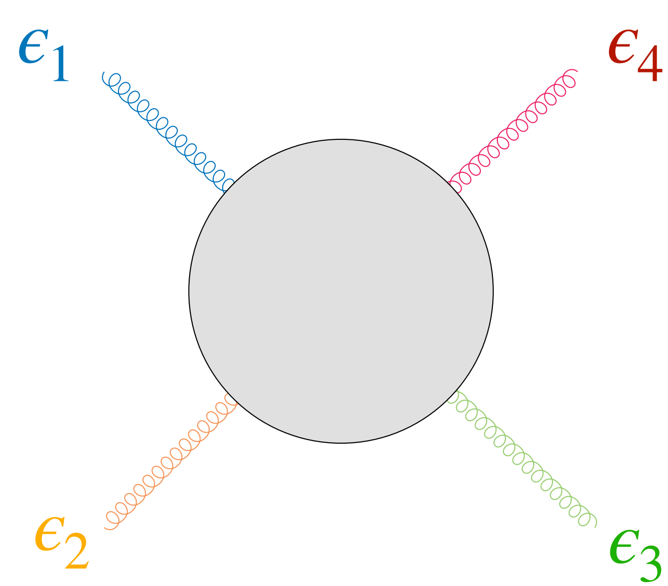
$$f^{a_1 b a_2} \, T^b_{ij} T^c_{ji} \, f^{a_3 a_4 c} \quad \epsilon^{\mu_1}_{h_1} \, \epsilon^{\mu_2}_{h_2} \, \epsilon^{\mu_3}_{h_3} \, \epsilon^{\mu_4}_{h_4} \quad \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathcal{N}_{\mu_1 \mu_2 \mu_3 \mu_4}}{D_1 D_2 \ldots D_M}$$

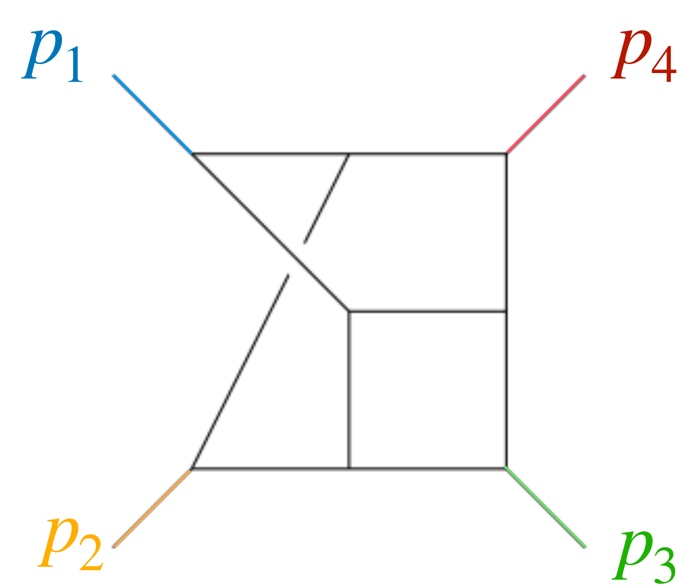


$= \sum$ Feynman Diagrams

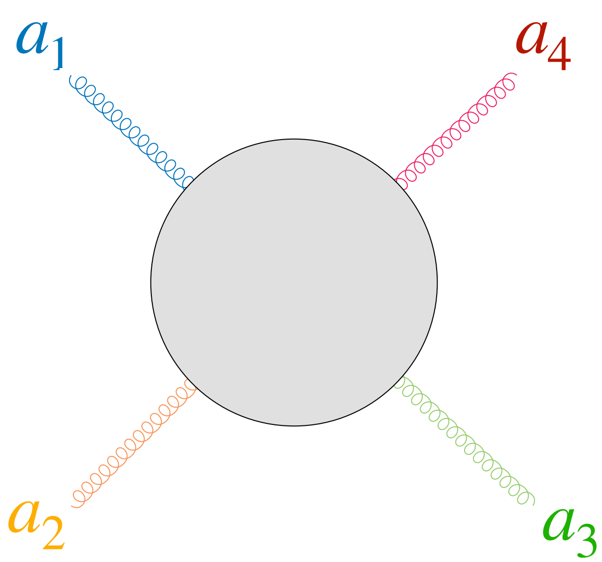
Colour **Spin** **Kinematics**





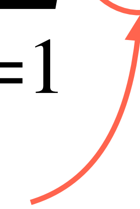


Colour

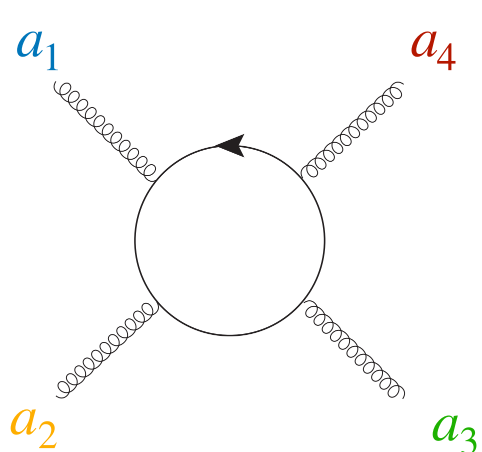


Colour Decomposition

$$A^{a_1a_2a_3a_4} = \sum_{c=1}^{6} A_c \, \mathcal{C}_c^{a_1a_2a_3a_4}$$

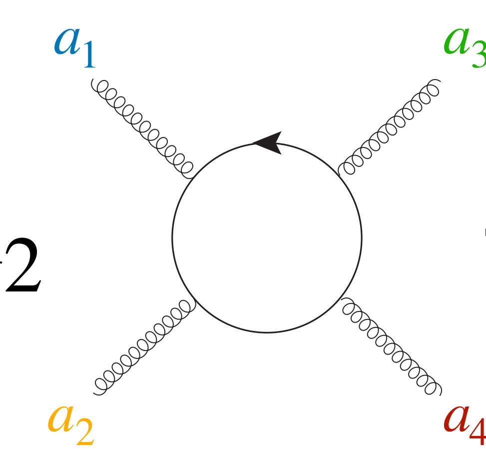


$= A_1$

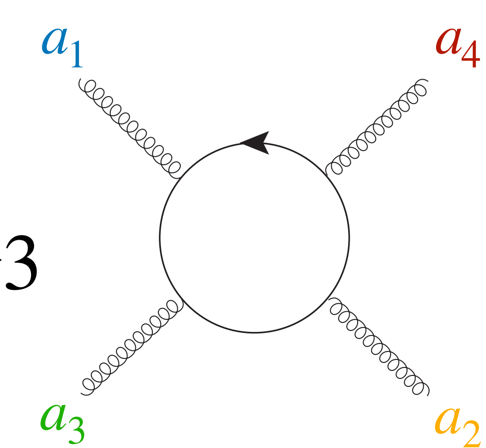


$Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_3}\mathbf{T}^{a_4})$

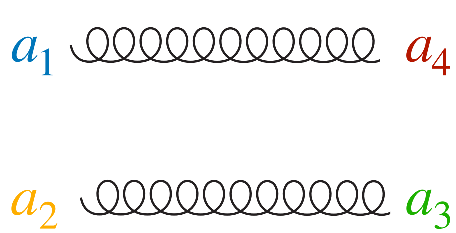
$+ A_2$



$+ A_3$

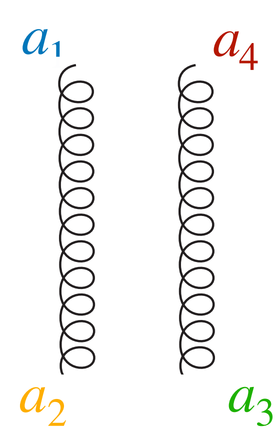


$+ A_4$

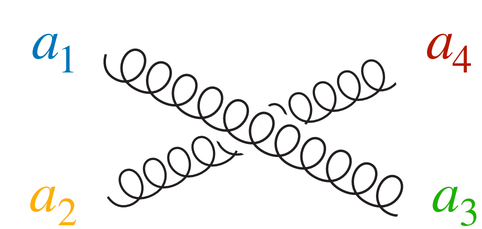


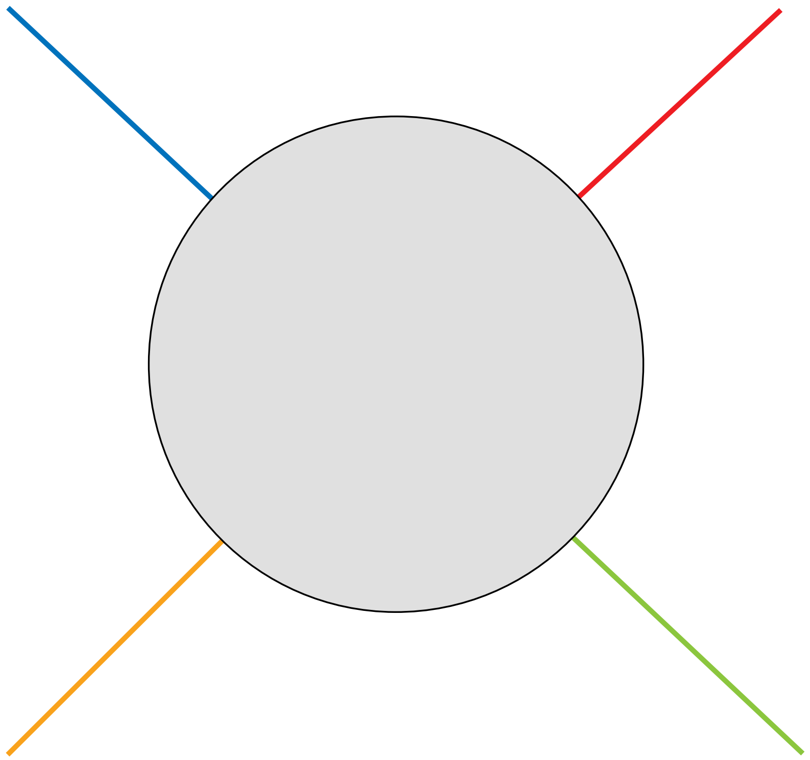
$\delta^{a_1a_4}\delta^{a_2a_3}$

$+ A_5$



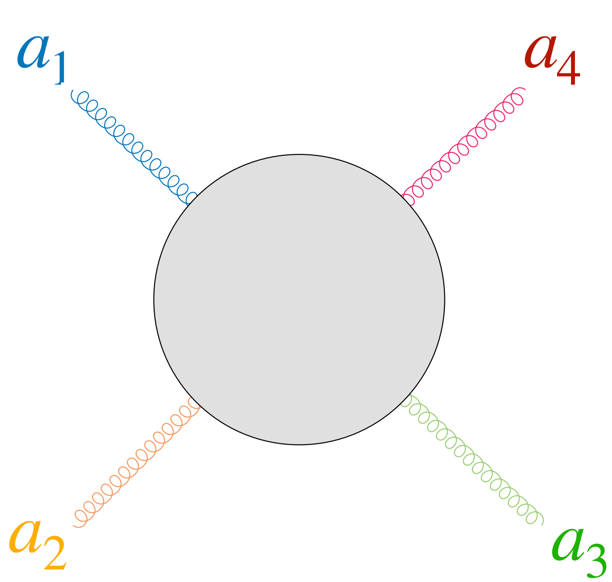
$+ A_6$



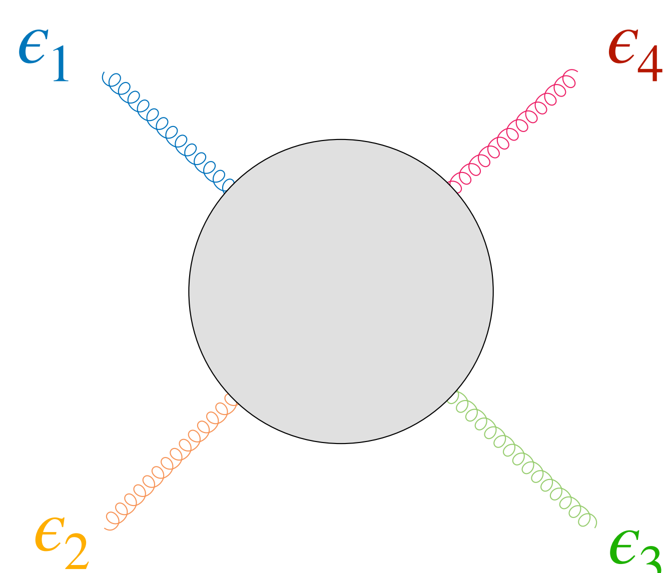


$$A^{a_1a_2a_3a_4} = \sum_{c=1}^{6} A_c \, \mathcal{C}_c^{a_1a_2a_3a_4}$$

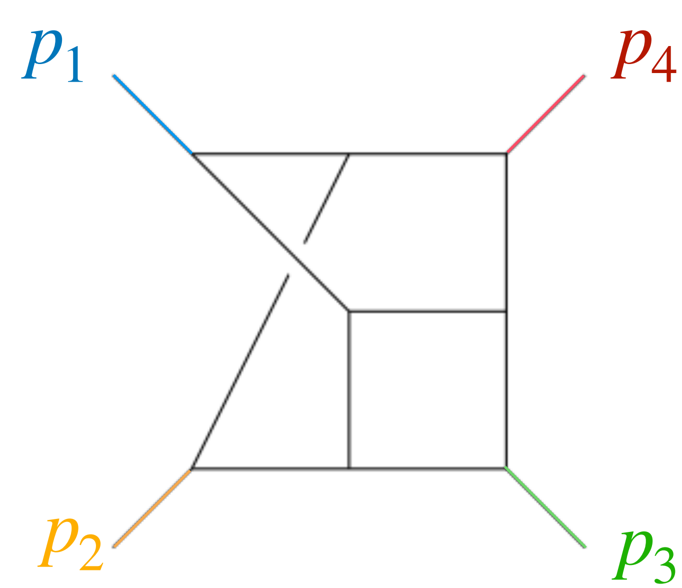
Colour



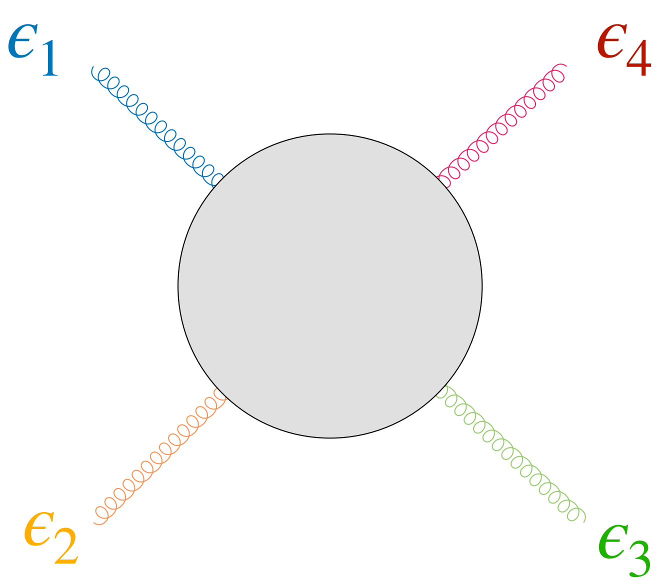
Spin



Kinematics



Spin



$$A_c = A_c^{\mu_1\mu_2\mu_3\mu_4} \; \epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

$$= \sum_j F_c^j \, T_j^{\mu_1\mu_2\mu_3\mu_4} \; \epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

Form Factors

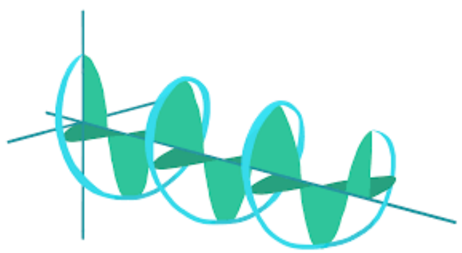
Explicitly…

$$T_j^{\mu_1\mu_2\mu_3\mu_4}\ \epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

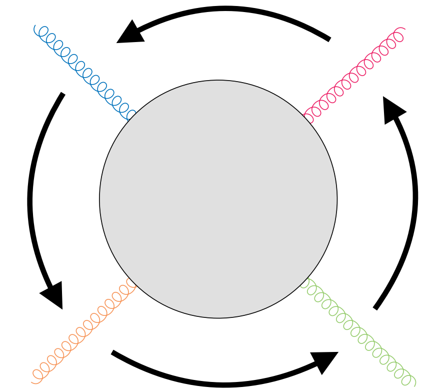
$$p_1 + p_2 + p_3 + p_4 = 0$$

$p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_3^{\mu_4}$	$p_3^{\mu_1}p_3^{\mu_2}g^{\mu_3\mu_4}$	$g^{\mu_1\mu_2}g^{\mu_3\mu_4}$
$p_2^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$	$g^{\mu_1\mu_2}p_1^{\mu_3}p_1^{\mu_4}$	$g^{\mu_1\mu_3}g^{\mu_2\mu_4}$
$p_3^{\mu_1}p_1^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$	$g^{\mu_1\mu_3}p_1^{\mu_2}p_2^{\mu_4}$	$g^{\mu_1\mu_4}g^{\mu_2\mu_3}$
$p_2^{\mu_1}p_1^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$	⋮ **54**	**3**
$p_3^{\mu_1}p_2^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$		
⋮ **81**		

138



$$T_j^{\mu_1\mu_2\mu_3\mu_4}\ \epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$

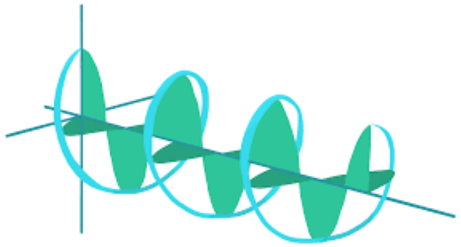


Transversality: $\epsilon \cdot p = 0$

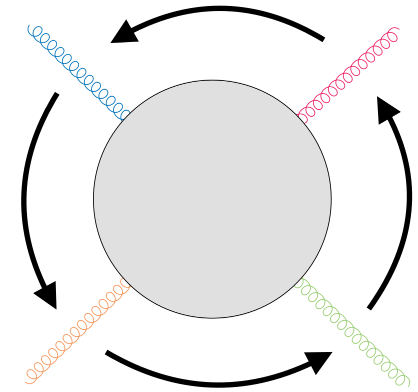
Gauge choice: $\epsilon_i \cdot p_{i+1} = 0$

~~$p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_3^{\mu_4}$~~	~~$p_3^{\mu_1}p_3^{\mu_2}g^{\mu_3\mu_4}$~~	$g^{\mu_1\mu_2}g^{\mu_3\mu_4}$
~~$p_2^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$~~	~~$g^{\mu_1\mu_2}p_1^{\mu_3}p_1^{\mu_4}$~~	$g^{\mu_1\mu_3}g^{\mu_2\mu_4}$
$p_3^{\mu_1}p_1^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$	$g^{\mu_1\mu_3}p_1^{\mu_2}p_2^{\mu_4}$	$g^{\mu_1\mu_4}g^{\mu_2\mu_3}$
~~$p_2^{\mu_1}p_1^{\mu_2}p_3^{\mu_3}p_2^{\mu_4}$~~	⋮ **54**	**3**
~~$p_3^{\mu_1}p_2^{\mu_2}p_1^{\mu_3}p_2^{\mu_4}$~~		
⋮ **81**		

~~**138**~~ **10**



$$T_j^{\mu_1\mu_2\mu_3\mu_4} \; \epsilon_{\mu_1}\epsilon_{\mu_2}\epsilon_{\mu_3}\epsilon_{\mu_4}$$



Transversality: $\epsilon \cdot p = 0$ **Gauge choice:** $\epsilon_i \cdot p_{i+1} = 0$

$$p_3^{\mu_1} p_1^{\mu_2} p_1^{\mu_3} p_2^{\mu_4}$$

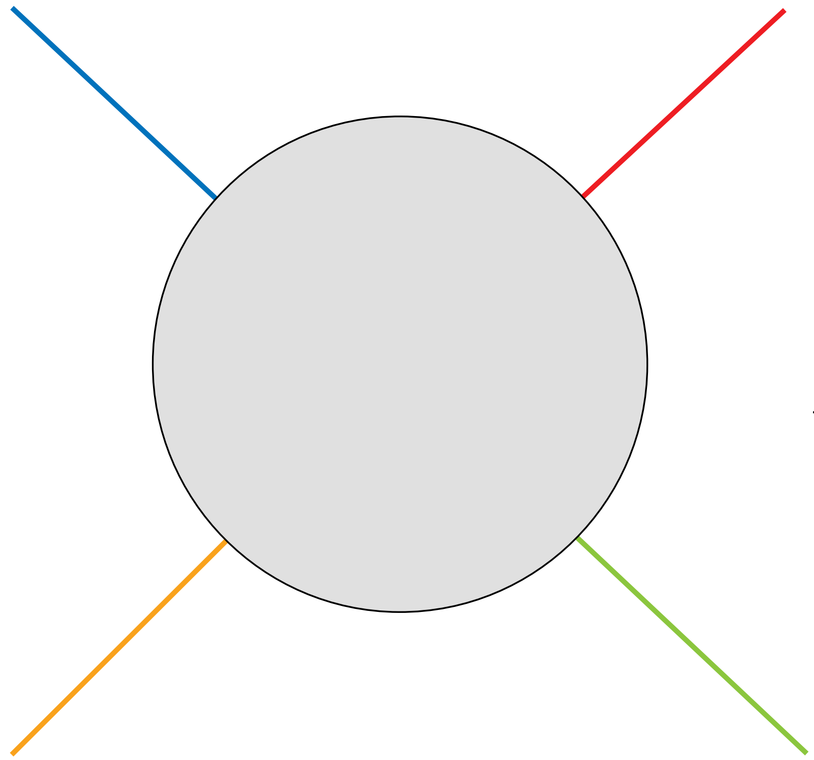
$$p_3^{\mu_1} p_1^{\mu_2} g^{\mu_3\mu_4} \qquad p_3^{\mu_1} p_2^{\mu_4} g^{\mu_2\mu_3} \qquad p_1^{\mu_2} p_1^{\mu_3} g^{\mu_1\mu_4}$$

$$p_3^{\mu_1} p_1^{\mu_3} g^{\mu_2\mu_4} \qquad p_1^{\mu_2} p_2^{\mu_4} g^{\mu_1\mu_3} \qquad p_1^{\mu_3} p_3^{\mu_4} g^{\mu_1\mu_2}$$

$$g^{\mu_1\mu_2} g^{\mu_3\mu_4} + g^{\mu_1\mu_3} g^{\mu_2\mu_4} + g^{\mu_1\mu_4} g^{\mu_2\mu_3}$$

8 tensors ➜ 8 helicity configurations!

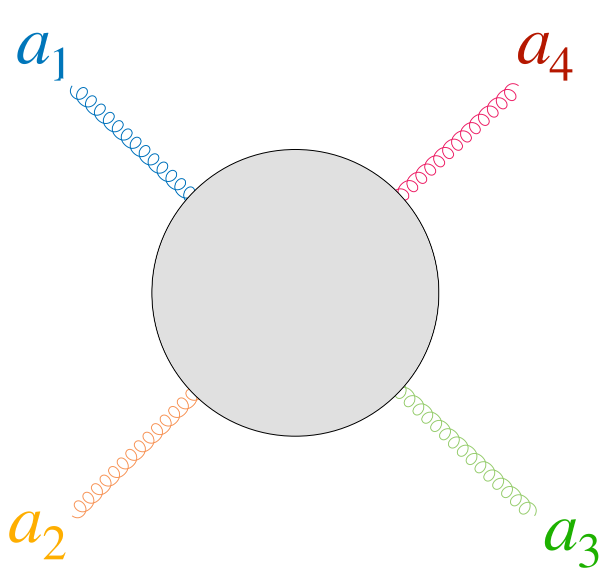
Tancredi, Peraro: 1906.03298, 2012.00820



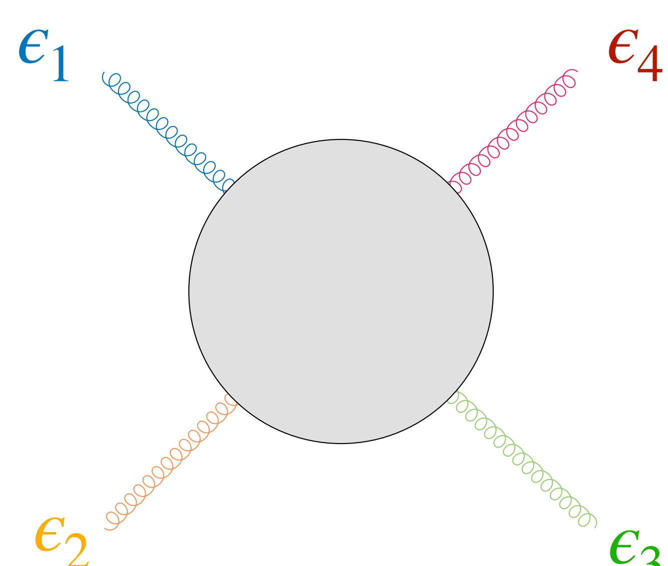
$$A_{\vec{h}}^{a_1 a_2 a_3 a_4} = \sum_{c,i} F_c^i \; T_i^{\vec{h}} \; \mathscr{C}_c^{a_1 a_2 a_3 a_4}$$

Non-perturbative !!

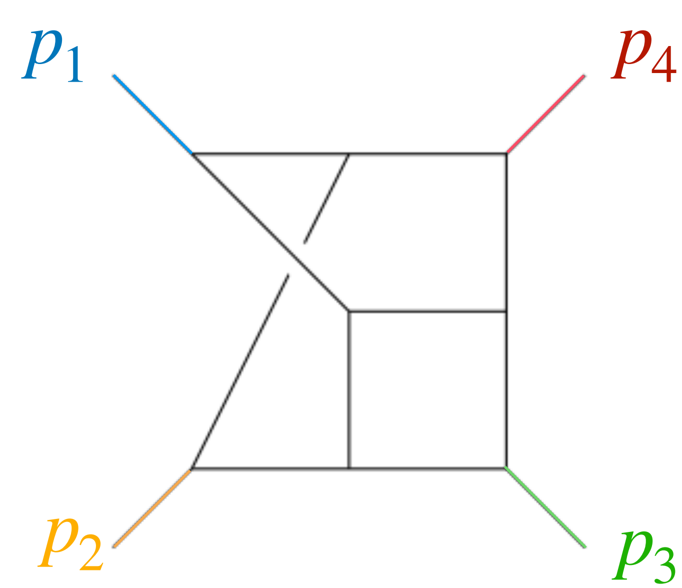
Colour



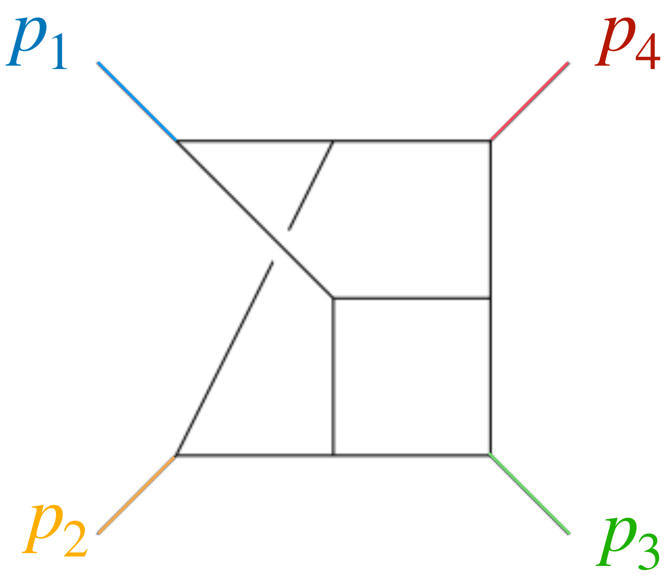
Spin



Kinematics



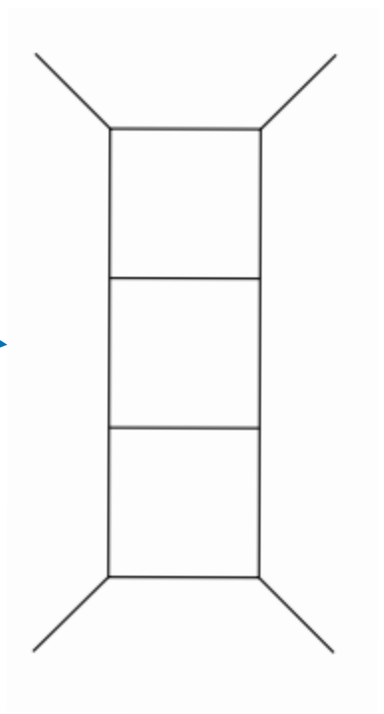
Kinematics



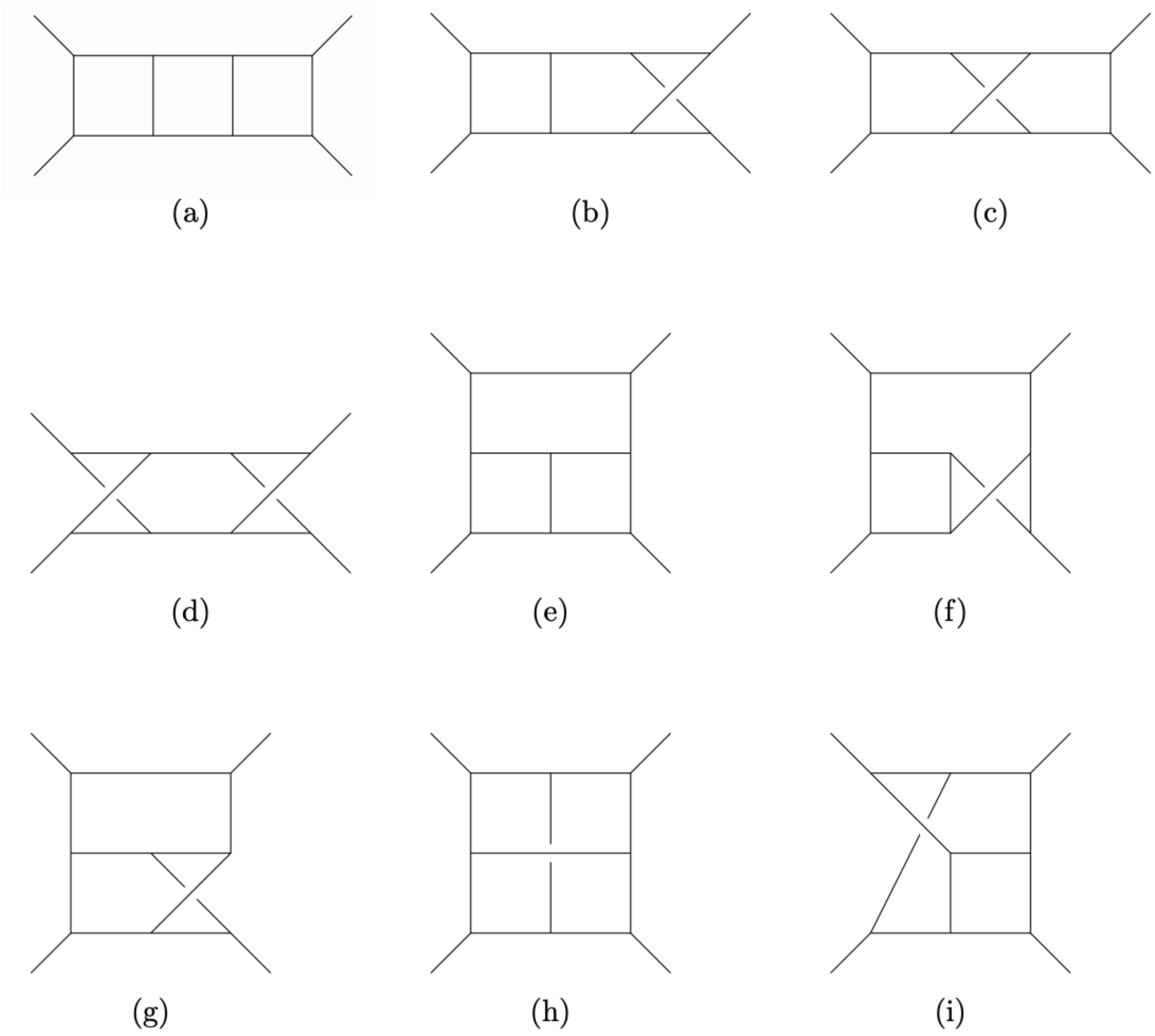
$$A^{a_1 a_2 a_3 a_4}_{\vec{h}} = \sum_{c,i} F^i_c \, T^{\vec{h}}_i \, \mathscr{C}^{a_1 a_2 a_3 a_4}_c$$

$$F^i_c \sim \int \frac{\mathrm{d}^d\{k_j\}}{(2\pi)^{dl}} \frac{\mathcal{N}(\{p\},\{k\})}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}}$$

$$(k_1 \cdot p_1)(k_2 \cdot k_3) \; + \; (k_1 \cdot p_2)(k_3 \cdot p_3) \; + \; \ldots$$



10 million integrals



Henn, Mistlberger, V.A. Smirnov, Wasser: **2002.09492**

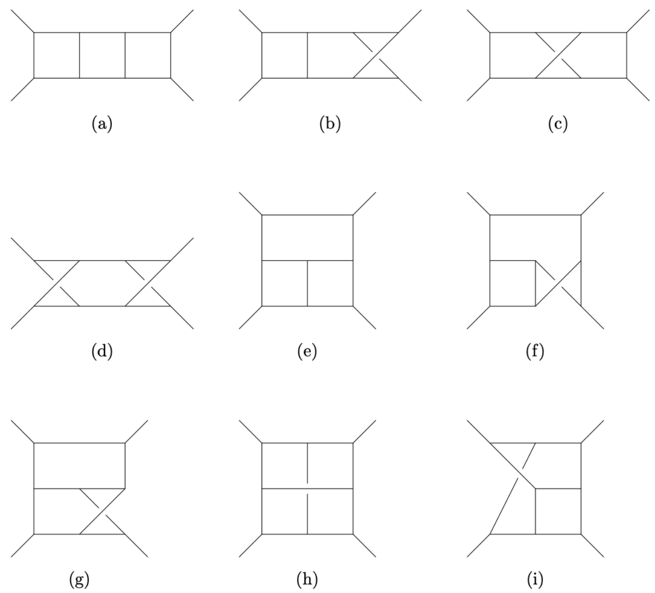
IBP reduction

$$\int_{\mathbb{R}^d} \mathrm{d}^d k \;\; \partial_\mu f^\mu(k) \;=\; 0 \qquad \Longrightarrow$$

Feynman Integrals
form a *finite dimensional* *vector space*

Master Integrals

$$I = \sum_{i \,\in\, basis} c_i M_i$$



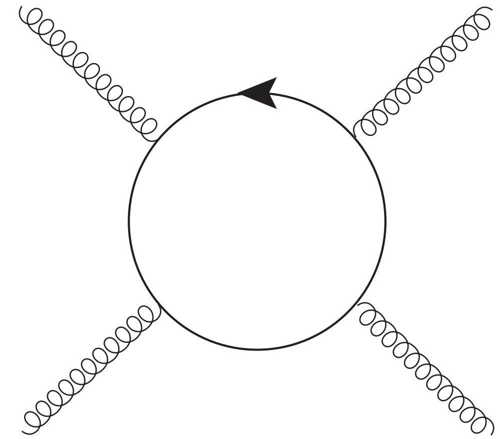
~ 500

(major bottleneck…)

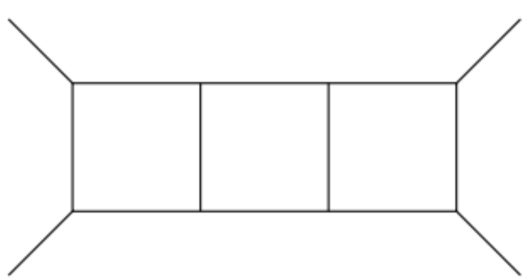
- **Algebraic geometry**
- **Physics**

computed with the
Differential Equations
method..

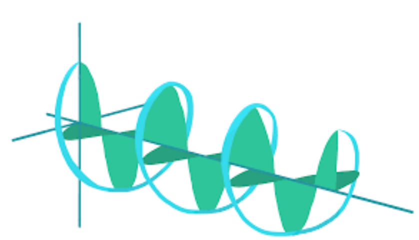
$$\frac{(p_1 \cdot p_2)(p_1 \cdot p_3) + (d-4)(p_2 \cdot p_3)(p_1 \cdot p_4)}{(p_2 \cdot p_4)}$$



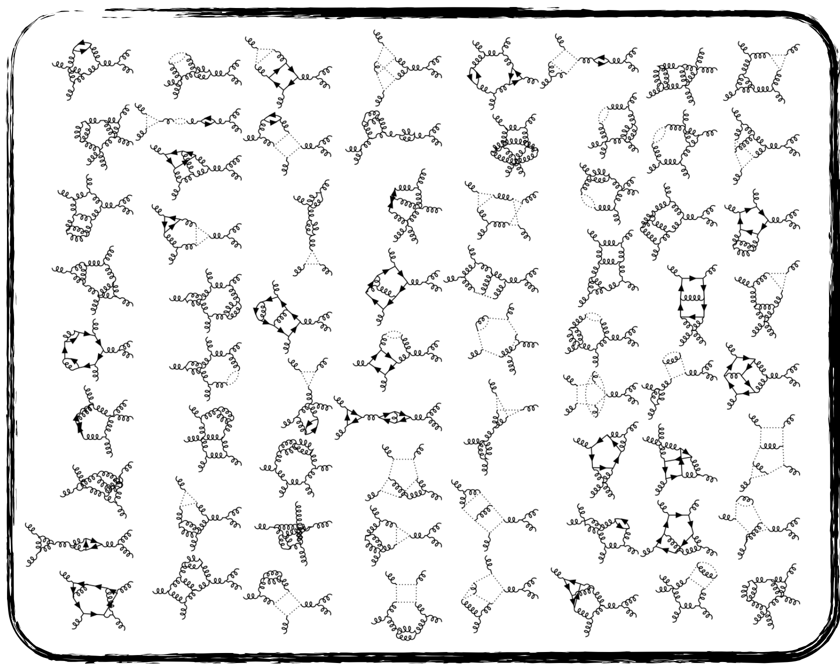
$$A_{\vec{h}}^{a_1 a_2 a_3 a_4} = \sum_{c,i,j} R_c^{i,j} \, M_j \, T_i^{\vec{h}} \, \mathscr{C}_c^{a_1 a_2 a_3 a_4}$$

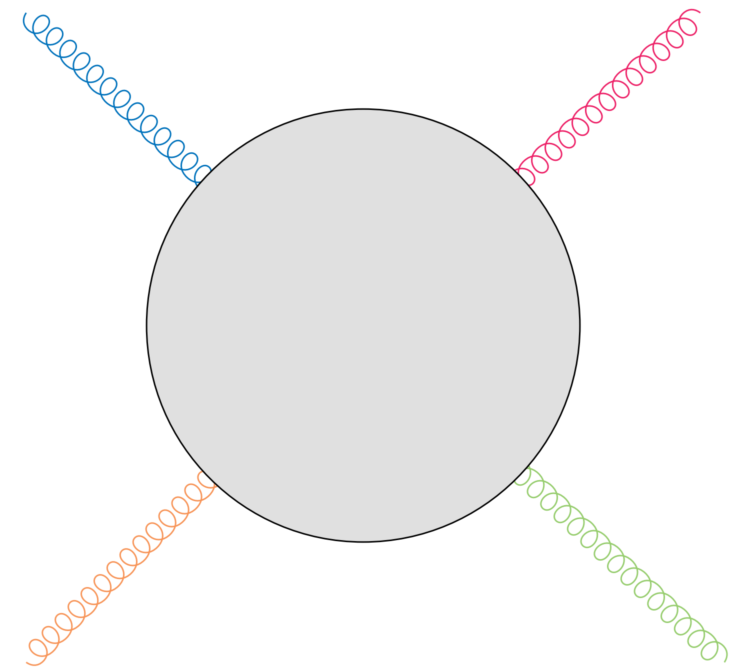


$$(p_3 \cdot \epsilon_1)\,(p_1 \cdot \epsilon_2)\,\epsilon_3 \cdot \epsilon_4$$



Infrared Structure





Renormalised Helicity Amplitudes

$$\mathbf{H}_{\text{ren}}$$

Renormalisation ←

Bare Helicity Amplitudes

$$\mathbf{H}^{a_1a_2a_3a_4}_{++++}$$

$$\mathbf{H}^{a_1a_2a_3a_4}_{+++-}$$

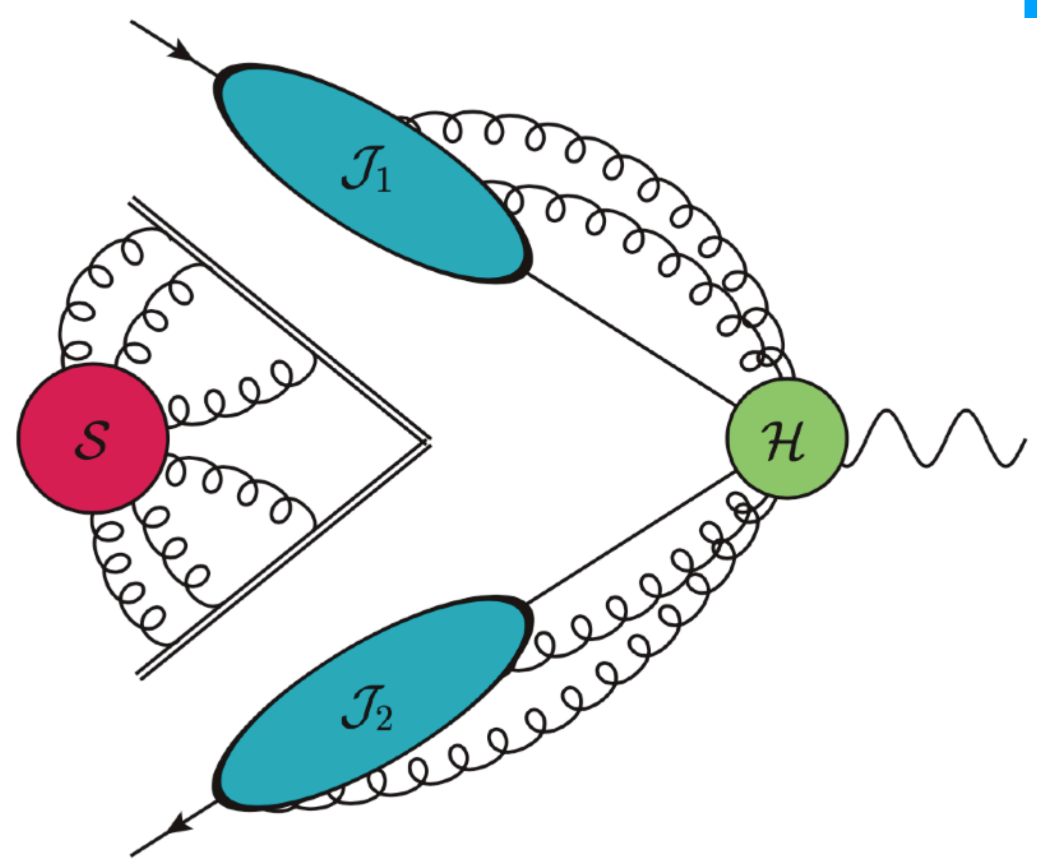
$$\mathbf{H}^{a_1a_2a_3a_4}_{++--} \quad \cdots \quad \mathbf{16}$$

IR physics does not care about the UV

IR "renormalisation"

$$\mathcal{H}_{\text{ren}}(\epsilon,\{p\}) = \mathcal{Z}(\epsilon,\{p\},\mu)\,\mathcal{H}_{\text{fin}}(\mu,\{p\})$$

- IR-divergent
- IR-finite



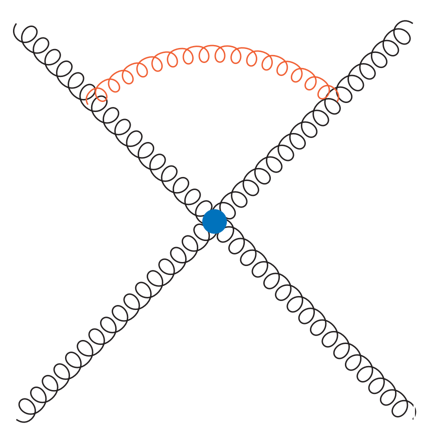
Gardi, Magnea: **0901.1091**, **0908.3273**; Becher, Neubert: **0903.1126**

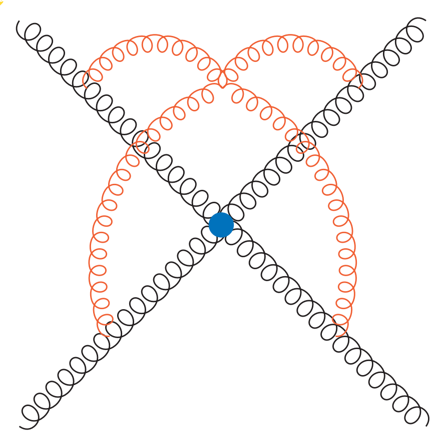
$$\boldsymbol{\mathcal{H}}_{\,\mathrm{ren}}(\epsilon, \{p\}) = \boldsymbol{\mathcal{Z}}(\epsilon, \{p\}, \mu)\, \boldsymbol{\mathcal{H}}_{\,\mathrm{fin}}(\mu, \{p\})$$

$$\frac{d}{d\log\mu}$$

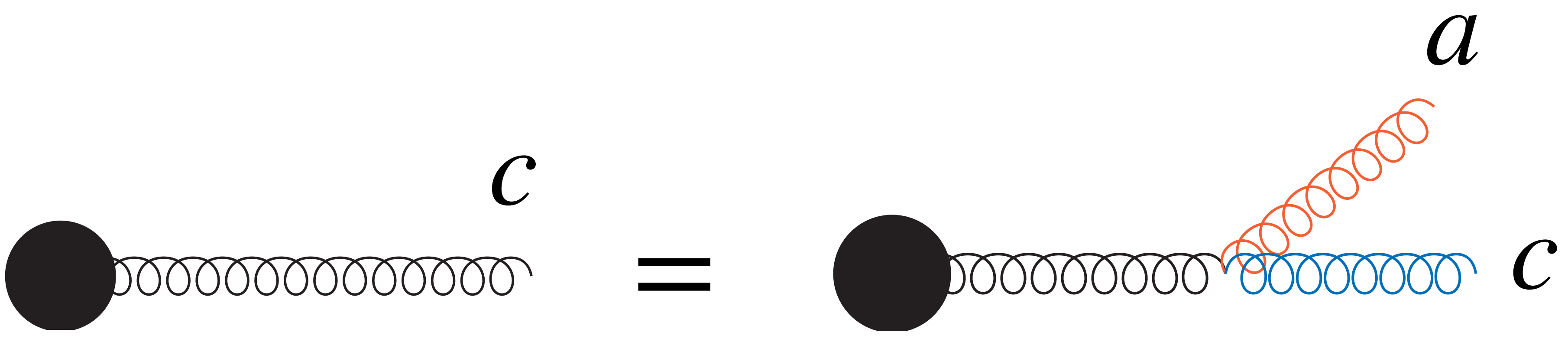
$$\frac{d}{d\log\mu}\boldsymbol{\mathcal{H}}_{\,\mathrm{fin}}(\mu) = \boldsymbol{\Gamma}(\mu)\, \boldsymbol{\mathcal{H}}_{\,\mathrm{fin}}(\mu)$$

$$\boldsymbol{\Gamma}(\{p\}, \mu) = \boldsymbol{\Gamma}_{\mathrm{dipole}}(\{p\}, \mu) + \boldsymbol{\Delta}_4(\{p\})$$





Universal across *all* gauge theories

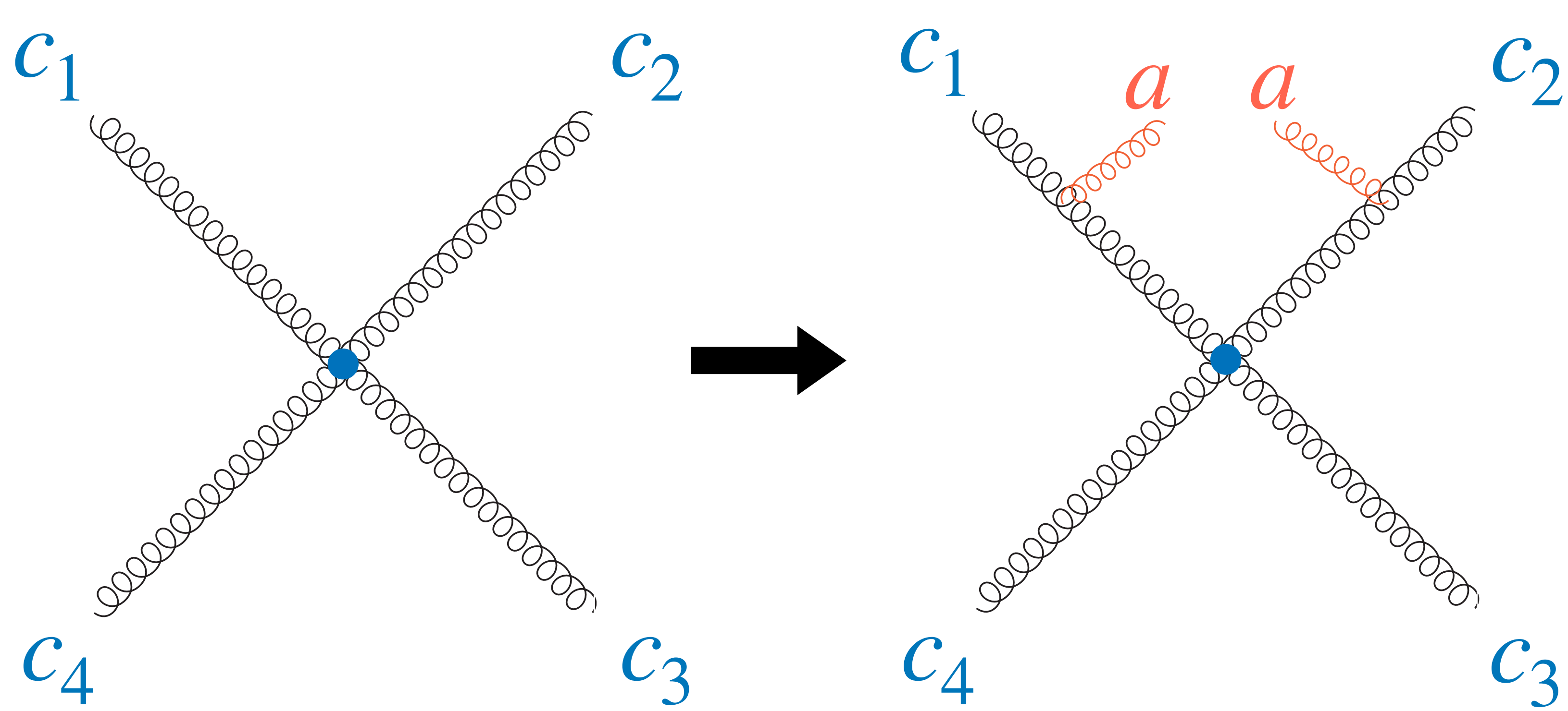
$\mathbf{T}^a$ 

$$\mathbf{T}^a X^c = -\, i f^a{}_{cc'} X^{c'}$$

Dipole

$$\mathbf{\Gamma}_{\text{dipole}}(\{p\},\mu) = \sum_{1\leq i<j\leq 4} \mathbf{T}_i^a\,\mathbf{T}_j^a\,\gamma^{\text{cusp}}(\alpha_{\text{s}})\,\log\left(\frac{\mu^2}{-s_{ij}-i\delta}\right) + \sum_{i=1}^{4}\gamma^i(\alpha_{\text{s}})$$

$$\mathbf{T}_1^a\,\mathbf{T}_2^a$$

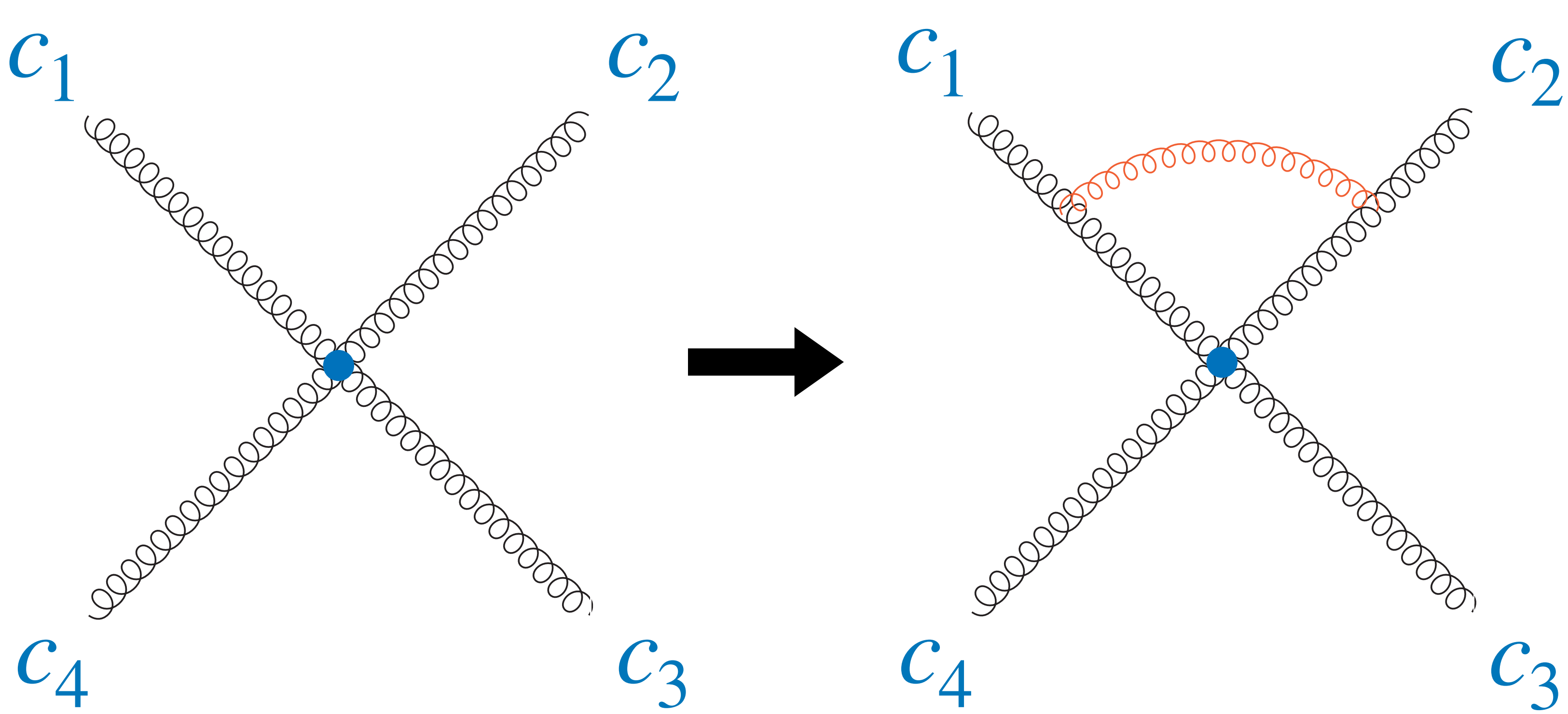


Catani: **9802439**

Dipole

$$\mathbf{\Gamma}_{\text{dipole}}(\{p\},\mu) = \sum_{1\le i<j\le 4} \mathbf{T}_i^a\,\mathbf{T}_j^a\,\gamma^{\text{cusp}}(\alpha_s)\,\log\left(\frac{\mu^2}{-s_{ij}-i\delta}\right) + \sum_{i=1}^{4}\gamma^i(\alpha_s)$$

$\mathbf{T}_1^a\,\mathbf{T}_2^a$



Catani: **9802439**

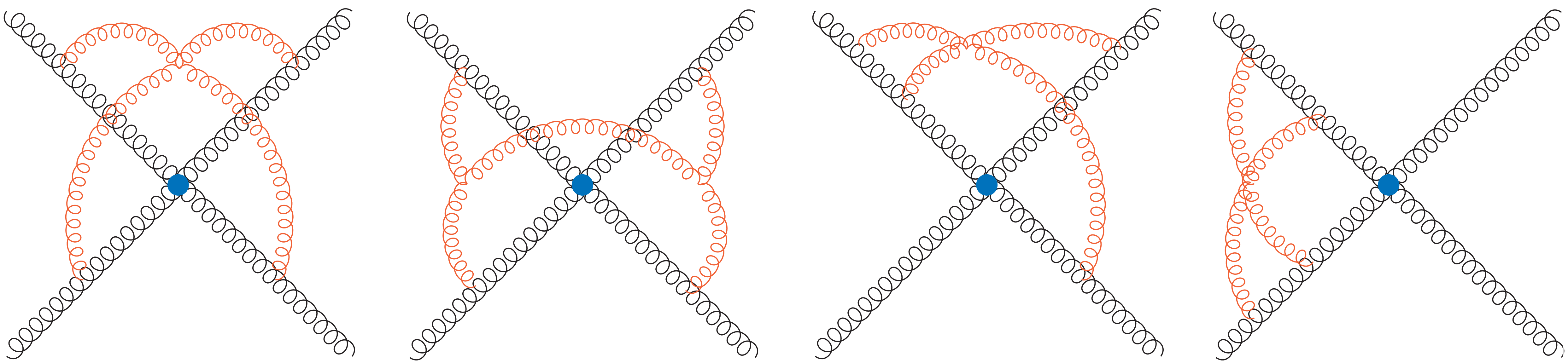
Quadrupole

$$\mathbf{\Delta}_4^{(3)} = 128\, f_{abe}\, f_{cde}\, \Big[\mathbf{T}_1^a\, \mathbf{T}_2^c\, \mathbf{T}_3^b\, \mathbf{T}_4^d\, D_1(x) - \mathbf{T}_4^a\, \mathbf{T}_1^b\, \mathbf{T}_2^c\, \mathbf{T}_3^d\, D_2(x)\Big]$$

$$- 16\, C\, f_{abe}\, f_{cde} \sum_{i=1}^{4} \sum_{\substack{1\leq j<k\leq 4 \\ j,k\neq i}} \Big\{\mathbf{T}_i^a, \mathbf{T}_i^d\Big\}\, \mathbf{T}_j^b\, \mathbf{T}_k^c\,,$$

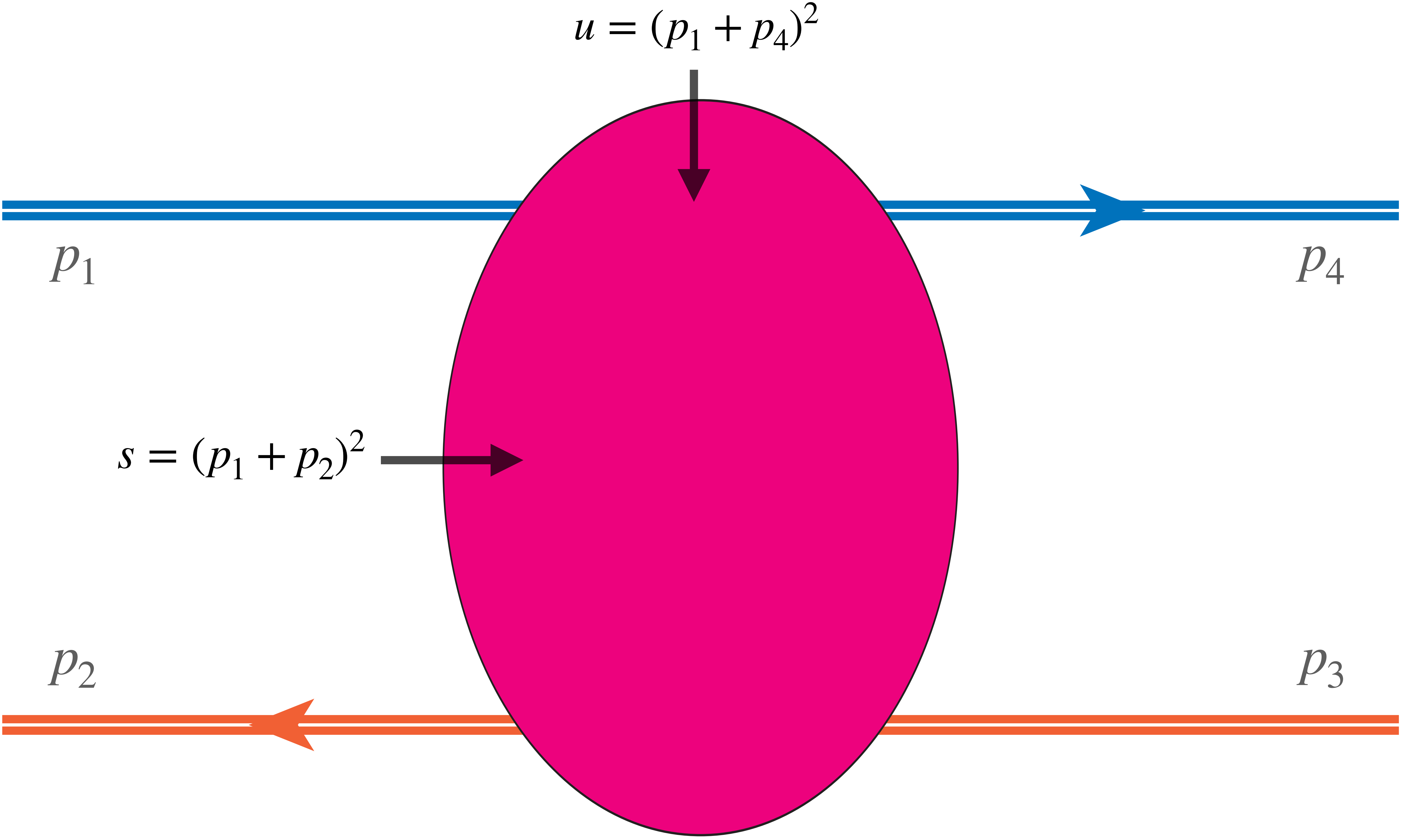
Almelid, Duhr, Gardi: **1507.00047**

Henn, Mistlberger: **1608.00850**



Now confirmed in QCD: Chakraborty, Caola, Gambuti, Tancredi, von Manteuffel: **2207.03503**, **2112.11097**

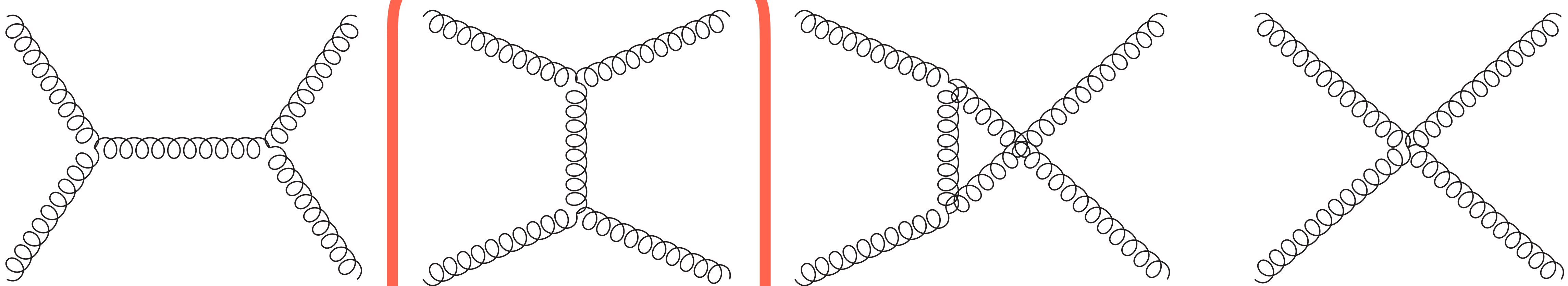
High Energy Limit



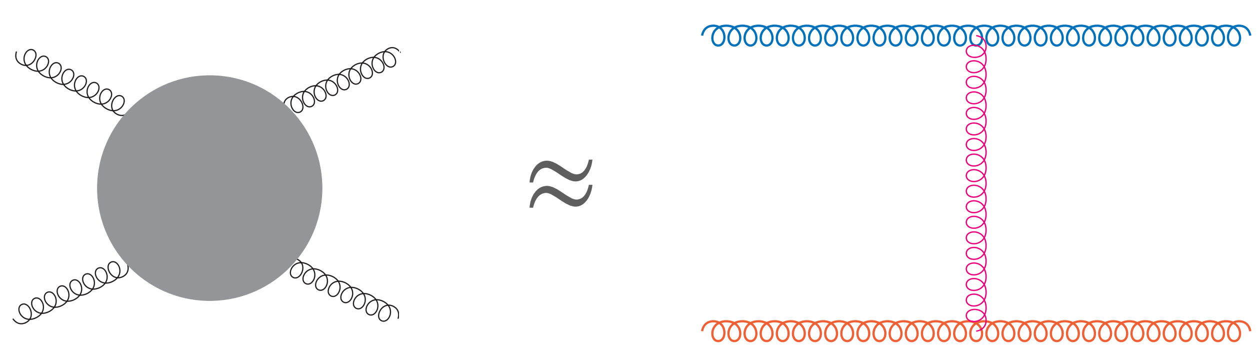
Regge Limit: $s \gg |u|$

gg → gg

$$s \approx -t \gg |u|$$



$\sim \frac{1}{s}$ $\sim \frac{1}{u}$ $\sim \frac{1}{t}$ constant



$$\mathcal{H}_{\text{fin}}(\mu) = \lim_{\epsilon\to 0} \mathcal{Z}^{-1}(\epsilon,\mu)\, \mathcal{H}_{\text{ren}}(\epsilon)$$

Harmonic Polylogarithms (HPLs)

Divergent Logarithms $\sim \log^{\#}\left(\frac{-u}{s}\right) = L^{\#}$

Spoiled expansion

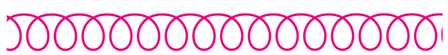
$$\mathbf{A}^{(0)} + \alpha\mathbf{A}^{(1)} + \alpha^2\mathbf{A}^{(2)} + \alpha^3\mathbf{A}^{(3)} + \dots$$

LL | NLL | NNLL

$$
\begin{aligned}
A^{(0)} &= \# \\
A^{(1)} &= \# L + \# \\
A^{(2)} &= \# L^2 + \# L + \# \\
A^{(3)} &= \# L^3 + \# L^2 + \# L + \# \\
&\ \ \vdots \qquad\quad \vdots \qquad\quad \vdots
\end{aligned}
$$

Reggeized gluon

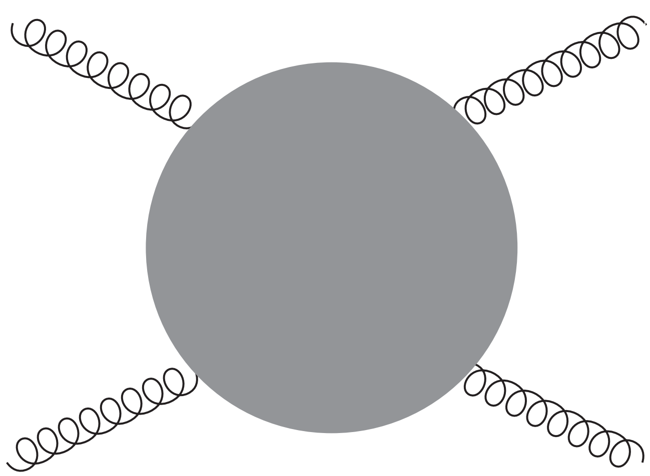
$$\frac{1}{u} \longrightarrow e^{L\,C_A\,\tau_g(\alpha)}\,\frac{1}{u}$$



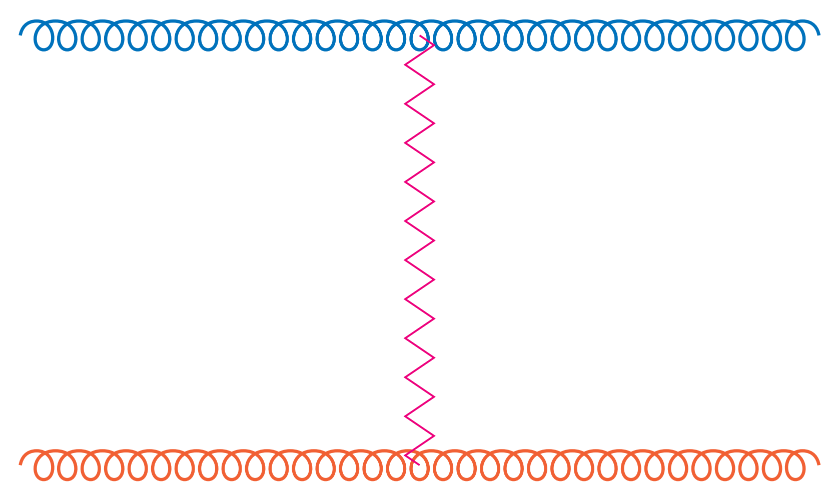
Regge Trajectory

$$\tau_g(\alpha) = \sum_{\ell=1} \alpha^{\ell}\,\tau^{(\ell)}$$

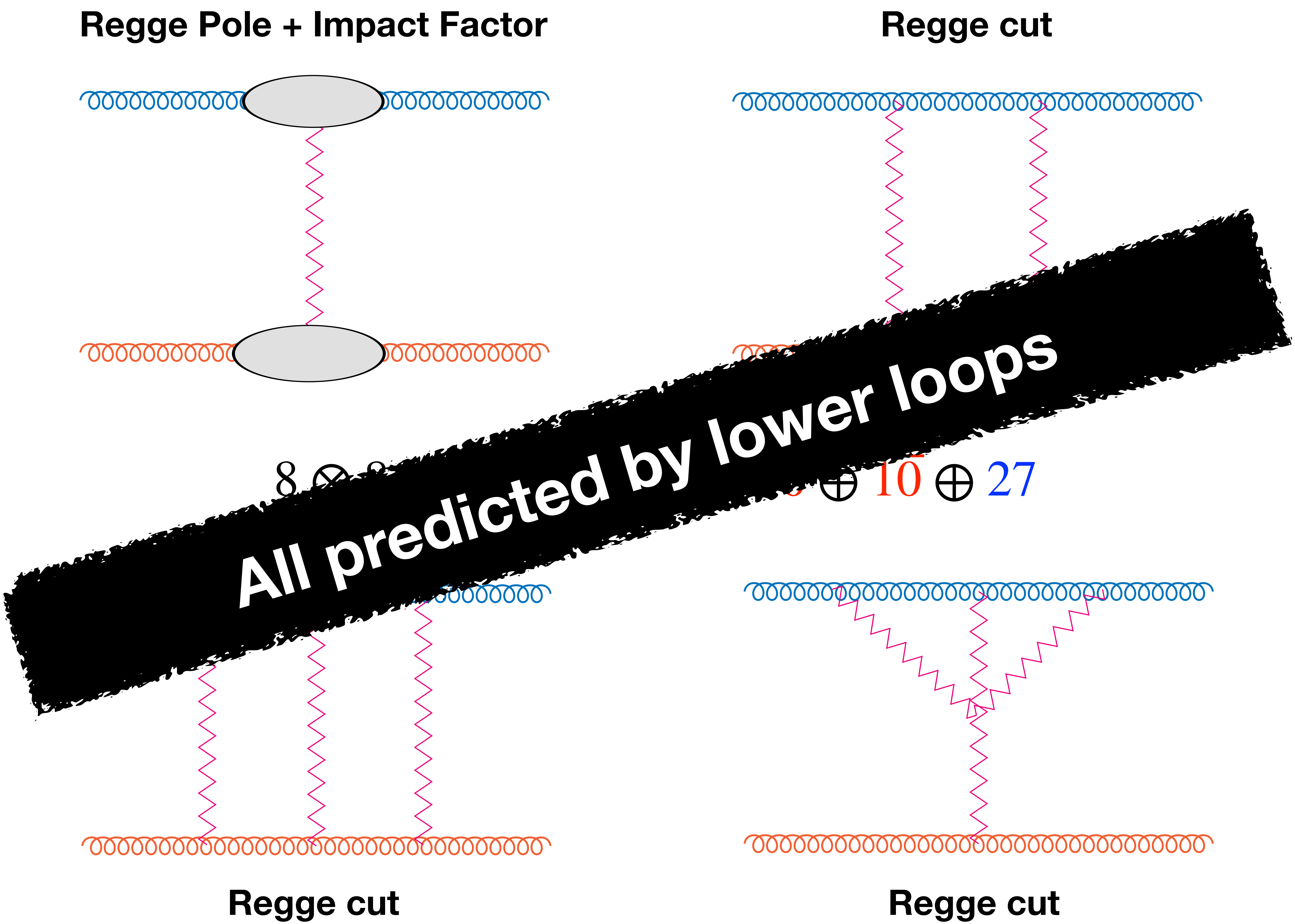
$$A_{LL} = e^{L\,C_A\,\tau_g(\alpha)}\,A^{tree}$$



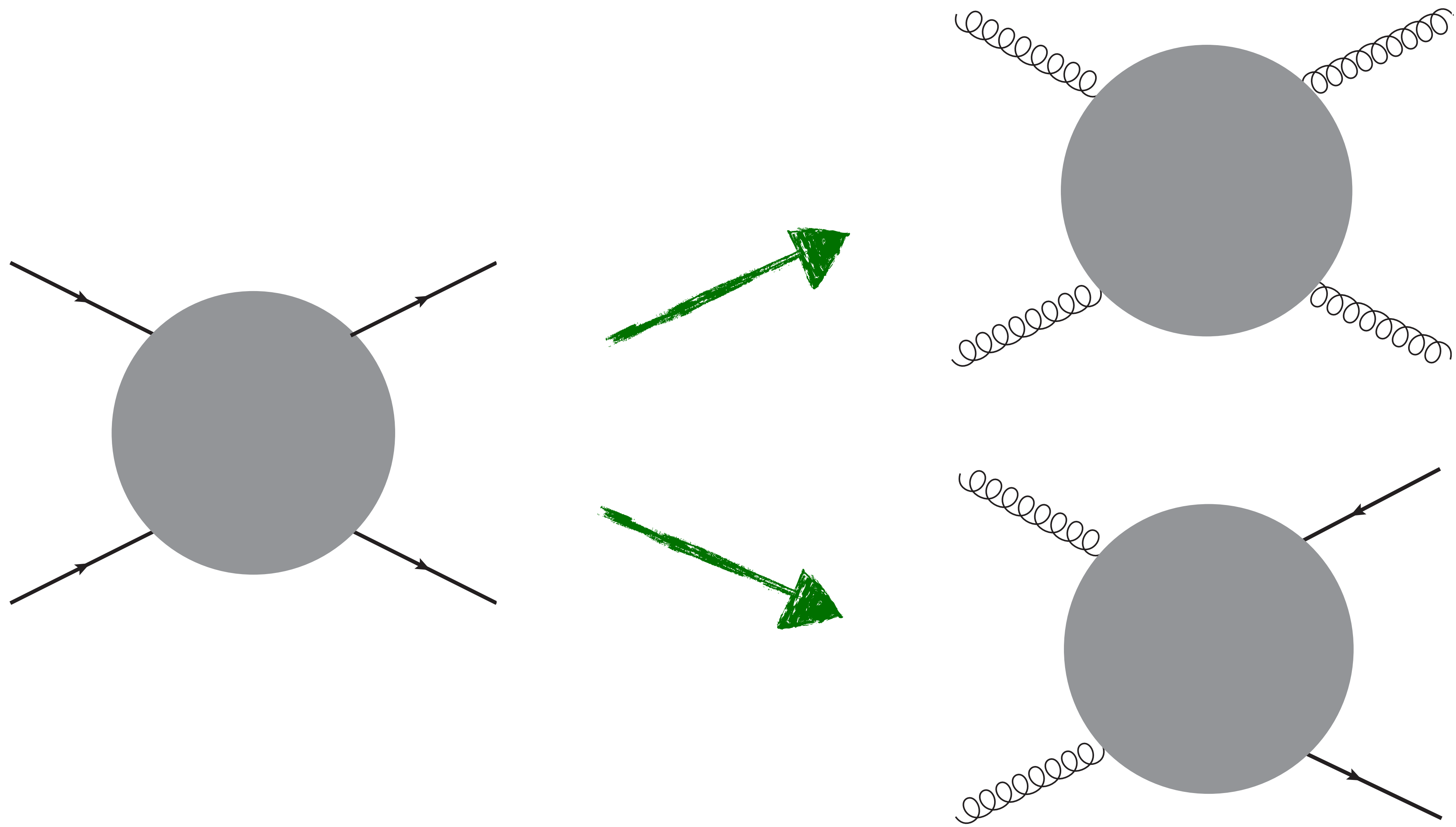
$\overset{\text{LL}}{=}$



Three-loop exact results $\Longrightarrow$ τ_g **at three loops**


Regge Pole + Impact Factor
Regge cut
All predicted by lower loops
8 ⊗ 8
⊕ 10̄ ⊕ 27
Regge cut
Regge cut

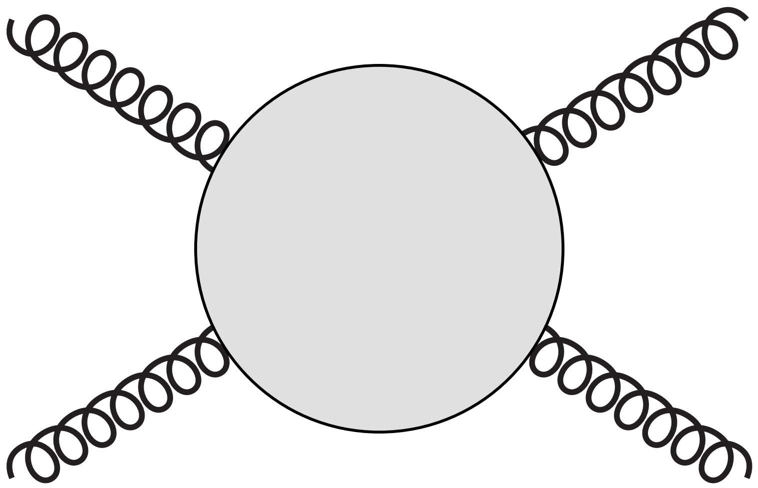
Regge limit as a check & constraint



Conclusion

Conclusion

- High-precision phenomenology
- Connection with formal QFT
- Intrinsic mathematical beauty



- General method in gauge theories
- All 4-point 3-loop QCD amplitudes
- Confirmation of quadrupole IR radiation
- Extraction of 3-loop gluon Regge trajectory

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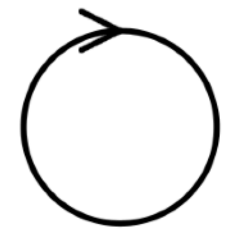
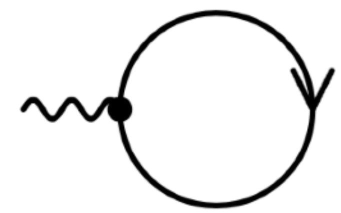
§

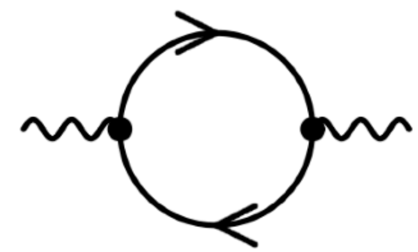
Backup Slides

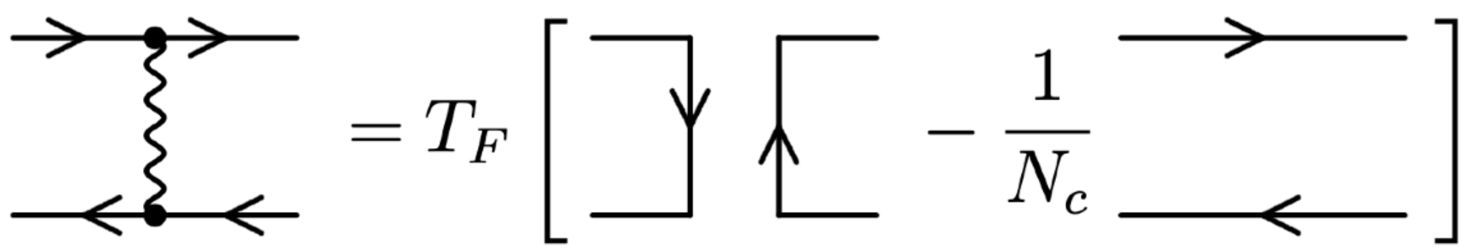
Colour Decomposition

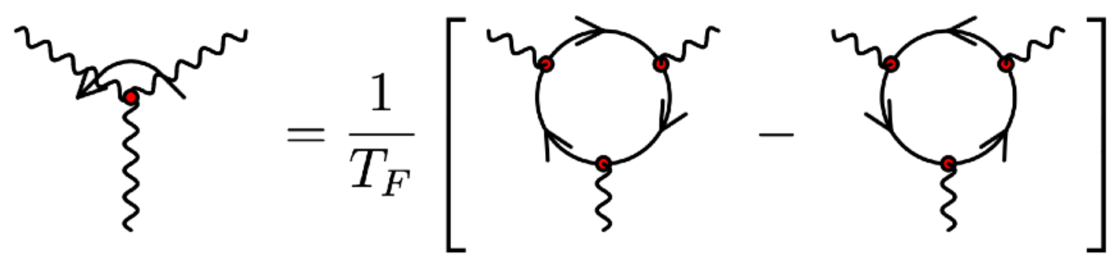
A SU(N) graphical game..

$\mathrm{Tr}\,1 = N_c$ or  $= N_c\,,$

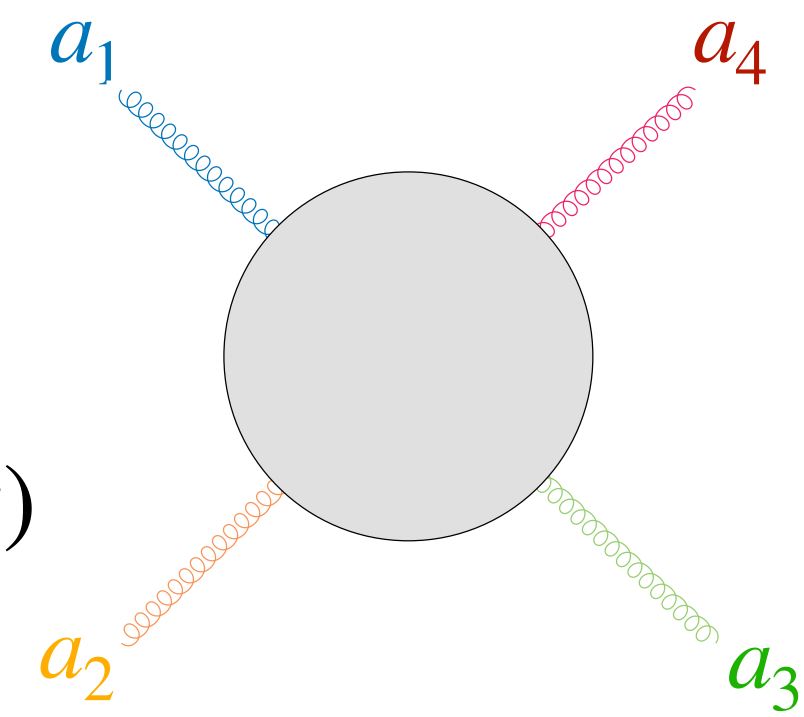
$\mathrm{Tr}\,t^a = 0$ or  $= 0\,,$

$\mathrm{Tr}\,t^a t^b = T_F \delta^{ab}$ or  $= T_F$ 





Colour Decomposition



$$A = \sum_{i=1}^{6} A_i \mathcal{C}_i$$

$$A = A_1 \mathcal{C}_1 + A_4 \mathcal{C}_4 + (perms)$$

$$\mathcal{C}_1 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_3}\mathbf{T}^{a_4}) + Tr(\mathbf{T}^{a_4}\mathbf{T}^{a_3}\mathbf{T}^{a_2}\mathbf{T}^{a_1})$$

$$\mathcal{C}_2 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_4}\mathbf{T}^{a_3}) + Tr(\mathbf{T}^{a_3}\mathbf{T}^{a_4}\mathbf{T}^{a_2}\mathbf{T}^{a_1})$$

$$\mathcal{C}_3 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_4}\mathbf{T}^{a_2}\mathbf{T}^{a_3}) + Tr(\mathbf{T}^{a_3}\mathbf{T}^{a_2}\mathbf{T}^{a_4}\mathbf{T}^{a_1})$$

$$\mathcal{C}_4 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2})\, Tr(\mathbf{T}^{a_3}\mathbf{T}^{a_4})$$

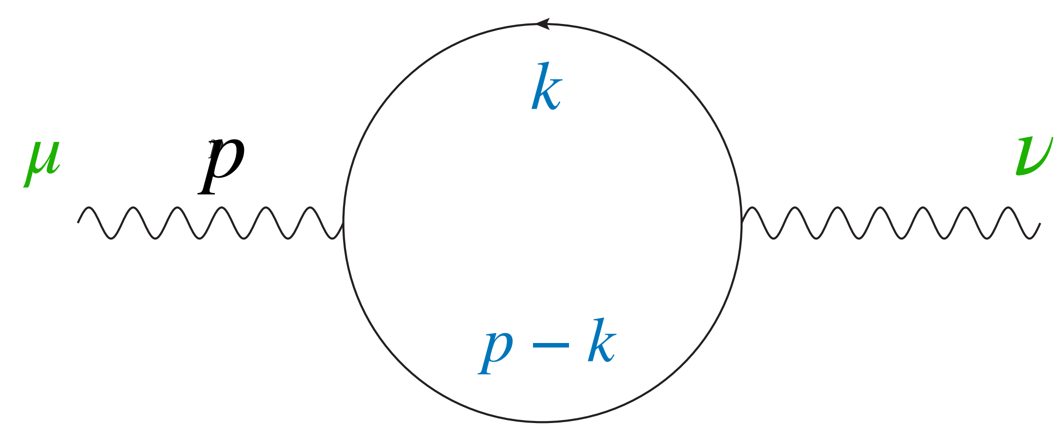
$$\mathcal{C}_5 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_3})\, Tr(\mathbf{T}^{a_2}\mathbf{T}^{a_4})$$

$$\mathcal{C}_6 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_4})\, Tr(\mathbf{T}^{a_2}\mathbf{T}^{a_3})$$

Tensor Projection

Example

QED vacuum polarisation



$$A^{\mu\nu} = e^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, \frac{4(k^\mu(p^\nu - k^\nu) + k^\nu(p^\mu - k^\mu) - g^{\mu\nu}(k \cdot p - k^2))}{(k^2 + i\epsilon)((k-p)^2 + i\epsilon)}$$

Tensor basis: $$T_1^{\mu\nu} = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \qquad T_2^{\mu\nu} = \frac{p^\mu p^\nu}{p^2}$$

$$\boxed{A^{\mu\nu} = F_1 T_1^{\mu\nu} + F_2 T_2^{\mu\nu}}$$

Lorentz indices are now outside the integrals!

Projectors: $$P_{1\,\mu\nu} = \frac{1}{d-1}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \qquad P_{2\,\mu\nu} = \frac{p^\mu p^\nu}{p^2}$$

General Case

- Ward Identities/gauge fixing
- ’t Hooft-Veltman scheme

$$A^X = \sum_{i=1}^{N} F_i T_i^X$$

Only contain **scalar integrals**!

Computed in Dim. Reg.

$$P_i \cdot T_i = \delta_{ij}$$

$$P_i \cdot A = F_i$$

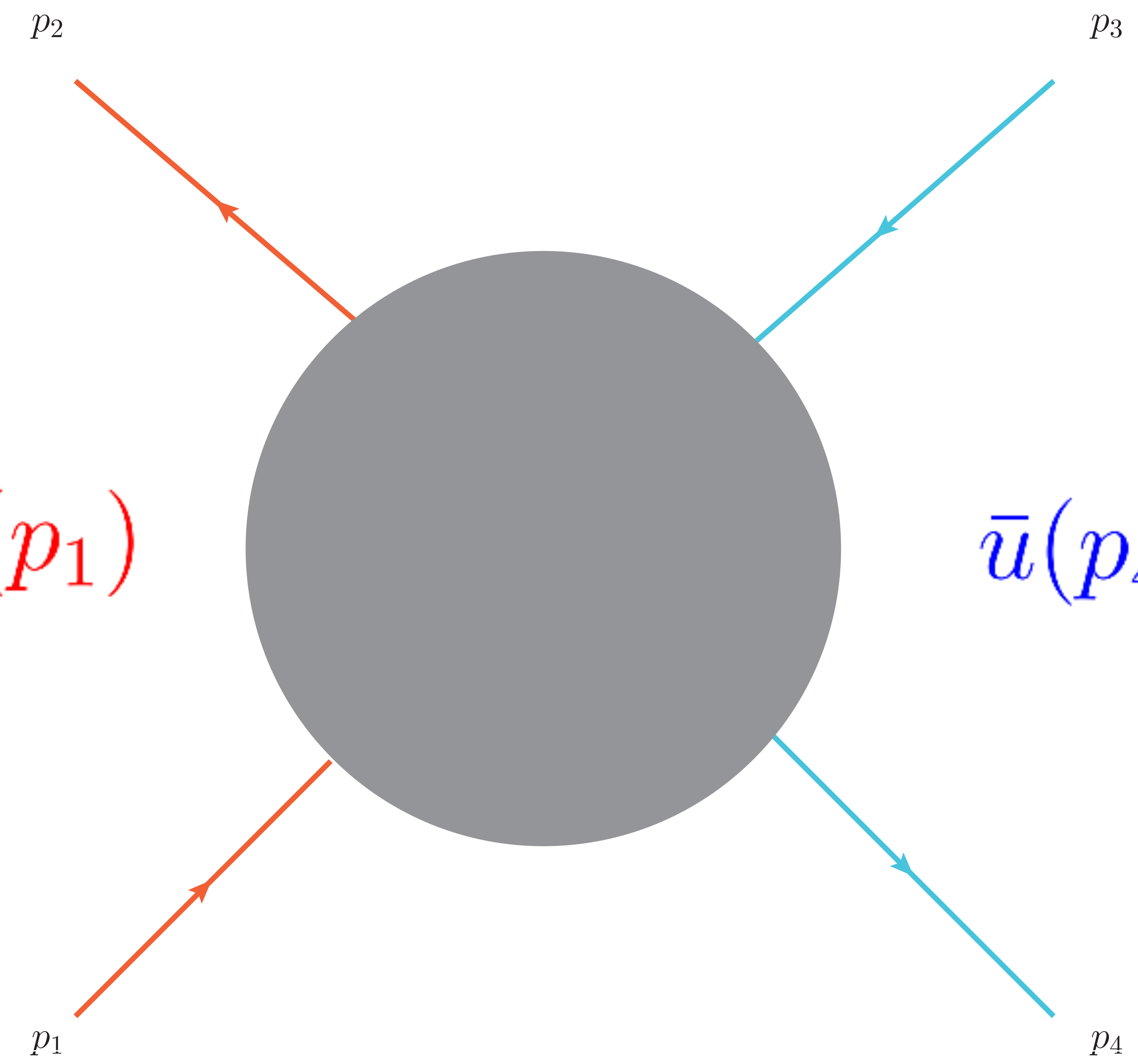
$$\bar{u}(p) \cdot u(p) = \gamma_\mu p^\mu$$

$$\epsilon^{*\mu}(p) \cdot \epsilon^\nu(p) = -\eta^{\mu\nu} + \ldots$$

Quarks Example

$qQ \to qQ$

$\bar{u}(p_2)\ldots u(p_1)$

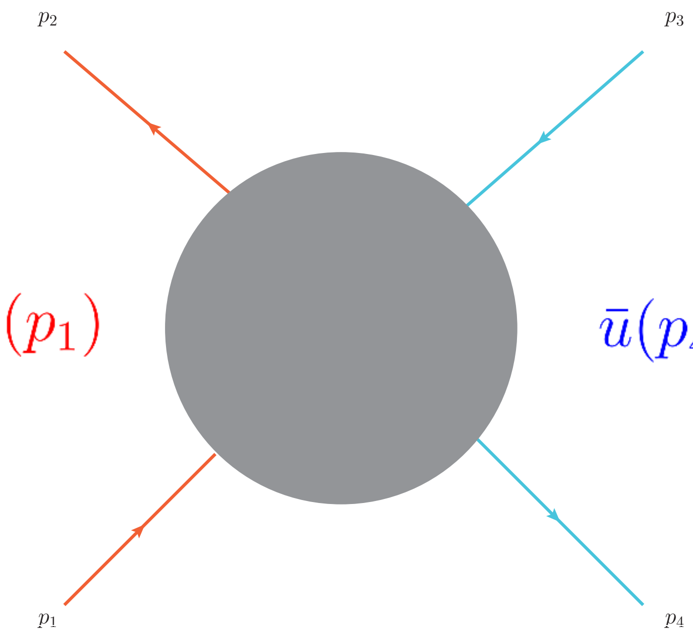


$\bar{u}(p_4)\ldots u(p_3)$

qQ → qQ

$$A^X = \sum_{i=1}^{N} F_i T_i^X$$

$\bar{u}(p_2)\ldots u(p_1)$



$\bar{u}(p_4)\ldots u(p_3)$

$$T_1 = \bar{u}(p_2)\gamma_{\mu_1}u(p_1) \;\times\; \bar{u}(p_4)\gamma^{\mu_1}u(p_3)$$
$$T_2 = \bar{u}(p_2)\not{p}_3 u(p_1) \;\times\; \bar{u}(p_4)\not{p}_1 u(p_3)$$

in d=4

$$T_3 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}u(p_1) \;\times\; \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}u(p_3)$$
$$T_4 = \bar{u}(p_2)\gamma_{\mu_1}\not{p}_3\gamma_{\mu_3}u(p_1) \;\times\; \bar{u}(p_4)\gamma^{\mu_1}\not{p}_1\gamma^{\mu_3}u(p_3)$$

$$T_5 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}u(p_1) \;\times\; \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_5}u(p_3)$$
$$T_6 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_3\gamma_{\mu_4}\gamma_{\mu_5}u(p_1) \;\times\; \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\not{p}_1\gamma^{\mu_4}\gamma^{\mu_5}u(p_3)$$

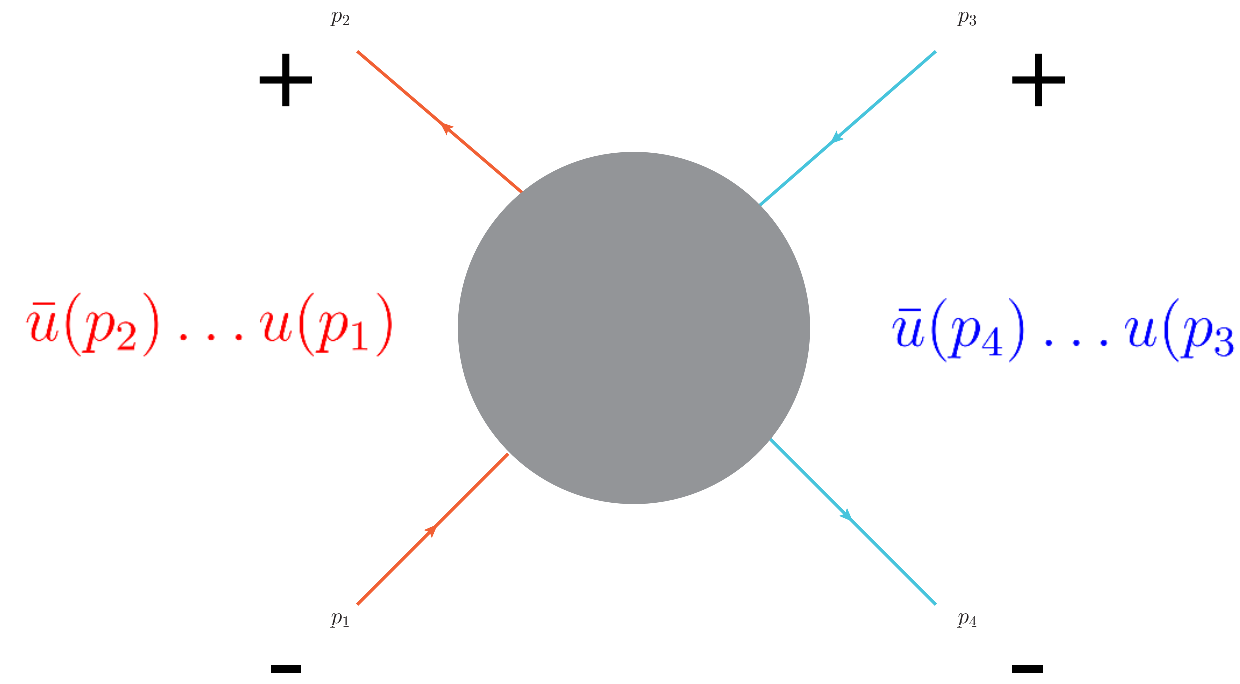
$$T_7 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}\gamma_{\mu_6}\gamma_{\mu_7}u(p_1) \;\times\; \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_5}\gamma^{\mu_6}\gamma^{\mu_7}u(p_3)$$
$$T_8 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\not{p}_3\gamma_{\mu_5}\gamma_{\mu_6}\gamma_{\mu_7}u(p_1) \;\times\; \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\not{p}_1\gamma^{\mu_5}\gamma^{\mu_6}\gamma^{\mu_7}u(p_3)$$

⋮

L. Tancredi, T. Peraro: **arXiv:1906.03298**, **arXiv:2012.00820**

qQ → qQ

$$A^X = \sum_{i=1}^{N} F_i T_i^X$$



$$T_1 = \bar{u}(p_2)\gamma_{\mu_1}u(p_1) \;\times\; \bar{u}(p_4)\gamma^{\mu_1}u(p_3)$$

$$T_2 = \bar{u}(p_2)\not{p}_3 u(p_1) \;\times\; \bar{u}(p_4)\not{p}_1 u(p_3)$$

$$\overline{T}_3 = T_3^4 + (d-4)T_3^{-2\epsilon}$$

$$\overline{T}_4 = T_4^4 + (d-4)T_4^{-2\epsilon}$$

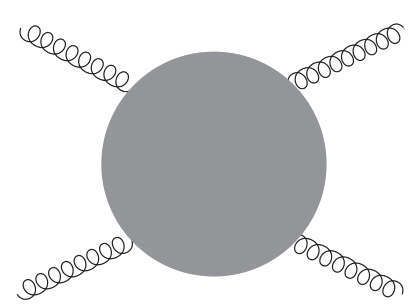
$$\overline{T}_5 = T_5^4 + (d-4)T_5^{-2\epsilon}$$

$$\overline{T}_6 = T_6^4 + (d-4)T_6^{-2\epsilon}$$

$$\overline{T}_7 = T_7^4 + (d-4)T_7^{-2\epsilon}$$

$$\overline{T}_8 = T_8^4 + (d-4)T_8^{-2\epsilon}$$

Orthogonal
&
zero **in d=4** !!



From 138 to 8 tensors!

L. Tancredi, T. Peraro: **arXiv:1906.03298**, **arXiv:2012.00820**

Integration by Parts

Integration by Parts

$$I(d,\{p\}) = \int_{(\mathbb{R}^d)^\ell} \mathrm{d}^d k_1 \;\dots\; \mathrm{d}^d k_\ell \quad f(k_1,\dots,k_\ell,p_1,\dots,p_n)$$

Invariance: $k_i \to A_{ij}\, k_j \,+\, B_{ih}\, p_h$ $\quad (k_1 \to k_1 + k_2 - k_3 + p_2)$

Generators: $k_i \to k_i \,+\, \alpha\, q_j$ $\quad (k_1 \to k_1 + \alpha k_1)$ $\quad (k_1 \to k_1 + \alpha p_1)$

$$(q_1,\dots,q_{l+n}) = (k_1,\dots,k_l,p_1,\dots,p_n)$$

$$I(d,\{p\}) \to \int_{(\mathbb{R}^d)^\ell} \mathrm{d}^d k_1 \dots (1{+}\alpha\, d\, \delta_{q_j k_i})\, \mathrm{d}^d k_i \dots \mathrm{d}^d k_\ell \;\left(f \,+\, \alpha\, q_j \cdot \frac{\partial}{\partial k_i} f \right)$$

Integration by Parts

$$I(d,\{p\}) = \int_{(\mathbb{R}^d)^\ell} \mathrm{d}^d k_1 \;\ldots\; \mathrm{d}^d k_\ell \quad f(k_1,\ldots,k_\ell,p_1,\ldots,p_n)$$

Invariance: $k_i \to A_{ij}\,k_j + B_{ih}\,p_h$

Generators: $k_i \to k_i + \alpha\, q_j$

$$(q_1,\ldots,q_{l+n}) = (k_1,\ldots,k_l,p_1,\ldots,p_n)$$

$$I(d,\{p\}) \to I(d,\{p\}) + \int_{(\mathbb{R}^d)^\ell} \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_\ell \left(\alpha\, d\, \delta_{q_j k_i} f + \alpha\, q_j \cdot \frac{\partial}{\partial k_i} f \right)$$

Integration by Parts

$$I(d, \{p\}) = \int_{(\mathbb{R}^d)^\ell} \mathrm{d}^d k_1 \; \dots \; \mathrm{d}^d k_\ell \quad f(k_1, \dots, k_\ell, p_1, \dots, p_n)$$

Invariance: $k_i \to A_{ij}\, k_j \,+\, B_{ih}\, p_h$

Generators: $k_i \to k_i \,+\, \alpha\, q_j$

$$(q_1, \dots, q_{l+n}) = (k_1, \dots, k_l, p_1, \dots, p_n)$$

$$I(d, \{p\}) \to I(d, \{p\}) \,+\, \alpha \int_{(\mathbb{R}^d)^\ell} \mathrm{d}^d k_1 \dots \mathrm{d}^d k_\ell \; \frac{\partial}{\partial k_i} \cdot (q_j f)$$

Integration by Parts

$$\int_{(\mathbb{R}^d)^\ell} \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_\ell \;\; \frac{\partial}{\partial k_i} \cdot (q_j f) \;=\; 0$$

↓

$$\frac{\partial}{\partial k_i} \cdot \left(q_j \frac{\mathcal{N}}{D_1^{n_1} \ldots D_N^{n_N}} \right) \;=\; 0$$

↓

$$\frac{\partial}{\partial k_1} \cdot \left(k_1 \frac{k_1 \cdot p_2}{k_1^2 (k_1 - p_1 - p_2)^2} \right) \;=\; 0$$

$$(d+1) \frac{k_1 \cdot p_2}{k^2 (k_1 - p_1 - p_2)^2} - 2 \frac{k_1 \cdot p_2}{k_1^2 (k_1 - p_1 - p_2)^2} - 2 \frac{k_1 \cdot p_2 (k^2 - k_1 \cdot p_1 - k_1 \cdot p_2)}{k^2 (k_1 - p_1 - p_2)^4} = 0$$

Linear system! ⟶ **Feynman Integrals form a *vector space***

Differential Equations

Invariants: $x_i \to I(d; x_1, \ldots, x_{4n-10})$

$$I(d; x_1, \ldots, x_{4n-10}) = \sum c_i \, M_i(d; x_1, \ldots, x_{4n-10})$$

$$\mathrm{d}\vec{M} = \mathbf{A}(d; x_i) \cdot \vec{M}$$

Canonical basis: $\mathrm{d}\vec{M}_c = (d-4)\, \mathbf{A}_{\mathbf{log}}(x_i) \cdot \vec{M}_c$

$$\begin{pmatrix} \frac{1}{x} - \frac{1}{1+x} & 0 \\ \frac{1}{x} & \frac{1}{1+x} \end{pmatrix} \mathrm{d}x \;=\; \begin{pmatrix} \mathrm{d}log(x) - \mathrm{d}log(1+x) & 0 \\ \mathrm{d}log(x) & \mathrm{d}log(1+x) \end{pmatrix}$$