Higher-dimensional Quantum Spin and Valley Hall Effects in Synthetic Matter

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Outline

- Topological phases: 2D QHE, QSHE and QVHE
- Abelian and Non-Abelian Berry connections
- Topological quantum field theories
- 4D generalization
- Ultracold-atom implementation

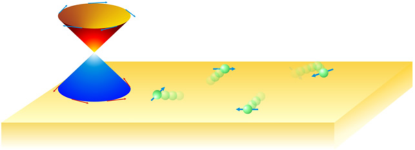
Topological Phases of Matter

- States of matter that are not described by the Landau's theory of phase transitions.
- Bulk-edge correspondence.
- Topological band theory:
  quantum numbers ($\mathbb{Z}$, $\mathbb{Z}_2$, etc.) are related to Chern, Stiefel-Whitney, Dixmier-Douady classes, etc.
- Several fermion models in the lattice can be described by effective Dirac Hamiltonians.
- Quantum field theory:
  in the low-energy regime, topological (Chern-Simons, BF, etc.) and conformal field theories describe their bulk and edge states, respectively.

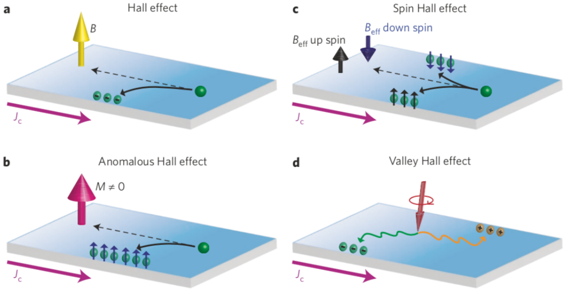
First-order topological insulators and superconductors

Symmetry				Spatial Dimension d								
Class	T	C	S	1	2	3	4	5	6	7	8	$\cdots$
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

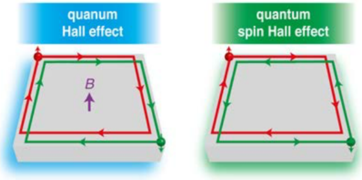
Topologically protected massless edge modes:
Dirac fermions in TIs and Majorana fermions in TSCs



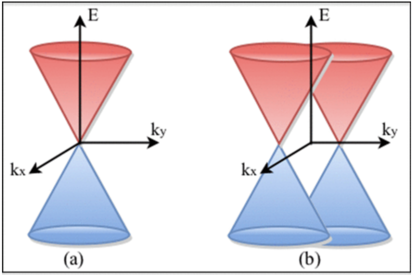
2D QHE, QSHE and QVHE

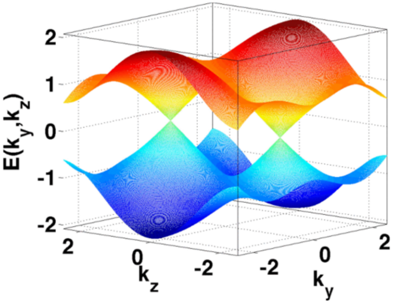


Edge states:



Valley = fermion doubling





Abelian Berry Connection

Bloch wave-vector: $|u(\mathbf{k})\rangle = (u^1(\mathbf{k}), u^2(\mathbf{k}), ..., u^{\aleph}(\mathbf{k}))^{\top}$.

The gauge redundancy in a non-degenerate Bloch state is encoded in the arbitrary phase in $|u\rangle$

$$|u\rangle \to e^{i\alpha(\mathbf{k})}|u\rangle,$$

where $\alpha(\mathbf{k})$ is a momentum-dependent function.

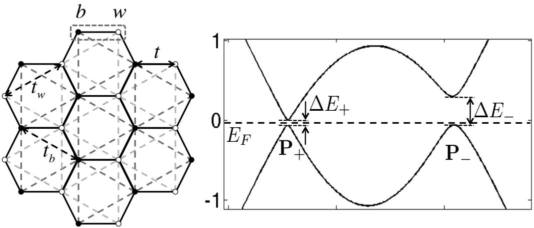
We can build an Abelian gauge connection in momentum space as follows

$$A_j = i\langle u|\partial_j|u\rangle, \quad A_j \to A_j - \partial_j\alpha$$

with $\partial_j \equiv \partial_{k_j}$, while the gauge-invariant Berry curvature is given by

$$\mathcal{F}_{jk} = \partial_j A_k - \partial_k A_j.$$

Haldane model and first Chern number



Each valley can be described in terms of an effective massive Dirac Hamiltonian:

$$H^{\pm}_{2\mathrm{D}} = k_x\sigma^x + k_y\sigma^y + m_{\pm}\sigma^z,$$

where $\boldsymbol{k} = (k_x, k_y)$ are the momenta, m is a mass term and $\sigma^{x,y,z}$ are the Pauli matrices.

$$C_1 = \frac{1}{2\pi}\int_{T^2} dk_x dk_y \epsilon^{jk}\mathcal{F}_{jk} \in \mathbb{Z}.$$

Spin Hall insulator and first spin Chern number

Bernevig-Hughes-Zhang Hamiltonian
(Bernevig, Hughes, Zhang, Science 2006).

$$H_{\mathrm{BHZ}} = \begin{pmatrix} H_+ & g\sigma_x \\ g\sigma_x & H_- \end{pmatrix},$$

The g describes the band coupling strength and this coupling term between blocks. Here, we assume $g \approx 0$.

Each pseudospin has an independent Chern number C_1^+ and C_1^-. The time-reversal symmetry gives rise to a vanished total Chern number, i.e. $C_1^+ + C_1^- = 0$.

The difference is not zero:

$$C_s = (C_1^+ - C_1^-)/2.$$

Cold-atom implementation, see Q.-X. Lv at al., PRL 2021.

Non-Abelian Berry connections

In the case of N-degenerate bands, the Abelian Berry connections are replaced by the SU(N) or U(N) Non-Abelian versions: (Wilczek and Zee, PRL 1984).

$$A_j^{ab} = i\langle u^a|\partial_j|u^b\rangle.$$

Curvature tensor:

$$\mathcal{F}_{ij} = \partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i - i[\mathbf{A}_i, \mathbf{A}_j].$$

Under gauge transformations $|u\rangle \to U|u\rangle$ we have that

$$\mathcal{F}_{ij} \to U\mathcal{F}_{ij}U^{-1},$$

where U is a Lie-algebra-valued matrix.

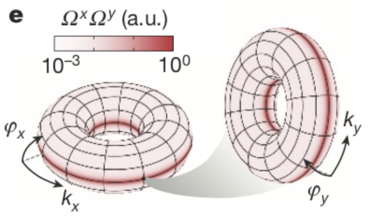
The Wilson line operator $\mathbf{W}$ is defined as $\mathbf{W} = \mathcal{P}\exp(i\int_{\mathbf{k}_i}^{\mathbf{k}_f} \mathbf{A}\cdot \mathrm{d}\mathbf{k})$.

Second Chern number

The second Chern number C_2 plays a central role in the 4D QHE and 4D Chern insulators.

$$C_2 = \frac{1}{8\pi^2} \int_{\mathbb{T}^4} \mathrm{tr}\, \mathcal{F} \wedge \mathcal{F},$$

where the first Brillouin zone is now a four-dimensional torus $\mathbb{T}^4$ spanned by $\boldsymbol{k} = (k_x, k_y, k_z, k_w)$.



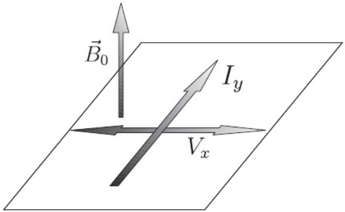
Cold-atom implementation: see H. Price at al., PRL 2015 and M. Lohse at al., Nature 2018.

Chern-Simons theory in the 2D QHE

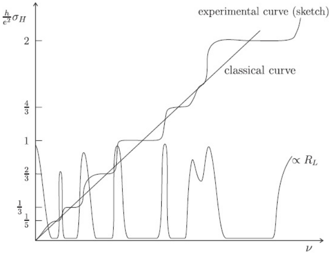
U(1) Chern-Simons theory in the IQHE

$$S_{CS} = \frac{\nu}{4\pi} \int d^3x \, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

- Quantized Hall conductivity with filling factor ν
- Bulk-edge correspondence: $\mathrm{CS}_{2+1}/\mathrm{CFT}_{1+1}$
- Physical observables: $\langle W_L \rangle = \langle \exp i \oint A \rangle$



(a) Quantum Hall Effect



(b) Hall conductivity

Axion electrodynamics in 3D TIs

Axion Electrodynamics describes the topological response of 3D topological insulators (TIs) (Qi, Hughes, Zhang, PRB 2008).

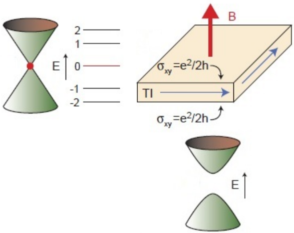
$$S_{axion} = \frac{\theta e^2}{32\pi^2} \int d^4x \, \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}.$$

For standard insulators $\theta = 0$ while for TIs $\theta = \pi$.

Gapped boundary states support an half-integer QHE:

$$S_{CS} = \frac{1}{8\pi} \int d^3x \, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda,$$

$$\nu = \frac{1}{2}.$$



Mixed Chern-Simons theory in the 2D QSHE

Each 2D block system is coupled to a gauge field $A_\mu^{(\pm)}$ and gives rise to a corresponding Abelian Chern-Simons theory,

$$S_{cs}^{(\pm)} = \frac{C_1^{\pm}}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} A_\mu^{(\pm)} \partial_\nu A_\lambda^{(\pm)}.$$

By taking $\tilde{A}_\mu = \frac{1}{2}\left[A^{(+)} - A^{(-)}\right]$, $A_\mu = \frac{1}{2}\left[A^{(+)} + A^{(-)}\right]$, we obtain

$$S_{s,eff} = \frac{C_s}{2\pi} \int d^3x \epsilon^{\mu\nu\lambda} \tilde{A}_\mu \partial_\nu A_\lambda,$$

with $C_s = C_1^+ - C_1^-$.
Currents:

$$J^\mu = \frac{C_s}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu \tilde{A}_\lambda, \; \tilde{J}^\mu = \frac{C_s}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda.$$

We here pick the gauge

$$A_\mu = (0, 0, -tE_y), \; \tilde{A}_\mu = 0,$$

$$J^x = 0, \; \tilde{J}^x = \frac{C_s}{2\pi} E_y.$$

4D model and Dirac-like Hamiltonian

Y.-Q. Zhu, Z. Zheng, G. P., Z. D. Wang, PRL (2022)

$$\mathcal{H}_0(k) = d_x\Gamma_1 + d_y\Gamma_2 + d_z\Gamma_3 + d_w\Gamma_4 + d_m\Gamma_0,$$

where the Bloch vector

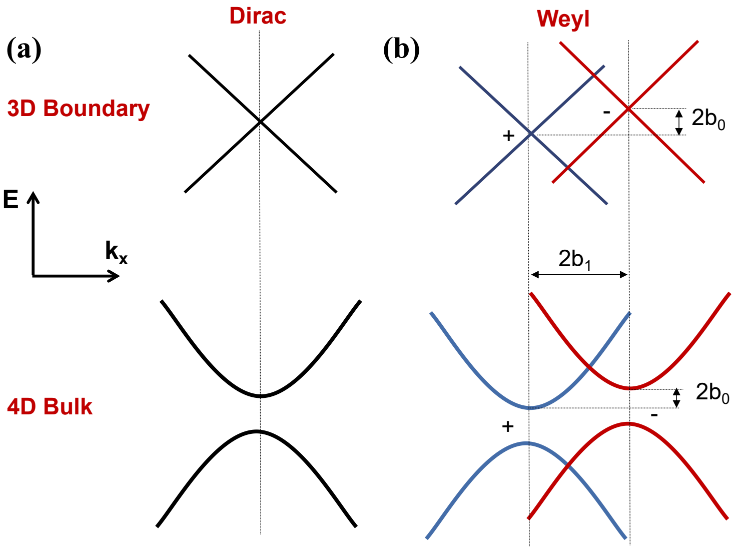
$$d_i = \sin k_i, d_m = m - \sum_i \cos k_i,$$

with $i = x, y, z, w$. The 8×8 matrices Γ_i satisfy the Clifford algebra. This system hosts two bands with the spectrum,

$$E_\pm = \pm\sqrt{d_x^2 + d_y^2 + d_z^2 + d_w^2 + d_m^2},$$

where each band has four-fold degeneracy.

Band structure and bulk-edge correspondence



Second spin Chern number

This model preserves the CP-symmetry, i.e., $\{CP, \mathcal{H}_0\} = 0$, with $CP = iG_{112}\mathcal{K}$ satisfies $(CP)^2 = -1$, with $G_{ijk} = \sigma_i \otimes \sigma_j \otimes \sigma_k$.

$$\mathcal{H}_{BD} = U\mathcal{H}_0 U^{-1} = H_+ \oplus H_-,$$

where each block Hamiltonian is given by

$$\mathcal{H}_\pm(k) = d_x G_{10} \pm d_y G_{21} \mp d_z G_{23} \mp d_w G_{22} + d_m G_{30},$$

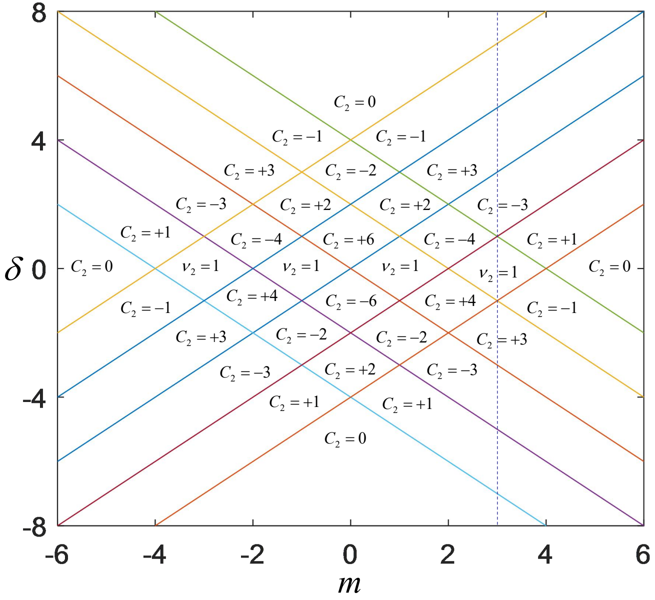
with $G_{ij} = \sigma_i \otimes \sigma_j$.

$$C_2^\pm = \frac{1}{8\pi^2} \int_{\mathbb{T}^4} \mathrm{tr}\, \mathcal{F}^\pm \wedge \mathcal{F}^\pm,$$

with the values $C_2^\pm = \pm 3\mathrm{sgn}(m)$ for $0 < |m| < 2$, $C_2^\pm = \mp\mathrm{sgn}(m)$ for $2 < |m| < 4$, and $C_2^\pm = 0$ elsewhere.

$$C_{2s} = (C_2^+ - C_2^-)/2.$$

Higher Second Chern number



4D QSHE

$$S_{\text{eff}}^{(a)} = \frac{C_2^{(a)}}{24\pi^2} \int d^5x \; \epsilon^{\mu\nu\lambda\rho\sigma} A_\mu^{(a)} \partial_\nu A_\lambda^{(a)} \partial_\rho A_\sigma^{(a)},$$

where $a = \pm$ for each block. We define two new gauge fields as

$$A_\mu = \frac{1}{2}\left[A_\mu^{(+)} + A_\mu^{(-)}\right], \; \tilde{A}_\mu = \frac{1}{2}\left[A_\mu^{(+)} - A_\mu^{(-)}\right].$$

$$\begin{aligned} S_{\text{eff}} &= \frac{C_{2s}}{4\pi^2} \int d^5x \epsilon^{\mu\nu\lambda\rho\sigma} \tilde{A}_\mu \partial_\nu A_\lambda \partial_\rho A_\sigma \\ &+ \frac{C_{2s}}{12\pi^2} \int d^5x \epsilon^{\mu\nu\lambda\rho\sigma} \tilde{A}_\mu \partial_\nu \tilde{A}_\lambda \partial_\rho \tilde{A}_\sigma. \end{aligned}$$

$$\tilde{A}_\mu = 0, \; A_\mu = (-zE_z, -yB_z, 0, 0, 0),$$

we obtain,

$$J^w = 0, \; \tilde{J}^w = \frac{C_{2s}}{4\pi^2} E_z B_z.$$

4D QVHE

Similarly to the previous case we have $A_\mu^{(\pm)} = A_\mu \pm b_\mu$, where $b_\mu = (b_0, \mathbf{b}, 0)$ and behaves like a axial gauge field.

$$S_{\text{v,eff}} = \frac{C_{2v}}{12\pi^2} \int d^5x \epsilon^{\mu\nu\lambda\rho\sigma} \left(3b_\mu \partial_\nu A_\lambda \partial_\rho A_\sigma + b_\mu \partial_\nu b_\lambda \partial_\rho b_\sigma\right).$$

Without loss of generality, we take $b_\mu = (b_0, b_1(y), b_2, b_3(t), 0)$. Defining the (pseudo)-magnetic and (pseudo)-electric fields as

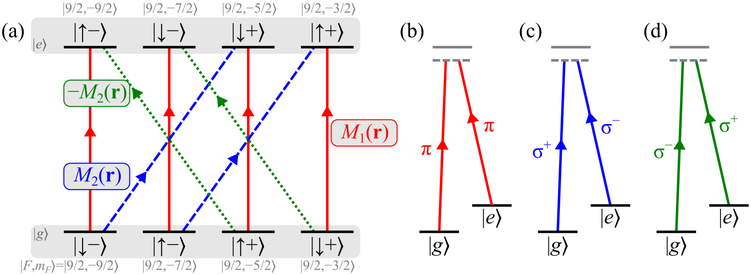
$$\mathbf{B} = \nabla \times \mathbf{A},\ \ \mathbf{E} = \partial_t \mathbf{A} - \nabla A_0,$$
$$\mathbf{B}^5 = \nabla \times \mathbf{b},\ \ \mathbf{E}^5 = \partial_t \mathbf{b} - \nabla b_0.$$

We obtain charge and valley currents,

$$J^w = \frac{C_{2v}}{2\pi^2}(E_z^5 B_z + E_z B_z^5),$$
$$J_5^w = \frac{C_{2v}}{4\pi^2}\left(E_z B_z + E_z^5 B_z^5\right)$$

J_5^w can be associated to the "*4D quantum valley Hall effect*".

Cold-atom implementation



Since there are eight degrees of freedom in the model Hamiltonian, a candidate choice is the use of the alkaline-earth-metal atoms. It has a meta-stable excited-state manifold $|e\rangle$ besides the ground-state manifold $|g\rangle$, from which we respectively select four hyperfine states as the pseudo-spins to construct the extra spaces. In particular, we specify the denotations to pseudo-spins as $\alpha = \uparrow\downarrow$ and $\beta = \pm$. We prepare the eight states are coupled by three groups of optical fields.

Cold-atom Hamiltonian

$$H_0 = \hat{H}_0 + H_{\rm so}^{(\alpha)} + H_{\rm so}^{(\beta)} \,.$$

The first part of Hamiltonian describes the local energy of the atoms and we employ the tight-binding approximation. $\hat{H}_0$ is then expressed as

$$\hat{H}_0 = \sum_j (-J\sigma_0 \otimes \sigma_0 \otimes \sigma_0 \psi_j^\dagger \psi_{j+1} + H.c.) + \hat{V}\sigma_3 \otimes \sigma_0 \otimes \sigma_0 \psi_j^\dagger \psi_j \,.$$

Here, $\psi_j = (e_{j\uparrow+}, e_{j\uparrow-}, e_{j\downarrow+}, e_{j\downarrow-}, g_{j\uparrow+}, g_{j\uparrow-}, g_{j\downarrow+}, g_{j\downarrow-})^T$ and λ_j denotes the atomic annihilation operator on the j-th site.

$H_{\rm so}^{(\alpha)}$ describes the coupling between the pseudo-spins $\alpha = \uparrow\downarrow$. Its form is given as follows,

$$H_{\rm so}^{(\alpha)} = \int d\mathbf{r} \sum_\beta M_1(\mathbf{r}) \Big[\psi_{e,\uparrow,\beta}^\dagger(\mathbf{r}) \psi_{g,\downarrow,\beta}(\mathbf{r}) + \psi_{e,\downarrow,\beta}^\dagger(\mathbf{r}) \psi_{g,\uparrow,\beta}(\mathbf{r}) \Big] + H.c.$$

Finally, $H_{\rm so}^{(\beta)}$ describes the coupling between the pseudo-spins $\beta = \pm$.

Conclusions and Outlook

- I have presented a novel 4D model that supports a quantized second spin Chern number and topologically protected edge states.
- I have shown that higher-dimensional mixed Chern-Simons theories can describe the 4D QSHE and VQHE in our 4D model in the low-energy regime.
- I have briefly presented a cold-atom implementation of our 4D Dirac Hamiltonian.
- In future work, we will explore thermal quantum effects in this 4D system that can be probed by introducing an effective curved background and can be described by higher-dimensional gravitational Chern-Simons theories.
- We are finalizing a work to implement the *second Euler number* in synthetic matter (it will appear on arXiv this month).