SOME POINTS ABOUT THE BMS GROUP

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Introduction

The possibility of replacing the Poincaré group (P from now on) with the Bondi-Metzner-Sachs group (BMS from now on) has a long story. P is the semidirect product of SL(2,C) (the group of the 2×2 matrices with complex entries and determinant equal to one) which is the universal covering group of the Lorentz group, with the space-time translations, which are the vector representation of SL(2,C) and make an abelian group with respect to vector adddition. As the space of translations is finite dimentional, there is essentially only one topology that makes P a continuous group without pathological features.

Analogously BMS is the semidirect product of SL(2,C) times the abelian group of supertranslations, i. e, an infinite-dimensional real vector space Λ of suitably smooth functions $f(\theta,\phi)$ defined on the sphere S^2. The elements of SL(2,C) act on Λ according to the $D_{2,2}$ representation of SL(2,C), following the classification of Gel'fand *et al.*[1]. The space Λ is infinite dimensional, so that the topology that makes BMS a continuous group is not unique at all.
Therefore the representations of BMS can be quite different according to the topology to be imposed upon supertranslations.

This situation is akin to that of the representations of the CCR in field theory.
Also, some models for current algebras and some gauge groups in General Relativity have a structure similar to that of BMS.
Therefore the study of the representations of BMS might be seen as a sidewise approach to quantum theories of infinitely many degrees of freedom.

The representations of P

We are interested in the unitary irreducible representations (UIRs) of P. Given the similarity in the structure between BMS and P, we first give a shortest account of the UIRs of P
They were built by E. P. Wigner[2] in 1939.
One first considers the dual space of translations, which is physically interpreted as the space of momenta. Choesen a particular momentum p_A, call it "standard momentum" the orbit Ω_A that contains it is the set of all momenta that can be obtained from p_A transforming it by a Lorentz (SL(2,C)) transformation:

$$p \in \Omega_A \iff \exists g \in SL(2,C)\,, p = g p_A\,. \tag{1}$$

Next one finds the "little groups" of the momenta; they are the subgroups of SL(2,C) that leave invariant some momentum. The little groups of the elements in one orbit are all isomorphic among them and to that of the "standard momentum" of each orbit.

Table 1: Standard momenta and their little groups, after S. Weinberg, *The Quantum Theory of Fields*, with some modification of the notation. The symbol κ is for an arbitrary positive quantity with the dimension of energy.

	Orbit	Standard momentum p_0	Little Group
(a)	$p^2 = m^2 > 0,\ p^0 > 0$	$(m,\ 0,\ 0,\ 0)$	SO(3)
(b)	$p^2 = m^2 > 0,\ p^0 < 0$	$(-m,\ 0,\ 0,\ 0)$	SO(3)
(c)	$p^2 = 0,\ p^0 > 0$	$(\kappa,\ 0,\ 0,\ \kappa)$	E_2
(d)	$p^2 = 0,\ p^0 < 0$	$(-\kappa,\ 0,\ 0,\ \kappa)$	E_2
(e)	$p^2 = -m^2 < 0,$	$(0,\ 0,\ 0,\ m)$	SO(2,1)
(f)	$p = 0,$	$(0,\ 0,\ 0,\ 0)$	SO(3,1)

E_2 is the group of rototranslations on the two-dimensional plane. Once we know the little groups and their UIRs, we can build the representations of P, or of BMS.

Infinite dimensional vs finite dimensional spaces

G. W. Mackey (see for example[8]) gave solid footing to the above with the induced representation theory, of which much can be used to build the UIRs of BMS.
What is remarkable is that if the space Λ is chosen to be countably normed, <u>*all*</u> the results of Mackey's apply [3]. This essentially happens because in such case the structure of BMS as a space with measure is not too different from that of the finite dimensional cases, say the Poincaré group.
P. J. McCarthy[5] first studied these representations giving the space Λ a Hilbert L^2

structure and in this case found that only compact little groups (or stability groups) arise, so that if the UIRs are associated to elementary particles only discrete spin appear. McCarthy however pointed out that non-compact little groups might appear with a different topology for Λ.

Little groups

The reason why this might happen is that a little group is defined as a subgroup of SL(2C) (in our case) that leaves invariant some element (or supermomentum) in Λ', the topological dual of Λ: if the topology of this last space changes, so its dual changes as well, and elements in it that are left invariant may disappear or new ones may appear. Accordingly, what was a little group for the previous topology is no longer a little group for the new topology or the other way around.
In particular, a refinement of the topology of Λ broadens Λ', so that new invariant supermomenta with associated little groups may appear.

Topology

It is interesting to consider the case of a "natural" topology for the supertranslation space (Λ is originally required to be C^2). In this sense one might consider topologies such that the completion of Λ is still smooth only. It is to be remembered that smoothness does not dictate a specific topology.

We considered[4] supertranslation spaces of functions on the sphere S^2 that are continuously differentiable up to order k, $0 \leq k \leq \infty$ ($C^k(S^2)$ spaces), each space with the topology of uniform convergence on compacta of the functions together with their first k derivatives, or to any order if $k = \infty$.
It is remarkable that with this topology non compact little groups are allowed for $C^2(S^2)$ and that if supertranslations are treated as $C^k(S^2)$ with the mentioned topology, the little groups, at least the connected ones, are the same as for the $k = 2$ case.

Differential equations

The little group of a supermomentum $F \in \Lambda'$ is, by definition, a subgroup of SL(2,C) that leaves F invariant, so that, if the little group is connected, its generators M_i annihilate F:

$$M_i F = 0 \quad i = 1, \dots n \,, \tag{2}$$

where n is the dimension of the Lie algebra.
These equations can be translated into a set of $2n$ differential equations in the following way.

Functions on the sphere

In order to define differentiable functions on the sphere S^2 at least two local charts are needed. These can be provided by two stereographic from two opposite poles, North and South, onto the respectively opposite tangent planes (from the North pole onto the tangent plane in the South pole and the other way around).
It is useful to pass from the real coordinates of these two planes, $\{x\,, y\}$ and $\{u\,, v\}$, to the corresponding complex conjugate coordinates $\{z\,, \overline{z}\}$ and $\{w\,, \overline{w}\}$, where

$$z = x + iy\,; \quad w = u + iv \tag{3}$$

and $\overline{a}$ is the complex conjugate of a.
A function Π on the sphere can be represented by a couple of functions $f^N(z\,, \overline{z})$ and $f^S(w\,, \overline{w})$ such that the equality holds:

$$f^S(w.\overline{w}) = f^N(\frac{1}{w}, \frac{1}{\overline{w}}) \quad \forall w \neq 0\,. \tag{4}$$

The function f^N is the representative of the function in the north pole chart, f^S is the representative for the South pole chart and condition (4) is the requirement that the two functions f^N and f^S define one and the same function Π on the sphere.
A function Π on S^2 is $\mathrm{C}^k(\mathrm{S}^2)$ iff both its local representatives f^N and f^S are $\mathrm{C}^k(\mathbb{R}^2)$. One can introduce a topology on the space of the functions on the sphere such that a sequence of functions Π_n converges to 0 iff both sequences f^N_n and f^S_n converge to 0 on any compact subset of $\mathbb{R}^2$. This can be implemented by introducing suitable norms that make $\mathrm{C}^k(\mathrm{S}^2)$ a Banach space or $\mathrm{C}^\infty(\mathrm{S}^2)$ a countably normed nuclear space, because the sphere S^2 is compact.

Distributions on the sphere

The dual spaces are built by means of a C^k ($k = \infty$ possibly) partition of unity (as suggested to us by G. Talenti[7], [6]), i. e. a couple of $C^k(S^2)$ functions Ψ_1 and Ψ_2 such that the North pole does not belong to the support of Ψ_2, the South pole does not belong to the support of Ψ_1 and for any point in S^2 the equality holds:

$$\Psi_1 + \Psi_2 = 1\,. \tag{5}$$

This condition is then transferred to the representatives of Ψ_1 and Ψ_2 in the North and South pole local charts.

Any distribution F on $C^k(S^2)$ defines two distributions F^N and F^S on $C^k(\mathbb{R}^2)$ in the following way:

$$(F, \Pi) = (F^N, \Psi_1^N f^N) + (F^S, \Psi_2^S f^S)\,, \tag{6}$$

where, if the suppport of Π contains neither the North nor the South pole, the equality must hold:

$$(F^N, f^N) = (F^S, f^S)\,. \tag{7}$$

On the other hand, if F^N and F^S are any two distributions on $C^k(\mathbb{R}^2)$ that satisfy condition (7) they define a unique distribution on $C^k(S^2)$ by means of equation (6) independently of the chosen partition of unity.
We remark that in this scheme the regularization of functions at infinity is not needed at all; this regularization is instead a shortcoming of other approaches.

Differential equations again

A generator M_i of the Lie algebra of a connected continuous subgroup of SL(2,C) is implemented by two differential operators, each one for a local chart, M_i^N and M_i^S, each acting on the distributions on $C^k(\mathbb{R}^2)$.

Therefore the problem of solving equation (2) for the generator M_i for distributions on $C^k(S^2)$ is transformed into that of solving two differential equations

$$M_i^N F^N = 0\,; \quad M_i^S F^S = 0 \tag{8}$$

for distributions on $C^k(\mathbb{R}^2)$ for the North and South pole charts, and looking for those solutions of the two equations that coincide on the functions whose support contains neither the North nor the South pole, i. e. that satisfy equation (7).

A connected little group is then generated by the maximal subset of generators that satisfy equations (8) for some couple F^N and F^S of distributions on $C^k(\mathbb{R}^2)$ that also satisfy equation (7).

Some notation

In the following we drop indices N and S understanding that a function depending on $\{z, \overline{z}\}$ is the North chart representative, a function depending on $\{w, \overline{w}\}$ is the South chart representative.

Some notation follows.

$M_{i,j}$	The generator of rotations in the $\{i, j\}$ plane.
$M_{0,i}$	The generator of boosts along the space direction i.
r	the absolute value of z: $r = \|z\|$.
α	the argument of z, $z = r\exp(i\alpha)$.
a, b, c	arbitrary real constants.
s	an arbitrary complex constant.
p, q	arbitrary integer numbers.
$F^{(n)}(x)$	the n-th derivative of the distribution F with respect to the argument x.
$F^{(p,q)}(z, \overline{z})$	the p-th derivative with respect to z and the q-th derivative with respect to $\overline{z}$ of F.
$\mathrm{Reg}\, f(z, \overline{z})$	the regularization of the function f in the origin.

List of the little groups

Madamina il catalogo è questo
delle belle che amò il padron mio.
Un catalogo egli è che ho fatt'io;
osservate leggete con me.

We list the little groups together with the implementation of their generators and the associated invariant distributions in the North pole chart.

1. Non compact.

$$M_{0,3} = -3i(1+r^2)^{-1}(r^2-1) + z\partial_{\overline{z}} + \overline{z}\partial_z\,;$$
$$F = (1+r^2)^3 G\,,$$

where G is an arbitrary real homogeneous distribution of degree -3.

2. Compact.

$$M_{1,2} = i(z\partial_z - \overline{z}\partial_{\overline{z}})\,;$$
$$(F,f) = (G,(1+r^2)^3\int_0^{2\pi} f(r,\alpha)\,\mathrm{d}\alpha)\,,$$

where G is an arbitrary distribution in the variable r only.

3. Non compact.

$$M_{0,2} - M_{3,2} = 3i(1+r^2)^{-1}(z-\overline{z}) - iz^2\partial_z + i\overline{z}^2\partial_{\overline{z}}\,;$$
$$F = (1+r^2)^3[a\mathrm{Reg}\,(z+\overline{z})^{-3} + b\delta^{(2)}(z+\overline{z})]\,.$$

4. Non compact. The infinitesimal generator is $M_{0,3} + \rho M_{1,2}$, with ρ real positive and the two generators are given in points 1 and 2. An invariant distribuiton for this group which is not invariant for larger connected groups is:

$$F = (1+r^2)^3 r^{-3}\{a\cos[p(\rho\ln r - c)] + b\sin[q(\rho\ln r - c)]\}\,,$$

5. Non compact.

$$M_{0,2} - M_{3,2}\,;$$
$$M_{0,1} - M_{3,1} = 3i(1+r^2)^{-1}(z+\overline{z}) - i(z^2\partial_z + \overline{z}^2\partial_{\overline{z}})\,;$$
$$F = (1+r^2)^3[s\delta^{(0,2)}(z,\overline{z}) + \overline{s}\delta^{(2,0)}(z,\overline{z})]\,.$$

6. Non compact.

$$M_{1,2}\,;\quad M_{0,3}\,;$$
$$F = a(1+r^2)^3 z^{-3/2}\overline{z}^{-3/2}\,.$$

7. Non compact.

$$M_{0,3}\,; \quad M_{0,1} - M_{3,1}\,;$$
$$F = a(1+r^2)^3(\delta^{(1,0)}(z,\overline{z}) - \delta^{(0,1)}(z,\overline{z}))\,.$$

8. Non compact. This little group is isomorphic to E_2, the Euclidean group in two dimensions.

$$M_{1,2}\,; \quad M_{0,1}-M_{3,1}\,; \quad M_{0,2} - M_{3,2}\,;$$
$$F = (1+r^2)^3[a\delta^{(2,2)}(z,\overline{z}) + b\delta(z,\overline{z})]\,.$$

9. Compact. The little group is SU(2) wjth associated invariant distributions the real constants.

10. Non compact. The little group is SL(2,R), the Lorentz group in two dimensions. The invariant distributions F act on the test functions f according to:

$$(F,f) = (a\mathrm{Reg}\, y^{-3} + b\delta^{(2)}(y), \int_{-\infty}^{\infty} (1+x^2+y^2)^3 f(x,y)\, \mathrm{d}x\,,$$

where it is more handy to express the distribution in terms of x and y.

11. Non compact. The last little group is SL(2,C) itself, with invariant distribution $F=0$. This has non physical relevance.

Comments

All the listed little groups have at least one invariant distribution for all spaces $C^k(S^2)$ for $k \geq 2$ (the solutions for SU(2) and a solution for E_2 are distributions for $k \geq 0$ as well).
The conclusions hold for all topologies on the supertranslations that are coarser than those used here and finer than the weak ones.
It is to be borne in mind that the topology on the supertranslations space should be such that BMS be at least a topological group. For example, the topology of pointwise convergence is appealing, but then BMS is not a topological group.

References

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Alphabet

SL(2,C)	the group of 2×2 matrices with complex entries. It is the universal covering group of the Lorentz group.
Λ	the space of "supertranslations ", suitably smooth functions Π on the sphere S^2.
$\{z, \overline{z}\}$	the complex coordinates on the stereographic projection of the sphere from the North pole upon the tangent plane at the South pole. By this projection a function
Π	on the sphere is represented by
$f^N(z, \overline{z})$	on the plane.
$\{w, \overline{w}\}$	the complex coordinates on the stereographic projection of the sphere from the South pole upon the tangent plane at the North pole. By this projection a function
Π	on the sphere is represented by

$f^S(w,\overline{w})$ on the plane.
L^2 a space of square summable functions. $\mathrm{C}^k(\mathrm{S}^2)$ the space of continuously differentiable functions up to order k on the sphere.
$\mathrm{C}^k(\mathbb{R}^2)$ the space of continuously differentiable functions up to order k on the real plane.
$f^N(z,\overline{z})$ is in $\mathrm{C}^k(\mathbb{R}^2)$.
$f^S(w,\overline{w})$ is in $\mathrm{C}^k(\mathbb{R}^2)$.
$\Psi_1\,,\Psi_2$ a partition of unity, a couple of functions on the sphere such that
$\Psi_1+\Psi_2=1$ holds, plus some additional requirements.
Λ' the linear topological dual space of Λ.
F an element of Λ', a "supermomentum ".
$F^N(z,\overline{z})$ the representative of a supermomentum in the North pole chart.
$F^S(w,\overline{w})$ the representative of a supermomentum in the South pole chart.
M_i the generators of the Lie algebra of a subgroup of SL(2,C).
little group , or stability group. A subgroup of SL(2,C) that leaves invariant some supermomentum $F\in\Lambda'$.

Transformation laws of supertranslations and supermomenta

If $f(z,\overline{z})$ is a supertranslation first perform the transformation:

$$f(z,\overline{z})=(1+|z|^2)^{-1}\phi(z,\overline{z})\,.$$

Then it can be seen that the element $g\in\mathrm{SL(2,C)}$:

$$g=\begin{vmatrix}\alpha & \beta\\ \gamma & \delta\end{vmatrix}$$

acts on the function ϕ according to:

$$(g\phi)(z,\overline{z})=|\beta z+\delta|^2\phi(\frac{\alpha z+\gamma}{\beta z+\delta},\frac{\overline{\alpha z}+\overline{\gamma}}{\overline{\beta z}+\overline{\delta}})$$

which is explicitly the representation D_{22} of Gel'fand et al.
A regular distribution represented by the function F acts upon a test function ϕ according to:

$$(F,\phi)=\int\frac{1}{(1+|z|^2)^2}F(z,\overline{z})\phi(z,\overline{z})\,\mathrm{d}z\mathrm{d}\overline{z}\,.$$

An element $g \in \mathrm{SL(2,C)}$ acts upon regular distributions according to:

$$(gF)(z,\overline{z}) = |\beta z + \delta|^{-6} F(\frac{\alpha z + \gamma}{\beta z + \delta}, \frac{\overline{\alpha z} + \overline{\gamma}}{\overline{\beta}\overline{z} + \overline{\delta}})$$

The induced representations

Here we show how to build the UIRs of BMS (or P) once the space of supermomenta and the little groups are known. We set aside measure theory technicalities that though very important can be dealt with easily in the case supertranslations are given a countably normed space structure.

We choose an orbit in Λ', say Ω_0, obtained by the application of SL(2,C) to some standard supermomentum F_0. Let G_0 be the little group of F_0. For any supermomentum $F \in \Omega_0$ let us choose a unique transformation $g_0(F)$ in SL(2,C) that transforms F_0 into F.

Consider the Hilbert space of the square summable functions $h(F)$ defined on Ω_0 with values in the carrier space H_0 of an UIR U_0 of G_0.

The action $D(\Pi)$ of a supertranslation Π upon the function h is represented by a phase factor:

$$[D(\Pi)h](F) = \exp[i(F,\Pi)]h(F)\,, \tag{9}$$

where (F,Π) represents the action of the distribution F on the function Π as given in equation (6).

Next the action $D(g)$ of an element $g \in \mathrm{SL(2,C)}$;.

$$[D(g)h](F) = U_0\left(g_0^{-1}(F) g g_0(g^{-1}(F))\right) h\left(g^{-1}F\right)\,, \tag{10}$$

The argument $g_0^{-1}(F) g g_0(g^{-1}(F)$ of U_0 is the "Wigner Rotation"; it is an element of G_0, the little group of F_0; as said above U_0 is a UIR of G_0, so that the transformation makes sense.