# Non invertible generalized symmetries in various dimensions

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with Jeremias Aguilera-Damia, Eduardo Garcia-Valdecasas and Luigi Tizzano.

# Motivation

Symmetries are an important ingredient to characterize a QFT.

The more symmetries we find, the better we specify/constrain the QFT.

It is then helpful to outline the most general concept of symmetry.

In other words, we would like not to confine ourselves to the vanilla symmetries acting on the lagrangian.

$\Rightarrow$ In order to do this, the vanilla symmetries need first to be formulated in an easily generalizable way. [Gaiotto, Kapustin, Seiberg, Willett, 14]

A continuous symmetry which transforms the fields but keeps the action invariant is associated by Noether's theorem to a conserved current $\partial_\mu j^\mu = 0 \quad \Leftrightarrow \quad d * j = 0$ with $j$ a 1-form

The action on the charged local objects is effectuated by

$$U_\alpha(\Sigma_{d-1}) = e^{i\alpha Q(\Sigma_{d-1})} = e^{i\alpha \int_{\Sigma_{d-1}} *j}$$

which depends on the codimension-1 surface $\Sigma_{d-1}$ only topologically.
$\Rightarrow$ They can be called topological symmetry defects.
Their action on a local operator $\mathcal{O}_q(x)$ of charge $q$ is given by

$$U_\alpha(\Sigma_{d-1})\mathcal{O}_q(x) = e^{iq\alpha}\mathcal{O}_q(x)U_\alpha(\Sigma'_{d-1})$$

where $\Sigma_{d-1}$ encloses the point $x$ and $\Sigma'_{d-1}$ does not.

The first immediate generalization is to discrete groups: the definition above no longer needs the existence of a conserved current.
$\Rightarrow$ Conservation of $j$ is traded for topological nature of $U$.

# Generalized symmetries [Gaiotto, Kapustin, Seiberg, Willett, 14]

Given the above formulation of symmetry defects, the most direct generalization is to change their (co)dimension.

The symmetry defect must link the charged operator defined on $\gamma_p$:

$$U_\alpha(\Sigma_{d-p-1})\mathcal{O}_q(\gamma_p) = e^{iq\alpha L(\Sigma_{d-p-1},\gamma_p)}\mathcal{O}_q(\gamma_p)U_\alpha(\Sigma'_{d-p-1})$$

otherwise there is no way to distinguish l.h.s. and r.h.s.
For continuous symmetries, one can define the charge in terms of a current:

$$Q(\Sigma_{d-p-1}) = \int_{\Sigma_{d-p-1}} *j^{(p+1)}$$

The current is now a $(p+1)$-form and it is conserved $d * j^{(p+1)} = 0$.

These are called $p$-form symmetries because they act on charged operators which are defined on $p$-dimensional surfaces.

The multiplication of elements of the symmetry group is represented by the fusion of overlapping defects:

$$U_\alpha(\Sigma_{d-p-1})U_\beta(\Sigma_{d-p-1}) = U_{\alpha+\beta}(\Sigma_{d-p-1})$$

For codimension $> 1$ the group must be abelian, because one cannot sensibly order the defects.

However some theories can have topological defects that satisfy more general fusion rules—in particular that do not allow for an inverse

These defects still act on the (extended) operators of the theory, implementing a non-invertible symmetry.

- There are many instances of theories with non-invertible defects, where they arise in a very different way.

The most celebrated example is the one of the Ising CFT in 2d.

This theory has an ordinary symmetry defect $\mathcal{U}$, the one that implements the flip of spins.

At the CFT point it has an additional defect $\mathcal{D}$, implementing the Kramers-Wannier (self)duality.

- The latter defect is non-invertible!

Indeed the fusion rules of these defects/lines are not group-like, except $\mathcal{U}^2 = \mathbb{I}$ :

$$\mathcal{D}\mathcal{U} = \mathcal{U}\mathcal{D} = \mathcal{D} \quad , \qquad \mathcal{D}^2 = \mathbb{I} + \mathcal{U}$$

Generalizations of this story to 4d have been recently discussed, starting from [Choi, Cordova, Hsin, Lam, Shao, 21] and [Kaidi, Ohmori, Zheng, 21].

They arise in theories where there is a duality, at the self-dual points. E.g.: Maxwell, and $\mathcal{N} = 4$ SYM. In these examples (at $\tau = i$ ) there is just one non-invertible defect.

• A different kind of non-trivial fusion rules: continuous non-invertible 1-form symmetries in 4d Maxwell with $O(2)$ gauge group.

[see Thorngren, Wang, 21; Antinucci, Galati, Rizi, 22]

• Yet another set-up: rational non-invertible symmetries instead of an ABJ anomaly. [starting from Choi, Lam, Shao, 22; Cordova, Ohmori, 22]

⇒ We now discuss generalizations of the latter in various dimensions.

# Maxwell-Chern-Simons in 5d

[Aguilera-Damia, RA, Garcia-Valdecasas, 22]

5d Maxwell has a 1-form electric symmetry $U(1)_{\rm e}^{(1)}$ with $j_{\rm e}^{(2)} = \frac{1}{e^2} f^{(2)}$ and a 2-form magnetic symmetry $U(1)_{\rm m}^{(2)}$ with $j_{\rm m}^{(3)} = \frac{1}{2\pi} * f^{(2)}$.

We now add the 5d Chern-Simons term

$$S_{\rm MCS} \supset \frac{k}{24\pi^2} \int_{M_5} a^{(1)} \wedge f^{(2)} \wedge f^{(2)}$$

The naive charge for the electric symmetry is no longer topological:

$$Q_{\rm e,naive}(\Sigma_3) = \int_{\Sigma_3} \frac{1}{e^2} * f^{(2)} \qquad , \qquad \frac{1}{e^2} d * f^{(2)} = \frac{k}{8\pi^2} f^{(2)} \wedge f^{(2)} \neq 0$$

The conservation equation

$$d * j^{(2)}_{\mathrm{e,naive}} = \frac{k}{8\pi^2} f^{(2)} \wedge f^{(2)}$$

has an ABJ-like anomaly!

We can improve the definition of the charge to make it topological

$$Q_{\mathrm{e,imp}}(\Sigma_3) = \int_{\Sigma_3} \frac{1}{e^2} * f^{(2)} - \frac{k}{8\pi^2} a^{(1)} \wedge f^{(2)}$$

However $U_\alpha(\Sigma_3) = e^{i\alpha Q_{\mathrm{e,imp}}(\Sigma_3)}$ is not gauge invariant for generic $\alpha$ !

$\Rightarrow$ Only for $\alpha = \frac{2\pi n}{k}$ the 3d CS action has the proper quantized level.

The electric symmetry group is thus broken: $U(1)^{(1)}_{\mathrm{e}} \rightarrow \mathbb{Z}^{(1)}_k$

*Is this the end of the story?*

Take for simplicity $k = 1$ and try a rotation by $\alpha = \frac{2\pi}{N}$:

$$U_{\frac{2\pi}{N}}(\Sigma_3) = \exp\left( i \int_{\Sigma_3} \frac{2\pi}{Ne^2} * f^{(2)} - \frac{1}{4\pi N} a^{(1)} \wedge f^{(2)} \right)$$

The 3d CS term still not gauge invariant, but the theory of the Fractional Quantum Hall state has taught us how to handle such terms!

A gauge invariant way to produce the same response is by introducing a purely 3d gauge connection $c^{(1)}$

$$e^{-i \int_{\Sigma_3} \frac{1}{4\pi N} a^{(1)} \wedge da^{(1)}} \rightarrow \int \mathcal{D}c^{(1)} e^{i \int_{\Sigma_3} \frac{N}{4\pi} c^{(1)} \wedge dc^{(1)} + \frac{1}{2\pi} c^{(1)} \wedge da^{(1)}}$$

A gauge invariant topological defect generating a rotation by $\frac{2\pi}{N}$ is thus

$$D_{\frac{2\pi}{N}}(\Sigma_3) = \int \mathcal{D}c^{(1)} \exp\left(i\int_{\Sigma_3} \frac{2\pi}{Ne^2} * f^{(2)} + \frac{N}{4\pi}c^{(1)} \wedge dc^{(1)} + \frac{1}{2\pi}c^{(1)} \wedge f^{(2)}\right)$$

It must carry a non-trivial TFT that couples to the bulk gauge field strength $f^{(2)}$.

- There is a generalization for any angle such that $\frac{\alpha}{2\pi} \in \mathbb{Q}/\mathbb{Z}$.

Are the symmetry defects $D_\alpha(\Sigma_3)$ unitary operators?

It is simple to see that $D_{\frac{2\pi}{N}}(\Sigma_3)^\dagger = D_{-\frac{2\pi}{N}}(\Sigma_3)$

The question is what is $D_{\frac{2\pi}{N}}(\Sigma_3) D_{\frac{2\pi}{N}}(\Sigma_3)^\dagger$ ?

Each factor carries a non-trivial TFT, i.e. a different path integral.

For $f^{(2)} = 0$, it is a $U(1)_N \times U(1)_{-N}$ 3d CS theory $\Rightarrow$ non-trivial!

We have then that

$$D_{\frac{2\pi}{N}}(\Sigma_3)D_{\frac{2\pi}{N}}(\Sigma_3)^\dagger = C_{\frac{2\pi}{N}}(\Sigma_3) \neq \mathbb{I}$$

The defects generate a non-invertible symmetry.

We can thus say that the $U(1)_{\rm e}^{(1)}$ invertible symmetry of Maxwell becomes, due to the CS term, a $(\mathbb{Q}/\mathbb{Z})^{(1)}$ non-invertible symmetry.

- How do we detect that the defect is non-invertible?

Alternative way to define the defects: gauge a $\mathbb{Z}_N$ discrete subgroup of the magnetic $U(1)_{\rm m}^{(2)}$ symmetry, and only along an open co-dimension 1 surface $\Sigma_4$ $\Rightarrow$ one generates the symmetry and condensation defects on $\Sigma_3 = \partial\Sigma_4$. [Roumpedakis, Seifnashri, Shao, 22]

It is then intuitive to see that the defects have a non-trivial effect on 't Hooft surfaces.

# Reductions to 4d and 3d

The 5d model with only a gauge field can be dimensionally reduced.

In 4d, it generates a model of axion-electrodynamics, which realizes classically the ABJ anomaly of massless QED. This is the original model of [Choi, Lam, Shao, 22] and [Cordova, Ohmori, 22].

The topological term in the action is

$$S \supset \frac{k}{8\pi^2} \int_{M_4} \chi^{(0)} f^{(2)} \wedge f^{(2)}$$

and the rest follows very similarly.

- There are more symmetries: for $k = 0$ we would have

$$U(1)^{(0)}_{\chi} \times U(1)^{(1)}_{\mathrm{e}} \times U(1)^{(1)}_{\mathrm{m}} \times U(1)^{(2)}_{\mathrm{dual}}$$

We can further reduce to 3d: axion-Goldstone-Maxwell.

[Aguilera-Damia, RA, Tizzano, 22]

The topological term is now

$$S \supset \frac{k}{4\pi^2} \int_{M_3} \chi^{(0)} d\phi^{(0)} \wedge f^{(2)}$$

and even more $U(1)$s for $k = 0$: 6 of them!

(In all these models there is actually a higher-group structure!)

Both the defects for the electric 1-form symmetry and the ones for the 0-form shift symmetries become non-invertible.

They are given respectively by:

$$D_{\frac{2\pi}{M}}(\Sigma_1) = \int_{\hat{\xi},\hat{\chi}} \exp\left(i\int_{\Sigma_1} \frac{2\pi}{M} * f^{(2)} + \frac{M}{2\pi}\hat{\xi}d\hat{\chi} + \frac{1}{2\pi}\hat{\xi}d\chi - \frac{1}{2\pi}\phi d\hat{\chi}\right)$$

$$D_{\frac{2\pi}{N}}(\Sigma_2) = \int_{\hat{\xi},\hat{v}^{(1)}} \exp\left(i\int_{\Sigma_2} \frac{2\pi}{N} * d\chi + \frac{N}{2\pi}\hat{\xi}d\hat{v}^{(1)} + \frac{1}{2\pi}\hat{\xi}f^{(2)} - \frac{1}{2\pi}\phi d\hat{v}^{(1)}\right)$$

They are the naive defects dressed by reductions/generalizations of the FQH state TFT.

# 5d model with a boundary

With the perspective of doing holography, one can ask what happens if the 5d spacetime has a 4d boundary.

As in holography, a lot depends on the boundary conditions that one imposes on the dynamical fields.

Then one can ask whether the bulk charged (extended) objects and the symmetry defects can end on the boundary, or be parallel to it.

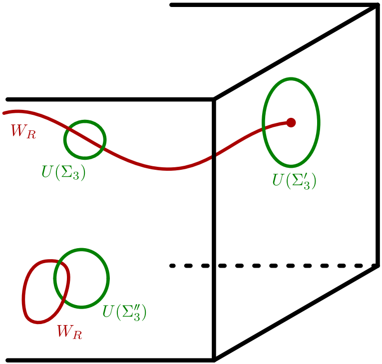
$\Rightarrow$ They can then be relevant to a (putative) boundary theory.

For Dirichlet b.c. for $a^{(1)}$, Wilson lines can end on the boundary.

$a^{(1)}$ becomes a background gauge field on the boundary, so that the end of the Wilson line carries a global charge there.

It is measured by the same defect $D(\Sigma_3)$, with $\Sigma_3 \subset \partial M_5$.

$D(\Sigma_3)$ becomes invertible there because $a^{(1)}$ is background.




It is actually impossible to impose Neumann b.c. on $a^{(1)}$!

- The global $U(1)$ on $\partial M_5$ has a cubic 't Hooft anomaly.

We need two bulk gauge fields with a mixed CS term to play a little more.

$$S \supset \int_{M_5} \frac{1}{8\pi^2} a_1^{(1)} \wedge da_2^{(1)} \wedge da_2^{(1)}$$

It is now possible to impose Neumann b.c. for $a_2^{(2)}$.

This has the effect of gauging the $U(1)_2^{(0)}$ symmetry on the boundary.

In this way one can engineer an ABJ anomaly for $U(1)_1^{(0)}$ on the boundary theory.

$\Rightarrow$ Then the non-invertible defects of the boundary theory can leave the boundary and live their life as non-invertible defects in the bulk.

# Outlook

Non invertible symmetries are fun to look for!

- A cornucopia of models, they appear in many different ways.

What are they good for?

- Constraints on RG flows.
- Indications on confinement or not, classification of confining strings and other (extended) objects.
- Symmetry=beauty?

*Thank you!*