Properties of average distances and emergent causality

Federico Piazza

With Andrew Tolley, 2212.06156
(see also 2108.12362)

Centre de Physique Théorique

Luciano Girardello Memorial, 16/1/2023

“Fatti dare un consiglio da zio”

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2005

Glimmers of a pre-geometric perspective

Federico Piazza

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Glimmers of a pre-geometric perspective

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Let me give you an ``uncle’s advice”

~ Work on problems that you can tackle with equations

“Fatti dare un consiglio da zio”

2021

~~2005~~

post-

Glimmers of a ~~pre~~-geometric perspective

Federico Piazza

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One Main Message

The average geodesic distance is not a geodesic distance.
It is generally **non-additive**

$$\bar{d}(x,y) \;\equiv\; \sqrt{\langle d^2(x,y)\rangle}$$

$\nexists\; g_{\mu\nu}(x)$ such that $\bar{d}(x,y)$ is the geodesic distance of $g_{\mu\nu}(x)$

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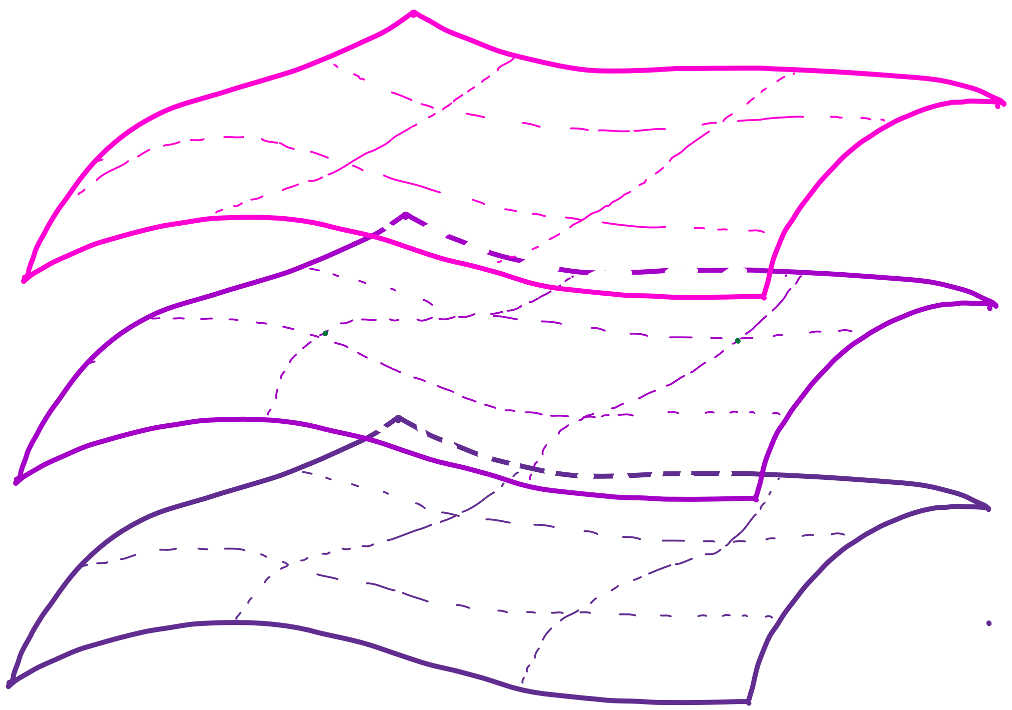
The metric is not enough!

In this talk: Quantum gravity $\neq$ UV

I just want to consider a superposition of metrics $\Psi[h_{ij}(x), \ldots]$



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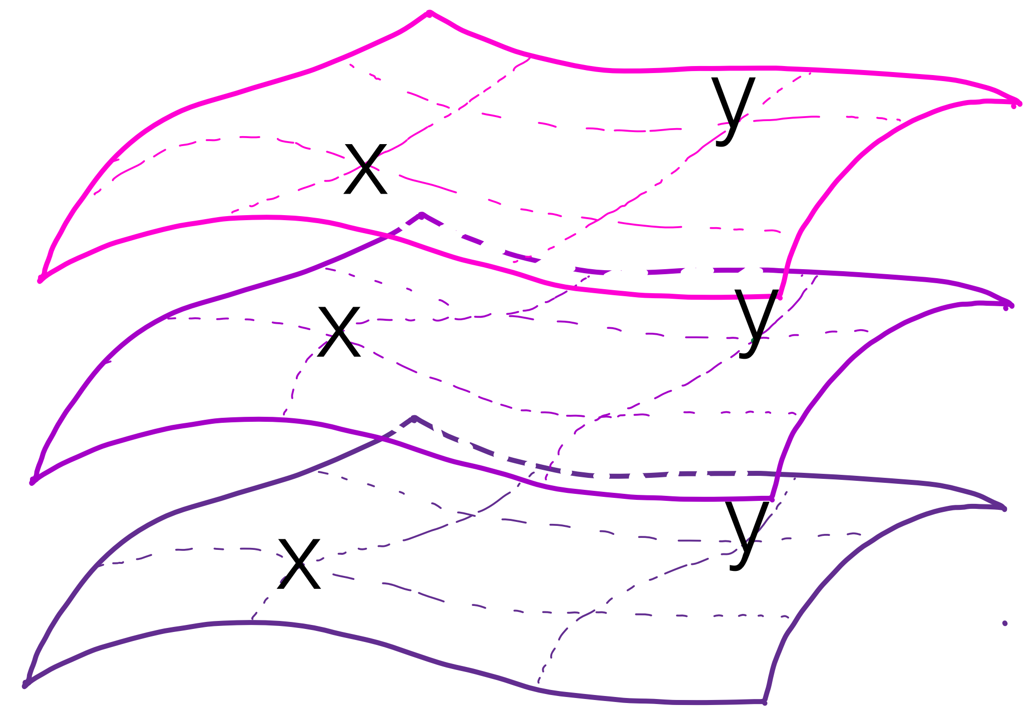
One known difficulty (gauge invariance) is that we need to establish *which point corresponds to which* within the elements of the statistical ensemble



Asymptotically flat: $\mathscr{A} \sim \langle \text{out} \,|\, \text{in} \rangle$

Asymptotically AdS: $\lim\limits_{r \to \infty} \langle \phi(x_1) \ldots \phi(x_n) \rangle$

$r \to \infty$

AdS boundary

$\delta g_{\mu\nu} \to 0$

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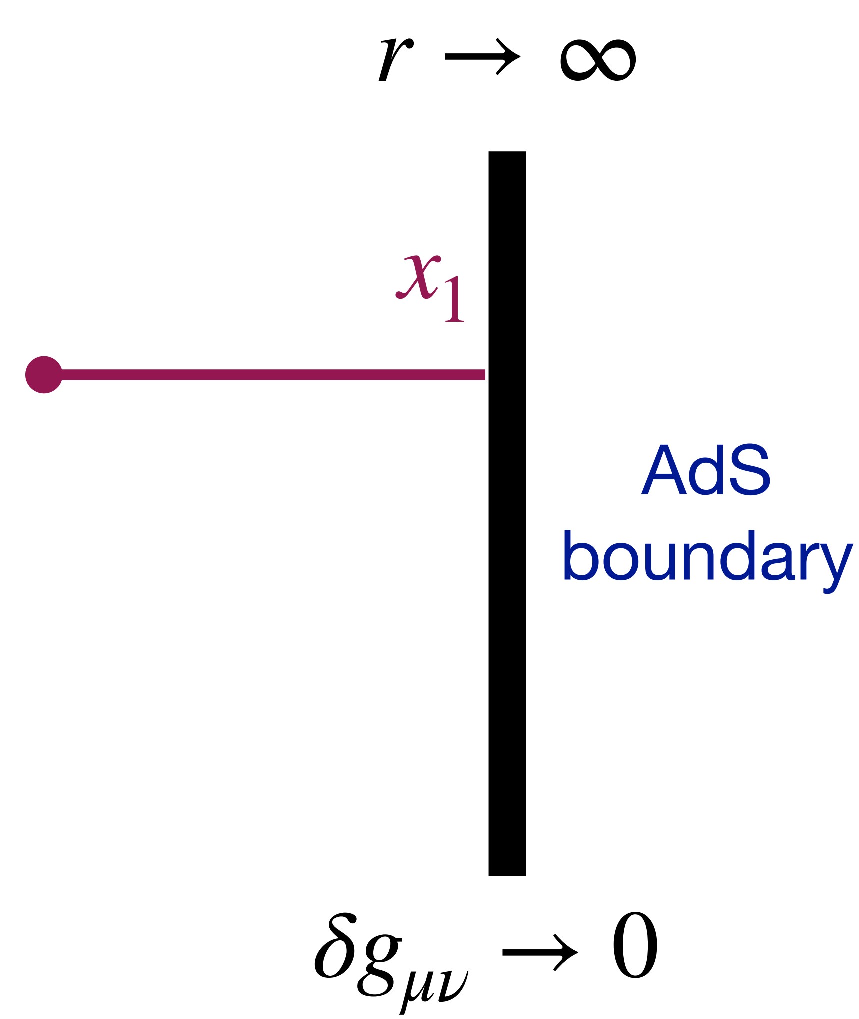
Venturing inside the bulk

Prescription: start from x_1 and move orthogonally inside AdS.
Follow the geodesic for 3.5 Km



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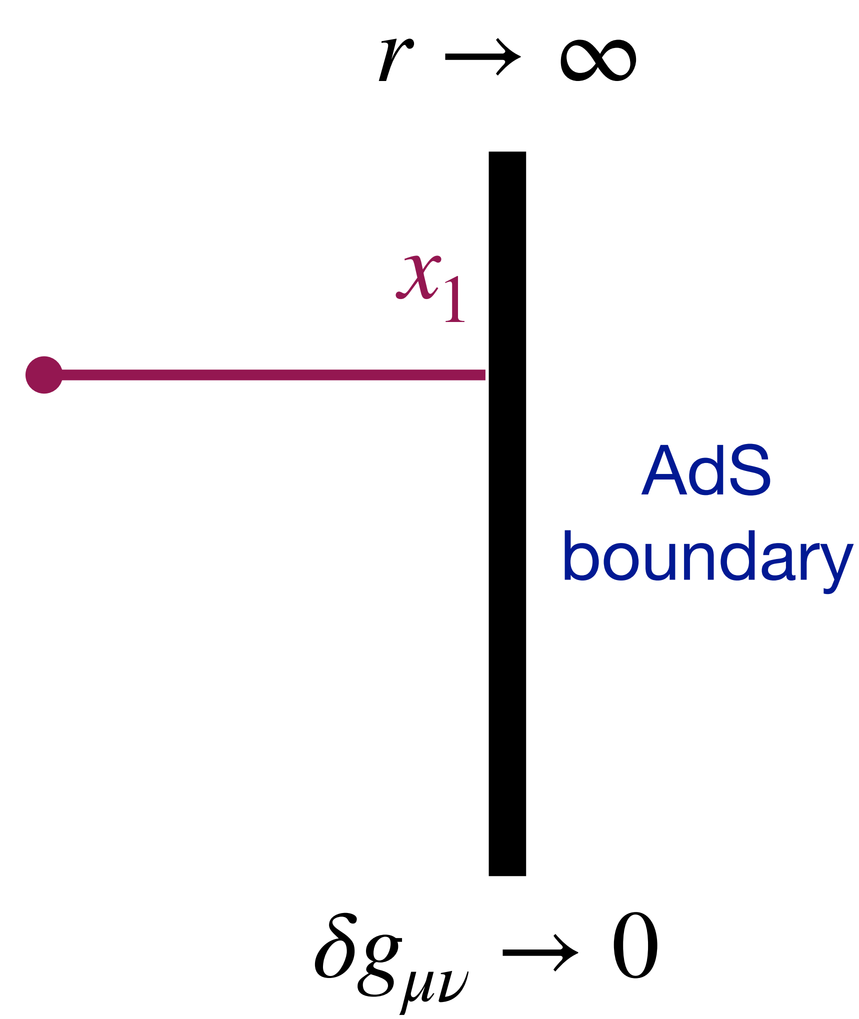

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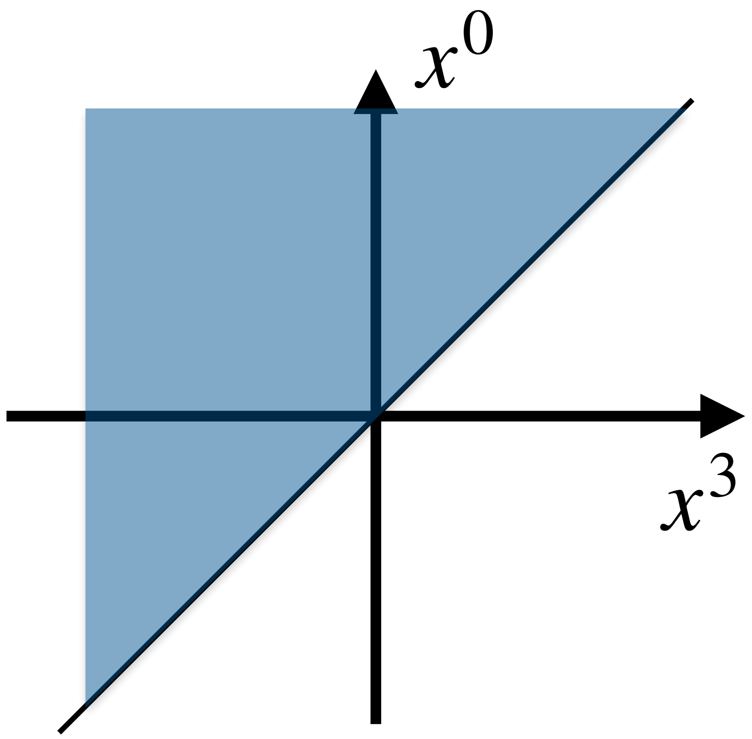
I will use timelike geodesics

Invitation: free falling into a GW superposition

Minkowski space is traversed by a gravitational wave at $x^0 = x^3$

Classical solution $\simeq$ coherent state $|\psi_1\rangle$



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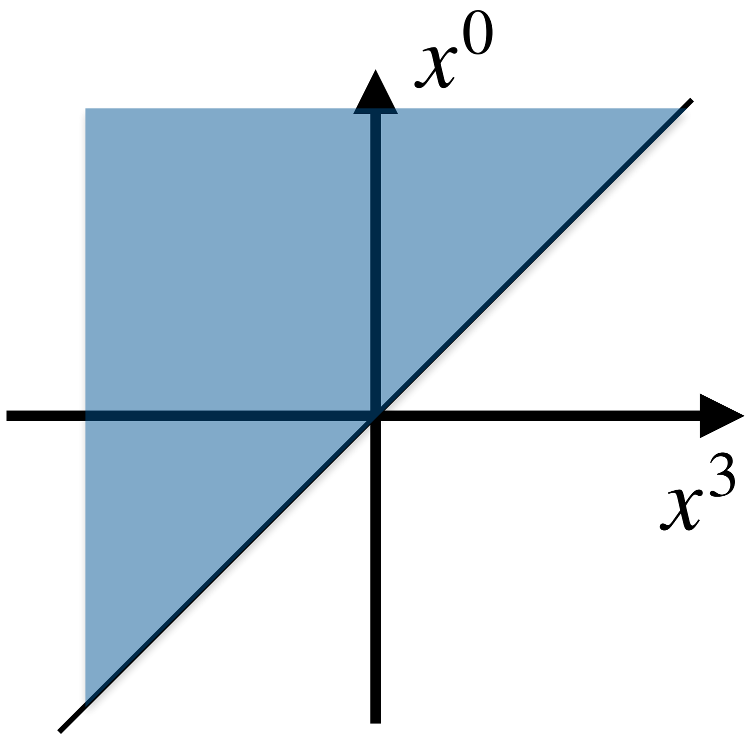

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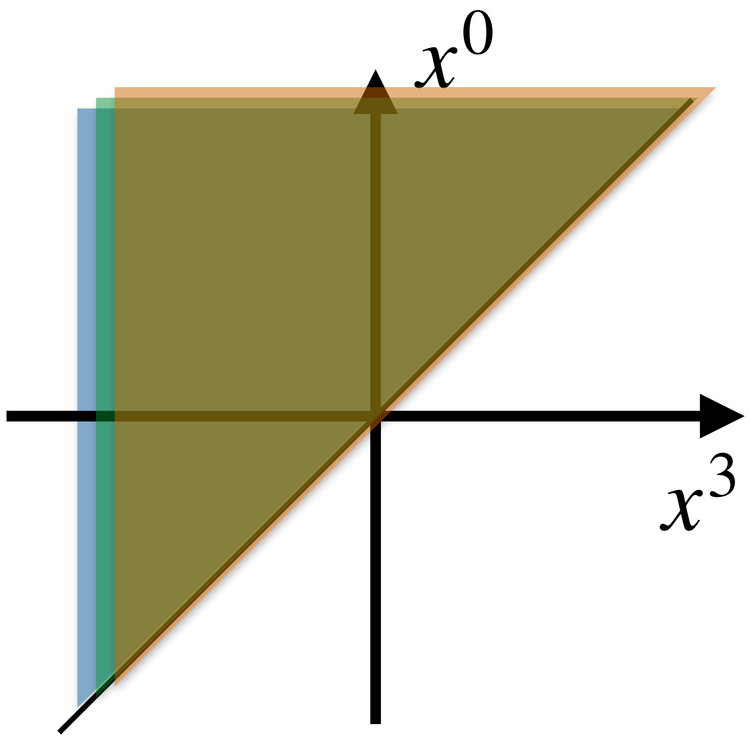
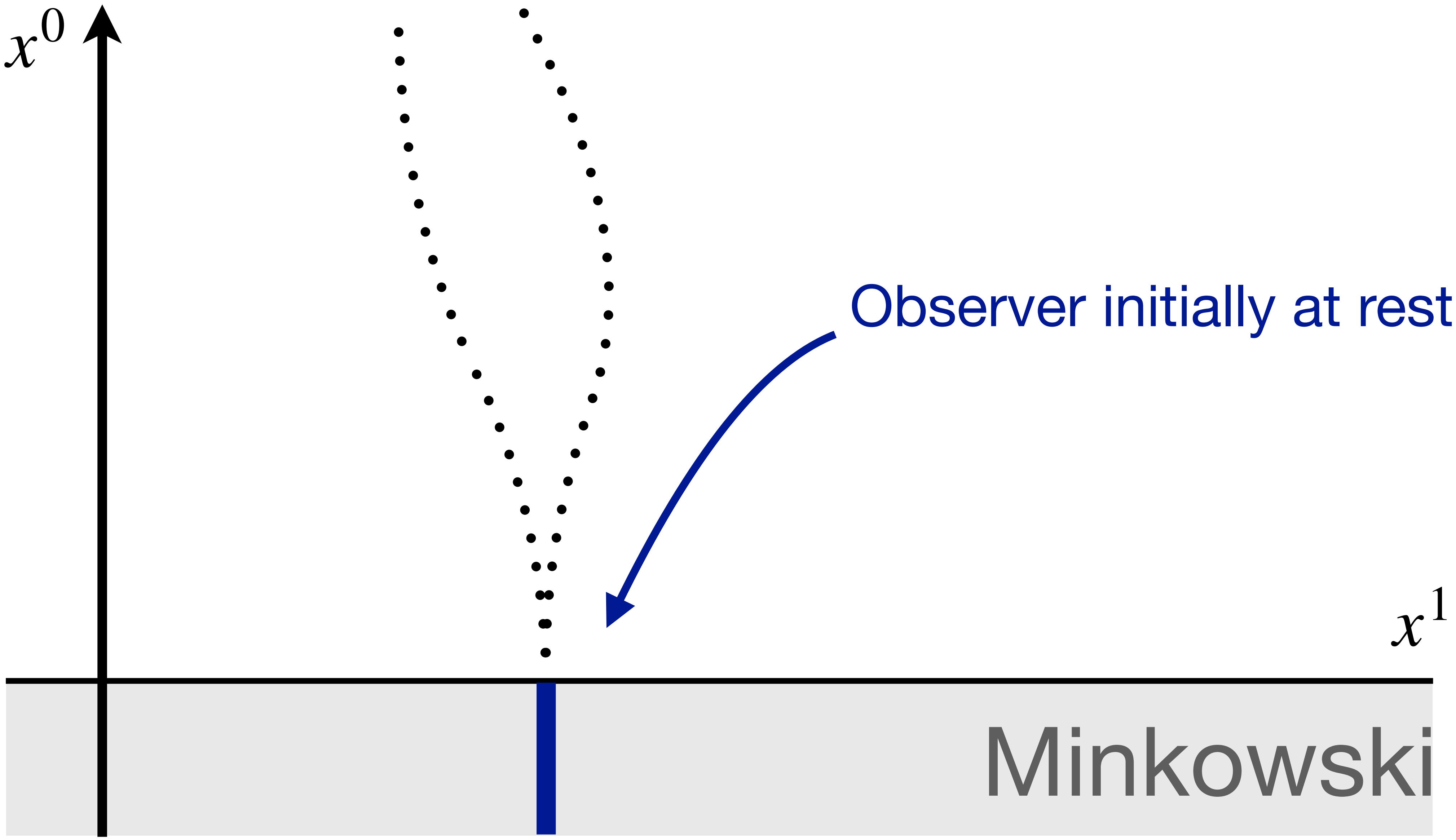
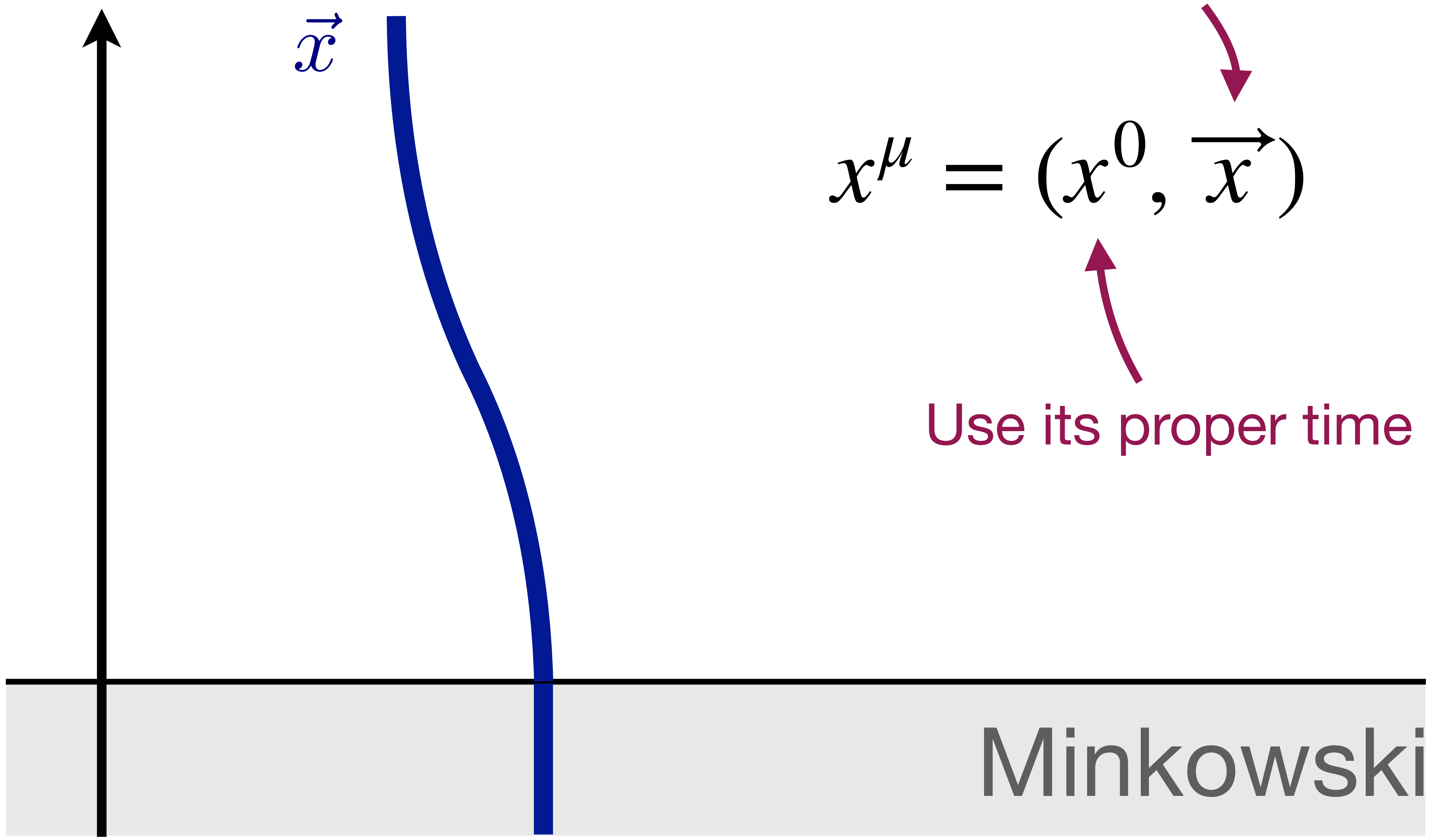


Quantum superposition $|\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle + \ldots$

Brinckmann coordinates: positions undetermined



Use Observers to define position!! (Rosen coord.)



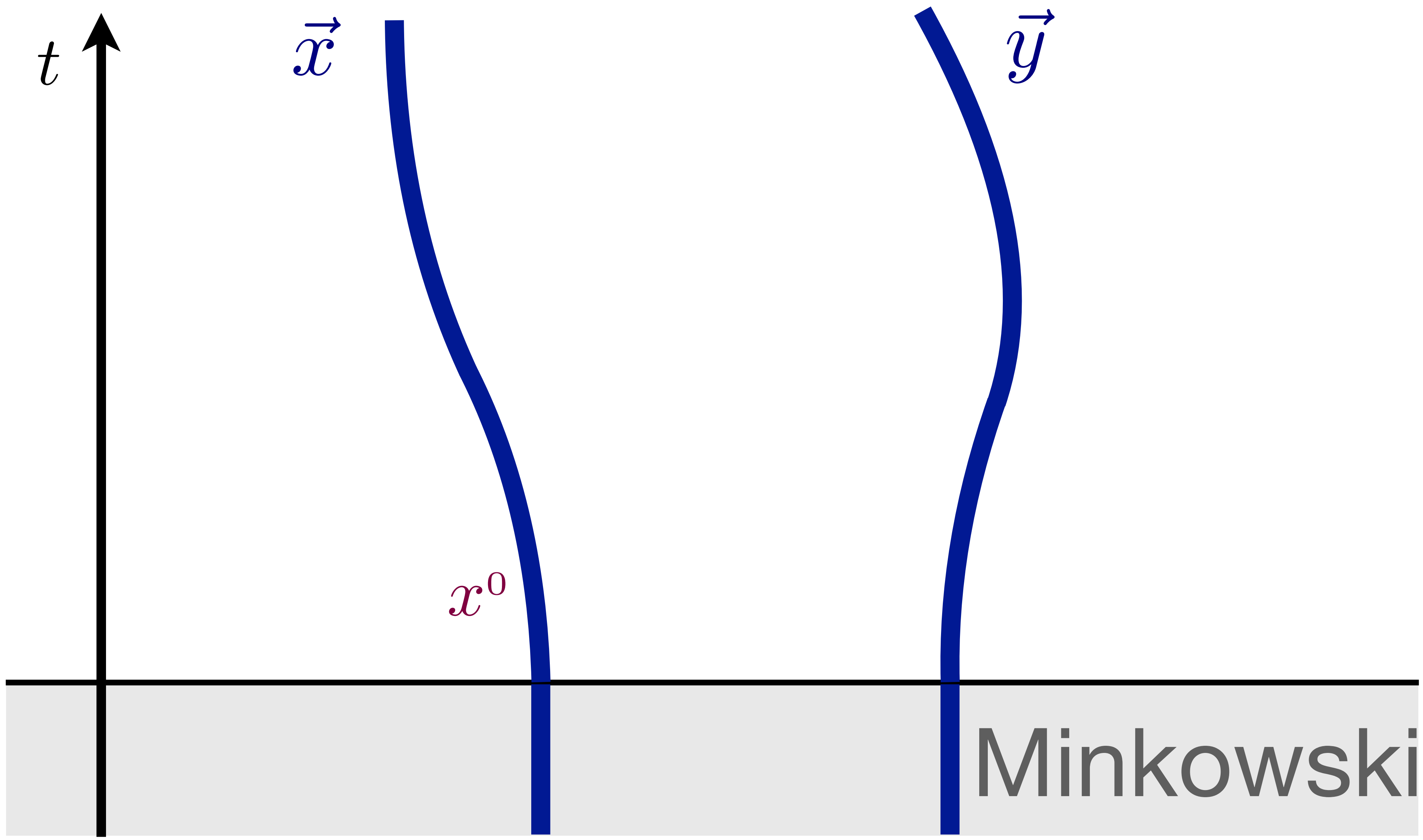
Label the observer with its initial Minkowski coordinate

$$x^\mu = (x^0, \overrightarrow{x})$$

Use its proper time

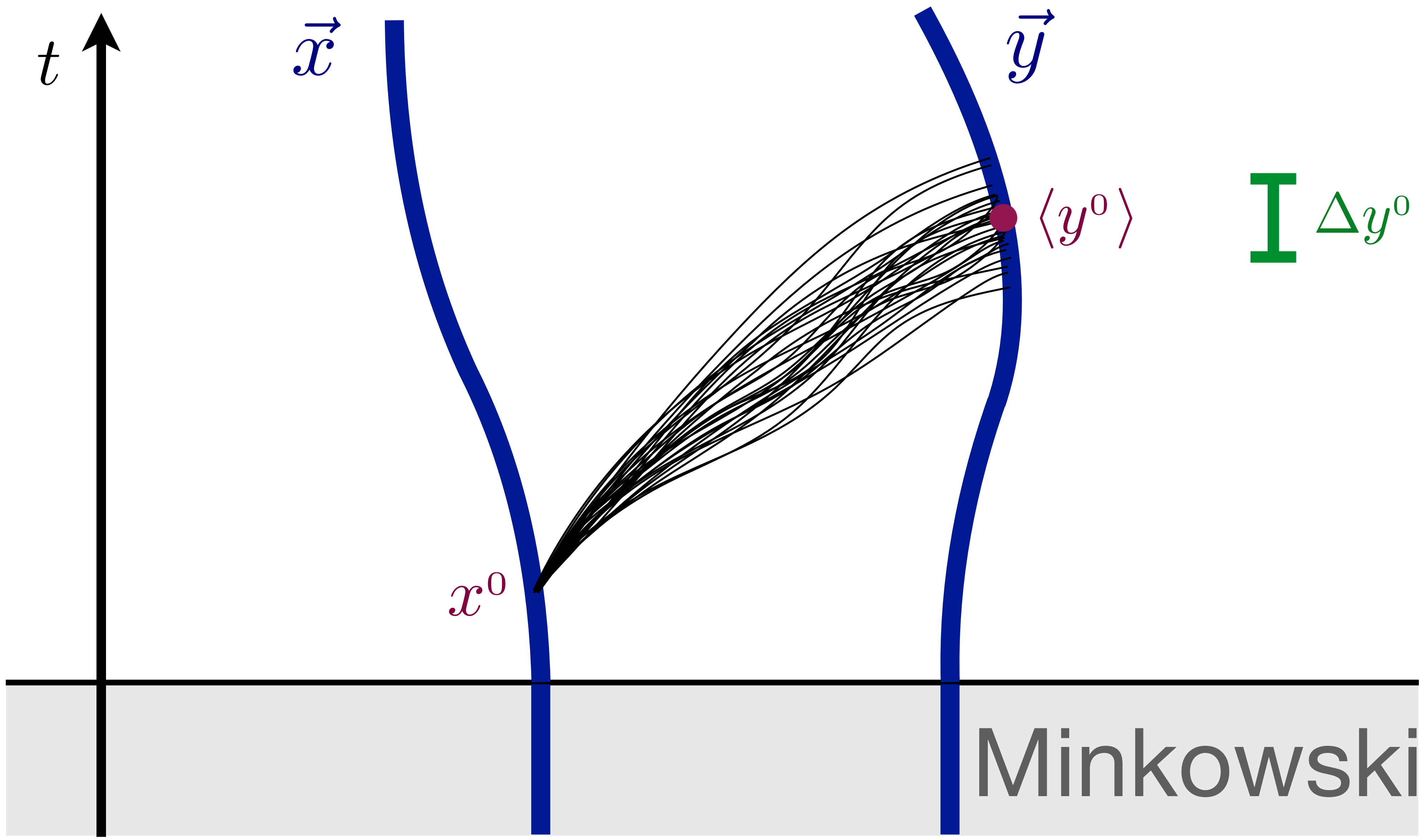
Gedanken experiments!

- $\overrightarrow{x}$ sends a photon at time x^0
- What's the probability that $\overrightarrow{y}$ detects it at time y^0?



Gedanken experiments!

Geometrical optics approx: photons follow geodesics



A proxy for causality:

$$\bar{d}(x,y) \;\equiv\; \sqrt{\langle d^2(x,y)\rangle}$$

"physical coordinates"

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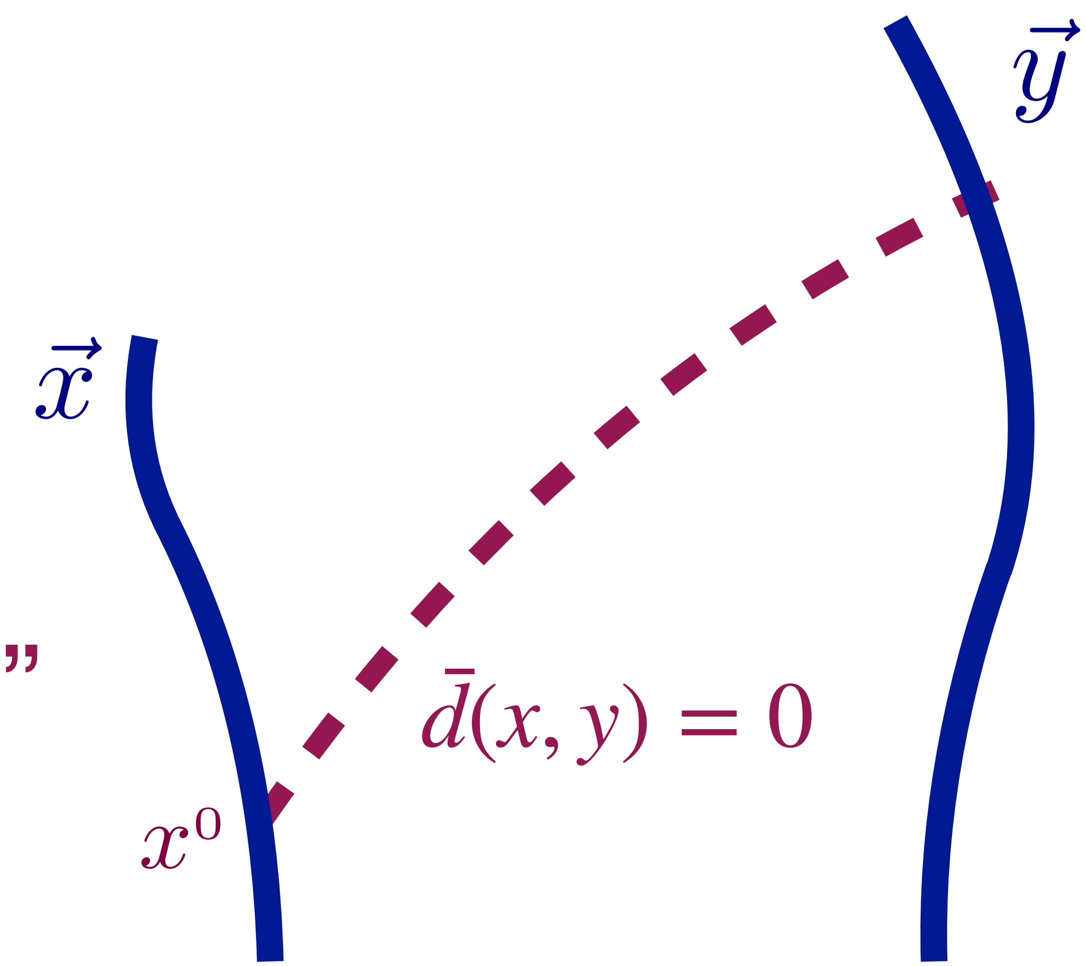
“physical coordinates”

“Average light cone structure”



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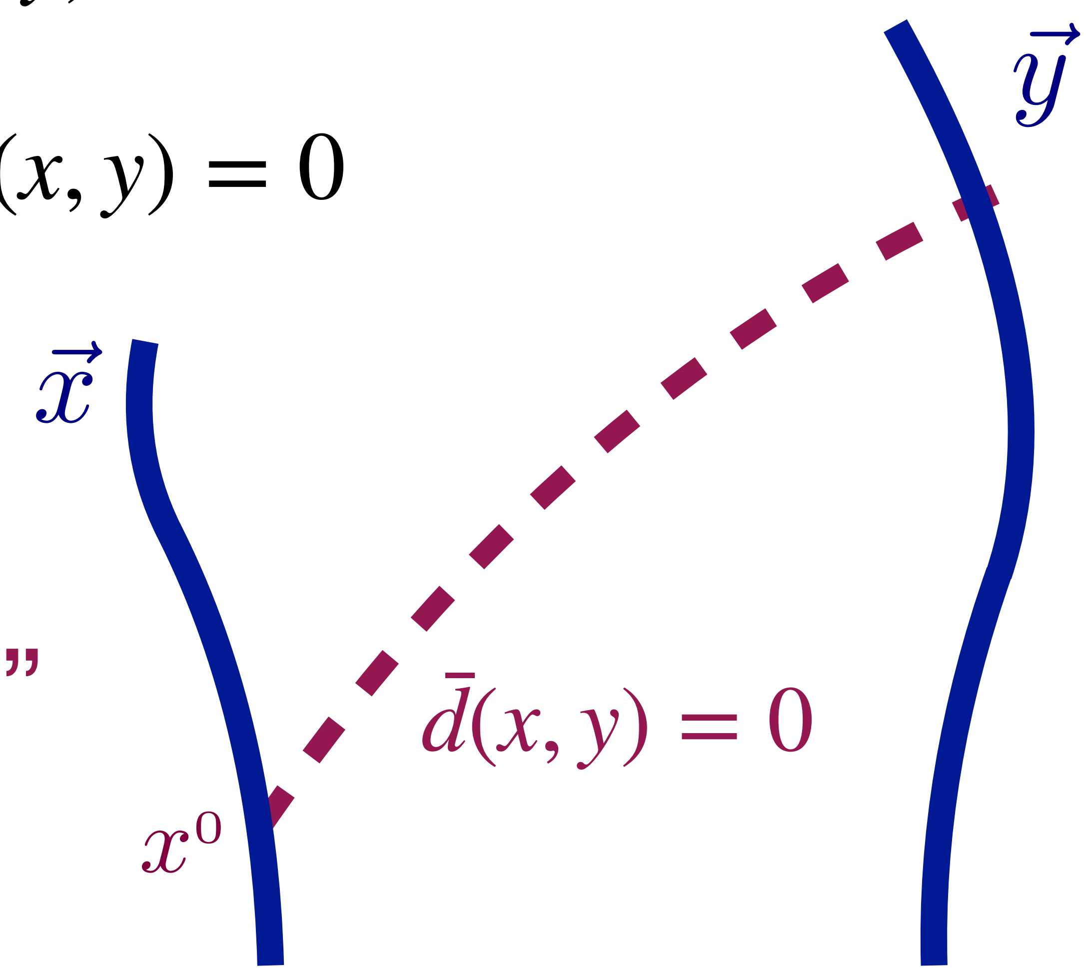
“physical coordinates”

Assumptions/hope:

(1) $\langle y^0 \rangle$ is well approximated by $\bar{d}(x,y) = 0$

(2) $\langle [A(x), B(y)] \rangle$ vanishes across $\bar{d}(x,y) = 0$

“Average light cone structure”



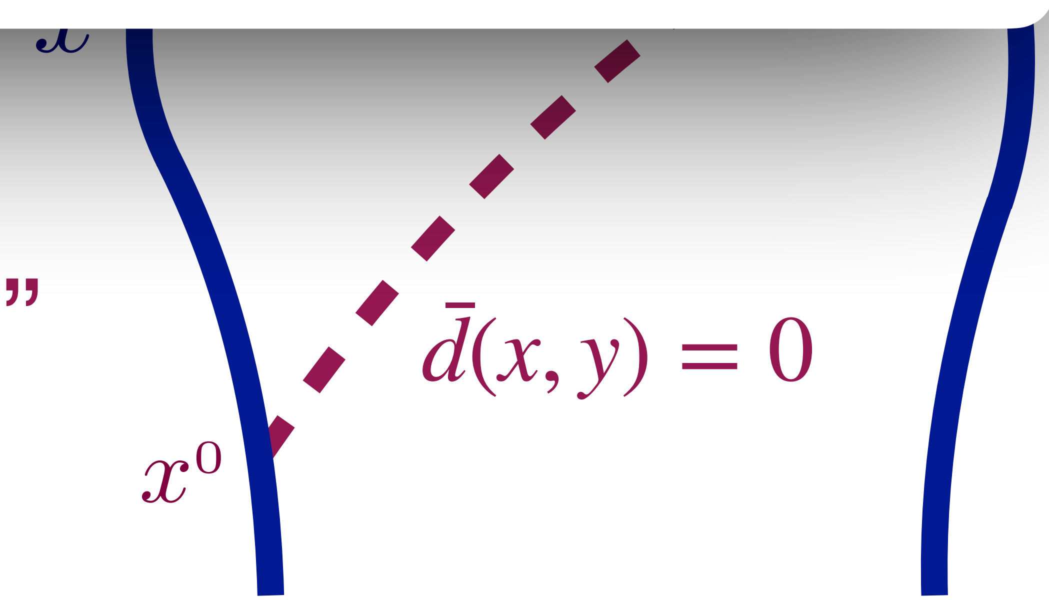
A proxy for causality:

$$\bar{d}(x,y) \equiv \sqrt{\langle d^2(x,y) \rangle}$$

More precisely, we should consider $\langle f(d^2(x,y)) \rangle$

e.g. $\langle G_{\mathrm{ret}}(x,y) \rangle$

“Average light cone structure”



A proxy for causality:

Anomalous geometry/causal structure:

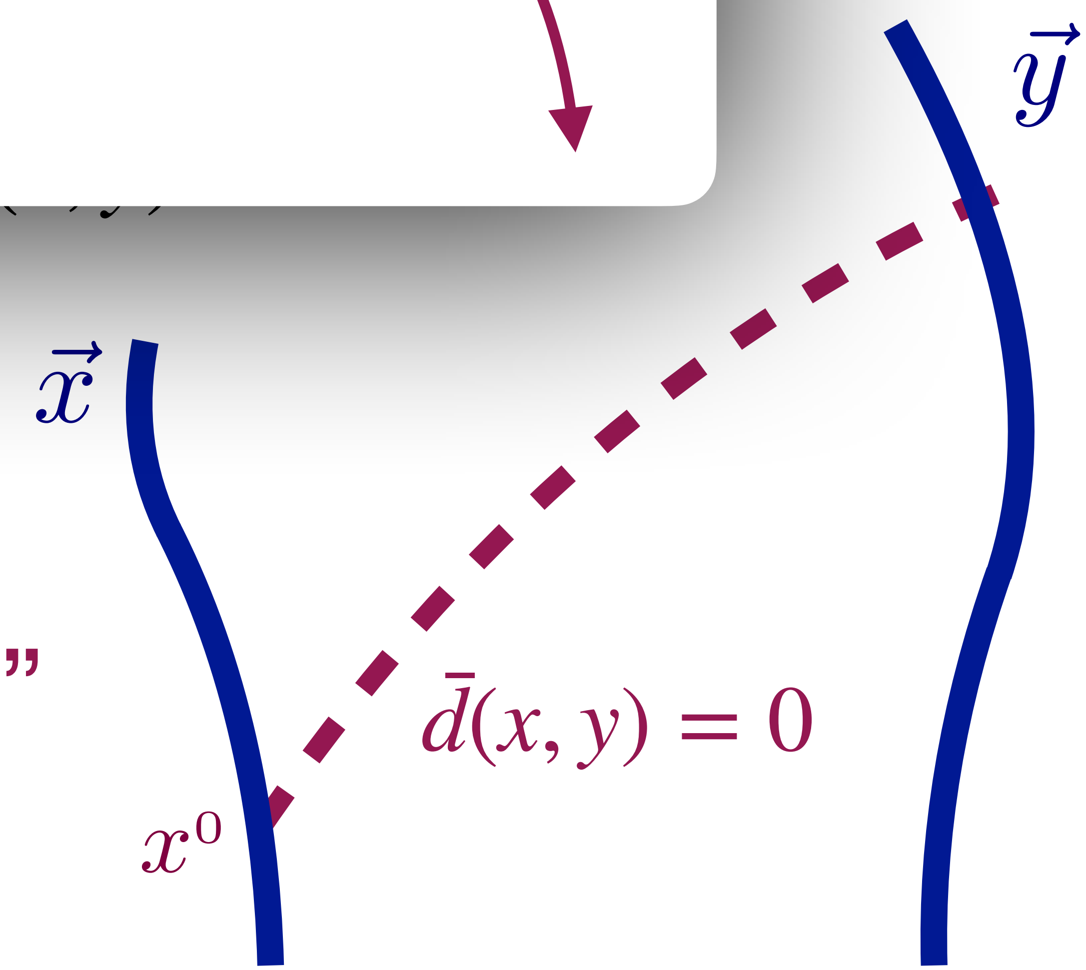
$\bar{d}(x, y)$ is **non-additive**

Assum … al coordinates”

(1) $\langle y^0 \rangle$ …

(2) $\langle [A(x$ …

“Average light cone structure”



A theory for the observers: pressure-less fluid

Dubovsky, Gregoire, Nicolis, Rattazzi, 2006

$$S \;=\; \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \;-\; \mu^4\int d^4x\sqrt{-g}\sqrt{\det(g^{\mu\nu}\partial_\mu x^I\partial_\nu x^J)} \;+\; S_m[\Phi] \;+\; \dots$$

– The three scalar fields x^1, x^2, x^3 label the observers.
– $x^I =$ const. is a geodesic on each classical solution
– $X^I = x^I$: unitary gauge. $\Psi[h_{ij}(X^i), x^i(X^k), \dots] \to \Psi_U[h_{ij}(x^I), \dots]$
– If no vorticity initially $\to$ $N^i = 0$, x^0 proper time of the observers

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \gamma_{ij}dx^i dx^j$$

Additivity and lack thereof

Basic idea: geodesic distances are integrals of a line element

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Third point problem (Euclidean signature):

given $d(x, z)$ and $0 < R < d(x, z)$: Find a third point y s.t.

$$d(x, y) = R, \qquad d(y, z) = d(x, z) - R$$

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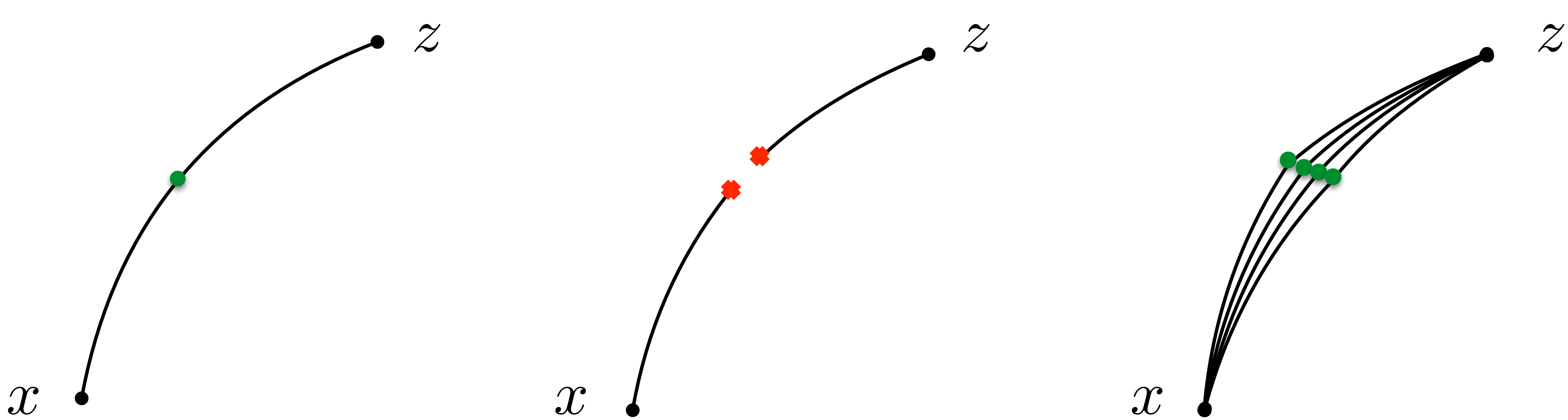

Additive:
only one solution

Subadditive:
no solution

Superadditive:
infinite solutions

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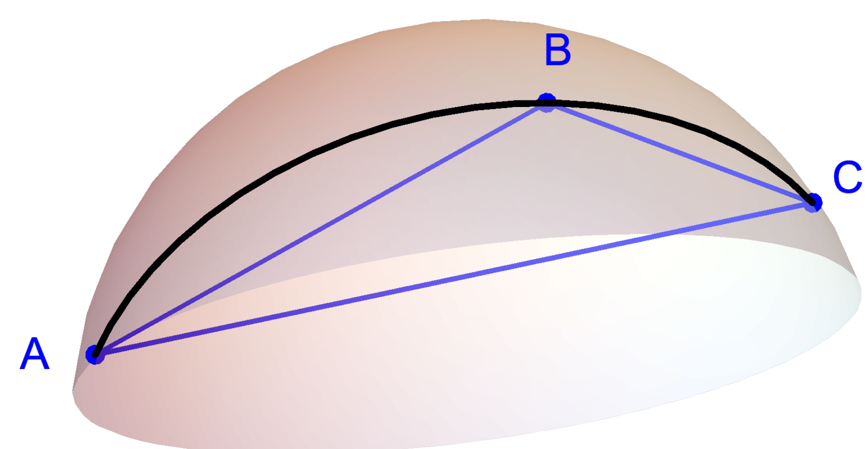
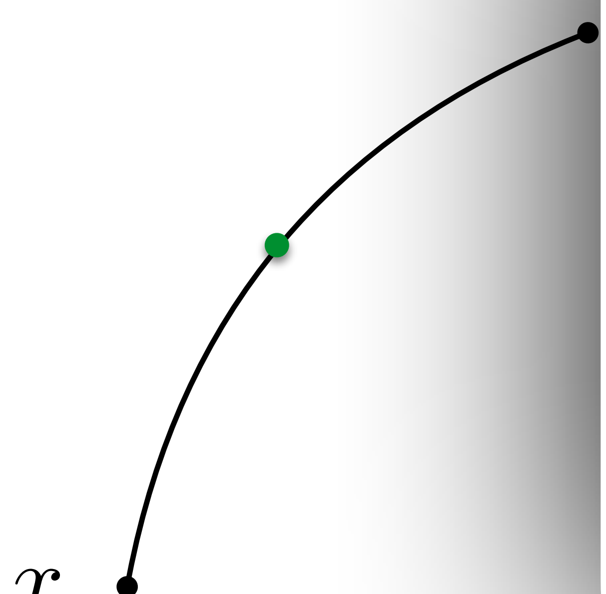
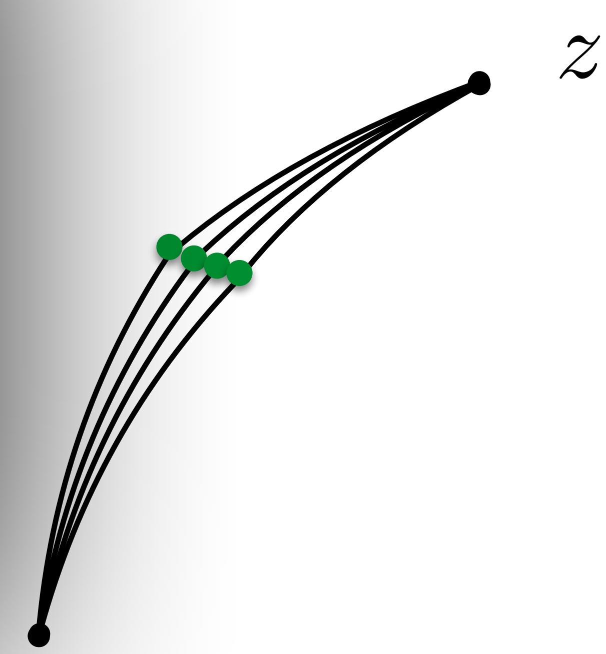
Similar to chordal distances







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Let me give you a distance $d(x, y)$

$d(x, y) \rightarrow g_{\mu\nu}(x)$ Always possible

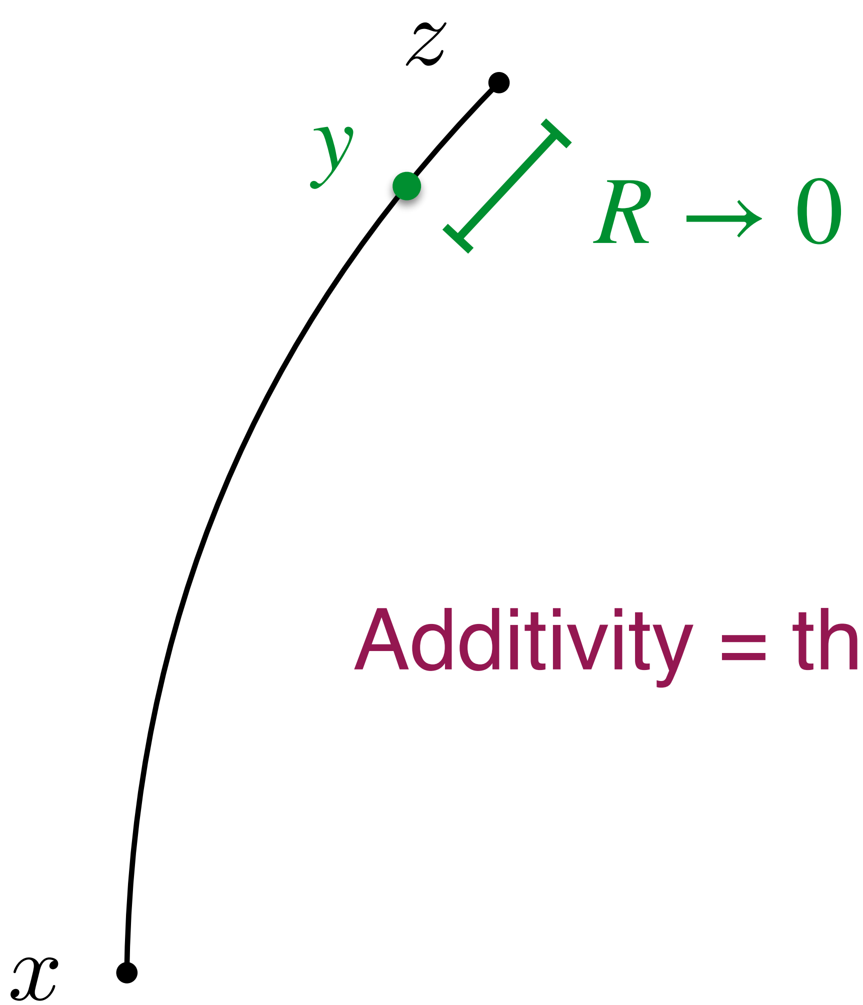
$$g_{\mu\nu}(x) \equiv -\frac{1}{2} \lim_{y \to x} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} d^2(x, y)$$

$g_{\mu\nu}(x) \rightarrow d(x, y)$ Only if $d(x, y)$ is additive

Chordal distance analogy

$g_{\mu\nu} \approx$ intrinsic geometry. But need more to calculate $d(x, y)$

The Third-Point-Problem: differential version



Additivity = the gradient of $d(x, z)$ in z has unit norm

A measure of non-additivity

$$C(x,y) \equiv \frac{1}{4}\,\frac{\partial\, d^2(x,y)}{\partial y^\mu}\frac{\partial\, d^2(x,y)}{\partial y^\nu}\; g^{\mu\nu}(y) - d^2(x,y)$$

Additive:
$C = 0$

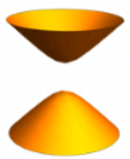
Subadditive:
$C < 0$

Superadditive:
$C > 0$

Solutions to the TPP: **Euclidean signature**

Character	Example ($\epsilon > 0$)	Solutions	Sketch
Additive	$d^2(x_1, x_2) = \overrightarrow{\Delta x}^{\,2}$	One point	
Subadditive	$d^2(x_1, x_2) = \overrightarrow{\Delta x}^{\,2}\left[1 - \epsilon\,\overrightarrow{\Delta x}^{\,2}\right]$	No solution	
Superadditive	$d^2(x_1, x_2) = \overrightarrow{\Delta x}^{\,2}\left[1 + \epsilon\,\overrightarrow{\Delta x}^{\,2}\right]$	Codimension-two surface	

Solutions to the TPP: **Lorentzian signature**

Character	First two points	Solutions	Sketch
Additive ($C = 0$) *e.g.* $d^2(x_1, x_2) = -(\Delta x^0)^2 + \overrightarrow{\Delta x}^2$	$d^2(x, z) > 0$	Codimension-two surface	
	$d^2(x, z) = 0$	One-dimensional curve	
	$d^2(x, z) < 0$	One point	
Subadditive ($C < 0$) $d^2(x_1, x_2) = -(\Delta x^0)^2 + \overrightarrow{\Delta x}^2 \left[1 - \epsilon \overrightarrow{\Delta x}^2\right]$	$d^2(x, z) > 0$	Codimension-two surface	
	$d^2(x, z) = 0$	No solution	
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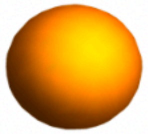
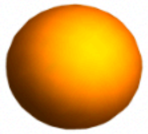
	$d^2(x, z) = 0$	Codimension-two surface	
	$d^2(x, z) < 0$	Codimension-two surface	

Solutions to the TPP: **Lorentzian signature**

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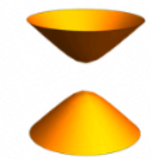
Subadditive ($C < 0$) $d^2(x_1, x_2) = -(\Delta x^0)^2 + \overrightarrow{\Delta x}^2 \left[1 - \epsilon \overrightarrow{\Delta x}^2\right]$	$d^2(x,z) > 0$	Codimension-two surface	
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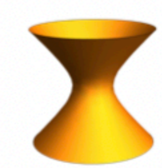
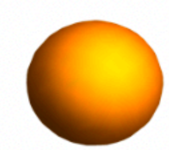
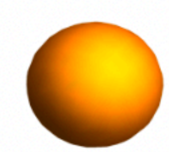
Superadditive ($C > 0$) $d^2(x_1, x_2) = -(\Delta x^0)^2 + \overrightarrow{\Delta x}^2 \left[1 + \epsilon \overrightarrow{\Delta x}^2\right]$	$d^2(x,z) > 0$	Codimension-two surface	
	$d^2(x,z) = 0$	Codimension-two surface	
	$d^2(x,z) < 0$	Codimension-two surface	

Distance within a normal neighborhood

Geodesic distance can be expressed in a coordinate expansion

$$d^2(0,x) = g_{\mu\nu}x^\mu x^\nu + \frac{1}{2}g_{\mu\nu,\rho}\,x^\mu x^\nu x^\rho - \frac{1}{12}\left(g_{\alpha\beta}\Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\rho\sigma} - 2g_{\mu\nu,\rho\sigma}\right)x^\mu x^\nu x^\rho x^\sigma + \mathcal{O}(x^5)$$

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The unitary gauge coordinates x drop from averages

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The unitary gauge coordinates x drop from averages

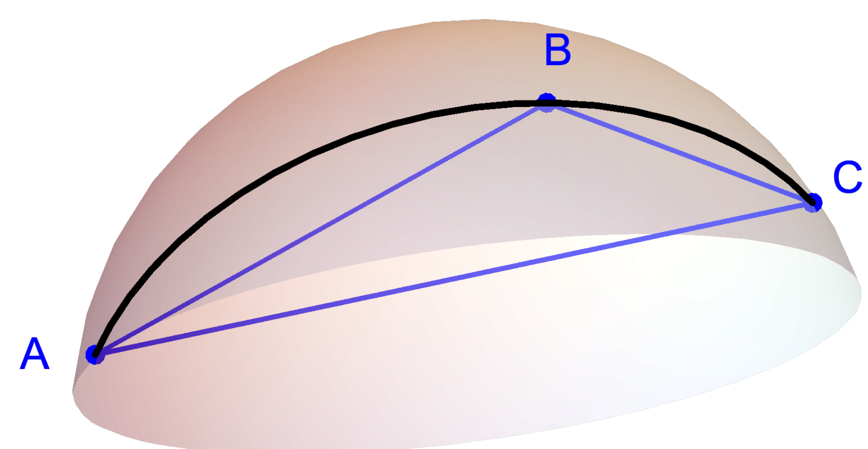
Terms higher than linear cannot be reproduced by an average metric

$$C(0,x) = \frac{1}{4}\left(\bar{g}^{\alpha\beta}\langle\Gamma_{\alpha\mu\nu}\rangle\langle\Gamma_{\beta\rho\sigma}\rangle - \langle g_{\alpha\beta}\Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\rho\sigma}\rangle\right) x^\mu x^\nu x^\rho x^\sigma + \mathcal{O}(x^5)$$

Result in Euclidean signature:

Average distances always *subadditive*

Similar to chordal distances



$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \gamma_{ij}dx^i dx^j$$

Lorentz signature (unitary gauge)

$$\begin{aligned}
C = \frac{1}{4}\Big[&\frac{1}{4}\left(\langle\dot\gamma_{ij}\dot\gamma_{lk}\rangle - \langle\dot\gamma_{ij}\rangle\langle\dot\gamma_{lk}\rangle\right)x^i x^j x^k x^l \\
&- \left(\langle\gamma^{pq}\dot\gamma_{pi}\dot\gamma_{qj}\rangle - \bar\gamma^{pq}\langle\dot\gamma_{pi}\rangle\langle\dot\gamma_{qj}\rangle\right)t^2 x^i x^j \\
&- 2\left(\langle\Gamma^p_{ij}\dot\gamma_{pk}\rangle - \bar\gamma^{pq}\langle\Gamma_{pij}\rangle\langle\dot\gamma_{qk}\rangle\right)t\, x^i x^j x^k \\
&- \left(\langle\gamma^{pq}\Gamma_{pij}\Gamma_{qkl}\rangle - \bar\gamma^{pq}\langle\Gamma_{pij}\rangle\langle\Gamma_{qkl}\rangle\right)x^i x^j x^k x^l\Big]\ ,
\end{aligned}$$

– No non-additivity along time ($\overrightarrow{x} = 0$).

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— Negative definite pieces

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Lorentz signature (unitary gauge)

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& - \left(\langle\gamma^{pq}\dot\gamma_{pi}\dot\gamma_{qj}\rangle - \bar\gamma^{pq}\langle\dot\gamma_{pi}\rangle\langle\dot\gamma_{qj}\rangle\right) t^2 x^i x^j \\
& - 2\left(\langle\Gamma^p_{ij}\dot\gamma_{pk}\rangle - \bar\gamma^{pq}\langle\Gamma_{pij}\rangle\langle\dot\gamma_{qk}\rangle\right) t\, x^i x^j x^k \\
& - \left(\langle\gamma^{pq}\Gamma_{pij}\Gamma_{qkl}\rangle - \bar\gamma^{pq}\langle\Gamma_{pij}\rangle\langle\Gamma_{qkl}\rangle\right) x^i x^j x^k x^l \Big] ,
\end{aligned}
$$

— No non-additivity along time ($\vec{x} = 0$).

— Negative definite pieces

— Positive definite

Examples:

– Superposition of plane waves: $C < 0$

– Fluctuations around homogeneous background: $C < 0$

Thermal state of gravitons: $$C(0,x) \simeq \frac{T^4}{M_P^2}\Delta x^4$$

effect important at $\ell \sim \frac{M_P}{T^2}$

– FRW: $C < 0$ if $w > -\frac{1}{3}$

Causality

Given $\langle d^2(x,y)\rangle$ one can define a metric tensor $\langle g_{\mu\nu}\rangle = \bar{g}_{\mu\nu}$.

$$\bar{g}_{\mu\nu}(x) \;\equiv\; -\frac{1}{2}\lim_{y\to x}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\langle d^2(x,y)\rangle$$

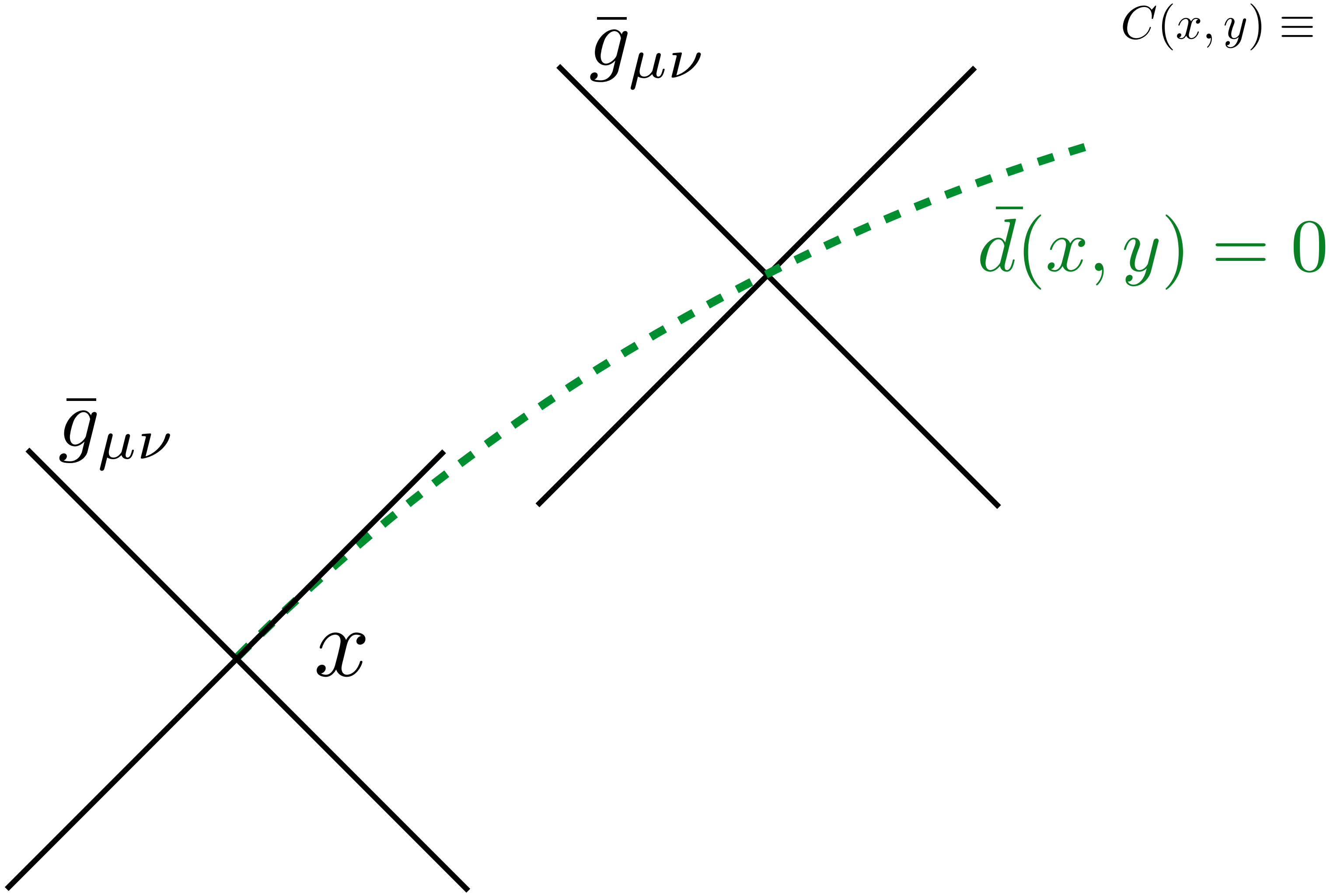
But there is more to $\langle d^2(x,y)\rangle$ than $\langle g_{\mu\nu}\rangle$!

$\langle g_{\mu\nu}\rangle \Delta x^\mu \Delta x^\nu = 0$: where we expect the photon to be detected in the immediate vicinity of the emission.

Further away: see where $\langle d^2(x,y)\rangle = 0$

Subadditive causality ($C < 0$)

$$C(x,y) \equiv \frac{1}{4}\,\frac{\partial\, d^2(x,y)}{\partial y^\mu}\,\frac{\partial\, d^2(x,y)}{\partial y^\nu}\; g^{\mu\nu}(y) - d^2(x,y)$$



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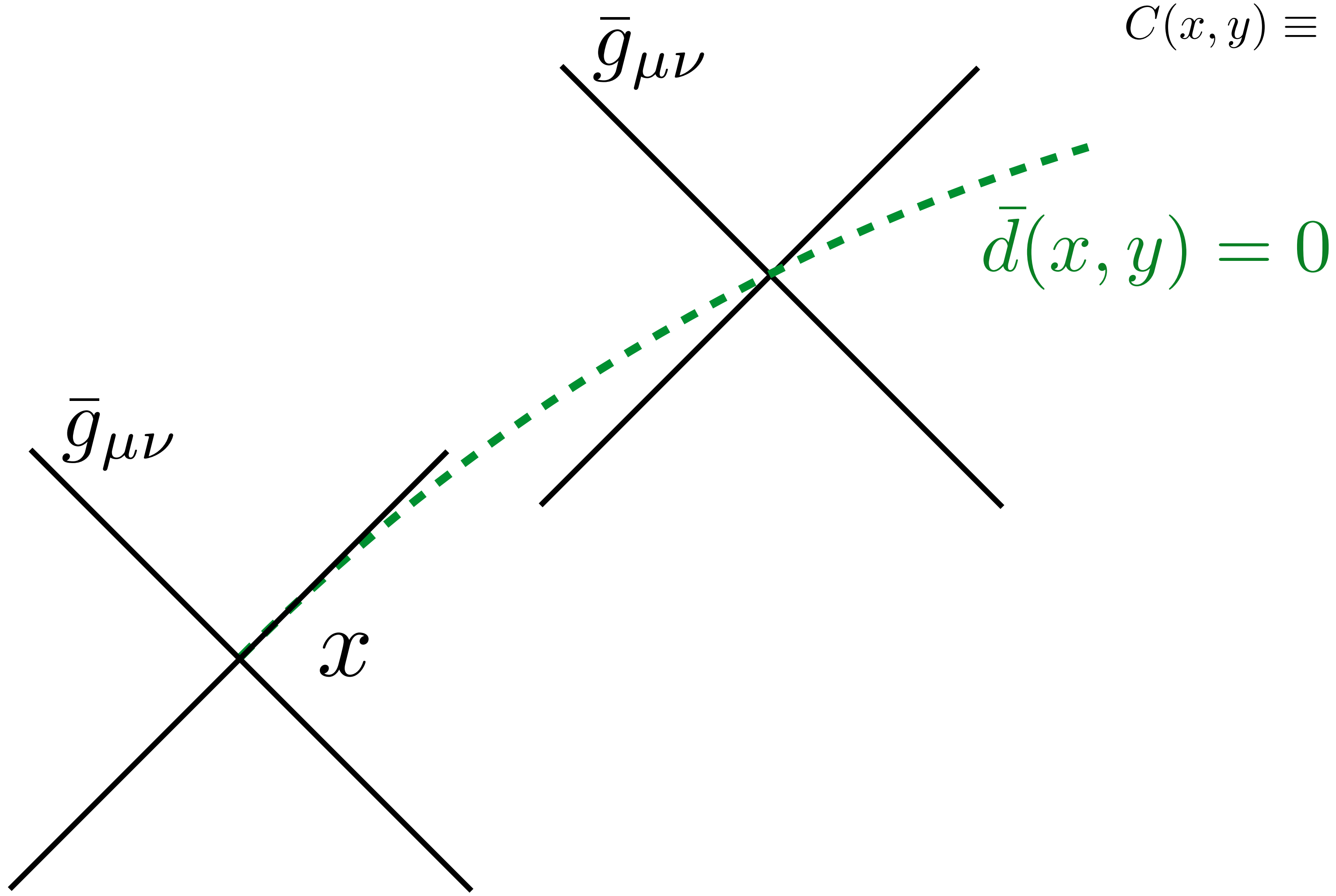

Two causal structures at play. One *rigid* defined at each point. One dependent on the two extremes x and y.

Photons are ``prompt” wrt the rigid structure given by $\bar{g}_{\mu\nu}$

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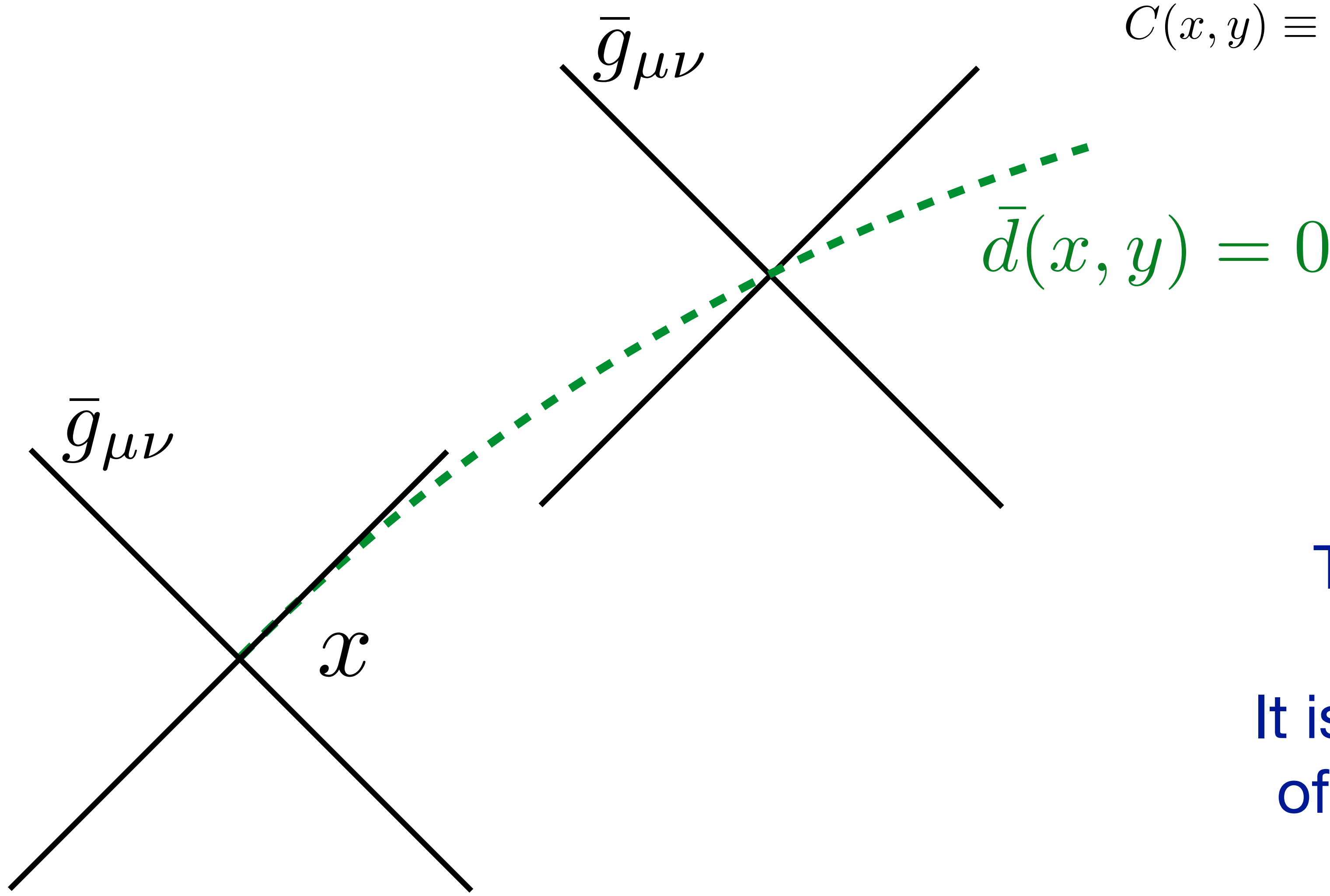


This is not the trajectory
of any light ray.
It is just where the ensemble
of events where we expect
to receive it

Two causal structures at play. One *rigid* defined at each point. One dependent on the two extremes x and y.

Photons are ``prompt” wrt the rigid structure given by $\bar g_{\mu\nu}$

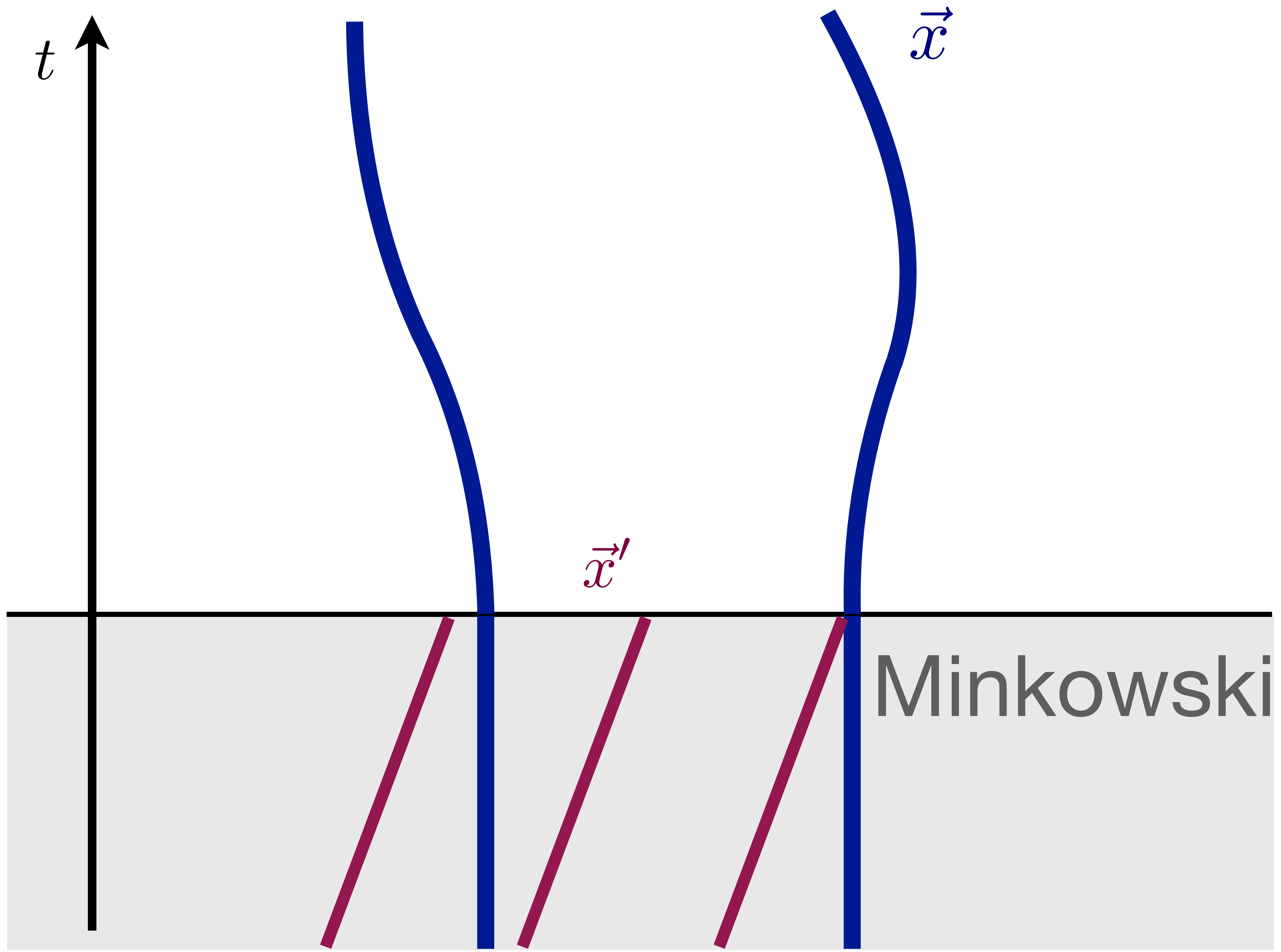
Superadditive causality ($C > 0$)

$$x \prec y \ \wedge \ y \prec z \quad \nRightarrow \quad x \prec z$$

$$\langle [\mathcal{A}(x), \mathcal{A}(y)] \rangle \neq 0, \quad \langle [\mathcal{A}(y), \mathcal{A}(z)] \rangle \neq 0, \quad \langle [\mathcal{A}(x), \mathcal{A}(z)] \rangle \approx 0$$

Conjecture: Subadditivity the outcome of evolution from relatively ``standard'' initial conditions

Aside: Special Relativity 2.0



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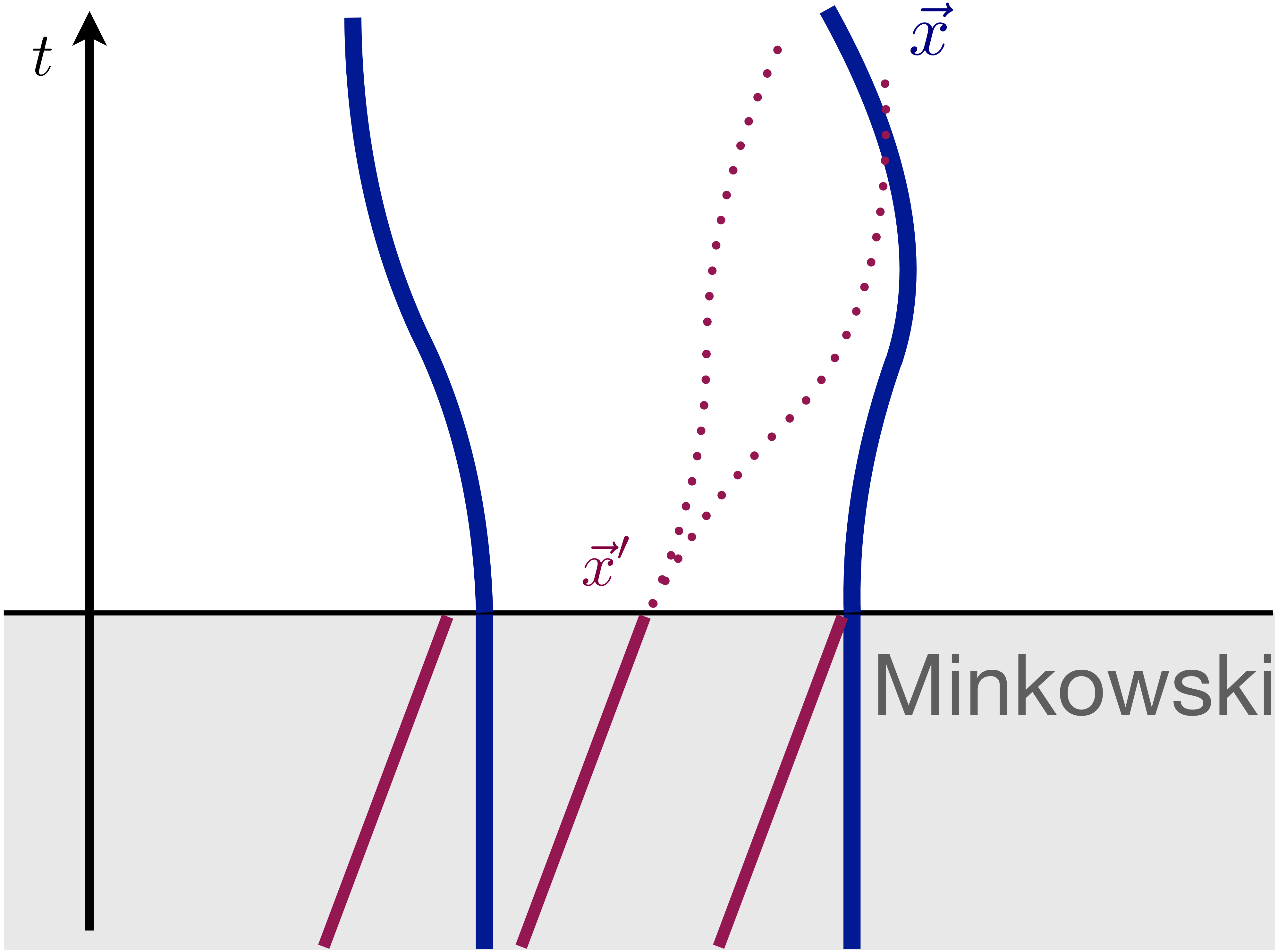

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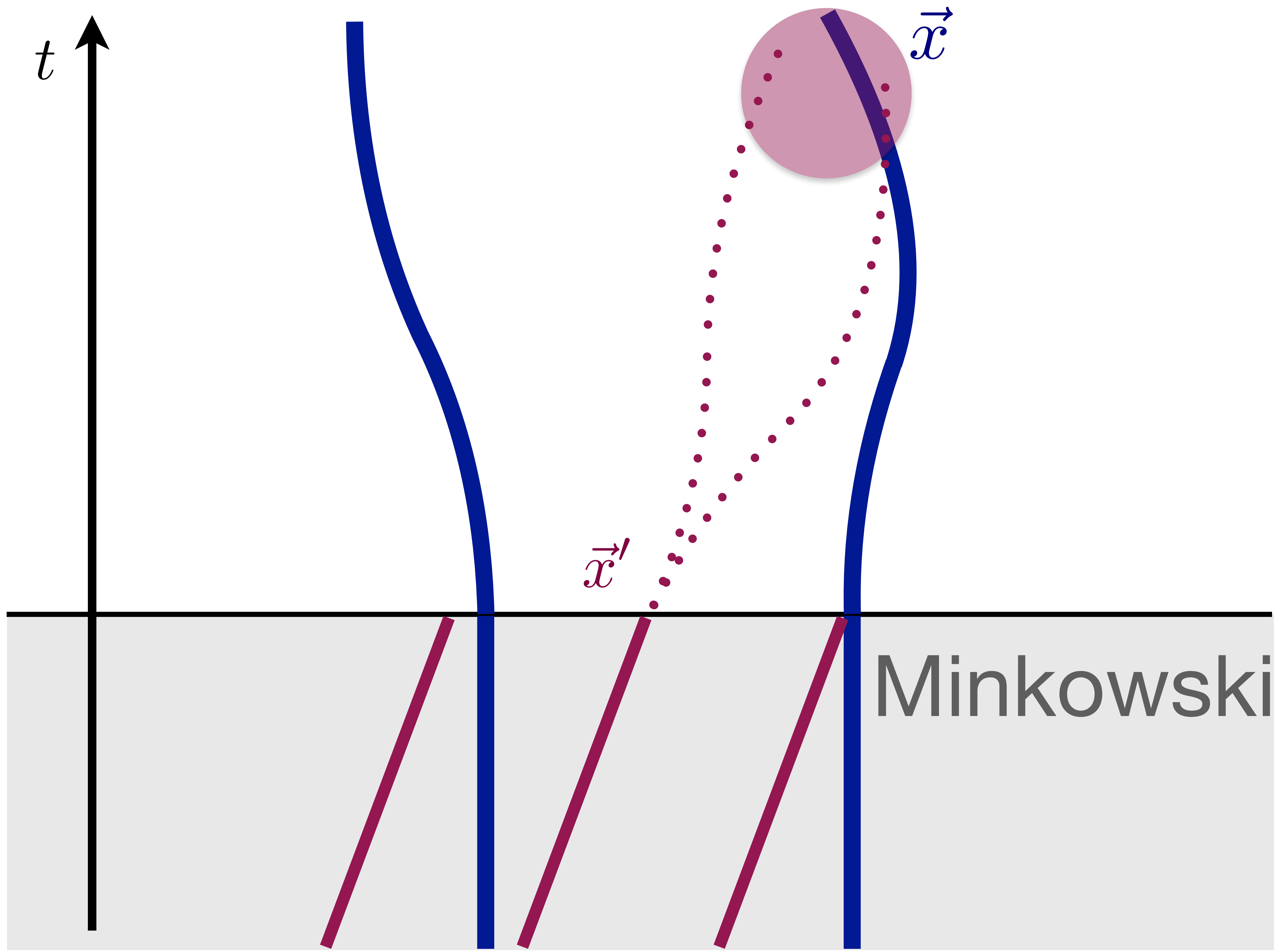

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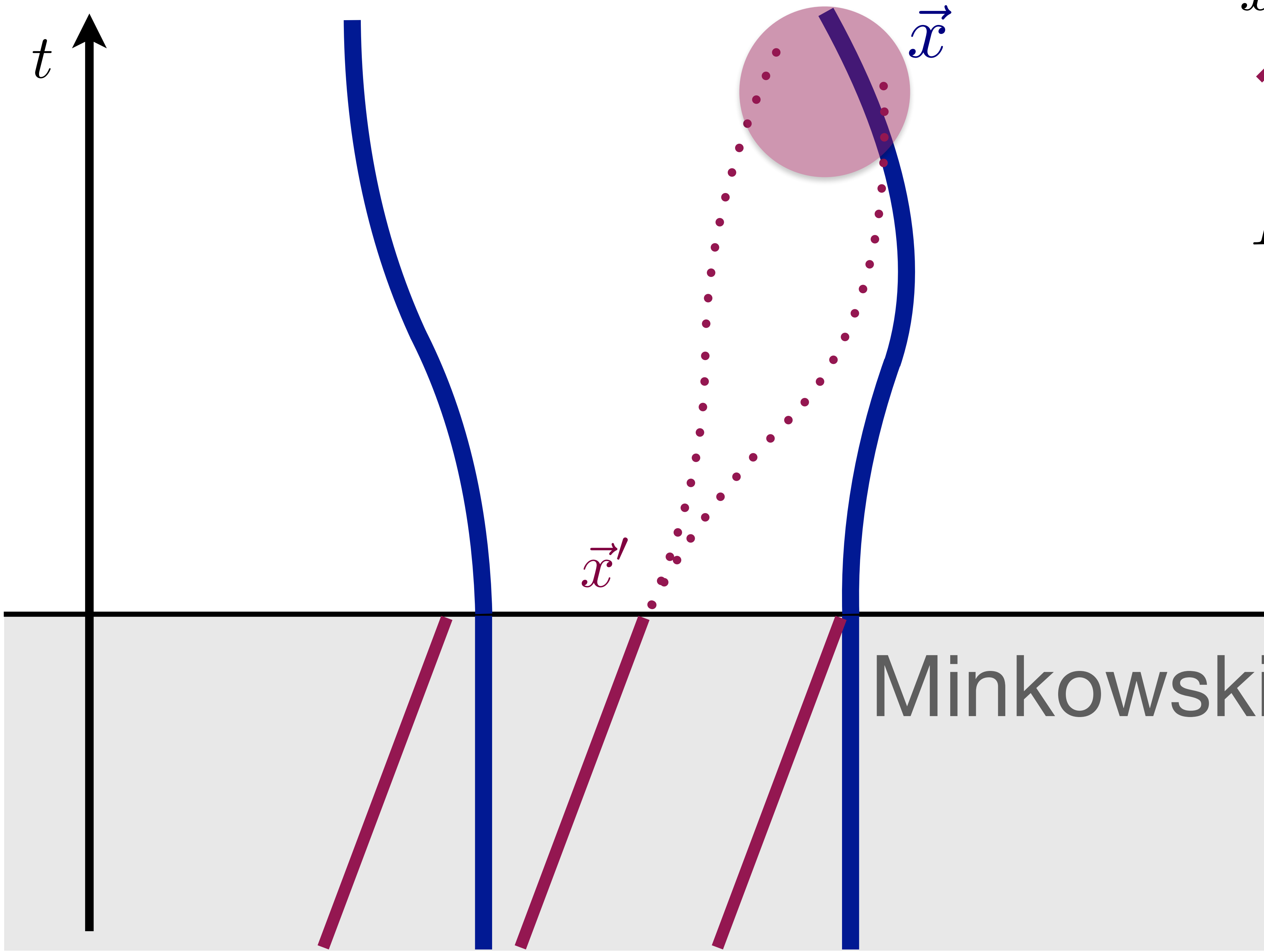
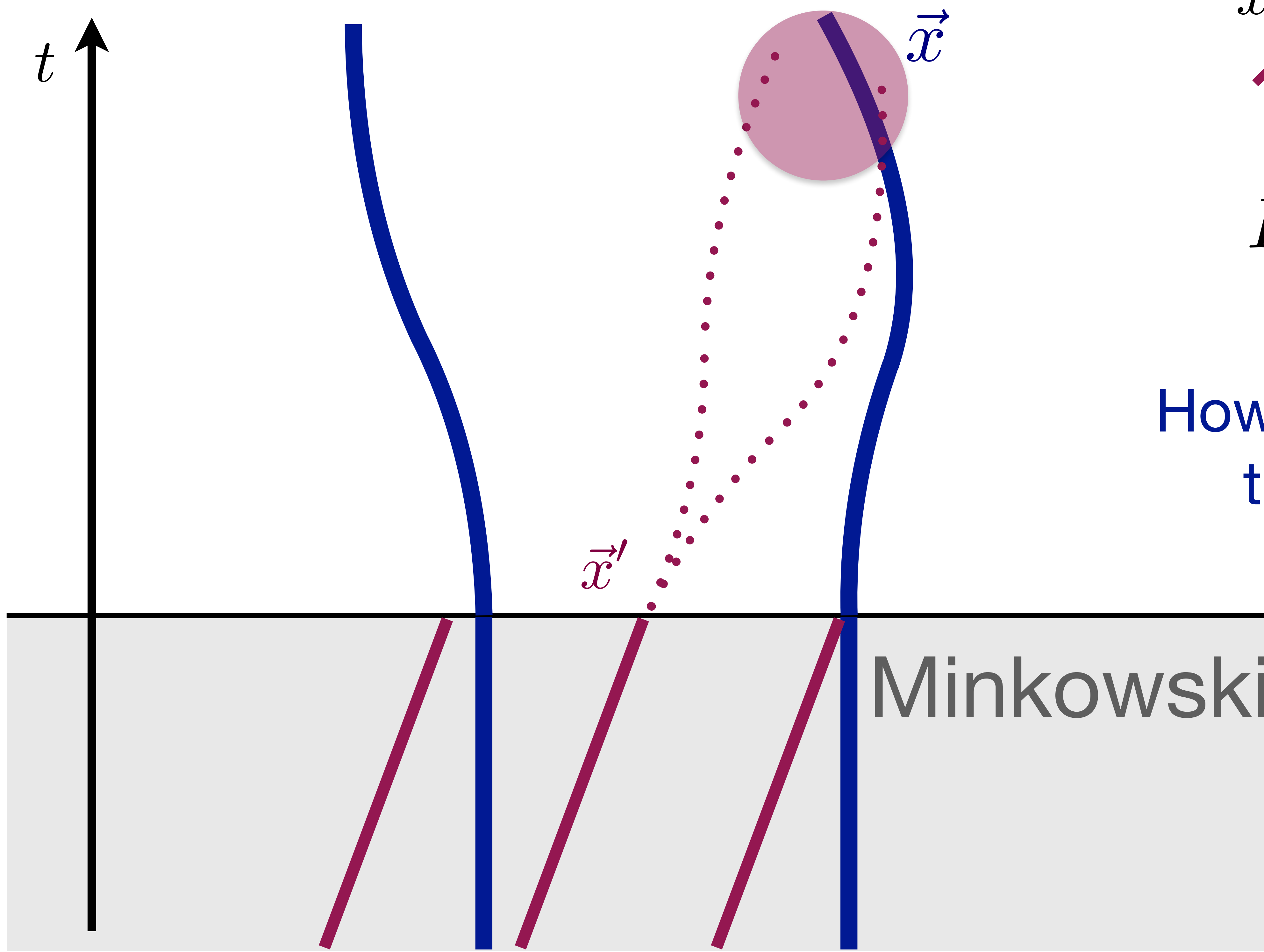


$$\cancel{x' = x'(x)}$$

$$P(x'|x)$$

Aside: Special Relativity 2.0



$$\cancel{x' = x'(x)}$$

$$P(x'|x)$$

How does $\bar{d}(x, y)$ transform?

Conclusions:

- The metric is not enough!
- Effect generically small in perturbative situations
- A lot of potential applications
- New mathematical structures…?

One might expect occasional violations of causality (because of the fluctuations of geometry) on top of an otherwise-classical causal structure.

We find instead that the very average $\bar{d}(x, y)$ is anomalous and *is not* the geodesic distance of any metric.

The reason is that $\bar{d}(x, y)$ is *non additive*

Distance within a normal neighborhood

Geodesic distance can be expressed in a coordinate expansion

$$d^2(0,x) = g_{\mu\nu}x^\mu x^\nu + \frac{1}{2}g_{\mu\nu,\rho}\, x^\mu x^\nu x^\rho - \frac{1}{12}\left(g_{\alpha\beta}\Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\rho\sigma} - 2g_{\mu\nu,\rho\sigma}\right) x^\mu x^\nu x^\rho x^\sigma + \mathcal{O}(x^5)$$

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The coordinates x are "physical" i.e. independently defined

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We want to evaluate $$\bar{d}(x,y) \;\equiv\; \sqrt{\langle d^2(x,y)\rangle}$$

The coordinates x are ``physical" i.e. independently defined

$$\Rightarrow \qquad \langle d^2(0,x)\rangle = \langle g_{\mu\nu}(0)\rangle x^\mu x^\nu + \ldots$$

$$\bar{g}_{\mu\nu}(x) \;\equiv\; -\frac{1}{2}\lim_{y\to x}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\bar{d}^2(x,y)$$

The metric tensor defined locally with $\bar{d}(x,y)$ is nothing else than $\langle g\rangle$!

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Terms higher than linear cannot be reproduced by an average metric

$$C(0,x) = \frac{1}{4} \left(\bar{g}^{\alpha\beta} \langle \Gamma_{\alpha\mu\nu} \rangle \langle \Gamma_{\beta\rho\sigma} \rangle - \langle g_{\alpha\beta} \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\rho\sigma} \rangle \right) x^\mu x^\nu x^\rho x^\sigma + \mathcal{O}(x^5)$$

Non-additivity builds up at large separation. Can we infer about the sign?

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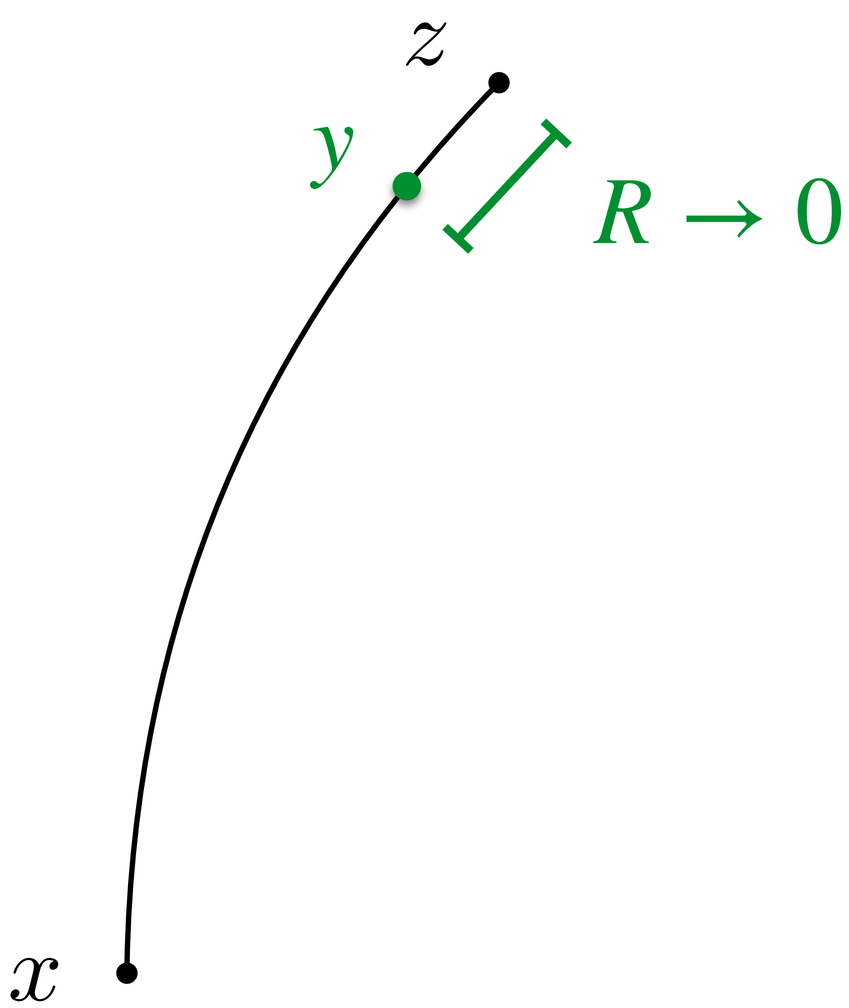
Non-additivity builds up at large separation. Can we infer about the sign?

$$C(0,x) = -\frac{1}{4} \langle Q_a \eta^{ab} Q_b \rangle \,,$$

$$Q_a = \left(e^\alpha_a\, \Gamma_{\alpha\mu\nu} - e_{\beta a}\, \bar{g}^{\alpha\beta} \langle \Gamma_{\alpha\mu\nu} \rangle \right) x^\mu x^\nu \,,$$

The Third-Point-Problem: differential version



$$d(y, z) = R$$

$$d(x, y) = d(y, z) + \frac{\partial d(x, z)}{\partial z^i} n^i R$$

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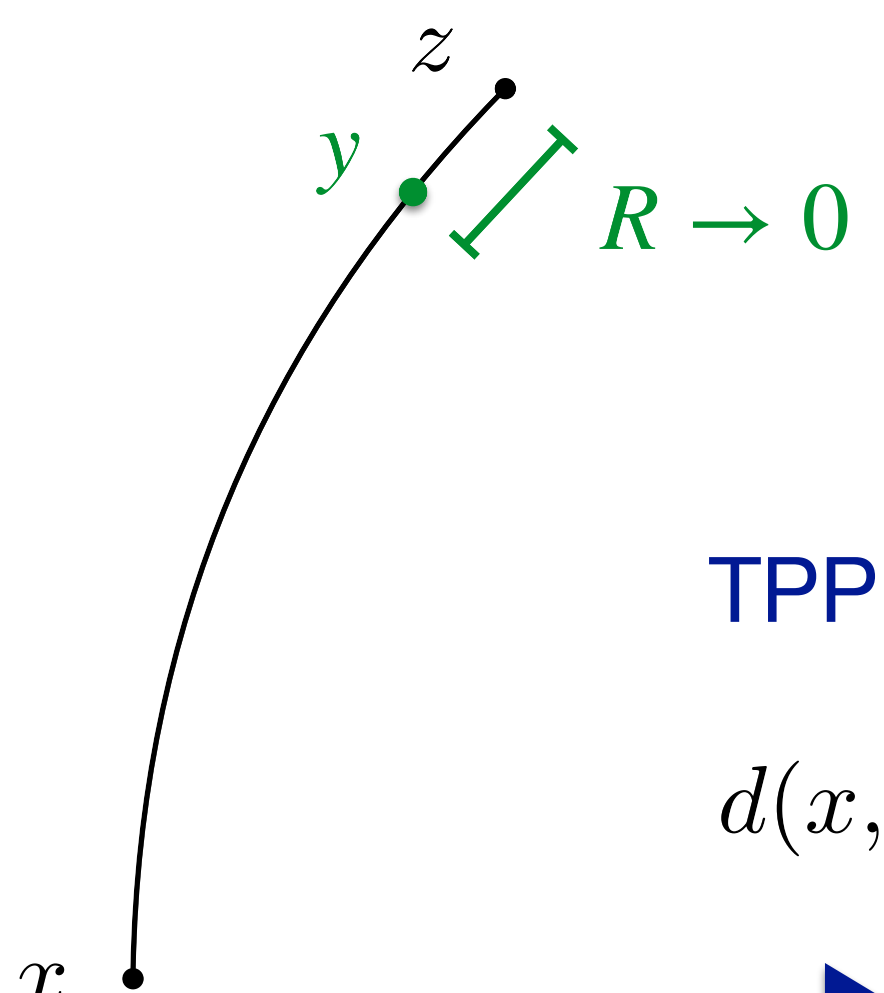

$$d(y,z) = R$$

$$d(x,y) = d(y,z) + \frac{\partial d(x,z)}{\partial z^i} n^i R$$

TPP:

$$d(x,y) + d(y,z) = d(x,z)$$

$$\Rightarrow \boxed{\frac{\partial d(x,z)}{\partial z^i} n^i = -1}$$

The size of the gradient of $d(x,z)$ in z determines how many solutions to the TPP: the character of $d(x,z)$