

Data-driven approximations to the Hadronic Light-by-Light scattering contribution to the muon ($g-2$)

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Summary. — We review recent progress on the numerical determination of the Hadronic Light-by-Light contribution to the anomalous magnetic moment of the muon. We advocate for a slight increase of the White Paper number for its Standard Model prediction, to $(102 \pm 17) \times 10^{-11}$, accounting for a revised contribution from axial-vector mesons and short-distance constraints. This $\sim 10\%$ larger result seems to be supported by the most recent lattice QCD evaluations.

1. – Why it matters?

The Standard Model (SM) uncertainty on the Muon $g-2$ ($2a_\mu = g_\mu - 2$) is dominated by the hadronic vacuum polarization (HVP) piece, amounting to 4.0×10^{-10} (for an overall error of 4.3×10^{-10}) [1] ⁽¹⁾. This is contributed very mildly by the error of the Hadronic light-by-light (HLbL) scattering part, 1.9×10^{-10} , that we will discuss here ⁽²⁾. Clearly, the most urgent thing is to clarify the discrepancy between the data-driven results [1, 2, 3, 4, 5] and the competitive lattice QCD evaluation, by the BMW collaboration [6], of a_μ^{HVP} . To this end, several approaches have been developed, exploiting the so-called windows in Euclidean time [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Ref. [16] (based on the isospin-breaking corrections computed in Ref. [18]), points to nice agreement between

⁽¹⁾ These and the following numbers are quoted -unless otherwise stated- from the Whitepaper of the Muon $g-2$ Theory Initiative [1] (WP), a collaboration which has been aiming for a community consensus value of the Standard Model prediction of the muon $g-2$, see <https://muon-gm2-theory.illinois.edu/>.

⁽²⁾ See talks focusing on diverse aspects of the HVP contribution by Matthia Bruno, Christoph Redmer, Francesca de Mori, Álex Miranda, Camilo Rojas and David Díaz Calderón.

data-driven predictions using $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ data [19, 20, 21, 22] (instead of $e^+e^- \rightarrow \pi^+\pi^-$ measurements) with lattice QCD evaluations. The barely acceptable discrepancy between KLOE [23, 24, 25, 26, 27] and BaBar [28, 29] $e^+e^- \rightarrow \pi^+\pi^-$ data has been aggravated by the new CMD-3 measurement [30], being this puzzle still not understood (see also e.g. the measurements [31, 32, 33]). Amid this conundrum, halving the error of the SM prediction for a_μ^{HLbL} [1] is still necessary, according to the final precision that the Fermilab experiment will achieve measuring a_μ , but not the top priority.

On the contrary, the experimental situation seems crystal-clear: FNAL measurements [34, 35] are extremely consistent with the BNL outcome [36] and their joint picture is fully convincing, yielding

$$(1) \quad a_\mu^{\text{Exp}} = 116592059(22) \times 10^{-11}.$$

This situation further enhances the pressing need for theoretical progress.

2. – Why such a large error for a_μ^{HLbL} ?

The outsider may wonder why the uncertainty of the a_μ^{HLbL} is $\mathcal{O}(20\%)$, while that of the a_μ^{HVP} is only $\mathcal{O}(0.6)\%$. This much better precision stems from its calculation via a single dispersive integral that is related to the accurately measured $\sigma(e^+e^- \rightarrow \text{hadrons})$ [37] plus a mild contribution from perturbative QCD. On the contrary, a data-driven approach to a_μ^{HLbL} is very much complicated by the additional loop and multi-scale nature of the problem. Despite enormous advances towards a fully-dispersive computation of a_μ^{HLbL} [38, 39, 40, 41, 42, 43], a completely dispersive evaluation is not feasible yet. This framework provided a rationale for the historical arrangement of the main contributions (starting from the dominance of the pseudoscalar-pole cuts [44]) and could in principle be used up to arbitrary complex multiparticle ones.

3. – Contributions

Amazingly, the whole a_μ^{HLbL} is basically saturated by the contribution from the lowest-multiplicity cut (even more so because of the approximate cancellations among the other contributions), corresponding to the lightest pseudoscalar (π^0, η, η') poles, yet it could be related to a combined chiral and large- N_C expansion [45]. This can be computed straightforwardly [44] knowing the corresponding pseudoscalar transition form factors (TFFs) as functions of both photons virtuality. See Christoph Redmer's talk on the precious experimental input to these (and others required for a_μ^{HLbL}) TFFs. In addition, there are some theoretical properties constraining these TFFs, like the chiral limit, the singly and doubly virtual asymptotic limits predicted by QCD, analyticity and unitarity ... The dispersive evaluation [46, 47] yields a very precise result for the π^0 contribution

$$(2) \quad a_\mu^{\pi^0, \text{HLbL}} = (63.0_{-2.1}^{+2.7}) \times 10^{-11},$$

confirming the rational approximants' determination [48]

$$(3) \quad a_\mu^{\pi^0, \text{HLbL}} = (63.6 \pm 2.7) \times 10^{-11}.$$

These results are also supported by e.g. Dyson-Schwinger eqs. evaluations, yielding $a_\mu^{\pi^0, \text{HLbL}} = (62.6 \pm 1.3) \times 10^{-11}$ [49], and $a_\mu^{\pi^0, \text{HLbL}} = (61.4 \pm 2.1) \times 10^{-11}$ [78] and by

holographic QCD results [51, 52, 53] (see, however, [54]) and chiral Lagrangians including resonances [55, 56]. For the $\eta^{(\prime)}$ contributions there is no dispersive computation yet. The rational approximants' calculation [57, 58, 59, 48] yields

$$(4) \quad a_{\mu}^{\eta, \text{HLbL}} = (16.3 \pm 1.4) \times 10^{-11}, \quad a_{\mu}^{\eta', \text{HLbL}} = (14.5 \pm 1.9) \times 10^{-11},$$

which are the reference values for this contribution. Again, they are supported by the different approaches mentioned before where, in particular, Dyson-Schwinger eqs. results in $a_{\mu}^{\eta, \text{HLbL}} = (15.8 \pm 1.2) \times 10^{-11}$, $a_{\mu}^{\eta', \text{HLbL}} = (14.7 \pm 1.9) \times 10^{-11}$ [49] and $a_{\mu}^{\eta, \text{HLbL}} = (13.3 \pm 0.9) \times 10^{-11}$, $a_{\mu}^{\eta', \text{HLbL}} = (13.6 \pm 0.8) \times 10^{-11}$ [78], respectively. From the dispersive and rational approximants calculations, the WP quotes

$$(5) \quad a_{\mu}^{\pi^0 + \eta + \eta', \text{HLbL}} = (93.8_{-3.6}^{+4.0}) \cdot 10^{-11},$$

still to be considered the data-driven SM prediction for this leading contribution to HLbL, coming from the lightest pseudoscalar poles.

The very well-known pseudoscalar electromagnetic form factors are the key objects to determine their box contributions to a_{μ}^{HLbL} . The dispersive result for the π case

$$(6) \quad a_{\mu}^{\pi\text{-box}, \text{HLbL}} = -(15.9 \pm 0.2) \times 10^{-11},$$

was later on confirmed by Schwinger-Dyson evaluations $a_{\mu}^{\pi\text{-box}, \text{HLbL}} = -(15.7 \pm 0.4) \times 10^{-11}$ [49], and $a_{\mu}^{\pi\text{-box}, \text{HLbL}} = -(15.6 \pm 0.2) \times 10^{-11}$ [60]. For the Kaon case, the early evaluation of ref. [61], $a_{\mu}^{K\text{-box}, \text{HLbL}} = -(0.46 \pm 0.02) \times 10^{-11}$ was slightly revised within Dyson-Schwinger and then also using a dispersive framework [62], both agreeing on

$$(7) \quad a_{\mu}^{K\text{-box}, \text{HLbL}} = -(0.48 \pm 0.02) \times 10^{-11}.$$

The SM prediction comes from eqs. (6) and (7), still coinciding with the WP number [1]

$$(8) \quad a_{\mu}^{(\pi/K)\text{-box}, \text{HLbL}} = -(16.4 \pm 0.2) \times 10^{-11}.$$

Now we turn to another contribution coming from two-particle cuts, that associated to pseudoscalars rescattering. For the pions case, the dispersive evaluation [42, 43] is quite precise for the contribution associated to the π -pole left-hand cut (LHC):

$$(9) \quad a_{\mu, J=0}^{\pi\pi, \pi\text{-poleLHC}} = -(8 \pm 1) \times 10^{-11},$$

where contributions from D - and higher orders partial waves were covered by the uncertainty. This agrees with other evaluations [63, 64, 65, 66] that include additional scalar contributions, converging to [66]

$$(10) \quad a_{\mu}^{\text{Scalars}} = -(9 \pm 1) \times 10^{-11},$$

again in accord with the WP [1]. Similarly, the tensors contribution [67]

$$(11) \quad a_{\mu}^{\text{Tensors}} = -(0.9 \pm 0.1) \times 10^{-11},$$

is unchanged with respect to Ref. [1].

The part which has been evolving less trivially since 2020 corresponds to the axial-vector contributions, which should be regarded together with the remaining perturbative QCD constraints.

Melnikov and Vainshtein [68] put forward that pseudoscalar poles alone cannot satisfy short-distance QCD restrictions and emphasized the importance of axial-vectors to fulfil this requirement. Modern studies coincide in smaller values for these contributions than initially advocated.

Ref. [69] clarified ambiguities about bases arising because of axials off-shellness and, together with ref. [77], emphasized the relationship between short-distance, axial anomaly constraints, and the axial contributions (with possible relevant role of pseudoscalar resonances, see also [78]), a hot topic since then. Refs. [40, 70, 69] gave rise to the WP number [1]

$$(12) \quad a_{\mu}^{Axials} = (6 \pm 6) \times 10^{-11} .$$

This was accompanied by the estimation of the contribution from light-quark loops and remaining QCD short-distance constraints (*SDCs*) [72, 73, 74]

$$(13) \quad a_{\mu}^{u/d/s-loops+SDCs} = (15 \pm 10) \times 10^{-11} .$$

Given their correlation, these two contributions were combined with errors added linearly (uncertainties are combined quadratically, unless otherwise stated) to [1]

$$(14) \quad a_{\mu}^{axials+SDCs} = (21 \pm 16) \times 10^{-11} .$$

Finally, the *c*-quark contribution (with uncertainty to be added linearly to the Eq. (14)) is [78, 71, 72, 73, 74]

$$(15) \quad a_{\mu}^{c-loop} = (3 \pm 1) \times 10^{-11} .$$

The leading-order a_{μ}^{HLbL} contributions is obtained from Eqs. (5), (8), (10), (11), Eqs. (14), and (15), yielding

$$(16) \quad a_{\mu}^{\text{HLbL},LO} = (92 \pm 19) \times 10^{-11} .$$

Progress since the WP on axials and/or SDCs has improved the understanding of the regime where all photon virtualities are large, and when one of them is much smaller than the other two [75, 76, 77, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89]. However, different model calculations considering axial-vector mesons and SDCs [90, 51, 77, 84, 52, 53] suggest a shift in the central value around

$$(17) \quad a_{\mu}^{axials+SDCs} = (31 \pm 10) \times 10^{-11} ,$$

larger than previously estimated, (14), but compatible within errors. Using Eq. (17), the overall contribution would then be

$$(18) \quad a_{\mu}^{\text{HLbL},LO} = (102 \pm 17) \times 10^{-11} ,$$

which is closer to the latest lattice QCD evaluations by the Mainz [91] $((109.6 \pm 15.9) \times 10^{-11})$ and RBC/UKQCD [92] $((124.7 \pm 14.9) \times 10^{-11})$ collaborations (to be compared to $(78.7 \pm 35.4) \times 10^{-11}$ [93] by RBC, used in the WP). At *NLO* [94] the central value and its uncertainty are increased by only $(2 \pm 1) \times 10^{-11}$.

These observations evince that a better understanding of the role of axial vector mesons and the intermediate energy region is an important step towards a more precise and reliable estimate for the HLbL contribution. Progress in this direction continues [75, 76, 77, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89].

4. – Conclusions

- The WP number, $a_\mu^{\text{HLbL},LO} = (92 \pm 19) \times 10^{-11}$ [1], still stands as the data-driven SM prediction for $a_\mu^{\text{HLbL},LO}$.
- The dominant uncertainty comes from short-distance + axial contributions (correlated uncertainties), with improved understanding since the WP, where still work needs to be done. This may shift the SM prediction slightly, to $a_\mu^{\text{HLbL},LO} = (102 \pm 17) \times 10^{-11}$.
- Measurement of di-photon resonance couplings (particularly for axials) would be very helpful.
- Lattice QCD has just reached a comparable uncertainty to the data-driven determinations of this piece, thereby reducing the uncertainty through their combination to $\leq 10 \times 10^{-11}$, in agreement with the sought accuracy by the time of the final publication of the a_μ measurement by the FNAL experiment. So the ball is on HVP's court.

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