Sill distribution: genesis and salient features

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Summary. — We present the so-called Sill distribution, both in the nonrelativistic and relativistic cases, as a natural and simple way to include the effect of threshold(s) on the energy line shapes of resonances. The Sill is correctly normalized (even for broad states), is continuous at threshold(s), does not require any modification to the 'mass part', is easily extendable to the multichannel case, and can be applied to both mesons and baryons. Here, as a novel example, we employ the Sill to describe the resonance $\psi(3770)$.

The description of line shapes (or energy distributions) of unstable states is an important element of both Quantum mechanics (QM) and Quantum Field Theory (QFT) [1]. This is especially true for the study of short-lived hadrons, in which broad states with relatively close thresholds are quite common [2]. The aim of this work is to present the genesis and the main properties of a relatively simple distribution, called Sill distribution, originally put forward in Ref. [3].

The famous Breit-Wigner (BW) distribution [4, 5] (see also [6] for modern applications),

$$d^{\mathrm{BW}}(E) = \frac{\Gamma}{2\pi}\left[(E-M)^2 + \frac{\Gamma^2}{4}\right]^{-1} , \tag{1}$$

describes (in the non-relativistic limit) the line shape of a resonance with mass M and decay width Γ that fulfills the normalization $\int_{-\infty}^{+\infty} \mathrm{dE} d^{\mathrm{BW}}(E) = 1$. According to BW, any energy in the range $(-\infty, +\infty)$ is admissible, what is clearly a nonphysical feature since any physical system has a minimal 'threshold' energy E_{th}, being the energy of the ground

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state ($E_{th} \geq 0$ in QFT with $E_{th} = 0$ only if all the decay products of the resonance are massless particles). One may include this feature by a simple rescaling

$$d^{\mathrm{BW}}(E) \to N d^{\mathrm{BW}}(E)\theta(E - E_{th}), \tag{2}$$

where $\theta(E)$ is the step function and $N \geq 1$ is necessary to cure the loss of normalization. While this procedure may be a good strategy in certain applications, it is clear that it does not follow from a rigorous treatment of the problem.

In general, the proper inclusion of the decay in a quantum context implies the calculation of the self-energy $\Pi(E)$ describing the process 'state $\to$ decay products $\to$ state' leading to

$$d(E) = \frac{\mathrm{Im}\,\Pi(E)}{\pi}\left[(E - M + \mathrm{Re}\,\Pi(E))^2 + \mathrm{Im}\,\Pi(E)^2\right]^{-1} \ , \tag{3}$$

where $\Gamma(E) = 2\,\mathrm{Im}\,\Pi(E)$ is the energy-dependent width. Indeed, $d(E) = -\frac{1}{\pi}\,\mathrm{Im}[G(E)]$, with $G(E) = (E - M + \Pi(E) + i\varepsilon)^{-1}$ being the propagator of the unstable state. The normalization $\int_{E_{th}}^{+\infty} \mathrm{dE}d(E) = 1$ is valid in general [7].

For the actual calculation of the loop function $\Pi(E)$, one needs a microscopic model that couples the unstable states to its decay products (it could be in the form of Lee-Friedrichs Hamiltonian, e.g. [8, 9, 10]). Often, what is actually known (or assumed) is the function $\Gamma(E)$, thus $\Pi(E)$ (for complex E) can be reconstructed by the dispersion relation:

$$\Pi(E) = -\frac{1}{2\pi}\int_{E_{th}}^{\Lambda} \frac{\Gamma(E')}{E' - E + i\varepsilon} dE' + C \ , \tag{4}$$

where Λ is a high-energy cutoff (to be sent to ∞) and C is a real subtraction constant that guarantees that the mass M remains unchanged (that is, $\mathrm{Re}\,\Pi(M) = 0$; note, for E real, $\mathrm{Re}\,\Pi(E)$ reduces to the principal part of the integral above).

First, we apply this procedure to the choice $\Gamma(E) = \Gamma\theta(E - E_{th})$, which is the simplest extension of BW upon introducing a threshold. The resulting spectral function reads

$$d(E) = \frac{\Gamma}{2\pi}\left[\left(E - M + \frac{\Gamma}{2\pi}\ln\left(\frac{E - E_{th}}{M - E_{th}}\right)\right)^2 + \frac{\Gamma^2}{4}\right]^{-1} \theta(E - E_{th}) \ , \tag{5}$$

which is correctly normalized, but contains a logarithm in the mass part of the distribution as well as an unphysical behavior at threshold due to the abrupt jump of the decay width at that value. Neglecting the log-term would spoil the normalization and bring back to Eq. 2. In the limit $E_{th} \to -\infty$ the BW-limit is, as expected, recovered.

As discussed in [3], the simple choice

$$\Gamma(E) = \gamma\sqrt{E - E_{th}}\theta(E - E_{th}) \tag{6}$$

with γ being a constant ($\gamma = \Gamma/\sqrt{M - E_{th}}$), leads to the nonrelativistic Sill distribution

$$d^{\mathrm{nrSill}}(E) = \frac{\gamma\sqrt{E - E_{th}}}{2\pi}\left[(E - M)^2 + \frac{1}{4}\left(\gamma\sqrt{E - E_{th}}\right)^2\right]^{-1} \theta(E - E_{th}) \ , \tag{7}$$

which is correctly normalized to one for any value of the involved parameters (as long as $M > E_{th}$). Since $\operatorname{Re}\Pi(E)$ vanishes above E_{th}, the Sill does not contain any modification to the 'mass part' of the spectral function. Moreover, it describes the left threshold without abrupt jumps. The on-shell width is obtained as $\Gamma \equiv \Gamma(M) = \gamma\sqrt{M - E_{th}}$.

In order to discuss the relativistic case, we need to perform the replacements $E \to s = E^2$ (as well as $M \to M^2$ and $E_{th} \to s_{th} \geq 0$ for the lowest energy threshold), and $\Pi(E) \to \Pi(s)$ with $\operatorname{Im}\Pi(s) = \sqrt{s}\Gamma(s)$. Thus, in analogy to the QM case, an unstable resonance is described by

$$d(s) = \frac{\operatorname{Im}\Pi(s)}{\pi}\left[(s - M^2 + \operatorname{Re}\Pi(s))^2 + \operatorname{Im}\Pi(s)^2\right]^{-1} , \tag{8}$$

which is correctly normalized, $\int_{s_{th}}^{+\infty} \mathrm{dE} d(s) = 1$. Moreover, $d(s) = -\frac{1}{\pi}\operatorname{Im}[G(s)]$ with the relativistic propagator $G(s) = (s - M^2 + \Pi(s) + i\varepsilon)^{-1}$. The loop function $\Pi(s)$ reads

$$\Pi(s) = -\frac{1}{\pi}\int_{s_{th}}^{\Lambda^2} \frac{\sqrt{s'}\Gamma(s')}{s' - s + i\varepsilon} dE' + C \tag{9}$$

where C is chosen such that $\operatorname{Re}\Pi(M^2) = 0$, thus the nominal mass is left unchanged.

The relativistic Sill distribution corresponds to the choice $\Gamma(s) = \tilde{\Gamma}\sqrt{(s - s_{th})/s}$, leading to

$$d^{\mathrm{Sill}}(s) = \frac{\tilde{\Gamma}\sqrt{s - s_{th}}}{\pi}\left[(s - M^2)^2 + \tilde{\Gamma}^2\,(s - s_{th})\right]^{-1}\theta(s - s_{th}) . \tag{10}$$

It is normalized to unity for any choice of M and $\tilde{\Gamma}$, the left-threshold is taken into account, $\operatorname{Re}\Pi(s) = 0$ above s_{th}, thus making it quite simple to use. Moreover, the width function $\Gamma(s)$ reduces to a constant for large values of s, thus heuristically one may consider the Sill distribution as the proper extension of the BW function to the relativistic case. This is also evident in the limit $s_{th} = 0$, for which $\Gamma(s) = \tilde{\Gamma}$ is a constant.

Various comments are important at this point:

1) For a two-body decay into particles with masses m_1 and m_2, one sets $s_{th} = (m_1 + m_2)^2$. In general, the Sill is different from the Flatté distribution [11] (see also e.g. [12, 13]) for which $\operatorname{Im}\Pi(s) \sim k(s)$ with $k(s)$ being the momentum.

2) Extension to the multichannel case and to decay chains is straightforward [3]. The Sill can be applied to both mesons and baryons.

3) One may obtain the relativistic BW (rBW) distribution by setting $\operatorname{Im}\Pi(s) = M\Gamma\theta(s)$ (a left threshold must exist in QFT). Interestingly, this corresponds to a non-constant width of the type $\Gamma(s) = \Gamma M/\sqrt{s}$. The resulting distribution

$$d(s) = \frac{\Gamma M}{\pi}\left[\left(s - M^2 + \frac{\Gamma M}{\pi}\ln\left(s/M^2\right)\right)^2 + \left(\Gamma M\right)^2\right]^{-1}\theta(s) \tag{11}$$

reduces to $d^{\mathrm{rBW}}(s) = N\frac{\Gamma M}{\pi}\left[\left(s - M^2\right)^2 + \left(\Gamma M\right)^2\right]^{-1}\theta(s)$ when the log-term is neglected (see e.g. [15] for applications).

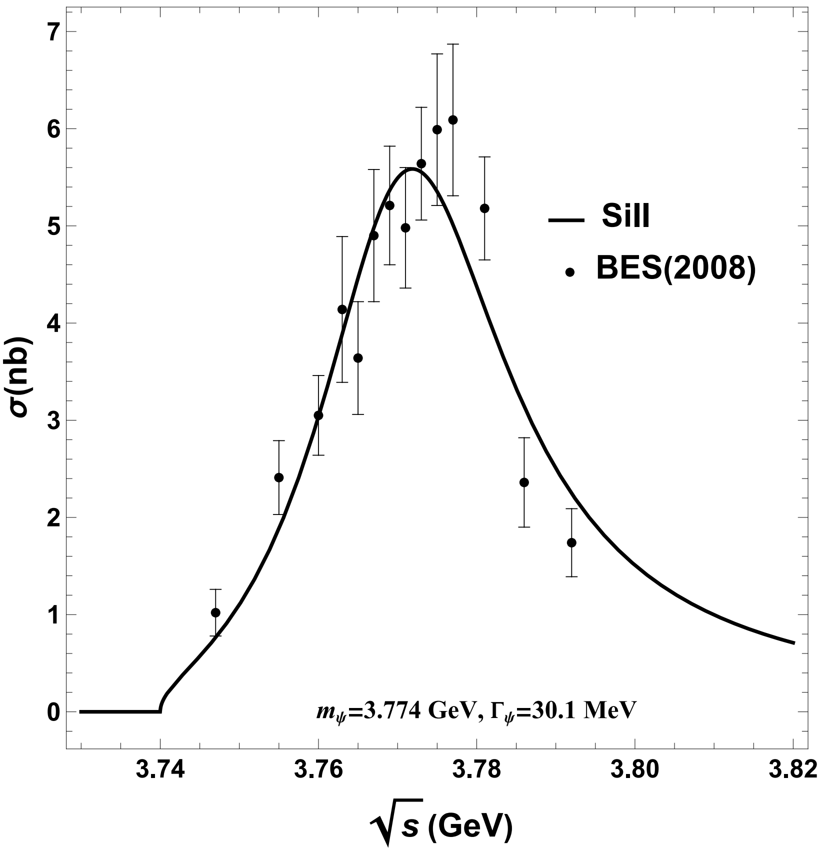

Fig. 1. – The $e^+e^- \to D\bar{D}$ data reported by BES (Table 2 of Ref. [16]) fitted with the Sill distribution (χ^2 per d.o.f is 1.59). The mass and width of $\psi(3770)$ obtained from the fit are shown at the bottom of the figure.

4) Although all distributions are peaked at the mass M, the high-E scaling are quite different. BW scales as E^{-2} , the nrSill goes as $E^{-3/2}$, the rBW as E^{-3} and the Sill as E^{-2}, as BW. (Note, due to the variable change $s = E^2$, one has $d^{\text{Sill}}(E) = 2Ed^{\text{Sill}}(s = E^2)$).

5) In the complex plane, the Sill loop $\Pi(s) = i\tilde{\Gamma}\sqrt{s - s_{th}}$ in the first Riemann sheet (RS) contains a (s_{th}, ∞)-cut. In the II RS, $\Pi_{II}(s) = \Pi(s) + 2i\tilde{\Gamma}(\sqrt{s - s_{th}})_{II} = -i\tilde{\Gamma}\sqrt{s - s_{th}}$ has a similar feature. While BW contains only one sheet, the Sill has a richer and more realistic (although not complete, see below) analytic structure.

6) An actual scalar QFT of the type $\mathcal{L} = gS\varphi^2$ leads to the decay width $\Gamma(s) = \frac{g^2\sqrt{s-s_{th}}}{8\pi s}$ [14], where $s_{th} = 4m^2$, with m being the φ mass. This expression has different analytical properties with respect to the Sill in Eq. 10. The self-energy takes the form:

$$\Pi(s) = \frac{g^2\sqrt{s - s_{th}}}{8\pi\sqrt{s}} \ln\left(\frac{\sqrt{s - s_{th}} - \sqrt{s}}{\sqrt{s - s_{th}} + \sqrt{s}}\right) + C \ . \tag{12}$$

In the I Riemann sheet it has a cut on the real axis that extends from s_{th} to ∞ (note, $s < 0$

is, thanks to the log-function, regular). In the II RS, $\Pi_{II}(s) = \Pi(s) + 2i\frac{g^2\left(\sqrt{s(s-s_{th})}\right)_{II}}{8\pi s}$, which contains also a branch cut on the negative s-axis, a feature not contemplated in the Sill. In this sense, the Sill simplifies certain analytic properties, but still retains the most important ones.

In Ref. [3] various examples of resonances described via the Sill were presented. Here, as a novel one, the line shape of the charmonium resonance $\psi(3770)$, with one dominating $\bar{D}D$ decay channel, is plotted in Fig. 1,. The mass and width of $\psi(3770)$ are obtained as $m = 3.774$ GeV and $\Gamma = 30.1$ MeV respectively, in good agreement with the PDG values. To arrive at these values, we fit the Sill distribution the data reported by BES collaboration (Table 2 of Ref. [16]) along with a normalization constant that takes care of the phase space factors. Quite remarkably, the Sill fits as well as the complete loop treatment of this resonance discussed in [17] (to which we refer for connecting the cross section to the spectral function)

Meanwhile, the Sill has been already used in the review paper for the description of the Δ baryon [18] as well as by the Alice collaboration [19] for the description of the baryon state $\Xi(1620)$ (see also [20]). In the future, one may search for Sill extensions that are better suited to describe higher angular momentum waves [21] and to three-body decay rates (see, Ref. [22] for a phenomenological exposition).

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