J/ψ and ω decays to 3π with Khuri–Treiman equations

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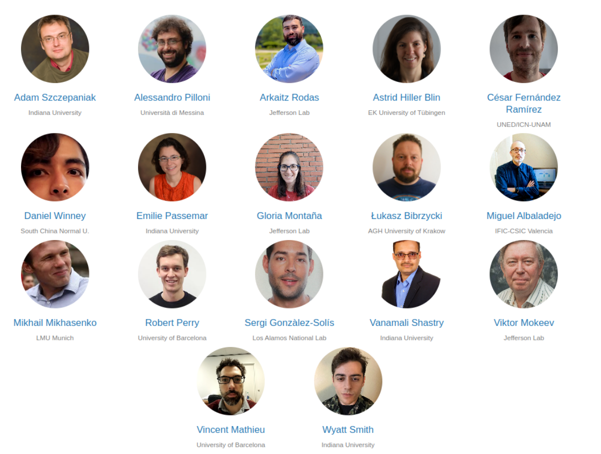
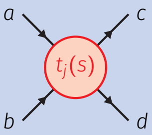
JPAC: Joint Physics Analysis Center

- Work in theoretical/experimental/phenomenological analysis
- Light/heavy meson spectroscopy
- Interaction with many experimental collaborations: (GlueX, CLAS, BES, ...) and LQCD groups
- Plenary talk by A. Szczepaniak [Tue., noon]
- Web site: https://www.jpac-physics.org/



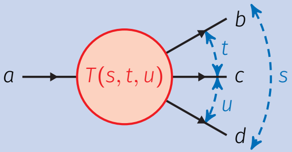
Introduction: Khuri-Treiman equations in a nutshell



- Partial wave expansion in the s-channel:

$$T(s,t,u) = \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z_s) t_\ell(s)$$

- Two main (connected) problems:
  - Infinite number of PW
  - PW have RHC and LHC
- Only RHC: BS equation, K-matrix, DR,...
- Problem with "truncation": $t_\ell(s)$ only depends on s, so singularities in the t-, u-channel can only appear suming an infinite number of PW.



- In many decay processes one wants to take into account unitarity/FSI interactions in the three possible channels.

Introduction: Khuri-Treiman equations in a nutshell

- Khuri-Treiman equations are a tool to achieve this two-body unitarity in the three channels
  [N. Khuri, S. Treiman, Phys. Rev. **119**, 1115 (1960)]
- Consider three (s-, t-, u-channels) truncated "isobar" expansions.
- Isobars $f_\ell^{(s)}(s)$ have only RHC: amenable for dispersion relations.

$$
\begin{aligned}
T(s,t,u) &= \sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(z_s)\; t_\ell(s) \\
&= \sum_{\ell=0}^{n_s}(2\ell+1)P_\ell(z_s)f_\ell^{(s)}(s) + \sum_{\ell=0}^{n_t}(2\ell+1)P_\ell(z_t)f_\ell^{(t)}(t) + \sum_{\ell=0}^{n_u}(2\ell+1)P_\ell(z_u)f_\ell^{(u)}(u)
\end{aligned}
$$

- s-channel singularities appear in the s-channel isobar, $t_\ell^{(s)}(s)$.
- Singularities in the t-, u-channel are recovered!
- The LHC of the partial waves are given by the RHC of the crossed channel isobars

$$
t_\ell(s) = \frac{1}{2}\int \mathrm{d}z\, P_\ell(z) T(s,t',u') = f_\ell^{(s)}(s) + \frac{1}{2}\int \mathrm{d}z\, Q_{\ell\ell'}(s,t') f_{\ell'}^{(t)}(t')\ .
$$

$\omega \longrightarrow 3\pi$ amplitude. Phenomenology

JPAC Collab., EPJ,C80,1107('20)

- Amplitude:

$$\mathcal{M}_+(s,t,u) = \frac{\sqrt{\phi(s,t,u)}}{2} F(s,t,u)\,. \qquad \left(\phi(s,t,u) = 4sp^2(s)q^2(s)\sin^2\theta_s\right)$$

- Decay width: $d^2\Gamma \sim \phi(s,t,u)\,|F(s,t,u)|^2$
- Dalitz plot parameters (α, β, γ) "equivalent" to bins... $(X,Y) \leftrightarrow (Z,\varphi) \leftrightarrow (s,t,u)$

$$|F(s,t,u)|^2 = |\mathcal{N}|^2 \left(1 + 2\alpha Z + 2\beta Z^{\frac{3}{2}} \sin 3\varphi + 2\gamma Z^2 + \cdots\right)$$

- Why revisit $\omega \longrightarrow 3\pi$?

$\omega \to 3\pi$ amplitude. Phenomenology

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	Bonn (2012)		JPAC (2015)		
	Eur. Phys. J., **C72**, 2014 (2012)		Phys. Rev., **D91**, 094029 (2015)		
	w/o KT	w KT	w/o KT	w KT	
α	130(5)	79(5)	125	84	
β	31(2)	26(2)	30	28	

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	Bonn (2012)		JPAC (2015)		BESIII (2018)
	Eur. Phys. J., **C72**, 2014 (2012)		Phys. Rev., **D91**, 094029 (2015)		Phys. Rev., **D98**, 112007 (2018)
	w/o KT	w KT	w/o KT	w KT	Exp.
α	130(5)	79(5)	125	84	120.2(7.1)(3.8)
β	31(2)	26(2)	30	28	29.5(8.0)(5.3)

- One (or more) out of three is wrong...

1) Experiment?
2) KT eqs., in general?
3) Something particular?

KT equations: DR, subtractions, solutions, and all that...

- PW decomposition: $F(s,t,u) = \sum_{j\,\mathrm{odd}} P_j'(\cos\theta_s)[p(s)q(s)]^{j-1} f_j(s) = f_1(s) + \cdots$
- KT/isobar decomposition: consider only $j=1$ (ρ) isobar, $F(s)$:

$$\boxed{F(s,t,u) = F(s) + F(t) + F(u)}$$

- PW projection of the KT decomposition:

$$f_1(s) = F(s) + \hat{F}(s)\,, \qquad \hat{F}(s) = \frac{3}{2}\int_{-1}^{1} dz_s (1-z_s^2) F(t(s,z_s))$$

- Discontinuity:

$$\Delta F(s) = \Delta f_1(s) = \rho(s) t_{11}^*(s) f_1(s) = \rho(s) t_{11}^*(s)\left(F(s) + \hat{F}(s)\right)$$

Unsubtracted DR

$$F(s) = a\, F_0(s)$$

$$F_0(s) = \Omega(s)\left[1 + \frac{s}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\sin\delta(s')\hat{F}_0(s')}{|\Omega(s')|(s'-s)}\right]$$

Once-subtracted DR

$$F(s) = a\left(F_a'(s) + b\, F_b(s)\right)$$

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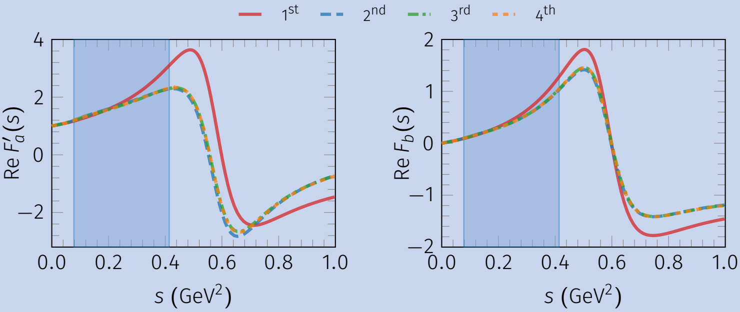
KT equations: DR, subtractions, solutions, and all that...



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Once-subtracted DR

$$F(s) = a\left(F'_a(s) + b\, F_b(s)\right)$$

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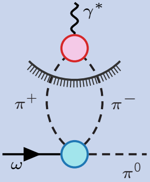
$\omega \to \pi^0$ transition form factor

- The decays $\omega(\to \pi^0\gamma^*) \to \pi^0\ell^+\ell^-$ and $\omega \to \pi^0\gamma$ governed by the TFF $f_{\omega\pi^0}(s)$.

$$\mathcal{M}(\omega \to \pi^0\ell^+\ell^-) = f_{\omega\pi^0}(s)\,\epsilon_{\mu\nu\alpha\beta}\epsilon^\mu(p_\omega,\lambda)p^\nu q^\alpha \frac{ie^2}{s}\bar{u}(p_-)\gamma^\beta v(p_+)$$

$$\Gamma(\omega \to \pi^0\gamma) = |f_{\omega\pi^0}(0)|^2 \frac{e^2(m_\omega^2 - m_{\pi^0}^2)^3}{96\pi m_\omega^3}$$

- Dispersive representation:



$$f_{\omega\pi^0}(s) = f_{\omega\pi^0}(0) + \frac{s}{12\pi^2}\int_{4m_\pi^2}^\infty \mathrm{d}s' \frac{q_\pi(s')^3}{s'^{\frac{3}{2}}(s'-s)}\left(F(s') + \hat{F}(s')\right)F_\pi^V(s')^*$$

- $f_{\omega\pi^0}(0) = |f_{\omega\pi^0}(0)|\, e^{i\phi_{\omega\pi^0}(0)}$
- Experimental information: $F_{\omega\pi^0}(s) = \frac{f_{\omega\pi^0}(s)}{f_{\omega\pi^0}(0)}$
- Only the relative phase $\frac{a}{f_{\omega\pi^0}(0)} = \frac{|a|}{\left|f_{\omega\pi^0}(0)\right|}\frac{1}{e^{i(\phi_{\omega\pi^0}(0)-\phi_a)}}$.

Summary of amplitudes/free parameters/exp. input

$\omega \to 3\pi$ amplitude $[F(s,t,u)]$

Free parameters: $|a|$, $|b|$, ϕ_b

Experimental input:

- $\Gamma_{3\pi}$
- Dalitz plot parameters

$\omega \to \gamma^{(*)}\pi^0$ TFF $[f_{\omega\pi^0}(s)]$

Free parameters: $|f_{\omega\pi^0}(0)|$, $\phi_{\omega\pi^0}(0)$
($\bigoplus$ $|a|$, $|b|$, ϕ_b)

Experimental input:

- $\Gamma_{\gamma\pi^0}$
- $\left|F_{\omega\pi^0}(s)\right|^2$

First analysis in three steps

JPAC Collab., EPJ,C80,1107('20)

1. Fix $|b| \simeq 2.9$, $\phi_b \simeq 1.9$ with the DP parameters.
2. Fix $|a| \simeq 280\ \text{GeV}^{-3}$, $|f_{\omega\pi^0}(0)| \simeq 2.3\ \text{GeV}^{-1}$ from $\Gamma_{\omega\to 3\pi}$, $\Gamma_{\omega\to\gamma\pi}$.
3. You are left with $\phi_{\omega\pi^0}(0)$ and the TFF Data.

1. $$\chi^2_{\text{DP}} = \left(\frac{\alpha^{(t)} - \alpha^{(e)}}{\sigma_\alpha}\right)^2 + \cdots$$
2. $$\chi^2_\Gamma = \left(\frac{\Gamma^{(t)}_{3\pi} - \Gamma^{(e)}_{3\pi}}{\sigma_{\Gamma_{3\pi}}}\right)^2 + \left(\frac{\Gamma^{(t)}_{\gamma\pi} - \Gamma^{(e)}_{\gamma\pi}}{\sigma_{\Gamma_{\gamma\pi}}}\right)^2$$
3. $$\chi^2_{\text{A2,NA60}} = \sum_i \left(\frac{|F_{\omega\pi}(s_i)|^2 - \left|F^{(i)}_{\omega\pi}\right|^2}{\sigma_{F^{(i)}_{\omega\pi}}}\right)^2$$

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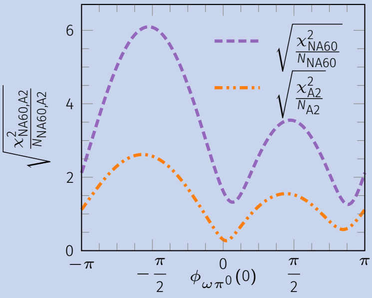
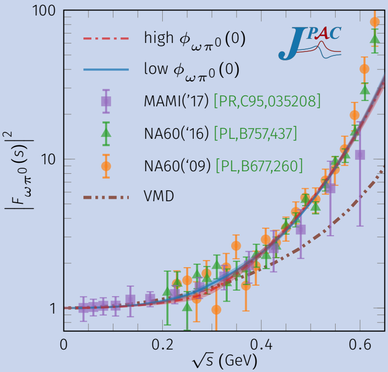


- Two different minima (low and high $\phi_{\omega\pi^0}(0)$) are found.
- Both have similar χ^2 of the TFF.

Make a global, simultaneous analysis

$$\overline{\chi}^2 = N\left(\frac{\chi^2_{\rm DP}}{N_{\rm DP}} + \frac{\chi^2_\Gamma}{N_\Gamma} + \frac{\chi^2_{\rm NA60}}{N_{\rm NA60}} + \frac{\chi^2_{\rm A2}}{N_{\rm A2}}\right)$$

Results



	α	β	γ
BESIII	111(18)	25(10)	22(29)
low	112(15)	23(6)	29(6)
high	109(14)	26(6)	19(5)

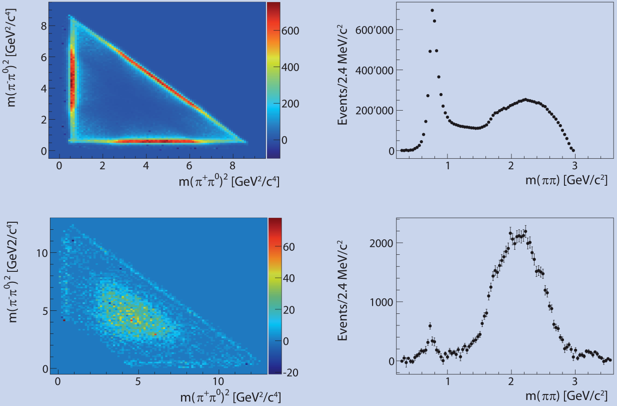
Using once-subtracted DR for KT:

- Agreement is restored with DP parameters by BESIII
- One can also describe the $\omega\pi^0$ TFF

	2 par.		3 par.	
	low $\phi_{\omega\pi^0}(0)$	high $\phi_{\omega\pi^0}(0)$	low $\phi_{\omega\pi^0}(0)$	high $\phi_{\omega\pi^0}(0)$
$10^{-2}\ \lvert a\rvert\ \left[\text{GeV}^{-3}\right]$	3.14(25)	2.63(25)	3.11(28)	2.70(30)
$\lvert b\rvert$	3.15(22)	2.59(35)	3.25(26)	2.65(35)
ϕ_b	2.03(14)	1.61(38)	2.03(13)	1.70(27)
$\lvert f_{\omega\pi^0}(0)\rvert\ \left[\text{GeV}^{-1}\right]$	2.314(32)	2.314(32)	2.314(32)	2.315(32)
$\phi_{\omega\pi^0}(0)$	0.207(60)	2.39(46)	0.195(76)	2.48(31)
$\chi^2_{\rm DP}$	0.19	< 0.01	0.10	0.03
$10^4\chi^2_{\Gamma}$	2.4	2.4	1.1	3.5
$\chi^2_{\rm A2}$	2.3	3.6	2.4	3.7
$\chi^2_{\rm NA60}$	31	35	31	35

$J/\psi \to 3\pi$ decays

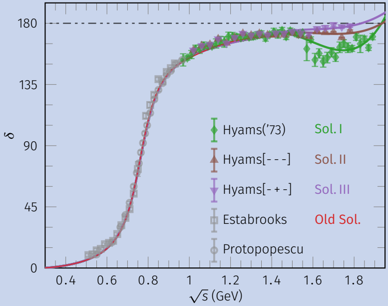
- Formalism for J/ψ is completely analogous to ω (V).
- BESIII data [Phys. Lett., **B710**, 594 (2012)] show ψ/ψ' puzzle:

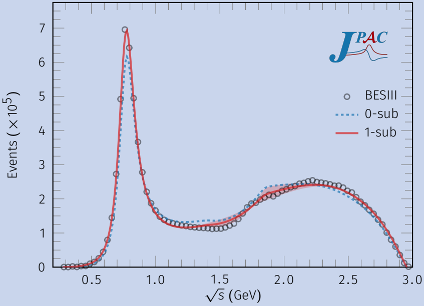


- The J/ψ decay seems to be dominated by ρ, despite the larger phase space
- One would expect that 0-sub (prediction) would get the basic features

$J/\psi \to 3\pi$ decays

[JPAC Collab., 2304.09736]





- $\delta_{\pi\pi}(s)$ taken as input:
  - Old solution: [Garcia-Martin, Kaminski, Pelaez, Ruiz de Elvira, Yndurain, Phys. Rev. **D83**, 074004 (2011)]
  - New solutions: [Pelaez, Rodas, Ruiz De Elvira, Eur. Phys. J. **C79**, 1008 (2019)]
- Take as central fit the one performed with solution I for $\delta_{\pi\pi}$
- The spread in the other solutions: theoretical uncertainty

- The J/ψ decay seems to be dominated by ρ, despite the larger phase space
- 0-sub (prediction) get the basic features
- 1-sub (fit) improves the description
- 1-sub + F-wave [$\rho_3(1690)$] describes better the movements above $\gtrsim$ 1.5 GeV.

$J/\psi \to 3\pi$ decays

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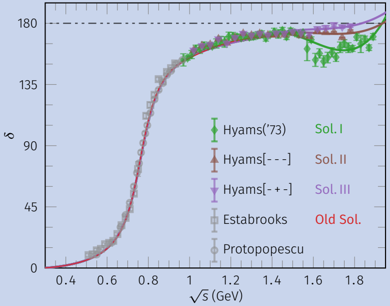

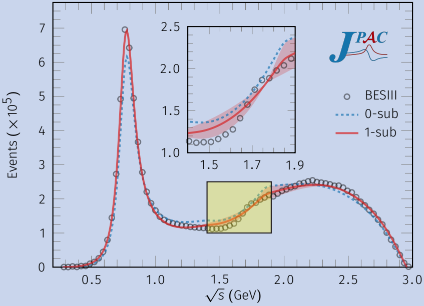

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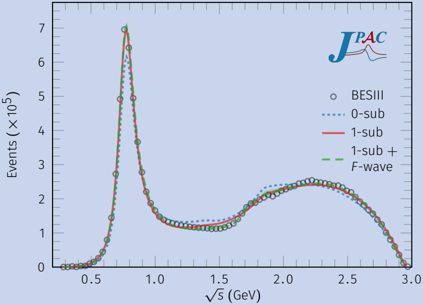
$J/\psi \to 3\pi$ decays: including F wave

[JPAC Collab., 2304.09736]

- How to improve the description? Include F-wave, $\rho_3(1690)$ (PDG values)

$$F(s,t,u) = F_1(s) + F_1(t) + F_1(u) + \frac{\kappa^2(s)}{16} P_3'(z_s) F_3(s) + \frac{\kappa^2(t)}{16} P_3'(z_t) F_3(t) + \frac{\kappa^2(u)}{16} P_3'(z_u) F_3(u)$$

- Neglect $\widehat{F}_3$, so that $F_3(s) = p_3(s)\,\Omega_3(s)$



- The fit improves significantly, especially around $\sqrt{s} \gtrsim 1.5$ GeV, the main contribution being the P–F-wave interference.

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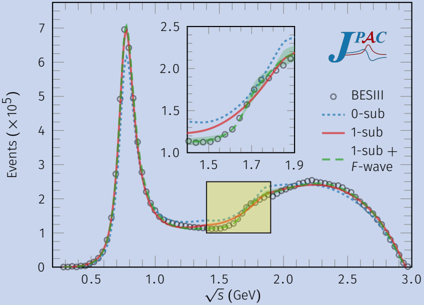

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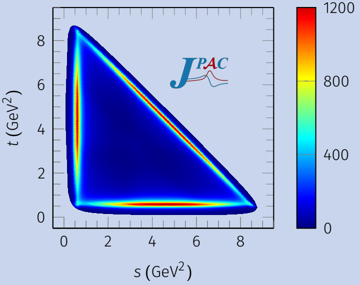
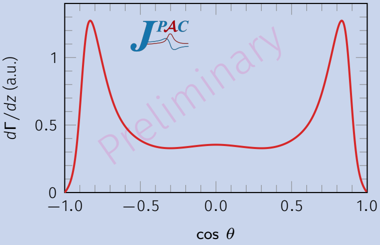
$J/\psi \to 3\pi$ decays: additional information

[JPAC Collab., 2304.09736]

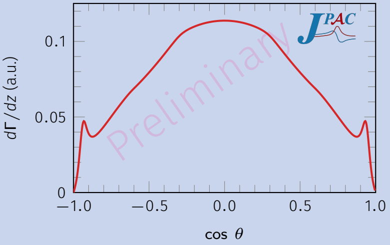
- Dalitz plot distribution similar to exp. one
- More statistics will allow to unveil other effects (resonances, interferences,...)
- Predictions can be done for angular [$z = \cos\theta_s$] distributions, specially restricted to ρ-mass region.



Full $\sqrt{s}$ range



$\left|\sqrt{s} - m_\rho\right| \leqslant 50$ MeV

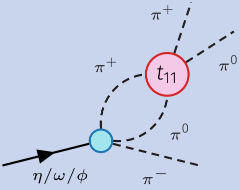


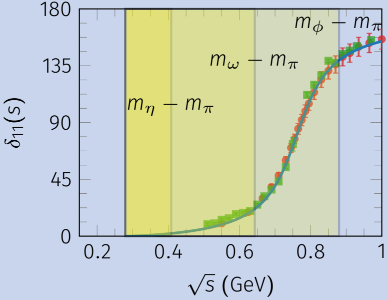
Summary

- KT equations are a powerful tool to study 3-body decays
- They allow to implement two-body unitarity in all the three channels (s, t, u).
- For $\omega \to 3\pi$ decays: JPAC Collab., EPJ,C80,1107('20)
  - Using once-subtracted DRs, we are able to reproduce the $\omega \to 3\pi$ DP parameters,
  - and the $\omega \to \pi^0\gamma^*$ transition form factor data.
- For $J/\psi \to 3\pi$ decays, good agreement with the data is found assuming elastic unitarity (P- and F-waves). JPAC Collab., 2304.09736

- KT equations are a powerful tool to study 3-body decays.
- They allow to implement two-body unitarity in all the three channels (s, t, u).
- Iterative solution converges fast, linear in subtraction constants.
- For $\eta \to 3\pi$: MA, B. Moussallam, EPJ,C77,508('17)
  - Not well described by the perturbative chiral amplitudes.
  - We have presented an extension of this approach to coupled channels. The extension is quite general.
  - Effects of $K\bar{K}$ and $\eta\pi$ amplitudes $[f_0(980), a_0(980)]$ play some role in the DP parameters, tend to improve.
- For $\pi\pi$ scattering: JPAC Collab., EPJ,C78,574('18)
  - We have applied KT equations to $\pi\pi$ scattering as benchmark.
  - Restricted to S- and P-waves, KT equations are equal to Roy equations.
  - When other waves are included, good comparison is obtained with GKPY equations.
- We have presented a generalization of the KT equations for arbitrary quantum numbers of the decaying particle. JPAC Collab., PR,D101,054018('20)
  - Not trivial, because of spin/crossing.
- For $\omega \to 3\pi$ decays: JPAC Collab., EPJ,C80,1107('20)
  - Using once-subtracted DRs, we are able to reproduce the $\omega \to 3\pi$ DP parameters,
  - and the $\omega \to \pi^0\gamma^*$ transition form factor data.
- For $J/\psi \to 3\pi$ decays, good agreement with the data is found assuming elastic unitarity (P- and F-waves). JPAC Collab., 2304.09736

KT and phase space



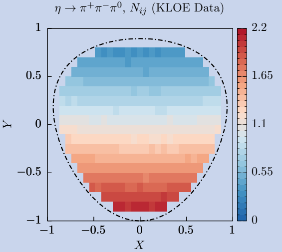


Generalities about $\eta \to 3\pi$

- In QCD isospin-breaking phenomena are driven by

$$\boxed{H_{IB} = -(m_u - m_d)\overline{\psi}\frac{\lambda_3}{2}\psi}$$

- Isospin-breaking induced by EM & strong interactions are similar in size, but
- $\eta \to 3\pi$ is special, since EM effects are smaller
- $\Gamma_{\eta\to 3\pi} \propto Q^4$, with $\boxed{Q^{-2} = \frac{m_d^2 - m_u^2}{m_s^2 - (m_u + m_d)^2/2}}$



- Experimental situation: Several high-statistics studies; $|T|^2$ well known across the Dalitz plot $\Rightarrow$ stringent tests for the amplitudes (before getting Q!)

$\eta \to 3\pi^0$	$\eta \to \pi^+\pi^-\pi^0$
Crys. Ball, PRL**87**,192001('01)	KLOE, JHEP**0805**,006('08)
Crys. Ball@MAMI, A2, PRC**79**,035204('09)	WASA-at-COSY, PRC**90**,045207('14)
Crys. Ball@MAMI, TAPS, A2, EPJA**39**,169('09)	BESIII, PRD**92**,012014('15)
WASA-at-COSY, PLB**677**,24('09)	KLOE-2, JHEP**1605**,019('16)
KLOE, PLB**694**,16('11)	

Previous dispersive approaches to $\eta \to 3\pi$

- Chiral $\mathcal{O}(p^4)$ amplitude fails in describing experiments.

  Gasser, Leutwyler, Nucl. Phys. B**250**, 539 (1985)
- Several attemps to include unitarity/FSI/rescattering effects.

  Neveu, Scherk, AP**57**, 39('70); Roiesnel, Truong, NPB**187**, 293('81); Kambor, Wiesendanger, Wyler, NPB**465**, 215('96); Anisovich, Leutwyler, PLB**375**, 335('96); Borasoy, R. Nißler, EPJA**26**, 383('05); Schneider, Kubis, Ditsche, JHEP**1102**, 028('11); Kampf, Knecht, Novotný, Zdráhal, PRD**84**, 114015('11); Colangelo, Lanz, Leutwyler, Passemar, PRL**118**, 022001('17); Guo, Danilkin, Fernández-Ramírez, Mathieu, Szczepaniak, PL**771**, 497('17).
- Here we reconsider the KT approach. N. Khuri, S. Treiman, Phys. Rev. **119**, 1115 (1960)
- $\pi\pi$ scattering elastic in the decay region. But dispersive approaches require higher energy T-matrix inputs:
  - $\pi\pi$ near 1 GeV rapid energy variation. $f_0(980)$, $(K\bar{K})_0$
  - Double resonance effect $\eta\pi$ ISI, $a_0(980)$, $(K\bar{K})_1$

  Abdel-Rehim, Black, Fariborz, Schechter, PRD**67**, 054001('03)

> We propose a generalization to coupled channels $[(K\bar{K})_{0,1}, \eta\pi, (\pi\pi)_{0,1,2}]$ of the KT equations, extending their validity up to the physical $\eta\pi \to \pi\pi$ region. Allows for the study of the influence of a_0, f_0 into the decay region.

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RESOLUTION OF THE $\eta \to 3\pi$ PROBLEM

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Isospin amplitudes

- Start with well-defined isospin amplitudes:

$$\mathcal{M}^{I,I_z}(s,t,u) = \left\langle \eta\pi; 1, I_z \left| \hat{T}_0^{(1)} \right| \pi\pi; I, I_z \right\rangle = \langle I, I_z; 1, 0 | 1, I_z \rangle \langle \eta\pi \| \hat{T}^{(1)} \| \pi\pi; I \rangle$$

- They can be written in terms of a single amplitude ($\eta\pi^0 \to \pi^+\pi^-$), $A(s,t,u)$ (like in $\pi\pi$ scattering):

$$\begin{bmatrix} -\sqrt{3}\mathcal{M}^0(s,t,u) \\ \sqrt{2}\mathcal{M}^1(s,t,u) \\ \sqrt{2}\mathcal{M}^2(s,t,u) \end{bmatrix} = \begin{bmatrix} -\sqrt{3}\mathcal{M}^{0,0}(s,t,u) \\ \sqrt{2}\mathcal{M}^{1,1}(s,t,u) \\ \sqrt{2}\mathcal{M}^{2,1}(s,t,u) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} A(s,t,u) \\ A(t,s,u) \\ A(u,t,s) \end{bmatrix}$$

- Reconstruction theorem (for Goldstone bosons):

J. Stern, H. Sazdjian, N. Fuchs, Phys. Rev. D**47**, 3814 (1993)

$$A(s,t,u) = -\epsilon_L \Big[M_0(s) - \frac{2}{3} M_2(s) + M_2(t) + M_2(u) \qquad \epsilon_L = \frac{1}{Q^2} \frac{m_K^2 - m_\pi^2}{3\sqrt{3} f_\pi^2} \frac{m_K^2}{m_\pi^2}$$
$$+ (s-u) M_1(t) + (s-t) M_1(u) \Big]$$

- Or in general, "the" KT approximation:

Infinite sum of s-channel PW $\to$ Truncated sums of s-, t-, and u-channels PWs

- Single variable functions: amenable for dispersion relations.

Partial wave amplitudes

- Summary of previous slide: $\mathcal{M}^I(s,t,u)$ is written in terms of $A(s,t,u)$ (and permutations), and $A(s,t,u)$ is written in terms of $M_I(w)$.
- Now, define partial waves: $\mathcal{M}^I(s,t,u) = 16\pi\sqrt{2}\sum_j (2j+1)\mathcal{M}^I_j(s)P_j(z)$

$$\mathcal{M}^0_0(s) = \epsilon_L \frac{\sqrt{6}}{32\pi}[M_0(s) + \hat{M}_0(s)] \,, \quad \mathcal{M}^2_0(s) = \epsilon_L \frac{-1}{32\pi}[M_2(s) + \hat{M}_2(s)] \,,$$
$$\mathcal{M}^1_1(s) = \epsilon_L \frac{\kappa(s)}{32\pi}[M_1(s) + \hat{M}_1(s)] \,,$$

LHC $[\hat{M}_I(s)]$

$\hat{M}_I(s)$ written as angular averages.
Take $M_0(s)$ as an example:

$$\hat{M}_0(s) = \frac{2}{3}\langle M_0\rangle + \frac{20}{9}\langle M_2\rangle$$
$$+\, 2(s-s_0)\langle M_1\rangle + \frac{2}{3}\kappa(s)\langle zM_1\rangle$$
$$\langle z^n M_I\rangle(s) = \frac{1}{2}\int_{-1}^{1} dz\, z^n M_I(t(s,z))$$
$$\kappa(s) = \sqrt{(1-4m_\pi^2/s)\lambda(s, m_\eta^2, m_\pi^2)}$$

RHC $[M_I(s)]$

$\hat{M}(s)$ no discontinuity along the RHC:

$$\text{disc}\, M_I(s) = \text{disc}\mathcal{M}^I_j(s) =$$
$$= \sigma_\pi(s) t^I(s)^* \mathcal{M}^I_j(s)$$
$$= \sigma_\pi(s) t^I(s)^* \left(M_I(s) + \hat{M}_I(s)\right)$$
$$\sigma_\pi(s) = \sqrt{1-4m_\pi^2/s}$$
$$\sigma_\pi(s) t^I(s) = \sin\delta_I(s)\, e^{i\delta_I(s)}$$

Muskhelisvili-Omnès representation

$$\boxed{\mathrm{disc}M_I(s) = \sigma_\pi(s) t_I^*(s)\left[M_I(s) + \hat{M}_I(s)\right]}$$

- MO (dispersive) representation of $M_I(s)$:

$$M_0(s) = \Omega_0(s)\left[\alpha_0 + \beta_0 s + \gamma_0 s^2 + \hat{I}_0(s) s^2\right] ,$$
$$M_1(s) = \Omega_1(s)\left[\beta_1 s + \hat{I}_1(s) s\right] ,$$
$$M_2(s) = \Omega_2(s)\left[\hat{I}_2(s) s^2\right] .$$

$$\Omega_I(s) = \exp\left[\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} \mathrm{d}s' \frac{\delta_I(s')}{s'(s'-s)}\right] \text{ (Omnès function/matrix)}$$

$$\hat{I}_I(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\sin\delta_I(s')\hat{M}_I(s')}{|\Omega_I(s')|\,(s')^{m_I}(s'-s)} , \quad (m_{0,2} = 2,\ m_1 = 1)$$

- $m_\eta^2 + i\varepsilon$ prescription needed. Integral equations solved iteratively.
- Subtraction constants: Most natural way is to match with ChPT:

$$\mathcal{M}(s,t,u) - \overline{\mathcal{M}}_\chi(s,t,u) = \mathcal{O}(p^6)$$

Descotes-Genon, Moussallam, EPJ,C74,2946(2014)

- Matching conditions: fix α_0, β_0, β_1, γ_0 in terms of ChPT amplitudes (no free parameters).

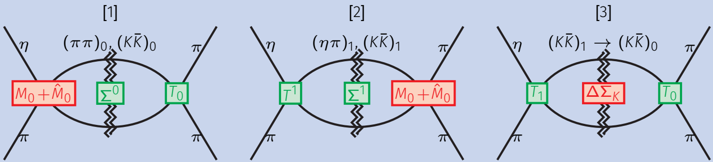
Coupled channels: take into account intermediate states other than $(\pi\pi)_I$.

$$\boldsymbol{M}_0 = \begin{bmatrix} M_0 & G_{10} \\ N_0 & H_{10} \end{bmatrix} = \begin{bmatrix} (\eta\pi)_1 \to (\pi\pi)_0 & (K\bar{K})_1 \to (\pi\pi)_0 \\ (\eta\pi)_1 \to (K\bar{K})_0 & (K\bar{K})_1 \to (K\bar{K})_0 \end{bmatrix},$$

$$\boldsymbol{T}_0 = \begin{bmatrix} t_{(\pi\pi)_0\to(\pi\pi)_0} & t_{(\pi\pi)_0\to(K\bar{K})_0} \\ t_{(\pi\pi)_0\to(K\bar{K})_0} & t_{(K\bar{K})_0\to(K\bar{K})_0} \end{bmatrix}, \boldsymbol{T}_1 = \begin{bmatrix} t_{(\eta\pi)_1\to(\eta\pi)_1} & t_{(\eta\pi)_1\to(K\bar{K})_1} \\ t_{(\eta\pi)_1\to(K\bar{K})_1} & t_{(K\bar{K})_1\to(K\bar{K})_1} \end{bmatrix}$$

$$\begin{aligned} \text{disc}\, \boldsymbol{M}_0(s) &= \boldsymbol{T}^{0*}(s)\boldsymbol{\Sigma}^0(s)\left[\boldsymbol{M}_0(s+i\epsilon) + \hat{\boldsymbol{M}}_0(s)\right] && \to [1] \\ &+ \left[(\boldsymbol{M}_0(s-i\epsilon) + \hat{\boldsymbol{M}}_0(s)\right]\boldsymbol{\Sigma}^1(s)\,\boldsymbol{T}^1(s) && \to [2] \\ &+ \boldsymbol{T}^{0*}(s)\Delta\boldsymbol{\Sigma}_K(s)\boldsymbol{T}^1(s) && \to [3] \end{aligned}$$

Schematically:



Coupled channels: MO representations

$$\begin{array}{rcll} \text{disc}\, M_0(s) & = & T^{0*}(s)\Sigma^0(s)\left[M_0(s+i\epsilon)+\hat{M}_0(s)\right] & \to [1] \\ & + & \left[(M_0(s-i\epsilon)+\hat{M}_0(s)\right]\Sigma^1(s)\,T^1(s) & \to [2] \\ & + & T^{0*}(s)\Delta\Sigma_K(s)T^1(s) & \to [3] \end{array}$$

- MO representation for $M_0(s)$:

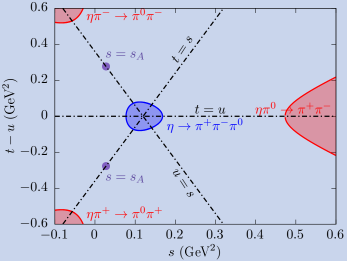
$$\begin{bmatrix} M_0(s) & G_{10}(s) \\ N_0(s) & H_{10}(s) \end{bmatrix} = \boldsymbol{\Omega}_0(s)\left[P_0(s)+s^2\left(\hat{I}_a(s)+\hat{I}_b(s)\right)\right]{}^t\boldsymbol{\Omega}_1(s)$$

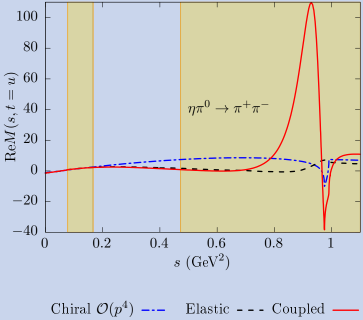
- $P_0(s)$ is a matrix of polynomials (subtractions matched to ChPT: no free parameters).
- The $\hat{I}(s)$ functions are:

$$\hat{I}_{a,b}(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{(s')^2(s'-s)}\,\Delta X_{a,b}(s')\,,$$

$$\Delta X_a = \boldsymbol{\Omega}_0^{-1}(s-i\epsilon)\left[\underbrace{T^{0*}(s)\,\Sigma^0(s)\,\hat{M}_0(s)}_{[1]}+\underbrace{\hat{M}_0(s)\,\Sigma^1(s)\,T^1(s)}_{[2]}\right]{}^t\boldsymbol{\Omega}_1^{-1}(s+i\epsilon)\,,$$

$$\Delta X_b = \underbrace{\boldsymbol{\Omega}_0^{-1}(s-i\epsilon)T^{0*}(s)\,\Delta\Sigma_K(s)\,T^1(s)\,{}^t\boldsymbol{\Omega}_1^{-1}(s+i\epsilon)}_{[3]}$$

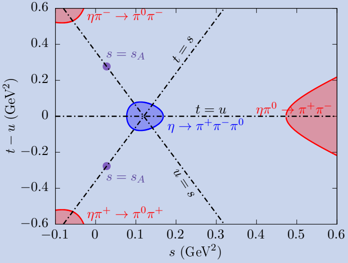


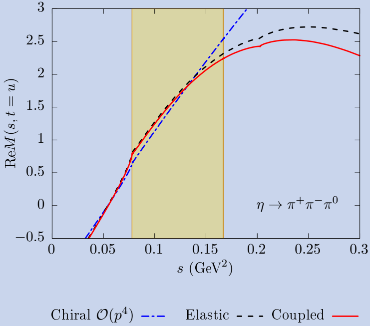


Behaviour in different regions:

- $\boxed{s \sim 1\ \text{GeV}^2}$ Very sharp energy variation,
  - $a_0(980)$ and $f_0(980)$ interference,
  - K^+K^- and $K^0\bar{K}^0$ thresholds.
- $\boxed{0.7 \lesssim s \lesssim 0.97\ \text{GeV}^2}$ Coupled channel largely enhanced compared with elastic amplitude.
- $\boxed{s \lesssim 0.7\ \text{GeV}^2}$ Effect of coupled channels is to reduce the amplitude.
- $\boxed{s \lesssim s_{\text{th}}}$ Elastic and inelastic amplitudes indistinguishable.

MA, B. Moussallam, EPJ,C77,508('17)





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Dalitz plot

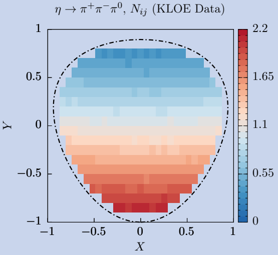
MA, B. Moussallam, EPJ,C77,508('17)

- DP variables X,Y: $X = \frac{\sqrt{3}}{2m_\eta Q_c}(u-t)$, $Y = \frac{3}{2m_\eta Q_c}\left((m_\eta - m_{\pi^0})^2 - s\right) - 1$
- Charged mode amplitude written as:

$$\frac{|M_c(X,Y)|^2}{|M_c(0,0)|^2} = \boxed{1 + a\,Y + b\,Y^2 + d\,X^2 + f\,Y^3 + g\,X^2 Y} + \cdots$$

- Neutral decay mode amplitude $[Q_c \to Q_n]$:

$$\frac{|M_n(X,Y)|^2}{|M_n(0,0)|^2} = \boxed{1 + 2\alpha\,|z|^2 + 2\beta\,\mathrm{Im}(z^3)} + \cdots$$



		$O(p^4)$	elastic	coupled	KLOE	BESIII
charged	a	-1.328	-1.156	-1.142(45)	-1.095(4)	-1.128(15)
	b	0.429	0.200	0.172(16)	0.145(6)	0.153(17)
	d	0.090	0.095	0.097(13)	0.081(7)	0.085(16)
	f	0.017	0.109	0.122(16)	0.141(10)	0.173(28)
	g	-0.081	-0.088	-0.089(10)	-0.044(16)	–
neutral					PDG	
	α	+0.0142	-0.0268	-0.0319(34)	-0.0318(15)	
	β	-0.0007	-0.0046	-0.0056	–	

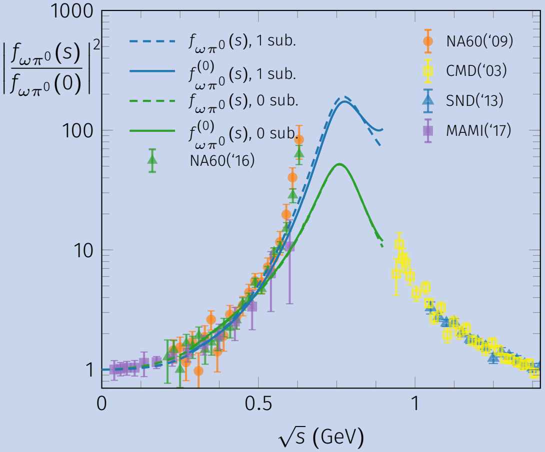
BESIII Collab., Phys. Rev. D**92**,012014 (2015)
KLOE-2 Collab., JHEP **1605**, 019 (2016)

- (Theory) uncertainty estimation:
  1. $\eta\pi$ interaction put to zero or to "large"
  2. $10^3 L_3^r = -3.82 \to -2.65$
- General trend: improve agreement $[\mathcal{O}(p^4) \to$ elastic $\to$ coupled$]$
- Particularly relevant: $\boldsymbol{\alpha}$.

Unitarity and analyticity bounds

- In works by Caprini *et al.* bounds (min and max) of the form factor have been derived. [EPJ,C74,3209('14); PR,D92,014014('15)]
- $f^{(\pm)}_{\omega\pi^0}(s) = f^{(0)}_{\omega\pi^0}(s) \pm \delta f^{(0)}_{\omega\pi^0}(s)$
- $f^{(0)}_{\omega\pi^0}(s)$ depends on $\Delta f_{\omega\pi^0}(s)$
- $\delta f^{(0)}_{\omega\pi^0}(s) \propto I'$, depends on the value of the TFF for $s \geqslant (m_\omega + m_\pi)^2$
- High energy data well above our scope...

Tension between low and high energy data?

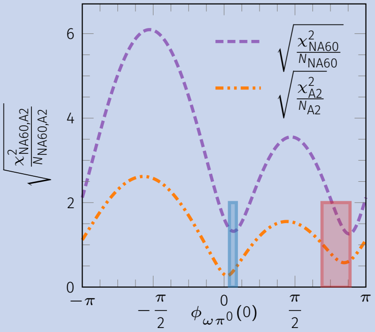


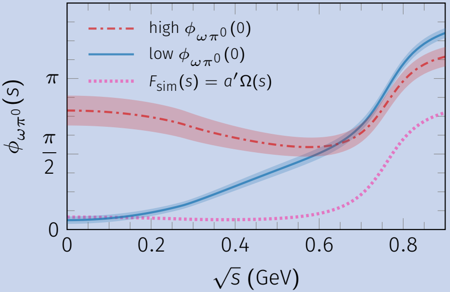
Meaning of the phase?

- Original solutions around $\phi_{\omega\pi^0}(0) \sim 0, \pi$
- Global fits remain near the original ones...

If $f_{\omega\pi^0}(0)$, a are considered as part of a microscopic (lagrangian) calculation, they would be real (hermiticity), and their relative phase would be ± 1 .

On the other hand, we find 2σ deviation: almost real, but not exactly...

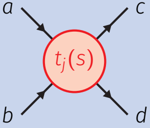


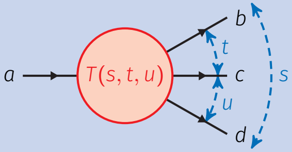


Khuri-Treiman equations for $\pi\pi$ scattering

JPAC Collab., EPJ,C78,574('18)

- KT equations for 3-body decays. Crossing: 2-to-2 scattering. Test: $\pi\pi$ scattering.

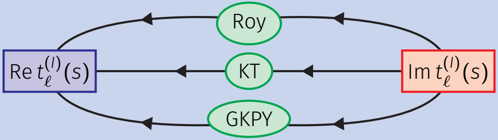




What happens if you apply KT equations to $\pi\pi$ scattering?

- KT equations for $\pi\pi$ scattering can be written as Roy-like equations:

$$t_\ell^{(I)}(s) = k_\ell^{(I)}(s) + \sum_{\ell', I'} \int_{s_{\text{th}}}^{\infty} dt' \, K_{\ell\ell'}^{II'}(s,t') \, \mathrm{Im}\, t_{\ell'}^{(I')}(t')$$



Results: Comparison with Roy equations

- Roy equations [PL,36B,353(1971)] and KT equations written as:

$$t_\ell^{(I)}(s) = k_\ell^{(I)}(s) + \sum_{\ell',I'} \int_{s_{\rm th}}^{\infty} dt' \, K_{\ell\ell'}^{II'}(s,t') \, \mathrm{Im}\, t_{\ell'}^{(I')}(t')$$

  They differ in the expressions for the polynomial ($k_\ell^{(I)}(s)$) and the kernel ($K_{\ell\ell'}^{II'}(s,t')$).
- Restrict KT to
  1. S, P-waves ($t_0^{(0)}$, $t_0^{(2)}$, $t_1^{(1)}$),
  2. one subtraction in each channel: only two subtraction constants.
- Difference between KT and Roy equations amplitudes:

$$(t_{\rm KT})_\ell^{(I)}(s) - (t_{\rm Roy})_\ell^{(I)}(s) = \tilde{k}_\ell^{(I)}(s) - k_\ell^{(I)}(s) + \sum_{\ell',I'} \int_{s_{\rm th}}^{\infty} dt' \, \Delta_{\ell\ell'}^{II'}(4m^2,t') \, \mathrm{Im} t_{\ell'}^{(I')}(t')$$

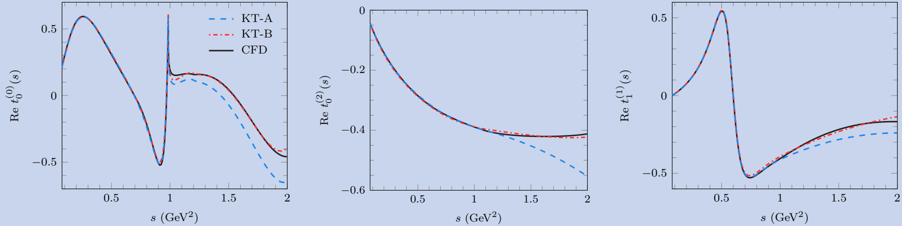
- $\Delta_{\ell\ell'}^{II'}(s,t')$: Difference of kernels is polynomial (logarithmic terms cancel).
- Five conditions that can be fulfilled with the two subtraction constants.

$\boxed{\text{KT equations and Roy equations are equal.}}$

Results: Comparison with GKPY

JPAC Collab., EPJ,C78,574('18)

- Take a succesful parameterization of the amplitude as **input for** $\mathrm{Im}\,t_\ell^{(I)}(s)$, and compare the **output** $\mathrm{Re}\,t_\ell^{(I)}(s)$ Madrid group, PR,D83,074004(2011)
  - A: one subtraction ($\times$ 6), but only 5 free constants. $s_{\max} = 1.0$ GeV^2
  - B: two subtractions ($\times$ 6), but only 7 free constants. $s_{\max} = 1.9$ GeV^2



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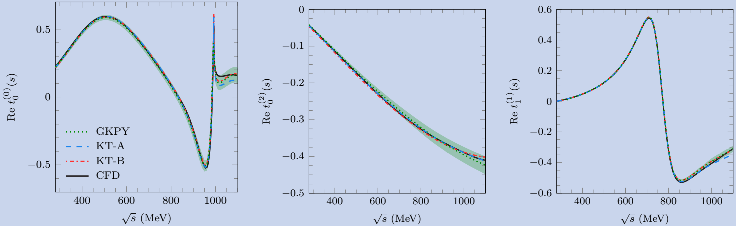

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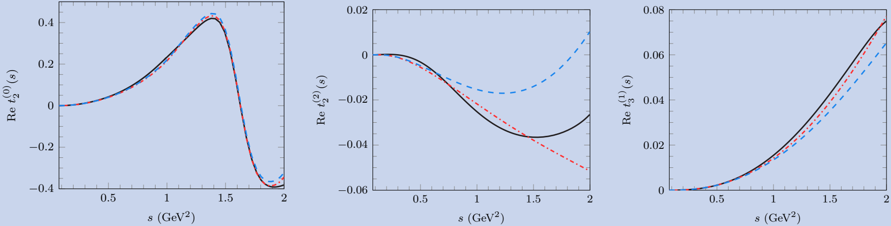

Results: Comparison with GKPY

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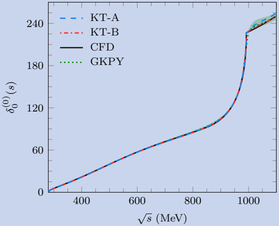

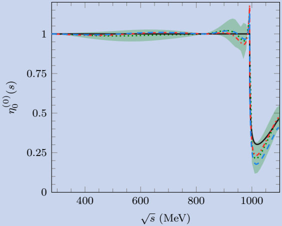

Results: Comparison with GKPY (II)

JPAC Collab., EPJ,C78,574('18)

- Threshold parameters (right):

$$\frac{m^{2\ell}}{p^{2\ell}(s)}\,\mathrm{Re}\, t_\ell^{(I)}(s) = a_\ell^{(I)} + b_\ell^{(I)} \frac{p^2(s)}{m^2} + \cdots \,.$$

- Poles and residues (bottom):

$$t_{II}^{-1}(s) = t_I^{-1}(s) + 2i\sigma(s)\,,$$

$$t_{II}(s) \simeq \frac{\tilde{g}_p^2}{s-s_p} + \cdots$$

PR,D83,074004('11); PRL,107,072001('11); PL,B749,399('15)

	KT-A	KT-B	GKPY—CFD
$a_0^{(0)}$	0.217	0.213	0.221(9)
$b_0^{(0)}$	0.274	0.275	0.278(7)
$a_0^{(2)}$	-0.044	-0.047	$-0.043(8)$
$b_0^{(2)}$	-0.078	-0.079	$-0.080(9)$
$10^3 \cdot a_1^{(1)}$	37.5	37.9	38.5(1.2)
$10^3 \cdot b_1^{(1)}$	5.6	5.7	5.1(3)
$10^4 \cdot a_2^{(0)}$	17.8	17.8	18.8(4)
$10^4 \cdot b_2^{(0)}$	-3.4	-3.4	$-4.2(1.0)$
$10^4 \cdot a_2^{(2)}$	1.9	1.8	2.8(1.0)
$10^4 \cdot b_2^{(2)}$	-3.2	-3.2	$-2.8(8)$
$10^5 \cdot a_3^{(1)}$	5.7	5.7	5.1(1.3)
$10^5 \cdot b_3^{(1)}$	-4.0	-4.0	$-4.6(2.5)$

	KT-A	KT-B	GKPY—CFD
$\sqrt{s_\sigma}$ (MeV)	$(448, 270)$	$(448, 269)$	$(457^{+14}_{-13}, 279^{+11}_{-7})$
$\lvert g_\sigma \rvert$ GeV	3.36	3.37	$3.59^{+0.11}_{-0.13}$
$\sqrt{s_\rho}$ (MeV)	$(762.2, 72.4)$	$(763.4, 73.5)$	$(763.7^{+1.7}_{-1.5}, 73.2^{+1.0}_{-1.1})$
$\lvert g_\rho \rvert$	5.95	6.01	$6.01^{+0.04}_{-0.07}$
$\sqrt{s_{f_0}}$ (MeV)	$(1000, 24)$	$(995, 26)$	$(996 \pm 7, 25^{+10}_{-6})$
$\lvert g_{f_0} \rvert$ (GeV)	2.4	2.3	2.3 ± 0.2
$\sqrt{s_{f_2}}$ (MeV)	$(1275.1, 89.5)$	$(1268.9, 86.4)$	$(1267.3^{+0.8}_{-0.9}, 87 \pm 9)$
$\lvert g_{f_2} \rvert$ (GeV^{-1})	5.6	5.5	5.0 ± 0.3

Khuri-Treiman equations for spin

- $\eta \to 3\pi$, $\pi\pi \to \pi\pi$: $J = 0$, no spin complications.
- $\omega \to 3\pi$: single amplitude, $F(s,t,u) = F(s,u,t) = F(t,s,u)$. $J = 1$ particular case.
- For general $J \neq 0$, there are more than a single amplitude, and the t-, u-isobar amplitudes related with s-isobar through crossing.

PC	J_{min}	I	notation (for $I = 0,1$)
$++$	1	odd	a_J
$+-$	1	even	h_J
$-+$	0	odd	π_J
$--$	0	even	ω_J/ϕ_J

JPAC Collab., PR,D101,054018('20)

$$\boxed{\mathcal{A}^{abcd}(\epsilon(p_X), p_3; p_1, p_2) = \langle \pi^c(p_1)\pi^d(p_2) \,|\, \hat{T} \,|\, X_J^a(\epsilon(p_X))\; \pi^b(p_3)\rangle}$$

- Definition of s- and t-channel helicity amplitudes:

$$\mathcal{A}_{\lambda}^{(s)abcd}(s,t,u) \equiv \mathcal{A}^{abcd}(\epsilon_{\lambda}^{(s)}(p_X), p_3; p_1, p_2)$$

$$\mathcal{A}_{\lambda'}^{(t)acbd}(t,s,u) \equiv \mathcal{A}^{abcd}(\epsilon'^{(t)}_{\lambda'}(p'_X), -p'_1, p'_2, -p'_3)$$

- Crossing, helicity amplitudes: $\mathcal{A}_{\lambda'}^{(t)acbd}(t,s,u) = \sum_{\lambda} d^J_{\lambda\lambda'}(\omega_t)\; \mathcal{A}_{\lambda}^{(s)abcd}(s,t,u)$

Jacob, Wick, Ann.Phys.,7,404('59); Trueman, Wick, Ann.Phys.,26,322('64); Hara, PTP,45,584('71); Martin & Spearman ('70);

- Crossing, Isospin: $\mathcal{A}_{\lambda'}^{(t)acbd}(t,s,u) = (-1)^{\lambda'}\; \mathcal{A}_{\lambda'}^{(s)acbd}(t,s,u)$
- Combining both results:

$$\boxed{\mathcal{A}_{\lambda}^{(s)abcd}(s,t,u) = \sum_{\lambda'} (-1)^{\lambda}\; d^J_{\lambda'\lambda}(\omega_t)\; \mathcal{A}_{\lambda'}^{(s)acbd}(t,s,u)}$$

Why is this relation so important?

It allows the relation between the same one-variable functions (helicity partial waves or helicity isobars) for s and t.

KT decomposition & equations

JPAC Collab., PR,D101,054018('20)

- Isospin projection:

$$\mathcal{A}_{\lambda\, I}(s,t,u) \equiv \frac{1}{(2I+1)} \sum_{a,b,c,d} P^{(I)}_{abcd}\, \mathcal{A}^{(s)abcd}_{\lambda}(s,t,u)$$

- KT decomposition in terms of isobars:

$$\begin{aligned}
\mathcal{A}_{\lambda\, I}(s,t,u) = &\sum_{j\geqslant|\lambda|}^{j_{\max}} (2j+1)\, d^{j}_{\lambda 0}(\theta_s)\, a_{j\lambda\, I}(s) \\
&+ \sum_{\lambda' j' I'} (-1)^{\lambda}\, (2j'+1)\, d^{J}_{\lambda'\lambda}(\omega_t)\, d^{j'}_{\lambda' 0}(\theta_t)\, a_{j'\lambda'\, I'}(t)\, \frac{1}{2} C_{II'} \\
&+ \sum_{\lambda' j' I'} (-1)^{\lambda'} (2j'+1)\, d^{J}_{\lambda'\lambda}(\omega_u)\, d^{j'}_{\lambda' 0}(\theta_u)\, a_{j'\lambda'\, I'}(u)\, \frac{1}{2} C_{II'}\, (-1)^{I+I'}
\end{aligned}$$

- Discontinuity:

$$\Delta a_{j\lambda I}(s) = \rho(s) t^{*}_{jI}(s) \left(a_{j\lambda I}(s) + \overline{a}_{j\lambda I}(s)\right) ,$$

- Inhomogeneity:

$$\overline{a}_{j\lambda I}(s) = (-1)^{\lambda} \sum_{I' j' \lambda'} \frac{1}{2} C_{II'} \int \mathrm{d}\cos\theta' d^{j}_{\lambda 0}(\theta') d^{J}_{\lambda'\lambda}(\omega_{t'}) d^{j'}_{\lambda' 0}(\theta'_t) a_{j\lambda' I'}(t')$$

- One last point: kinematical singularities and constraints fully taken into account in the paper.

$\pi\pi$ solutions