$J/\psi$ and $\omega$ decays to $3\pi$ with Khuri–Treiman equations

Miguel Albaladejo (IFIC)

HADRON 2023
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Genova Jun. 5-9, 2023
JPAC: Joint Physics Analysis Center

- Work in theoretical/experimental/phenomenological analysis
- Light/heavy meson spectroscopy
- Interaction with many experimental collaborations: (GlueX, CLAS, BES, ...) and LQCD groups
- Plenary talk by A. Szczepaniak [Tue., noon]
- Web site: https://www.jpac-physics.org/

Miguel Albaladejo (IFIC) \( J/\psi \) and \( \omega \) decays to \( 3\pi \) with KT equations
Introduction: Khuri-Treiman equations in a nutshell

- Partial wave expansion in the $s$-channel:

$$T(s, t, u) = \sum_{\ell=0}^{\infty} (2\ell + 1) P_\ell(z_s) t_\ell(s)$$

- Two main (connected) problems:
  - Infinite number of PW
  - PW have RHC and LHC
- Only RHC: BS equation, $K$-matrix, DR,...
- Problem with “truncation”: $t_\ell(s)$ only depends on $s$, so singularities in the $t$-, $u$-channel can only appear suming an infinite number of PW.

- In many decay processes one wants to take into account unitarity/FSI interactions in the three possible channels.
Introduction: Khuri-Treiman equations in a nutshell

- Khuri-Treiman equations are a tool to achieve this **two-body unitarity** in the three channels
  \[ [N. Khuri, S. Treiman, Phys. Rev. 119, 1115 (1960)] \]
- Consider three \((s-, t-, u-\text{channels})\) **truncated** "isobar" expansions.
- Isobars \(f^{(s)}_{\ell}(s)\) have only RHC: amenable for dispersion relations.

\[
T(s, t, u) = \sum_{\ell=0}^{\infty} (2\ell + 1)P_{\ell}(z_s) \ t_{\ell}(s) \\
= \sum_{\ell=0}^{n_s} (2\ell + 1)P_{\ell}(z_s)f^{(s)}_{\ell}(s) + \sum_{\ell=0}^{n_t} (2\ell + 1)P_{\ell}(z_t)f^{(t)}_{\ell}(t) + \sum_{\ell=0}^{n_u} (2\ell + 1)P_{\ell}(z_u)f^{(u)}_{\ell}(u)
\]

- \(s\)-channel singularities appear in the \(s\)-channel isobar, \(t^{(s)}_{\ell}(s)\).
- Singularities in the \(t\)-, \(u\)-channel are recovered!
- The LHC of the partial waves are given by the RHC of the crossed channel isobars

\[
t_{\ell}(s) = \frac{1}{2} \int dz \ P_{\ell}(z)T(s, t', u') = f^{(s)}_{\ell}(s) + \frac{1}{2} \int dz \ Q_{\ell\ell'}(s, t')f^{(t)}_{\ell'}(t') .
\]
$\omega \rightarrow 3\pi$ amplitude. Phenomenology

- Amplitude:
  \[
  \mathcal{M}_+(s, t, u) = \frac{\sqrt{\phi(s, t, u)}}{2} F(s, t, u). \quad \left(\phi(s, t, u) = 4sp^2(s)q^2(s)\sin^2 \theta_s\right)
  \]

- Decay width: $d^2\Gamma \sim \phi(s, t, u) |F(s, t, u)|^2$

- Dalitz plot parameters ($\alpha, \beta, \gamma$) “equivalent” to bins...
  \[
  (X, Y) \leftrightarrow (Z, \varphi) \leftrightarrow (s, t, u)
  \]
  \[
  |F(s, t, u)|^2 = |\mathcal{N}|^2 \left(1 + 2\alpha Z + 2\beta Z^{3/2} \sin 3\varphi + 2\gamma Z^2 + \cdots\right)
  \]

- Why revisit $\omega \rightarrow 3\pi$?
$\omega \to 3\pi$ amplitude. Phenomenology

- Amplitude:
  \[ M_+(s, t, u) = \frac{\sqrt{\phi(s, t, u)}}{2} F(s, t, u) \cdot (\phi(s, t, u) = 4sp^2(s)q^2(s)\sin^2 \theta_s) \]

- Decay width: \( d^2\Gamma \sim \phi(s, t, u)|F(s, t, u)|^2 \)

- Dalitz plot parameters \((\alpha, \beta, \gamma)\) “equivalent” to bins...
  \[ (X, Y) \leftrightarrow (Z, \varphi) \leftrightarrow (s, t, u) \]
  \[ |F(s, t, u)|^2 = |N|^2 \left( 1 + 2\alpha Z + 2\beta Z^3 \sin 3\varphi + 2\gamma Z^2 + \cdots \right) \]

- Why revisit $\omega \to 3\pi$?

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<tbody>
<tr>
<td>w/o KT</td>
<td>w KT</td>
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<tr>
<td>\alpha</td>
<td>130(5)</td>
<td>125</td>
</tr>
<tr>
<td>\beta</td>
<td>31(2)</td>
<td>30</td>
</tr>
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$\omega \rightarrow 3\pi$ amplitude. Phenomenology

- **Amplitude:**

$$M_+(s, t, u) = \frac{\sqrt{\phi(s, t, u)}}{2} F(s, t, u) \cdot \left(\phi(s, t, u) = 4sp^2(s)q^2(s) \sin^2 \theta_s\right)$$

- **Decay width:**

$$d^2\Gamma \sim \phi(s, t, u) |F(s, t, u)|^2$$

- **Dalitz plot parameters** ($\alpha, \beta, \gamma$) “equivalent” to bins...

$$|F(s, t, u)|^2 = |N|^2 \left(1 + 2\alpha Z + 2\beta Z^2 \sin 3\varphi + 2\gamma Z^2 + \cdots\right)$$

- **Why revisit $\omega \rightarrow 3\pi$?**

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<td>Exp.</td>
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<tr>
<td>$\alpha$</td>
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<tr>
<td>$\beta$</td>
<td>31(2)</td>
<td>26(2)</td>
<td>30</td>
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</table>

- One (or more) out of three is wrong...

1) Experiment?
2) KT eqs., in general?
3) Something particular?
KT equations: DR, subtractions, solutions, and all that...

- PW decomposition: \( F(s, t, u) = \sum_{j\text{odd}} P'_j(\cos \theta_s)[\rho(s)q(s)]^{j-1}f_j(s) = f_1(s) + \cdots \)

- KT/isobar decomposition: consider only \( j = 1 (\rho) \) isobar, \( F(s) \):
  \[
  F(s, t, u) = F(s) + F(t) + F(u)
  \]

- PW projection of the KT decomposition:
  \[
  f_1(s) = F(s) + \hat{F}(s), \quad \hat{F}(s) = \frac{3}{2} \int_{-1}^{1} dz_s (1 - z^2_s) F(t(s, z_s))
  \]

- Discontinuity:
  \[
  \Delta F(s) = \Delta f_1(s) = \rho(s)t_{11}^*(s)f_1(s) = \rho(s)t_{11}^*(s) \left( F(s) + \hat{F}(s) \right)
  \]

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<table>
<thead>
<tr>
<th>Unsubtracted DR</th>
<th>Once-subtracted DR</th>
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<tbody>
<tr>
<td>( F(s) = a F_0(s) )</td>
<td>( F(s) = a \left( F'_a(s) + b F_b(s) \right) )</td>
</tr>
</tbody>
</table>
| \( F_0(s) = \Omega(s) \left[ 1 + \frac{s}{\pi} \int_{4m^2_{\pi}}^{\infty} \frac{ds'}{s'} \frac{\sin \delta(s')}{|\Omega(s')|(s' - s)} \right] \) | \( F'_a(s) = \Omega(s) \left[ 1 + \frac{s^2}{\pi} \int_{4m^2_{\pi}}^{\infty} \frac{ds'}{s'^2} \frac{\sin \delta(s')}{|\Omega(s')|(s' - s)} \right] \)
| \( F_b(s) = \Omega(s) \left[ s + \frac{s^2}{\pi} \int_{4m^2_{\pi}}^{\infty} \frac{ds'}{s'^2} \frac{\sin \delta(s')}{|\Omega(s')|(s' - s)} \right] \) | |
KT equations: DR, subtractions, solutions, and all that...

**Unsubtracted DR**

\[
F(s) = a F_0(s) \\
F_0(s) = \Omega(s) \left[ 1 + \frac{s}{\pi} \int_{4m^2_\pi}^{\infty} \frac{ds'}{s'} \frac{\sin \delta(s')\hat{F}_0(s')}{|\Omega(s')|(s' - s)} \right]
\]

**Once-subtracted DR**

\[
F(s) = a \left( F'_a(s) + b F_b(s) \right) \\
F'_a(s) = \Omega(s) \left[ 1 + \frac{s^2}{\pi} \int_{4m^2_\pi}^{\infty} \frac{ds'}{s'^2} \frac{\sin \delta(s')\hat{F}_a(s')}{|\Omega(s')|(s' - s)} \right] \\
F_b(s) = \Omega(s) \left[ s + \frac{s^2}{\pi} \int_{4m^2_\pi}^{\infty} \frac{ds'}{s'^2} \frac{\sin \delta(s')\hat{F}_b(s')}{|\Omega(s')|(s' - s)} \right]
\]
\[ \omega \to \pi^0 \text{ transition form factor} \]

- The decays \( \omega(\to \pi^0 \gamma^*) \to \pi^0 \ell^+ \ell^- \) and \( \omega \to \pi^0 \gamma \) governed by the TFF \( f_{\omega \pi^0}(s) \).

\[
\mathcal{M}(\omega \to \pi^0 \ell^+ \ell^-) = f_{\omega \pi^0}(s) \epsilon_{\mu \nu \alpha \beta} \epsilon^{\mu}(p_\omega, \lambda) \rho^\nu q^\alpha \frac{i e^2}{s} \bar{u}(p_-) \gamma^\beta \nu(p_+) \\
\Gamma(\omega \to \pi^0 \gamma) = |f_{\omega \pi^0}(0)|^2 \frac{e^2 (m_\omega^2 - m_{\pi^0}^2)^3}{96 \pi m_\omega^3}
\]

- Dispersive representation:

\[
f_{\omega \pi^0}(s) = f_{\omega \pi^0}(0) + \frac{s}{12 \pi^2} \int_{4m_\pi^2}^\infty ds' \frac{q_\pi(s')^3}{s'} \frac{3}{(s' - s)} (F(s') + \hat{F}(s')) F_V(s')^*
\]

- \( f_{\omega \pi^0}(0) = |f_{\omega \pi^0}(0)| e^{i \phi_{\omega \pi^0}(0)} \)

- Experimental information: \( F_{\omega \pi^0}(s) = \frac{f_{\omega \pi^0}(s)}{f_{\omega \pi^0}(0)} \)

- Only the relative phase \( \frac{a}{f_{\omega \pi^0}(0)} = \left| \frac{a}{f_{\omega \pi^0}(0)} \right| e^{i(\phi_{\omega \pi^0}(0) - \phi_0)} \).
Summary of amplitudes/free parameters/exp. input

\[ \omega \rightarrow 3\pi \text{ amplitude } [F(s, t, u)] \]

Free parameters: \(|a|, |b|, \phi_b\)

Experimental input:
- \(\Gamma_{3\pi}\)
- Dalitz plot parameters

\[ \omega \rightarrow \gamma(\ast)\pi^0 \text{ TFF } [f_{\omega\pi^0}(s)] \]

Free parameters: \(|f_{\omega\pi^0}(0)|, \phi_{\omega\pi^0}(0)\)

\(\bigoplus |a|, |b|, \phi_b\)

Experimental input:
- \(\Gamma_{\gamma\pi^0}\)
- \(|F_{\omega\pi^0}(s)|^2\)
First analysis in three steps

1. Fix $|b| \simeq 2.9$, $\phi_b \simeq 1.9$ with the DP parameters.
2. Fix $|a| \simeq 280$ GeV\(^{-3}\), $|f_{\omega \pi^0}(0)| \simeq 2.3$ GeV\(^{-1}\) from $\Gamma_{\omega \rightarrow 3\pi}$, $\Gamma_{\omega \rightarrow \gamma \pi}$.
3. You are left with $\phi_{\omega \pi^0}(0)$ and the TFF Data.

1. $\chi_{\text{DP}}^2 = \left( \frac{\alpha(t) - \alpha(e)}{\sigma_\alpha} \right)^2 + \cdots$
2. $\chi_{\text{T}}^2 = \left( \frac{\Gamma(t) - \Gamma(e)}{\sigma_{3\pi}} \right)^2 + \left( \frac{\Gamma(t) - \Gamma(e)}{\sigma_{\gamma \pi}} \right)^2$
3. $\chi_{A_2, \text{NA60}}^2 = \sum_i \left( \frac{|F_{\omega \pi}(s_i)|^2 - |F_{\omega \pi}(t_i)|^2}{\sigma_{F_{\omega \pi}}} \right)^2$
First analysis in three steps

1. Fix $|b| \simeq 2.9$, $\phi_b \simeq 1.9$ with the DP parameters.

2. Fix $|a| \simeq 280$ GeV$^{-3}$, $|f_{\omega\pi^0}(0)| \simeq 2.3$ GeV$^{-1}$ from $\Gamma_{\omega \rightarrow 3\pi}$, $\Gamma_{\omega \rightarrow \gamma\pi}$.

3. You are left with $\phi_{\omega\pi^0}(0)$ and the TFF Data.

\[ \chi^2_{\text{DP}} = \left( \frac{\alpha(t) - \alpha(e)}{\sigma_\alpha} \right)^2 + \cdots \]

\[ \chi^2_{\Gamma} = \left( \frac{\Gamma(t) - \Gamma(e)}{\sigma_{\Gamma 3\pi}} \right)^2 + \left( \frac{\Gamma(t) - \Gamma(e)}{\sigma_{\Gamma \gamma\pi}} \right)^2 \]

\[ \chi^2_{A2,NA60} = \sum_i \left( \frac{|F_{\omega\pi}(s_i)|^2 - |F_{\omega\pi}^{(i)}|^2}{\sigma_{F_{\omega\pi}^{(i)}}} \right)^2 \]

- Two different minima (low and high $\phi_{\omega\pi^0}(0)$) are found.
- Both have similar $\chi^2$ of the TFF.

Make a global, simultaneous analysis
Results

Using once-subtracted DR for KT:
- Agreement is restored with DP parameters by BESIII
- One can also describe the $\omega\pi^0$ TFF

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
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<tbody>
<tr>
<td>BESIII</td>
<td>111(18)</td>
<td>25(10)</td>
<td>22(29)</td>
</tr>
<tr>
<td>low</td>
<td>112(15)</td>
<td>23(6)</td>
<td>29(6)</td>
</tr>
<tr>
<td>high</td>
<td>109(14)</td>
<td>26(6)</td>
<td>19(5)</td>
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Using KT equations:

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<th>2 par.</th>
<th>3 par.</th>
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<tbody>
<tr>
<td>low $\phi_{\omega\pi^0}(0)$</td>
<td>high $\phi_{\omega\pi^0}(0)$</td>
<td>low $\phi_{\omega\pi^0}(0)$</td>
</tr>
<tr>
<td>$10^{-2}</td>
<td>a</td>
<td>$ [GeV$^{-3}$]</td>
</tr>
<tr>
<td>$</td>
<td>b</td>
<td>$</td>
</tr>
<tr>
<td>$\phi_b$</td>
<td>2.03(14)</td>
<td>1.61(38)</td>
</tr>
<tr>
<td>$</td>
<td>f_{\omega\pi^0}(0)</td>
<td>$ [GeV$^{-1}$]</td>
</tr>
<tr>
<td>$\phi_{\omega\pi^0}(0)$</td>
<td>0.207(60)</td>
<td>2.39(46)</td>
</tr>
</tbody>
</table>

$\chi^2_{DP}$ 0.19 $< 0.01$ 0.10 0.03
$10^4 \chi^2_F$ 2.4 2.4 1.1 3.5
$\chi^2_{A2}$ 2.3 3.6 2.4 3.7
$\chi^2_{NA60}$ 31 35 31 35
$J/\psi \rightarrow 3\pi$ decays

- Formalism for $J/\psi$ is completely analogous to $\omega$ $(V)$.
- BESIII data [Phys. Lett., B710, 594 (2012)] show $\psi/\psi'$ puzzle:

- The $J/\psi$ decay seems to be dominated by $\rho$, despite the larger phase space
- One would expect that $0$-sub (prediction) would get the basic features
$J/\psi \rightarrow 3\pi$ decays

- $\delta_{\pi\pi}(s)$ taken as input:
- Take as central fit the one performed with solution I for $\delta_{\pi\pi}$
- The spread in the other solutions: theoretical uncertainty

- The $J/\psi$ decay seems to be dominated by $\rho$, despite the larger phase space
- 0-sub (prediction) get the basic features
- 1-sub (fit) improves the description
- 1-sub + $F$-wave [$\rho_3(1690)$] describes better the movements above $\gtrsim 1.5$ GeV.
$J/\psi \rightarrow 3\pi$ decays

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- 0-sub (prediction) get the basic features
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- 1-sub + $F$-wave [$\rho_3(1690)$] describes better the movements above $\gtrsim 1.5$ GeV.
\( J/\psi \rightarrow 3\pi \) decays: including \( F \) wave

- How to improve the description? Include \( F \)-wave, \( \rho_3(1690) \) (PDG values)

\[
F(s, t, u) = F_1(s) + F_1(t) + F_1(u) + \frac{\kappa^2(s)}{16} P'_3(z_s) F_3(s) + \frac{\kappa^2(t)}{16} P'_3(z_t) F_3(t) + \frac{\kappa^2(u)}{16} P'_3(z_u) F_3(u)
\]

- Neglect \( \hat{F}_3 \), so that \( F_3(s) = p_3(s) \Omega_3(s) \)

- The fit improves significantly, especially around \( \sqrt{s} \gtrsim 1.5 \text{ GeV} \), the main contribution being the \( P-F \)-wave interference.
$J/\psi \rightarrow 3\pi$ decays: including $F$ wave

- How to improve the description? Include $F$-wave, $\rho_3(1690)$ (PDG values)

$$F(s, t, u) = F_1(s) + F_1(t) + F_1(u) + \frac{\kappa^2(s)}{16} P'_3(z_s) F_3(s) + \frac{\kappa^2(t)}{16} P'_3(z_t) F_3(t) + \frac{\kappa^2(u)}{16} P'_3(z_u) F_3(u)$$

- Neglect $\hat{F}_3$, so that $F_3(s) = p_3(s) \Omega_3(s)$

- The fit improves significantly, especially around $\sqrt{s} \gtrsim 1.5$ GeV, the main contribution being the $P-F$-wave interference.
$J/\psi \rightarrow 3\pi$ decays: additional information

- Dalitz plot distribution similar to exp. one
- More statistics will allow to unveil other effects (resonances, interferences,…)
- Predictions can be done for angular $[z = \cos \theta_s]$ distributions, specially restricted to $\rho$-mass region.

Miguel Albaladejo (IFIC) $J/\psi$ and $\omega$ decays to $3\pi$ with KT equations
Summary

- KT equations are a powerful tool to study 3-body decays
- They allow to implement two-body unitarity in all the three channels \((s, t, u)\).
- For \(\omega \to 3\pi\) decays:
  - Using once-subtracted DRs, we are able to reproduce the \(\omega \to 3\pi\) DP parameters,
  - and the \(\omega \to \pi^0\gamma^*\) transition form factor data.
- For \(J/\psi \to 3\pi\) decays, good agreement with the data is found assuming elastic unitarity \((P-\text{ and } F\text{-waves})\).
• KT equations are a powerful tool to study 3-body decays.
• They allow to implement two-body unitarity in all the three channels \((s, t, u)\).
• Iterative solution converges fast, linear in subtraction constants.
• For \(\eta \rightarrow 3\pi\):
  ▶ Not well described by the perturbative chiral amplitudes.
  ▶ We have presented an extension of this approach to coupled channels. The extension is quite general.
  ▶ Effects of \(K\bar{K}\) and \(\eta\pi\) amplitudes \([f_0(980), a_0(980)]\) play some role in the DP parameters, tend to improve.
• For \(\pi\pi\) scattering:
  ▶ We have applied KT equations to \(\pi\pi\) scattering as benchmark.
  ▶ Restricted to \(S\)- and \(P\)-waves, KT equations are equal to Roy equations.
  ▶ When other waves are included, good comparison is obtained with GKPY equations.
• We have presented a generalization of the KT equations for arbitrary quantum numbers of the decaying particle.
  ▶ Not trivial, because of spin/crossing.
• For \(\omega \rightarrow 3\pi\) decays:
  ▶ Using once-subtracted DRs, we are able to reproduce the \(\omega \rightarrow 3\pi\) DP parameters,
  ▶ and the \(\omega \rightarrow \pi^0\gamma^*\) transition form factor data.
• For \(J/\psi \rightarrow 3\pi\) decays, good agreement with the data is found assuming elastic unitarity \((P-\text{and } F\text{-waves})\).
KT and phase space

\[
\begin{align*}
\delta_{11}(s) &\approx m_{\phi} - m_{\pi} \\
m_{\omega} - m_{\pi} &\quad m_{\eta} - m_{\pi}
\end{align*}
\]

\[\sqrt{s} \text{ (GeV)}\]

Miguel Albaladejo (IFIC)  \(J/\psi\) and \(\omega\) decays to \(3\pi\) with KT equations
Generalities about $\eta \rightarrow 3\pi$

- In QCD isospin-breaking phenomena are driven by
  \[ H_{IB} = -(m_u - m_d)\bar{\psi}\frac{\lambda_3}{2}\psi \]

- Isospin-breaking induced by EM & strong interactions are similar in size, but $\eta \rightarrow 3\pi$ is special, since EM effects are smaller

- $\Gamma_{\eta \rightarrow 3\pi} \propto Q^4$, with
  \[ Q^{-2} = \frac{m_d^2 - m_u^2}{m_s^2 - (m_u + m_d)^2/2} \]

- Experimental situation: Several high-statistics studies; $|T|^2$ well known across the Dalitz plot ⇒ stringent tests for the amplitudes (before getting $Q$!)

<table>
<thead>
<tr>
<th>$\eta \rightarrow 3\pi^0$</th>
<th>$\eta \rightarrow \pi^+\pi^-\pi^0$</th>
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<tbody>
<tr>
<td>Crys. Ball, PRL87,192001('01)</td>
<td>KLOE, JHEP0805,006('08)</td>
</tr>
<tr>
<td>Crys. Ball@MAMI, A2, PRC79,035204('09)</td>
<td>WASA-at-COSY, PRC90,045207('14)</td>
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<td>Crys. Ball@MAMI, TAPS, A2, EPJA39,169('09)</td>
<td>BESIII, PRD92,012014('15)</td>
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<tr>
<td>WASA-at-COSY, PLB677,24('09)</td>
<td>KLOE, PLB694,16('11)</td>
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<tr>
<td>KLOE-2, JHEP1605,019('16)</td>
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Previous dispersive approaches to $\eta \rightarrow 3\pi$

- **Chiral $\mathcal{O}(p^4)$** amplitude fails in describing experiments.  

- Several attempts to include **unitarity/FSI/rescattering** effects.  
  Neveu, Scherk, AP57, 39(’70); Roiesnel, Truong, NPB187, 293(’81); Kambor, Wiesendanger, Wyler, NPB465, 215(’96); Anisovich, Leutwyler, PLB375, 335(’96); Borasoy, R. Nißler, EPJA26, 383(’05); Schneider, Kubis, Ditsche, JHEP1102, 028(’11); Kampf, Knecht, Novotný, Zdráhal, PRD84, 114015(’11); Colangelo, Lanz, Leutwyler, Passemar, PRL118, 022001(’17); Guo, Danilkin, Fernández-Ramírez, Mathieu, Szczepaniak, PL771, 497(’17).

- Here we reconsider the **KT approach**.  

- $\pi\pi$ scattering **elastic** in the decay region. But **dispersive approaches** require higher energy $T$-matrix inputs:

  - $\pi\pi$ near 1 GeV rapid energy variation. $f_0(980), (K\bar{K})_0$
  - Double resonance effect $\eta\pi$ ISI, $a_0(980), (K\bar{K})_1$

  Abdel-Rehim, Black, Fariborz, Schechter, PRD67, 054001(’03)

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We propose a **generalization to coupled channels** $[(K\bar{K})_{0,1}, \eta\pi, (\pi\pi)_{0,1,2}]$ of the KT equations, extending their validity up to the physical $\eta\pi \rightarrow \pi\pi$ region. Allows for the study of the influence of $a_0, f_0$ into the decay region.
Previous dispersive approaches to $\eta \to 3\pi$

- Chiral $\mathcal{O}(p^4)$ amplitude fails in describing experiments.

- Several attempts to include unitarity/FSI/rescattering effects.
  
  Neveu, Scherk, AP57, 39('70); Roiesnel, Truong, NPB187, 293('81); Kambor, Wiesendanger, Wyler, NPB465, 215('96); Anisovich, Leutwyler, PLB375, 335('96); Borasoy, R. Nißler, EPJA26, 383('05); Schneider, Kubis, Ditsche, PRD84, 114015('11); Colangelo, Lanz, Leutwyler, Passemar, PRL118, 022001('17); Guo, Danilkin, Fernandez-Ramírez, Mathieu, Szczepaniak, PL771, 497('17).

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  ▶ $\pi\pi$ near 1 GeV rapid energy variation. $f_0(980), (K\bar{K})_0$
  ▶ Double resonance effect $\eta\pi$ ISI, $a_0(980), (K\bar{K})_1$

We propose a generalization to coupled channels $[(K\bar{K})_{0,1}, \eta\pi, (\pi\pi)_{0,1,2}]$ of the KT equations, extending their validity up to the physical $\eta\pi \to \pi\pi$ region. Allows for the study of the influence of $a_0, f_0$ into the decay region.
Previous dispersive approaches to \( \eta \to 3\pi \)

- Chiral \( \mathcal{O}(p^4) \) amplitude fails in describing experiments.

- Several attempts to include unitarity/FSI/rescattering effects:
  - Neveu, Scherk, AP57, 39(‘70); Roiesnel, Truong, NPB187, 293(‘81); Kambor, Wiesendanger, Wyler, NPB465, 215(‘96); Anisovich, Leutwyler, PLB375, 335(‘96); Borasoy, R. Nißler, EPJA26, 383(‘05); Schneider, Kubis, Ditsche, PRD84, 114015(‘11); Colangelo, Lanz, Leutwyler, Passemar, PRL118, 022001(‘17); Guo, Danilkin, Fernández-Ramírez, Mathieu, Szczepaniak, PL771, 497(‘17).

- Here we reconsider the KT approach.

- \( \pi\pi \) scattering elastic in the decay region. But dispersive approaches require higher energy \( T \)-matrix inputs:
  - \( \pi\pi \) near 1 GeV rapid energy variation. \( f_0(980), (K\bar{K})_0 \)
  - Double resonance effect \( \eta\pi \) ISI, \( a_0(980), (K\bar{K})_1 \)

We propose a generalization to coupled channels \( [(K\bar{K})_{0,1}, \eta\pi, (\pi\pi)_{0,1,2}] \) of the KT equations, extending their validity up to the physical \( \eta\pi \to \pi\pi \) region. Allows for the study of the influence of \( a_0, f_0 \) into the decay region.
**Isospin amplitudes**

- Start with well-defined *isospin amplitudes*:

\[
\mathcal{M}^{I, l_z}(s, t, u) = \langle \eta \pi; 1, l_z | \hat{T}^{(1)}_0 | \pi \pi; l, l_z \rangle = \langle l, l_z; 1, 0 | 1, l_z \rangle \langle \eta \pi | \hat{T}^{(1)} | \pi \pi; l \rangle
\]

- They can be written in terms of a *single amplitude* \((\eta \pi^0 \to \pi^+ \pi^-), A(s, t, u)\) (like in \(\pi \pi\) scattering):

\[
\begin{bmatrix}
-\sqrt{3}M^0(s, t, u) \\
\sqrt{2}M^1(s, t, u) \\
\sqrt{2}M^2(s, t, u)
\end{bmatrix} = \begin{bmatrix}
-\sqrt{3}M^{0,0}(s, t, u) \\
\sqrt{2}M^{1,1}(s, t, u) \\
\sqrt{2}M^{2,1}(s, t, u)
\end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} A(s, t, u) \\
A(t, s, u) \\
A(u, t, s)
\end{bmatrix}
\]

- Reconstruction theorem (for Goldstone bosons):

\[
A(s, t, u) = -\epsilon_L[M_0(s) - \frac{2}{3}M_2(s) + M_2(t) + M_2(u)] + (s - u)M_1(t) + (s - t)M_1(u)
\]

\[
\epsilon_L = \frac{1}{Q^2} \frac{m_K^2 - m_\pi^2}{3\sqrt{3}f_\pi^2} \frac{m_K^2}{m_\pi^2}
\]

- Or in general, “the” KT approximation:

**Infinite sum of s-channel PW \(\to\) Truncated sums of s-, t-, and u-channels PWs**

- Single variable functions: amenable for dispersion relations.
Partial wave amplitudes

- Summary of previous slide: \( \mathcal{M}^l(s, t, u) \) is written in terms of \( A(s, t, u) \) (and permutations), and \( A(s, t, u) \) is written in terms of \( M_l(w) \).
- Now, define partial waves: 
  \[
  \mathcal{M}^l(s, t, u) = 16\pi\sqrt{2} \sum_j (2j + 1) M_j^l(s) P_j(z)
  \]

\[
\mathcal{M}_0^0(s) = \epsilon_L \frac{\sqrt{6}}{32\pi} [M_0(s) + \hat{M}_0(s)],
\]
\[
\mathcal{M}_0^2(s) = \epsilon_L \frac{-1}{32\pi} [M_2(s) + \hat{M}_2(s)],
\]
\[
\mathcal{M}_1^1(s) = \epsilon_L \frac{\kappa(s)}{32\pi} [M_1(s) + \hat{M}_1(s)],
\]

**LHC \[\hat{M}_l(s)\]**

\( \hat{M}_l(s) \) written as angular averages.

Take \( M_0(s) \) as an example:

\[
\hat{M}_0(s) = \frac{2}{3} \langle M_0 \rangle + \frac{20}{9} \langle M_2 \rangle
+ 2(s - s_0) \langle M_1 \rangle + \frac{2}{3} \kappa(s) \langle zM_1 \rangle
\]
\[
\langle z^n M_l \rangle(s) = \frac{1}{2} \int_{-1}^{1} dz \ z^n M_l(t(s, z))
\]
\[
\kappa(s) = \sqrt{(1 - 4m^2_{\pi}/s)} \lambda(s, m^2_{\eta}, m^2_{\pi})
\]

**RHC \[M_l(s)\]**

\( \hat{M}(s) \) no discontinuity along the RHC:

\[
\text{disc } M_l(s) = \text{disc } M_j^l(s) = 
\]
\[
= \sigma_{\pi}(s) t^l(s)^* M_j^l(s)
\]
\[
= \sigma_{\pi}(s) t^l(s)^* \left( M_l(s) + \hat{M}_l(s) \right)
\]
\[
\sigma_{\pi}(s) = \sqrt{1 - 4m^2_{\pi}/s}
\]
\[
\sigma_{\pi}(s) t^l(s) = \sin \delta_l(s) e^{i\delta_l(s)}
\]
Muskhelisvili-Omnès representation

\[
\text{disc} M_i(s) = \sigma_\pi(s) t_i^*(s) [M_i(s) + \hat{M}_i(s)]
\]

- MO (dispersive) representation of \( M_i(s) \):
  \[
  M_0(s) = \Omega_0(s) [\alpha_0 + \beta_0 s + \gamma_0 s^2 + \hat{I}_0(s)s^2] , \\
  M_1(s) = \Omega_1(s) [\beta_1 s + \hat{I}_1(s)s] , \\
  M_2(s) = \Omega_2(s) [\hat{I}_2(s)s^2] .
  \]

\[
\Omega_i(s) = \exp \left[ \frac{s}{\pi} \int_{4m_i^2}^{\infty} ds' \frac{\delta_i(s')}{s'(s' - s)} \right] \quad \text{(Omnès function/matrix)}
\]

\[
\hat{I}_i(s) = \frac{1}{\pi} \int_{4m_i^2}^{\infty} \frac{\sin \delta_i(s') \hat{M}_i(s')}{|\Omega_i(s')|(s')^{m_i}(s' - s)} , \quad (m_{0,2} = 2, \; m_1 = 1)
\]

- \( m_{\eta}^2 + i\varepsilon \) prescription needed. Integral equations solved iteratively.
- Subtraction constants: Most natural way is to match with ChPT:
  \[
  \mathcal{M}(s, t, u) - \overline{\mathcal{M}}_\chi(s, t, u) = \mathcal{O}(p^6)
  \]
  Descotes-Genon, Moussallam, EPJ,C74,2946(2014)

- Matching conditions: fix \( \alpha_0, \beta_0, \beta_1, \gamma_0 \) in terms of ChPT amplitudes (no free parameters).
Coupled channels: take into account intermediate states other than \((\pi\pi)_I\).

\[
M_0 = \begin{bmatrix}
M_0 G_{10} \\
N_0 H_{10}
\end{bmatrix} = \begin{bmatrix}
(\eta\pi)_1 \to (\pi\pi)_0 (K\bar{K})_1 \to (\pi\pi)_0 \\
(\eta\pi)_1 \to (K\bar{K})_0 (K\bar{K})_1 \to (K\bar{K})_0
\end{bmatrix},
\]

\[
T_0 = \begin{bmatrix}
t_{(\pi\pi)_0 \to (\pi\pi)_0} & t_{(\pi\pi)_0 \to (K\bar{K})_0} \\
t_{(\pi\pi)_0 \to (K\bar{K})_0} & t_{(K\bar{K})_0 \to (K\bar{K})_0}
\end{bmatrix},
T_1 = \begin{bmatrix}
t_{(\eta\pi)_1 \to (\eta\pi)_1} & t_{(\eta\pi)_1 \to (K\bar{K})_1} \\
t_{(\eta\pi)_1 \to (K\bar{K})_1} & t_{(K\bar{K})_1 \to (K\bar{K})_1}
\end{bmatrix}
\]

\[
\text{disc } M_0(s) = T_0^*(s)\Sigma_0^0(s) \left[M_0(s + i\epsilon) + \hat{M}_0(s)\right] \to [1] + \left[(M_0(s - i\epsilon) + \hat{M}_0(s))\Sigma_1^1(s) T_1(s)\right] \to [2] + T_0^*(s)\Delta\Sigma_K(s) T_1(s) \to [3]
\]

Schematically:
Coupled channels: MO representations

\[
\text{disc } M_0(s) = T^0(s) \Sigma^0(s) [M_0(s + i\epsilon) + \hat{M}_0(s)] \rightarrow [1] \\
+ (M_0(s - i\epsilon) + \hat{M}_0(s)) \Sigma^1(s) T^1(s) \rightarrow [2] \\
+ T^0(s) \Delta \Sigma_K(s) T^1(s) \rightarrow [3]
\]

- **MO representation for** \( M_0(s) \):
  \[
  \begin{bmatrix}
  M_0(s) & G_{10}(s) \\
  N_0(s) & H_{10}(s)
  \end{bmatrix} = \Omega_0(s) \begin{bmatrix}
  P_0(s) + s^2 (\hat{i}_a(s) + \hat{i}_b(s))
  \end{bmatrix}^t \Omega_1(s)
  \]

- **\( P_0(s) \)** is a matrix of polynomials (subtractions matched to ChPT: no free parameters).
- The \( \hat{i}(s) \) functions are:
  \[
  \hat{i}_{a,b}(s) = \frac{1}{\pi} \int_{4m^2}^{\infty} ds' \frac{ds'}{(s')^2(s' - s)} \Delta X_{a,b}(s')
  \]
  \[
  \Delta X_a = \Omega_0^{-1}(s - i\epsilon) \begin{bmatrix}
  T^0(s) \Sigma^0(s) \hat{M}_0(s) + \hat{M}_0(s) \Sigma^1(s) T^1(s)
  \end{bmatrix}^t \Omega_1^{-1}(s + i\epsilon)
  \]
  \[
  \Delta X_b = \Omega_0^{-1}(s - i\epsilon) T^0(s) \Delta \Sigma_K(s) T^1(s) \Omega_1^{-1}(s + i\epsilon)
  \]
Results

Behaviour in different regions:

- $s \sim 1 \text{ GeV}^2$ Very sharp energy variation,
  - $a_0(980)$ and $f_0(980)$ interference,
  - $K^+K^-$ and $K^0\bar{K}^0$ thresholds.

- $0.7 \lesssim s \lesssim 0.97 \text{ GeV}^2$ Coupled channel largely enhanced compared with elastic amplitude.

- $s \lesssim 0.7 \text{ GeV}^2$ Effect of coupled channels is to reduce the amplitude.

- $s \lesssim s_{\text{th}}$ Elastic and inelastic amplitudes indistinguishable.
Results

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- $s \lesssim s_{\text{th}}$ Elastic and inelastic amplitudes indistinguishable.
Dalitz plot

- **DP variables** $X, Y$: 
  \[
  X = \sqrt{\frac{3}{2m_\eta Q_c}} (u - t), \quad Y = \frac{3}{2m_\eta Q_c} \left( (m_\eta - m_{\pi^0})^2 - s \right) - 1
  \]

- **Charged mode amplitude written as:**
  \[
  \frac{|M_c(X, Y)|^2}{|M_c(0, 0)|^2} = 1 + a Y + b Y^2 + d X^2 + f Y^3 + g X^2 Y + \cdots
  \]

- **Neutral decay mode amplitude** $[Q_c \to Q_n]$:
  \[
  \frac{|M_n(X, Y)|^2}{|M_n(0, 0)|^2} = 1 + 2\alpha |z|^2 + 2\beta \text{Im}(z^3) + \cdots
  \]

<table>
<thead>
<tr>
<th><strong>$O(p^4)$</strong></th>
<th><strong>elastic</strong></th>
<th><strong>coupled</strong></th>
<th><strong>KLOE</strong></th>
<th><strong>BESIII</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>-1.328</td>
<td>-1.156</td>
<td>-1.142(45)</td>
<td>-1.095(4)</td>
</tr>
<tr>
<td>b</td>
<td>0.429</td>
<td>0.200</td>
<td>0.172(16)</td>
<td>0.145(6)</td>
</tr>
<tr>
<td>d</td>
<td>0.090</td>
<td>0.095</td>
<td>0.097(13)</td>
<td>0.081(7)</td>
</tr>
<tr>
<td>f</td>
<td>0.017</td>
<td>0.109</td>
<td>0.122(16)</td>
<td>0.141(10)</td>
</tr>
<tr>
<td>g</td>
<td>-0.081</td>
<td>-0.088</td>
<td>-0.089(10)</td>
<td>-0.044(16)</td>
</tr>
<tr>
<td><strong>PDG</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td>+0.0142</td>
<td>-0.0268</td>
<td>-0.0319(34)</td>
<td>-0.0318(15)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>-0.0007</td>
<td>-0.0046</td>
<td>-0.0056</td>
<td>-</td>
</tr>
</tbody>
</table>

(Theory) **uncertainty estimation**:
1. $\eta\pi$ interaction put to zero or to “large”
2. $10^3 L_3 = -3.82 \rightarrow -2.65$

**General trend**: improve agreement $[O(p^4) \rightarrow \text{elastic} \rightarrow \text{coupled}]$

**Particularly relevant**: $\alpha$.

---

Miguel Albaladejo (IFIC) / $J/\psi$ and $\omega$ decays to $3\pi$ with KT equations
Unitarity and analyticity bounds

- In works by Caprini et al. bounds (min and max) of the form factor have been derived.
  \[ \text{[EPJ,C74,3209(‘14); PR,D92,014014(‘15)]} \]
- \( f(±)_{\omega\pi^0}(s) = f^{(0)}_{\omega\pi^0}(s) \pm \delta f^{(0)}_{\omega\pi^0}(s) \)
- \( f^{(0)}_{\omega\pi^0}(s) \) depends on \( \Delta f_{\omega\pi^0}(s) \)
- \( \delta f^{(0)}_{\omega\pi^0}(s) \propto l' \), depends on the value of the TFF for \( s \geq (m_\omega + m_\pi)^2 \)
- High energy data well above our scope...

Tension between low and high energy data?
Meaning of the phase?

• Original solutions around $\phi_{\omega\pi^0}(0) \sim 0, \pi$
• Global fits remain near the original ones...

If $f_{\omega\pi^0}(0)$, $a$ are considered as part of a microscopic (lagrangian) calculation, they would be real (hermiticity), and their relative phase would be $\pm 1$.

On the other hand, we find $2\sigma$ deviation: almost real, but not exactly...

Miguel Albaladejo (IFIC) $J/\psi$ and $\omega$ decays to $3\pi$ with KT equations
Khuri-Treiman equations for $\pi\pi$ scattering

- KT equations for 3-body decays. Crossing: 2-to-2 scattering. Test: $\pi\pi$ scattering.

What happens if you apply KT equations to $\pi\pi$ scattering?

- KT equations for $\pi\pi$ scattering can be written as Roy-like equations:

$$t^{(i)}_{\ell}(s) = k^{(i)}_{\ell}(s) + \sum_{\ell',i'} \int_{s_{th}}^{\infty} dt' K_{\ell\ell'}^{ii'}(s, t') \text{Im} t^{(i')}_{\ell'}(t')$$

Miguel Albaladejo (IFIC) $J/\psi$ and $\omega$ decays to $3\pi$ with KT equations
Results: Comparison with Roy equations

- Roy equations \([PL,36B,353(1971)]\) and KT equations written as:

\[
t^{(l)}_{\ell}(s) = k^{(l)}_{\ell}(s) + \sum_{\ell', l'} \int_{s_{th}}^{\infty} dt' K^{ll'}_{\ell\ell'}(s, t') \text{Im} t^{(l')}_{\ell'}(t')
\]

They differ in the expressions for the polynomial \((k^{(l)}_{\ell}(s))\) and the kernel \((K^{ll'}_{\ell\ell'}(s, t'))\).

- Restrict KT to \(S, P\)-waves \((t^{(0)}_0, t^{(2)}_0, t^{(1)}_1)\),
  1. one subtraction in each channel: only two subtraction constants.

- Difference between KT and Roy equations amplitudes:

\[
(t_{KT})^{(l)}_{\ell}(s) - (t_{Roy})^{(l)}_{\ell}(s) = \tilde{k}^{(l)}_{\ell}(s) - k^{(l)}_{\ell}(s) + \sum_{\ell', l'} \int_{s_{th}}^{\infty} dt' \Delta^{ll'}_{\ell\ell'}(4m^2, t') \text{Im} t^{(l')}_{\ell'}(t')
\]

- \(\Delta^{ll'}_{\ell\ell'}(s, t')\): Difference of kernels is polynomial (logarithmic terms cancel).
- Five conditions that can be fulfilled with the two subtraction constants.

KT equations and Roy equations are equal.
Results: Comparison with GKPY

- Take a successful parameterization of the amplitude as input for $\text{Im} t^{(l)}_\ell(s)$, and compare the output $\text{Re} t^{(l)}_\ell(s)$

  A: one subtraction ($\times 6$), but only 5 free constants. $s_{\text{max}} = 1.0 \text{ GeV}^2$
  B: two subtractions ($\times 6$), but only 7 free constants. $s_{\text{max}} = 1.9 \text{ GeV}^2$

Madrid group, PR,D83,074004(2011)
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Miguel Albaladejo (IFIC) $J/\psi$ and $\omega$ decays to $3\pi$ with KT equations
Results: Comparison with GKPY

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  B: two subtractions ($\times 6$), but only 7 free constants. $s_{\text{max}} = 1.9 \text{ GeV}^2$
Results: Comparison with GKPY (II)

- Threshold parameters (right):
  \[ \frac{m^2 \ell}{p^2(s)} \text{Re} \ell^{(i)}(s) = a_\ell^{(i)} + b_\ell^{(i)} \frac{p^2(s)}{m^2} + \ldots. \]

- Poles and residues (bottom):
  \[ t_{ll}^{-1}(s) = t_{l}^{-1}(s) + 2i\sigma(s), \]
  \[ t_{ll}(s) \sim \frac{g^2}{s - s_p} + \ldots \]

---

PR,D83,074004(‘11); PRL,107,072001(‘11); PL,B749,399(‘15)

<table>
<thead>
<tr>
<th>KT-A</th>
<th>KT-B</th>
<th>GKPY—CFD</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_0^{(0)})</td>
<td>0.217</td>
<td>0.213</td>
</tr>
<tr>
<td>(b_0^{(0)})</td>
<td>0.274</td>
<td>0.275</td>
</tr>
<tr>
<td>(a_0^{(2)})</td>
<td>−0.044</td>
<td>−0.047</td>
</tr>
<tr>
<td>(b_0^{(2)})</td>
<td>−0.078</td>
<td>−0.079</td>
</tr>
</tbody>
</table>

| \(10^3 \cdot a_1^{(1)}\) | 37.5          | 37.9          | 38.5(1.2)     |
| \(10^3 \cdot b_1^{(1)}\) | 5.6           | 5.7           | 5.1(3)        |
| \(10^4 \cdot a_2^{(0)}\) | 17.8          | 17.8          | 18.8(4)       |
| \(10^4 \cdot b_2^{(0)}\) | −3.4          | −3.4          | −4.2(1.0)     |
| \(10^4 \cdot a_2^{(2)}\) | 1.9           | 1.8           | 2.8(1.0)      |
| \(10^4 \cdot b_2^{(2)}\) | −3.2          | −3.2          | −2.8(8)       |
| \(10^5 \cdot a_3^{(1)}\) | 5.7           | 5.7           | 5.1(1.3)      |
| \(10^5 \cdot b_3^{(1)}\) | −4.0          | −4.0          | −4.6(2.5)     |
Khuri-Treiman equations for spin

- \( \eta \to 3\pi, \pi\pi \to \pi\pi: J = 0, \) no spin complications.
- \( \omega \to 3\pi: \) single amplitude, \( F(s, t, u) = F(s, u, t) = F(t, s, u). \) \( J = 1 \) particular case.
- For general \( J \neq 0, \) there are more than a single amplitude, and the \( t-, u- \)isobar amplitudes related with \( s- \)isobar through crossing.

<table>
<thead>
<tr>
<th>( PC )</th>
<th>( J_{\text{min}} )</th>
<th>( l )</th>
<th>notation (for ( l = 0, 1 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>++</td>
<td>1</td>
<td>odd</td>
<td>( a_j )</td>
</tr>
<tr>
<td>+-</td>
<td>1</td>
<td>even</td>
<td>( h_j )</td>
</tr>
<tr>
<td>-+</td>
<td>0</td>
<td>odd</td>
<td>( \pi_j )</td>
</tr>
<tr>
<td>--</td>
<td>0</td>
<td>even</td>
<td>( \omega_j/\phi_j )</td>
</tr>
</tbody>
</table>
Crossing symmetry

\[ A^{abcd}(\epsilon(p_X), p_3; p_1, p_2) = \langle \pi^c(p_1)\pi^d(p_2) | \hat{T} | \chi_j^a(\epsilon(p_X)) \pi^b(p_3) \rangle \]

- Definition of \( s \)- and \( t \)-channel helicity amplitudes:

\[ A^{(s)abcd}(s, t, u) \equiv A^{abcd}(\epsilon^{(s)}(p_X), p_3; p_1, p_2) \]

\[ A^{(t)acbd}(t, s, u) \equiv A^{abcd}(\epsilon^{(t)}(p'_X), -p'_1, p'_2, -p'_3) \]

- Crossing, helicity amplitudes:

\[ A^{(t)acbd}(t, s, u) = \sum_{\lambda} d_{\lambda\lambda'}^l(\omega_t) A^{(s)abcd}(s, t, u) \]

Jacob, Wick, Ann.Phys.,7,404('59); Trueman, Wick, Ann.Phys.,26,322('64);
Hara, PTP,45,584('71); Martin & Spearman ('70);

- Crossing, Isospin:

\[ A^{(t)acbd}(t, s, u) = (-1)^{\lambda'} A^{(s)abcd}(t, s, u) \]

- Combining both results:

\[ A^{(s)abcd}(s, t, u) = \sum_{\lambda'}(-1)\lambda' d_{\lambda'\lambda}^l(\omega_t) A^{(s)acbd}(t, s, u) \]

Why is this relation so important?

It allows the relation between the same one-variable functions (helicity partial waves or helicity isobars) for \( s \) and \( t \).
KT decomposition & equations

- Isospin projection:

\[ \mathcal{A}_{\lambda I}(s, t, u) \equiv \frac{1}{(2l + 1)} \sum_{a, b, c, d} P_{abcd}^{(l)} \mathcal{A}_{\lambda}^{(s)abcd}(s, t, u) \]

- KT decomposition in terms of isobars:

\[ \mathcal{A}_{\lambda I}(s, t, u) = \sum_{j \geq |\lambda|} (2j + 1) d_{\lambda 0}^j(\theta_s) a_{j \lambda I}(s) \]

\[ + \sum_{\lambda' j' l'} (-1)^{\lambda} (2j' + 1) d_{\lambda' \lambda}^j(\omega_t) d_{\lambda' 0}^{j'}(\theta_t) a_{j' \lambda' I'}(t) \frac{1}{2} C_{l'l'} \]

\[ + \sum_{\lambda' j' l''} (-1)^{\lambda'} (2j' + 1) d_{\lambda' \lambda}^j(\omega_u) d_{\lambda' 0}^{j'}(\theta_u) a_{j' \lambda' I''}(u) \frac{1}{2} C_{l''} (-1)^{l+l''} \]

- Discontinuity:

\[ \Delta a_{j \lambda I}(s) = \rho(s) t_{j l}^*(s) \left( a_{j \lambda I}(s) + \bar{a}_{j \lambda I}(s) \right) \]

- Inhomogeneity:

\[ \bar{a}_{j \lambda I}(s) = (-1)^{\lambda} \sum_{l' j' \lambda'} \frac{1}{2} C_{l'l'} \int d\cos\theta' d_{\lambda 0}^j(\theta') d_{\lambda' \lambda}^{j'}(\omega_{t'}) d_{\lambda' 0}^{j'}(\theta_t') a_{j' \lambda' I'}(t') \]

- One last point: kinematical singularities and constraints fully taken into account in the paper.
ππ solutions

Miguel Albaladejo (IFIC)  $J/\psi$ and $\omega$ decays to $3\pi$ with KT equations