AD POLOSA, SAPIENZA UNIVERSITY OF ROME

ON THE COMPOSITION OF EXOTIC HADRON RESONANCES

`NUCLEAR DEMOCRACY`

"A proton could be obtained from a neutron and a pion, or from a Λ and a K, or from two nucleons and one anti-nucleon, and so on. Could we therefore say that a proton consists of continuous matter? [. . .] *There is no difference in principle between elementary particles and compound systems.*"

–WERNER HEISENBER, 1975 TALK AT GERMAN PHYSICAL SOCIETY

ELEMENTARY VS COMPOSITE PARTICLES

The fields of **elementary** particles appear in $\mathscr{L}$.

As opposite, a **composite** particle is one whose field $\boldsymbol{\Phi}$ does not appear in $\mathscr{L}$: it can be created/destroyed by operators constructed by (functions of) other fields, e.g. those appearing in $\mathscr{L}$.

Consider the complete propagator for $\boldsymbol{\Phi}$ which may, or may not, be elementary

$$\Delta'(p) = \int_0^\infty \frac{\rho(\mu^2)}{p^2 + \mu^2 - i\epsilon} d\mu^2$$

where the spectral function is defined by ($\rho = 0$ for $p^2 > 0$)

$$\theta(p_0)\,\rho(-p^2) = \sum_n \delta^4(p - p_n)\,|\langle 0\,|\,\Phi(0)\,|\,n\rangle\,|^2$$

and $|\,n\rangle = |\,\boldsymbol{k}\rangle$ or $|\,\boldsymbol{k}_1, \boldsymbol{k}_2\rangle \ldots$

ELEMENTARY VS COMPOSITE PARTICLES

Let $|\boldsymbol{k}\rangle$ be a one-particle state with mass $\boldsymbol{m}$.
Suppose $\langle\boldsymbol{k}|$ has a non-zero amplitude with $\boldsymbol{\Phi}^\dagger(0)|0\rangle$.
According to a general result, the complete propagator $\Delta'(p)$ of the bare field Φ has a **pole** at $-m^2$ with residue $Z = |N|^2 > 0$ where (Lorentz)

$$\langle 0|\Phi(0)|\boldsymbol{k}\rangle = \frac{N}{\sqrt{2E}} \qquad E = \sqrt{\boldsymbol{k}^2 + m^2}$$

As a consequence of this, it must be $\rho(\mu^2) = Z\,\delta(\mu^2 - m^2)$

$$\Delta'(p) = \frac{Z}{p^2 + m^2 - i\epsilon}$$

ELEMENTARY VS COMPOSITE PARTICLES

However the spectral function also includes multiparticle states in $|n\rangle$. The contribution of states like $|\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots\rangle$ is incorporated in the function $\sigma \geq 0$

$$\rho(\mu^2) = Z\,\delta(\mu^2 - m^2) + \sigma(\mu^2)$$

Consider the case $Z = 0$ which corresponds to non-zero amplitudes of $\langle \boldsymbol{k}_1, \boldsymbol{k}_2, \ldots |$ with $\Phi^\dagger(0)|0\rangle$ only. Then

$$\Delta'(p) = \int_0^\infty \frac{\sigma(\mu^2)}{p^2 + \mu^2 - i\epsilon}\, d\mu^2$$

The complete propagator is described only by the coupling of Φ to **multi-particle states**, namely $\displaystyle\int_0^\infty \sigma(\mu^2)\, d\mu^2$

ELEMENTARY DEUTERON

Say that the Lagrangian $\mathscr{L}$ of the nuclear theory contains only the elementary fields of the proton $\boldsymbol{p}$ and the neutron $\boldsymbol{n}$.

Add to $\mathscr{L}$ another elementary field, $\mathfrak{d}$ (it can be composite, but of something else than $\boldsymbol{p}, \boldsymbol{n}$, like six quarks). Call it *elementary deuteron*.

Assume that $\langle \boldsymbol{k} |$ is a one-particle state of mass $\boldsymbol{m}$ having non-zero amplitude with $\mathfrak{d}^\dagger(0)|0\rangle$ – can't be $\langle \boldsymbol{n}, \boldsymbol{k} |$ nor $\langle \boldsymbol{p}, \boldsymbol{k} |$ – must be the elementary deuteron one-particle state.

The complete propagator of $\mathfrak{d}$ has a pole at $-\boldsymbol{m}^2$ with residue $\boldsymbol{Z}$: the manifestation of the elementary deuteron.

COMPOSITE DEUTERON

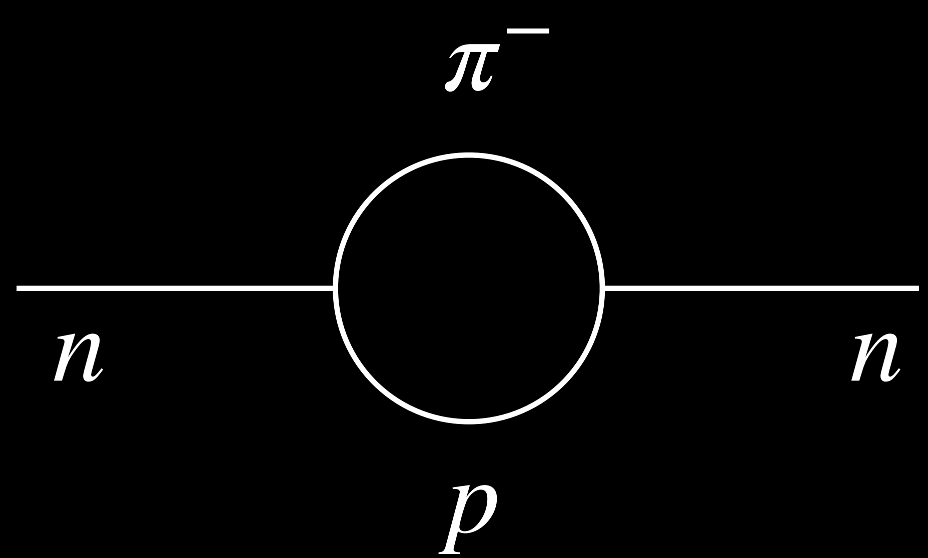
If $Z = 1$ we are making the case of the free theory, $\Delta'(p) = \Delta(p)$.

(Trivial case: if there is an elementary deuteron it must interact with n and p)

If $Z = 0$ we are in the case in which the complete propagator is due only to the coupling of $\mathfrak{d}$ to np continuum, $|np, k_1, k_2\rangle$.

(Composite case: the $\mathfrak{d}$ field in $\mathscr{L}$ can be *substituted* by function $F(n,p)$ of the elementary fields n, p. We can introduce a field Φ for the composite deuteron by adding to $\mathscr{L}$ a term of the form $\Delta\mathscr{L} = \lambda(F(n,p) - \Phi)^2$ and integrating over Φ in the path integral. This opens the way (but does not correspond) to the description of deuteron as a np bound state. Bound states can be counted with phase shifts in elastic scattering but their number N is $N = (\delta_\ell(0) - \delta_\ell(E = \infty))$. This formula is not \`practical\` since, at $E = \infty$, all the inelastic channels are open and Levinson theorem is proved for the elastic scattering only, and not even for shallow bound states.)

THE LEE MODEL



$$|\, n, \mathrm{in}\rangle = \sqrt{Z}\,|\, n, \mathrm{bare}\rangle + \int_{\boldsymbol{k}} C_{\boldsymbol{k}}\,|\, p\,\pi^{-}(\boldsymbol{k})\rangle$$

$$Z + \int_{\boldsymbol{k}} |\, C_{\boldsymbol{k}}\,|^2 = 1$$

See the “Lee-model” (’54) in Henley & Thirring, Elementary Quantum Field Theory, McGraw-Hill
T.D. Lee, Phys. Rev. 95, 1329 (1954)

WEINBERG'S ANALYSIS OF THE DEUTERON

The analysis is done in NRQM. The starting point is the same of that in the Lee model

$$|d\rangle = \sqrt{Z}\,|\mathfrak{d}\rangle + \int_{\boldsymbol{k}} C_{\boldsymbol{k}}\,|np(\boldsymbol{k})\rangle$$

$$Z + \int_{\boldsymbol{k}} |C_{\boldsymbol{k}}|^2 = 1$$

Is it possible to extract $\boldsymbol{Z}$ from data?

See Weinberg Phys. Rev. 137, B672 (1965)

WEINBERG'S ANALYSIS OF THE DEUTERON

$$r_0 = -\frac{Z}{1-Z}R + O\left(\frac{1}{m_\pi}\right)$$ (effective range)

$$R = \frac{1}{\sqrt{2mB}}$$ ($B =$ binding energy)

$$a = \frac{2(1-Z)}{2-Z}R + O\left(\frac{1}{m_\pi}\right)$$ (scattering length > 0)

where the effective range expansion is

$$k \cot \delta \simeq -\frac{1}{a} + \frac{1}{2} r_0 k^2$$ ($\delta =$ phase-shift in pn)

BETHE/LANDAU-SMORODINSKY

Scattering in the presence of shallow bound states generated by purely attractive potentials in NRQM are characterized by

$$r_0 \geq 0$$

even if there is a repulsive core, but in a *very* narrow region around the origin. In this case $O(1/m_\pi) \geq 0$ once $Z = 0$.

Esposito et al. 2108.11413

So a nuclear deuteron would need an r_0 small (≈ 1 fm) and positive, whereas an elementary deuteron should involve an r_0 large ($\gg 1$ fm) and negative. Data on np scattering say

$$r_0^{\text{expt.}} = +1.74 \text{ fm}$$

THE CASE OF THE X(3872)

The vicinity of the X(3872) to $D\bar{D}^*$ threshold is considered by many authors as –the proof– of its nuclear nature: a loosely bound state of a D and a $\bar{D}^*$ meson. The term molecule is used.

No $D\bar{D}^*$ scattering experiments are possible, yet the experimental determination of r_0 can proceed through the `lineshape` of the X(3872) using the connection between scattering amplitude (S-wave, low k)

$$f = \frac{1}{k\cot\delta(k) - ik} = \frac{1}{-\frac{1}{a} + \frac{1}{2}r_0 k^2 - ik}$$

and BW formula.

Assumption: the $D\bar{D}^*$ decay channel is the dominating one for the X.

THE CASE OF THE X(3872)

For small kinetic energies (and using LHCb analysis)

$$f(X \to J/\psi\pi\pi) = -\frac{(2N/g)}{(2/g)(E - m_X^0) - \sqrt{2\mu_+\delta} + E\sqrt{\mu_+/2\delta} + ik}$$

$$\delta = m_{D^{*-}} + m_{D^+} - m_{\bar{D}^{*0}} - m_{D^0}$$

$$E = m_{J/\psi\pi\pi} - m_D - m_{\bar{D}^*}$$

and μ_+ is the reduced mass of the charged $D\bar{D}^*$ pair.

Esposito et al. 2108.11413

THE CASE OF THE X(3872)

For small kinetic energies

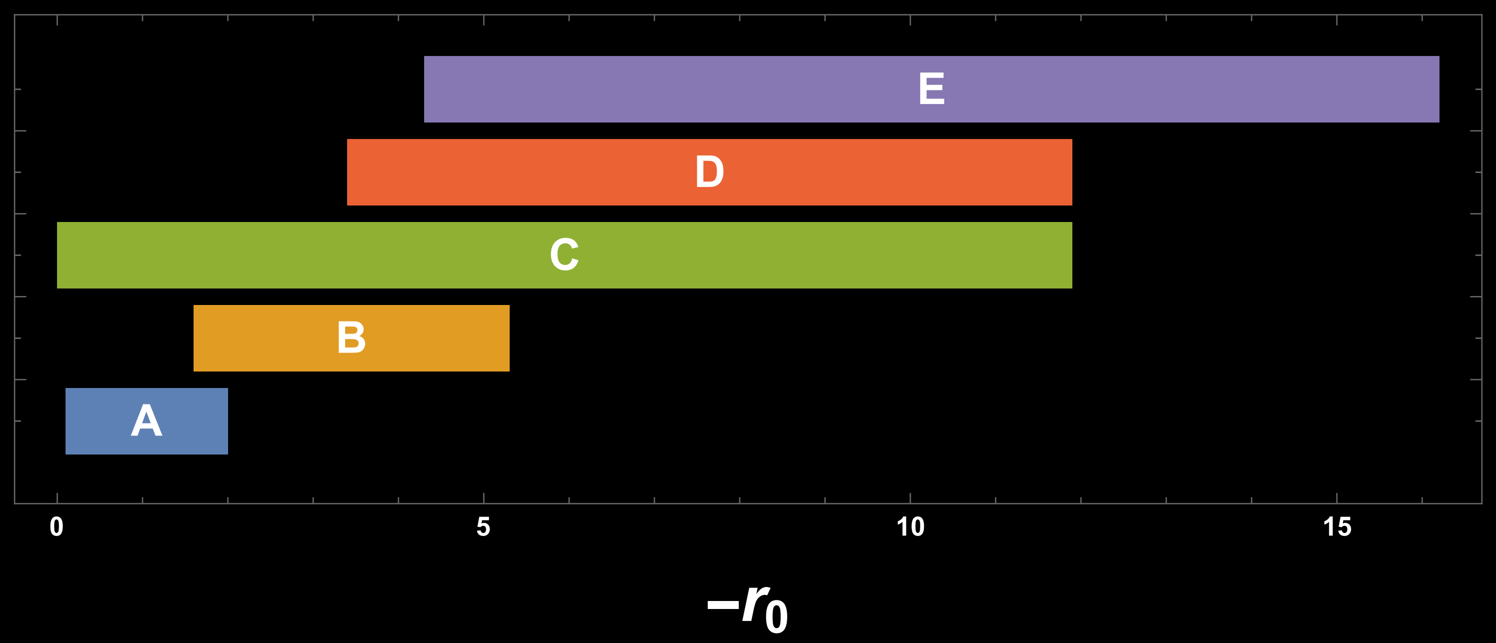
$$f(X \to J/\psi\pi\pi) = -\frac{(2N/g)}{(2/g)(E - m_X^0) - \sqrt{2\mu_+\delta} + E\sqrt{\mu_+/2\delta} + ik}$$

$$-\frac{1}{a} = \frac{2m_X^0}{g} + \sqrt{2\mu_+\delta} \simeq -6.92 \text{ fm} \qquad \text{positive } a$$

$$r_0 = -\frac{2}{\mu g} - \sqrt{\frac{2\mu_+}{2\mu^2\delta}} \simeq -5.34 \text{ fm} \qquad \text{negative } r_0$$

using $E = k^2/2\mu$, μ being the reduced mass of the neutral $D\bar{D}^*$ pair, and taking g (shaky…) and m_X^0 (stable determination) from the experimental analysis. Since g can be larger, $r_0 \leq -2$ fm.

$(-r_0)$ ACCORDING TO SOME ESTIMATES



A: Baru et al., 2110.07484

B: Esposito et al., 2108.11413

C: LHCb, 2109.01056

D: Maiani & Pilloni GGI-Lects

E: Mikhasenko, 2203.04622

COMPACT X

Having a negative r_0 means having a **finite** Z, which in turn means that there is an **elementary** X field in the Lagrangian.

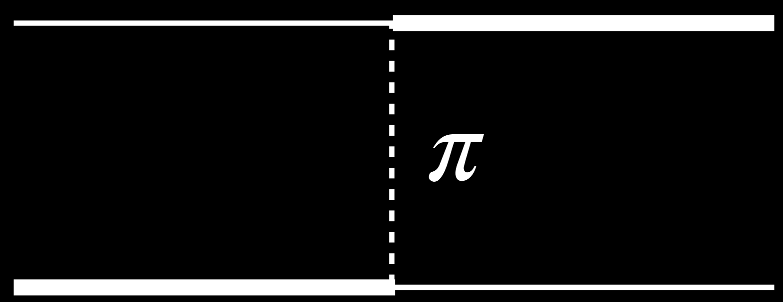
The X can interact as strongly as possible to the $D\bar{D}^*$ continuum, but the very fact that there is an elementary field of X, with whatever Z value, is an indication that it might be appropriate to work with an elementary X.

Does the Weinberg analysis apply to the X(3872)?

MOLECULAR PICTURE

$$H_{DD*} = \frac{\boldsymbol{p}_{D*}^2}{2m_{D*}} + \frac{\boldsymbol{p}_D^2}{2m_D} - \lambda_0\, \delta^3(\boldsymbol{r})$$

A perturbation to the $\delta^3(\boldsymbol{r})$ potential derives from



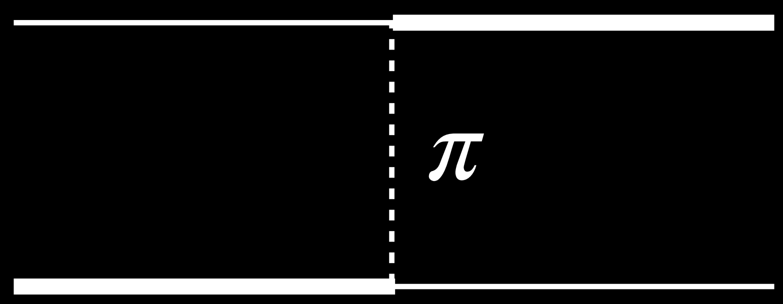
Potential = FT of the propagator in no-recoil approximation

$$\int \frac{q_i q_j\, e^{i\mathbf{q}\cdot\mathbf{r}}}{q^2 + m_\pi^2 - i\epsilon}\, d^3q \underset{\text{no rec.}}{\longrightarrow} \int \frac{q_i q_j\, e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2 - \mu^2 - i\epsilon}\, d^3q \approx \int \frac{q_i q_j\, e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2 - i\epsilon}\, d^3q = \nabla_i \nabla_j \int \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2 - i\epsilon}\, d^3q$$

$\mu^2 = (m_{D*} - m_D)^2 - m_\pi^2 \simeq 43$ MeV

and $1/r^3$ falls to the center

MOLECULAR PICTURE

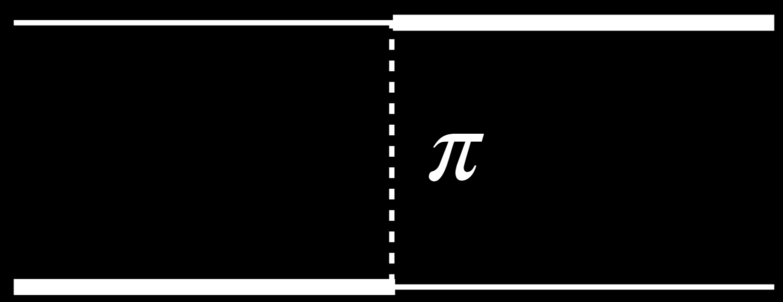


Keep μ finite! Are the corrections to r_0 of the size $O(1/m_\pi)$ or $O(1/\mu)$? Notice that $(197\ \text{MeV fm})/\mu \sim 5$ fm which is right where the bars in the previous figure mostly fall.

In principle the π-exchange contribution to r_0 might be negative and ≈ -5 fm, or smaller, the $D\bar{D}^*$ bound state being due to V_s only (not contributing to r_0).

If so the `Weinberg criterion`, which is fine for the deuteron, would just fail for the $X(3872)$. Difficult to judge without a calculation, even in consideration that V_w is small.

MOLECULAR PICTURE



Keep μ finite! Are the corrections to r_0 of the size $O(1/m_\pi)$ or $O(1/\mu)$?

$$\frac{g^2}{2f_\pi^2}\int \frac{q_i q_j\, e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2-\mu^2-i\epsilon}\,\frac{d^3q}{(2\pi)^3} = \frac{g^2}{6f_\pi^2}\left(\delta^3(r)+\mu^2\frac{e^{i\mu r}}{4\pi r}\right)\delta_{ij}$$

where the integral is decomposed as $A\delta_{ij} + B\, r^2\, n_i n_j$ and we use the S-wave relation

$$\langle n_i n_j\rangle = \frac{1}{3}\delta_{ij}$$

the contraction with non-rel. polarizations $e_i^{(\lambda)}\bar{e}_j^{(\lambda')}$ gives $\delta_{\lambda\lambda'}$

MOLECULAR PICTURE

So we have the case in which $\boldsymbol{V}$ itself is not small enough to be considered as a perturbation, but it can be divided in

$$V = V_s + V_w = -\left(\lambda_0 + 4\pi\alpha\right)\delta^3(\boldsymbol{r}) - \alpha\mu^2\frac{e^{i\mu r}}{r}$$

To compute any amplitude, all orders in $\boldsymbol{V_s}$ are needed, and possibly only the first order in $\boldsymbol{V_w}$.
The contribution deriving from $\boldsymbol{V_w}$ is calculated in the DWBA (Distorted-Wave-Born-Approximation) which amounts to use ($\pm$ = in/out)

$$T_{\beta\alpha} = \left(\Psi^-_{s\,\beta}, V_w \Psi^+_{s\,\alpha}\right)$$

THE IMAGINARY PART OF $V_w(r)$

How to take into account that there are unstable particles in the amplitudes T ? We should add `by hand` the D^* decay width to $V_s + V_w$. A derivation of this is possible.

$$-\frac{\nabla^2}{2m}\psi(r) - \left[\left(\lambda_0 + 4\pi\alpha\right)\delta^3(\mathbf{r}) + \alpha\mu^2\frac{e^{i\mu r}}{r} + i\frac{\Gamma}{2}\right]\psi(r) = E\,\psi(r)$$

Indeed the complex potential V_w alone will not allow any imaginary part in the positive spectrum $E > 0$ (exception made for ψs' exponentially blowing up).

$$\left(\lim_{r\to 0}\Im(V(r)) = \lim_{r\to 0}\Im\,\alpha\mu^2\frac{e^{i\mu r}}{r} = \frac{g^2\mu^3}{24\pi f_\pi^2} \equiv \frac{\Gamma}{2}\right)$$

Esposito, Glioti, Germani, ADP, Rattazzi, Tarquini draft in preparation

CALCULATION OF r_0

$$f = \frac{1}{k\cot\delta(k) - ik} = f_s + f_w = \frac{1}{-\frac{1}{a} - ik} + f_w$$

$$f_w = -\frac{2m}{4k^2}\int V_w(r)\,\chi_s^2(r)\,dr$$

Where $\chi_s(r)$ are scattering w.f. of the $\delta^3(\boldsymbol{r})$ potential, and $\boldsymbol{m}$ is the invariant $\boldsymbol{DD^*}$ mass. Thus r_0 is determined by the $\boldsymbol{k}^2$ coefficient in the *double expansion* around $r_0 = 0$ and $\alpha = 0$ of the expression

$$f^{-1} = \left(\frac{1}{-\frac{1}{a} - ik} - \frac{2m}{4k^2}\int V_w(r)\,\chi_s^2(r)\,dr\right)^{-1}$$

Esposito, Glioti, Germani, ADP, Rattazzi, Tarquini draft in preparation

CALCULATION OF r_0

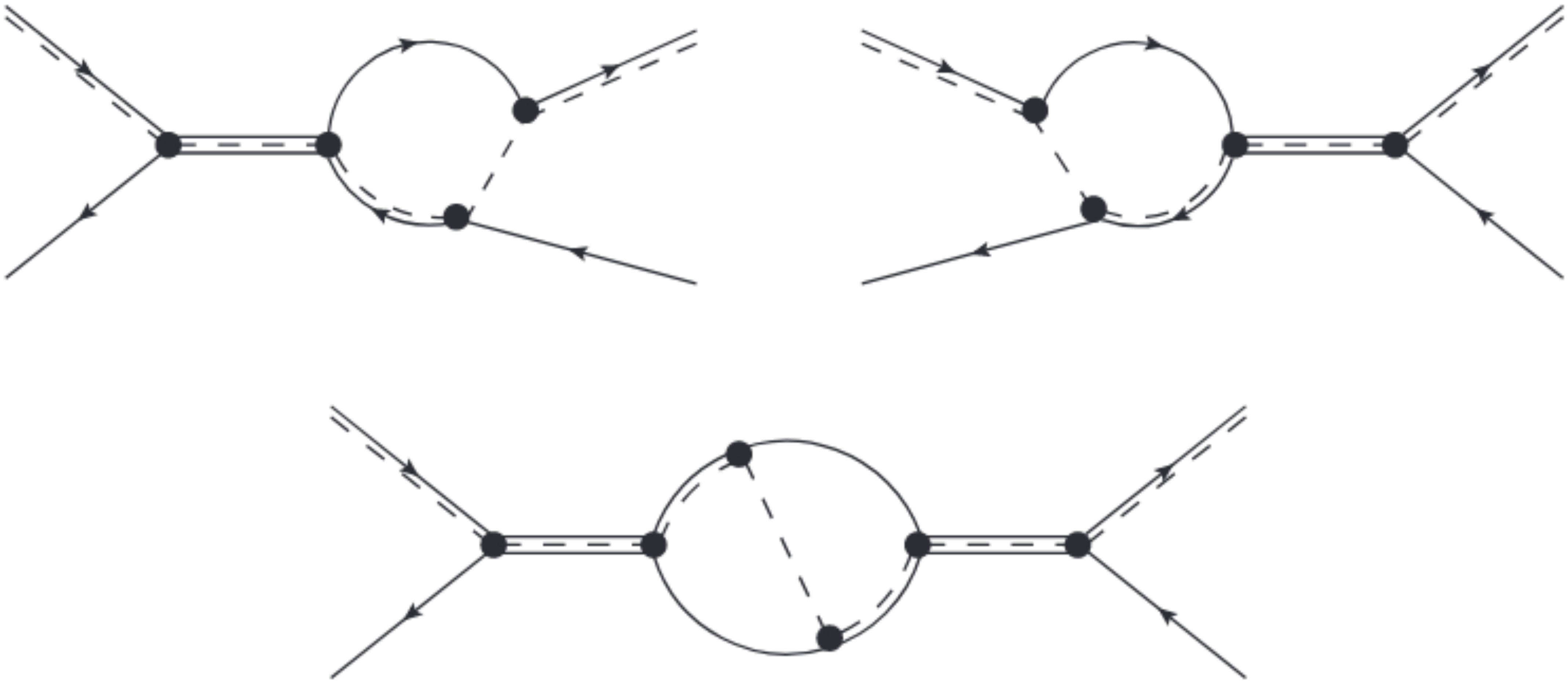
$$r_0 = 2m\alpha\left(\frac{2}{\mu^2 a^2} + \frac{8i}{3\mu a} - 1\right)$$

$$-0.20 \text{ fm} \lesssim \text{Re}\, r_0 \lesssim -0.15 \text{ fm}$$

$$0 \text{ fm} \lesssim \text{Im}\, r_0 \lesssim 0.17 \text{ fm}$$

$$\alpha\mu^2 = \frac{g^2}{24\pi f_\pi^2}\mu^2 = 5 \times 10^{-4}$$

These results agree, analytically, with what found by Braaten et al. using EFT. It turns out that the real part of r_0 is just a tiny (negative!) fraction of a Fermi. This confirms the fact that the Weinberg criterion can be extended to the $X(3872)$ too.



Braaten, Galilean invariant XEFT, Phys. Rev. D 103, 036014 (2021), arXiv:2010.05801 [hep-ph]

r_0 FROM LATTICE

M. Padmanath and S. Prelovsek, Phys. Rev. Lett. 2202.10110

Applying the lattice Lüscher method, the authors study the DD^* scattering amplitude and make a determination of the scattering length and of the effective range for $\mathcal{T}_{cc}$

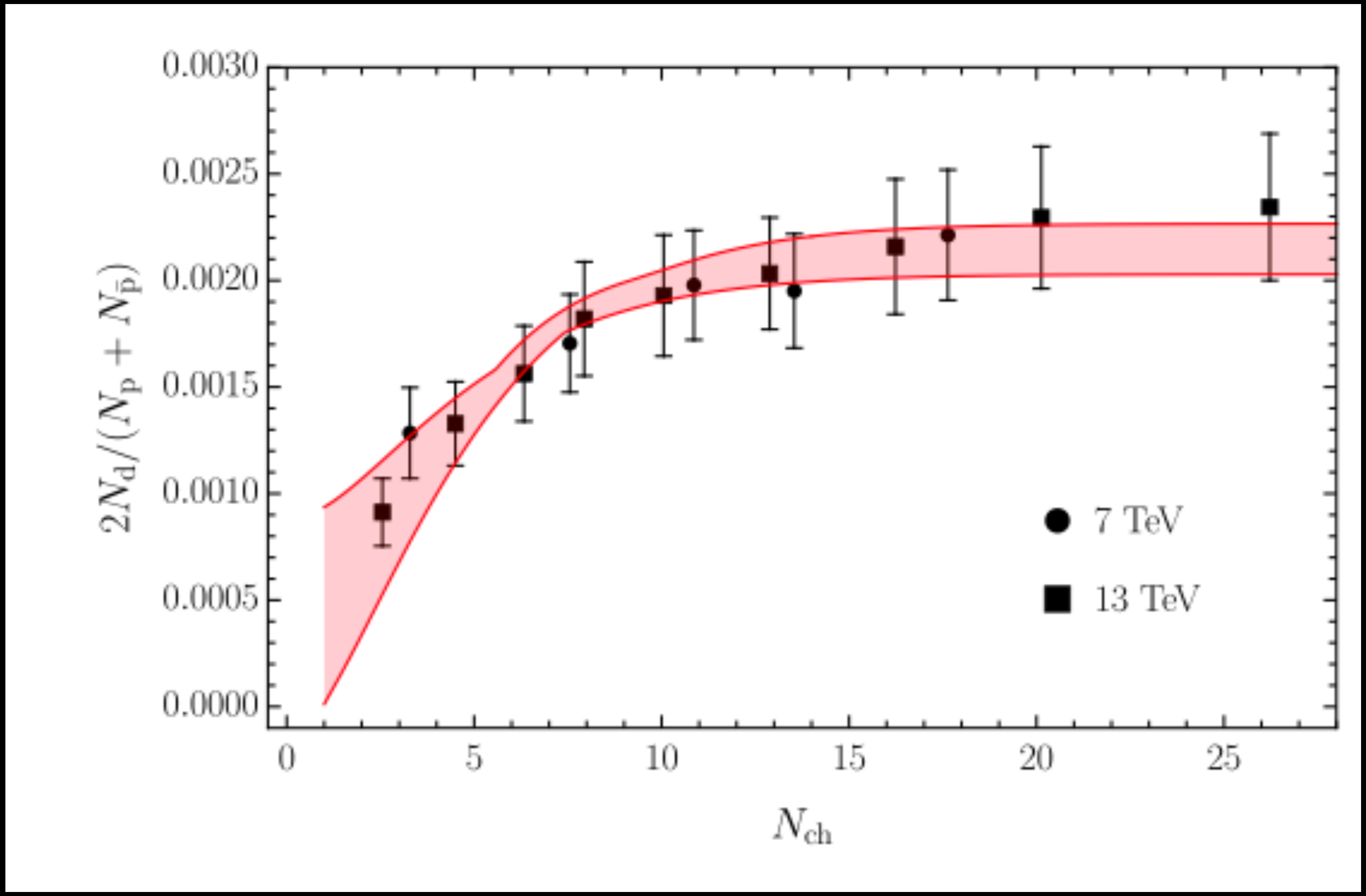
$$a = -1.04(29) \text{ fm}$$
$$r_0 = +0.96^{+0.18}_{-0.20} \text{ fm}$$

The mass of the pion is $m_\pi = 280$ MeV, to keep the D^* stable. This result, for the moment, is compatible with a *virtual state* because of the negative a – like the singlet deuteron.
As for LHCb (2109.01056 p.12)

$$a = +7.16 \text{ fm}$$
$$-11.9 \leq r_0 \leq 0 \text{ fm}$$

DOES THE X(3872) BEHAVE AS THE DEUTERON?

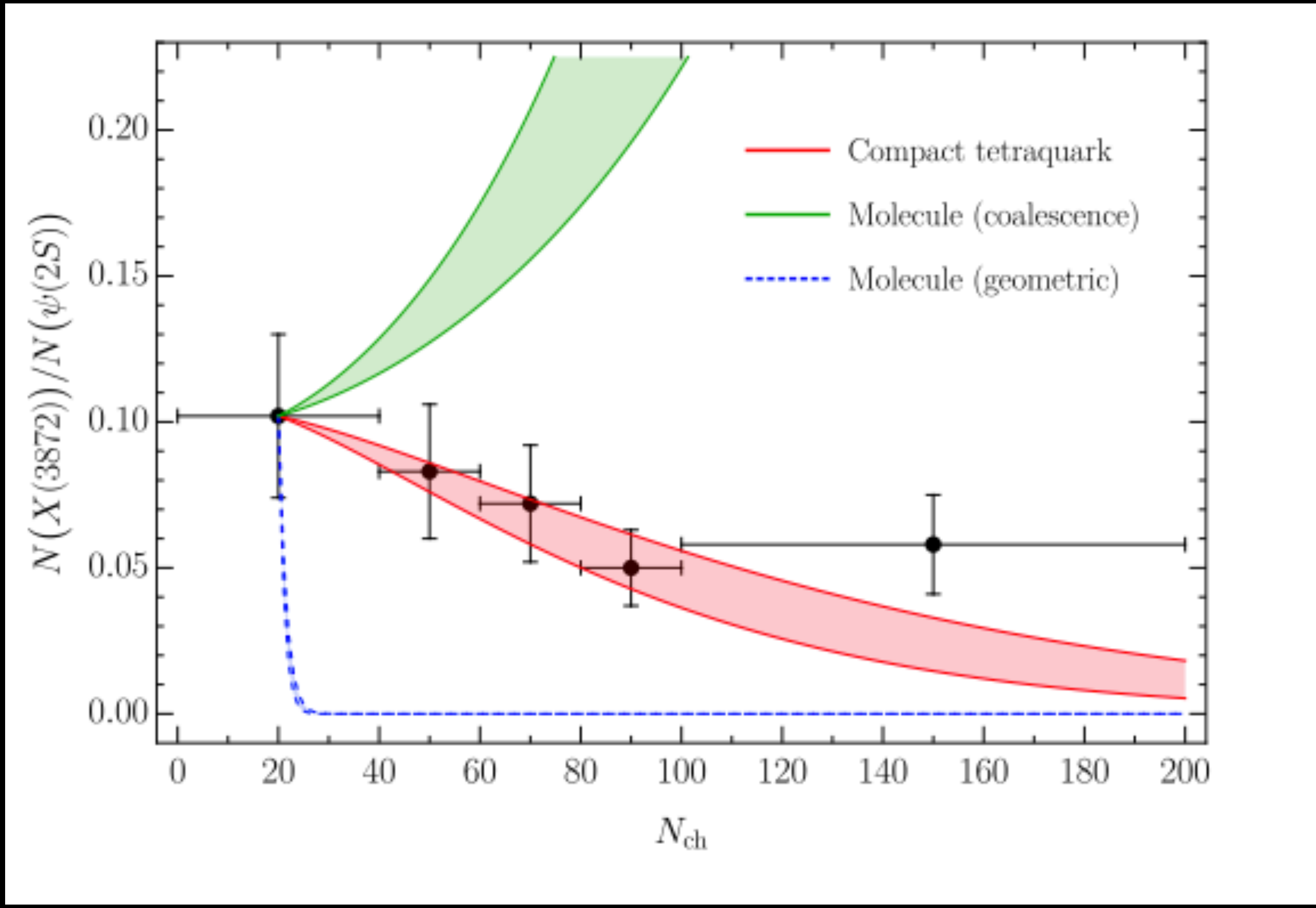
ALICE: 1902.09290; 2003.03184



Esposito, Ferreiro, Pilloni, ADP, Salgado *Eur. Phys. J. C* 81 (2021) 669

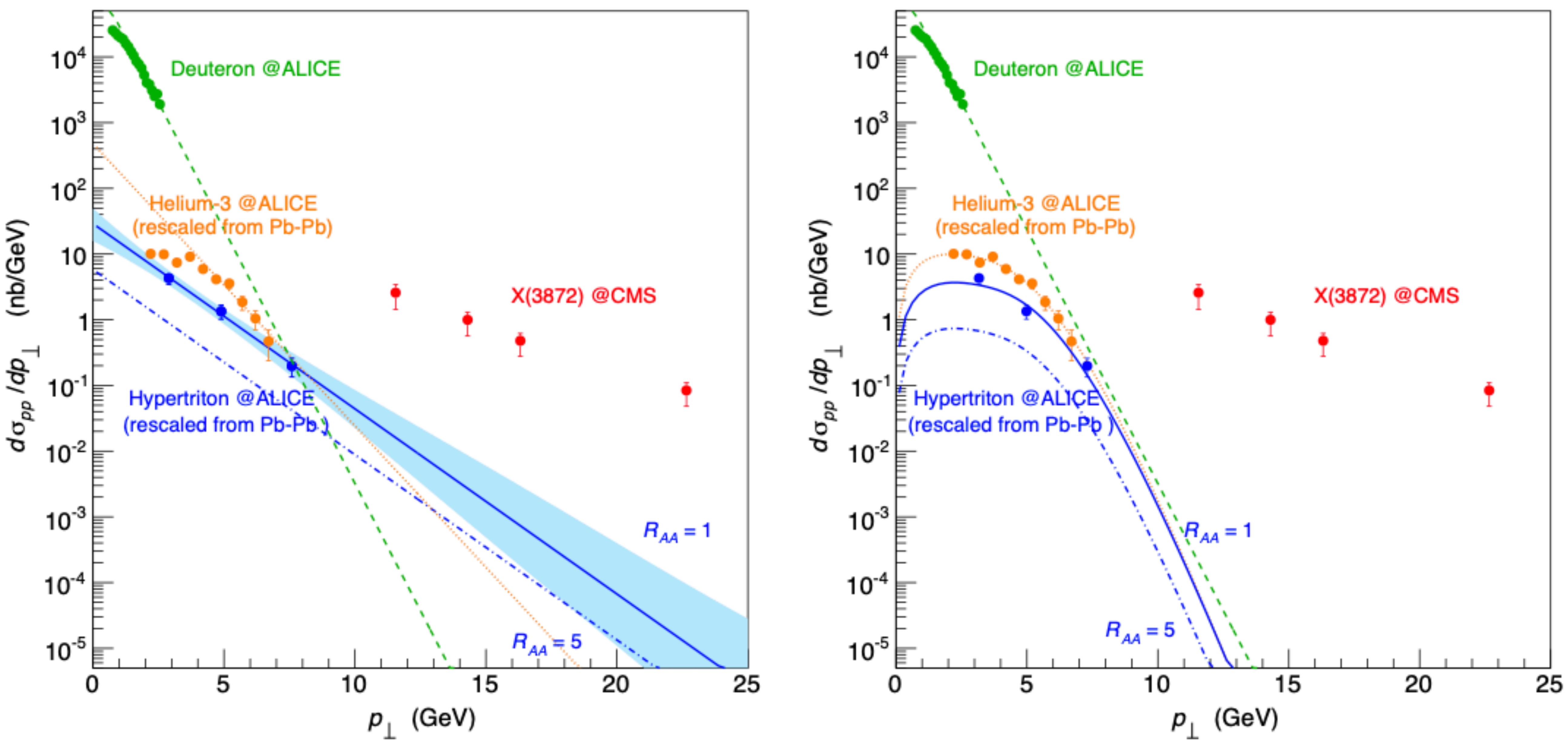
Number of deuterons as a function of the multiplicity computed with Boltzmann equation in a coalescence model.

DOES THE X(3872) BEHAVE AS THE DEUTERON?



The coalescence picture predicts a behavior (green band) qualitatively different from data.

NUCLEI AT HIGH p_T ?



Esposito, Guerrieri, Maiani, Piccinini, Pilloni, ADP, Riquer, *Phys. Rev. D* 92 (2015) 3, 034028

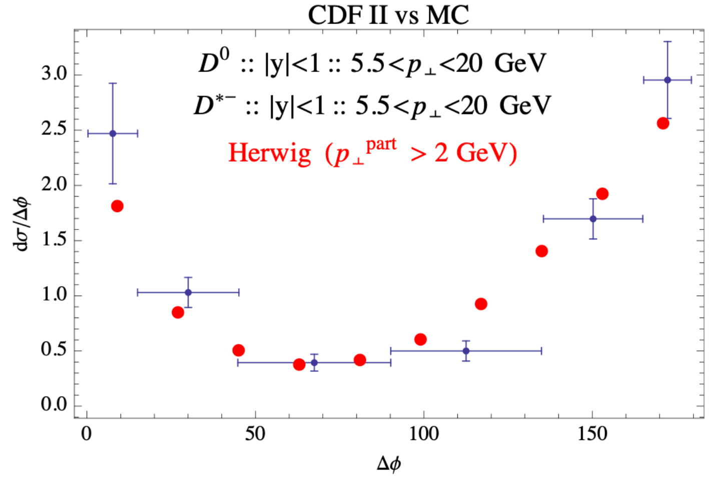


FIG. 1: The D^0D^{*-} pair cross section as function of $\Delta\phi$ at CDF Run II. The transverse momentum, $p_\perp$, and rapidity, y, ranges are indicated. Data points with error bars, are compared to the leading order event generator Herwig. The cuts on parton generation are $p_\perp^{\rm part} > 2$ GeV and $|y^{\rm part}| < 6$. We have checked that the dependency on these cuts is not significative. We find that we have to rescale the Herwig cross section values by a factor $K_{\rm Herwig} \simeq 1.8$ to best fit the data on open charm production.

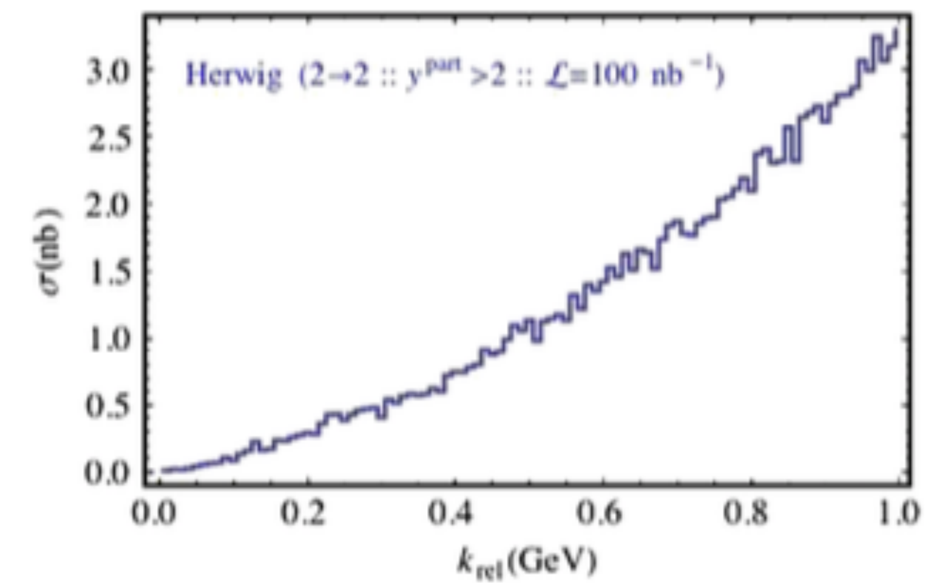
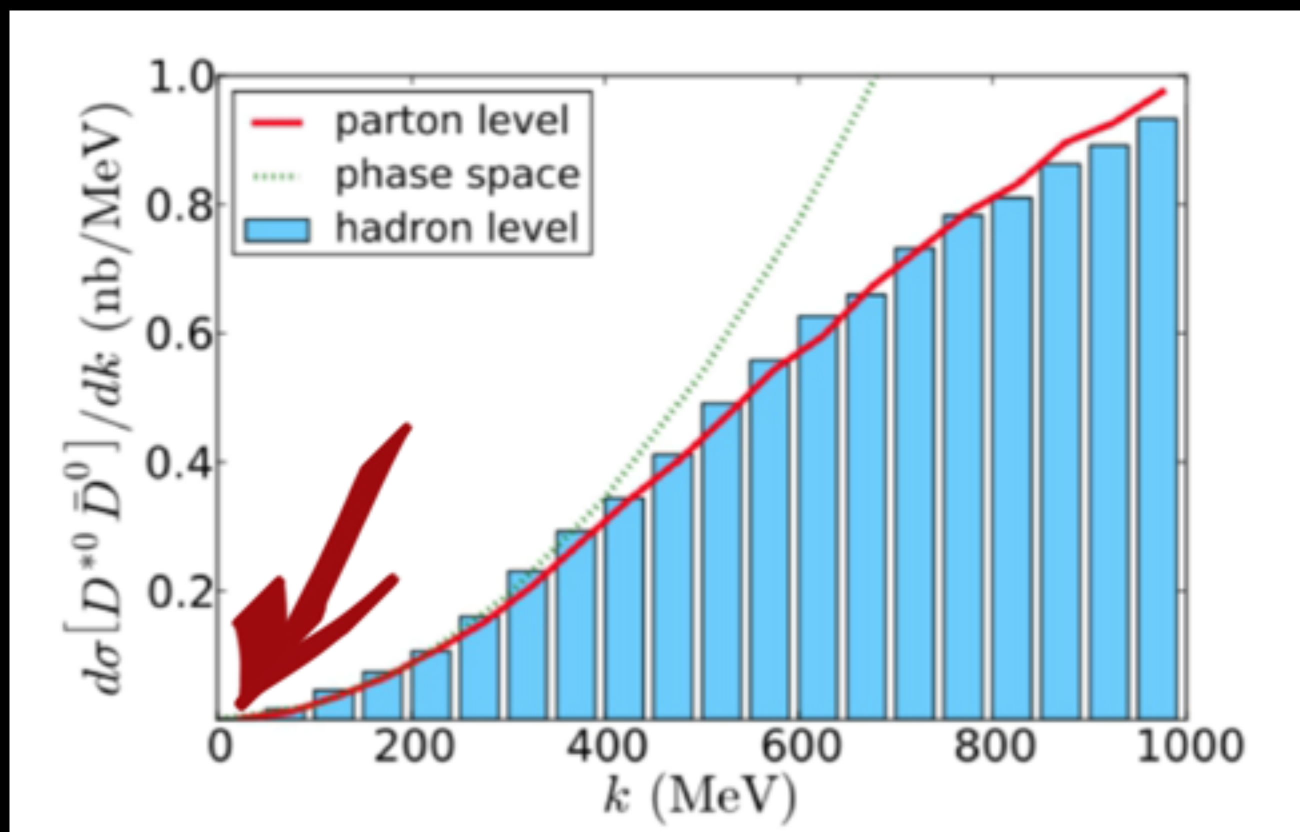


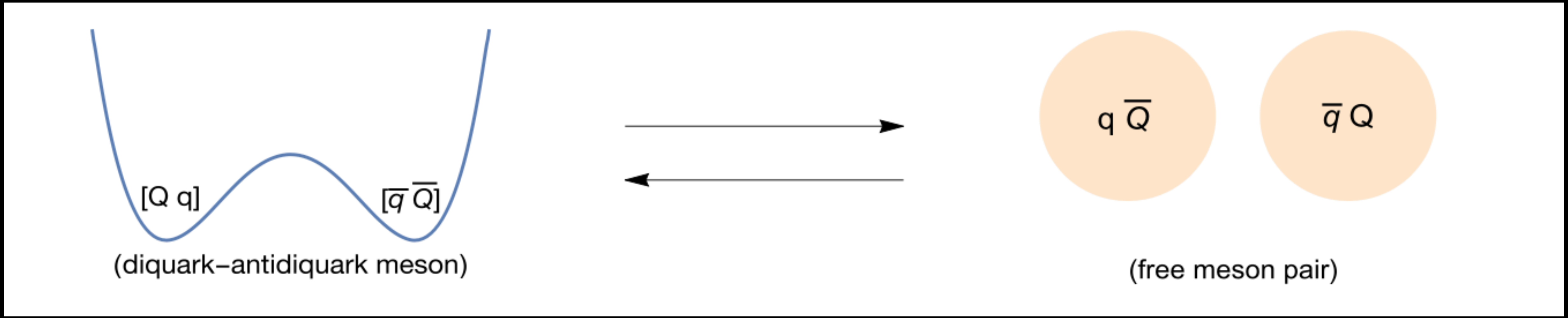
FIG. 3 (color online). The integrated cross section obtained with HERWIG as a function of the center of mass relative momentum of the mesons in the $D^0\bar{D}^{*0}$ molecule. This plot is obtained after the generation of 55×10^9 events with parton cuts $p_\perp^{\rm part} > 2$ GeV and $|y^{\rm part}| < 6$. The cuts on the final D mesons are such that the molecule produced has a $p_\perp > 5$ GeV and $|y| < 0.6$.

Bignamini, Grinstein, Piccinini, ADP, Sabelli, PRL103 (2009) 162001



Braaten and Artoisenet,
PRD81103 (2010) 114018

`SEGREGATED` DIQUARKS



Maiani, ADP, Riquer PLB 778 (2018) 247

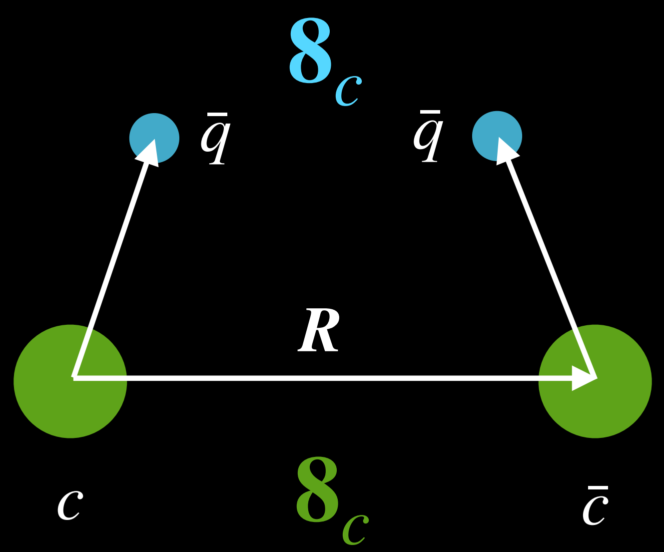
Maiani, Piccinini, ADP, Riquer PRD71 (2005) 014028

If $X^{\pm}$ is degenerate with X^{0} it can't decay in $D^{\pm}\bar{D}^{*}$ – it is forced to decay in $J/\psi\rho^{\pm}$, tunneling the heavy quark at a higher price in rate.

The $X^{\pm}$ might still be hiding in $J/\psi\rho^{\pm}$ decays.

This picture of `segregated diquarks` inspired the idea of `segregated hevay-quarks`, kept away by color repulsion in the octet.

THE BORN-OPPENHEIMER PICTURE



The fast motion of light quarks, in the field of heavy quarks (slow), generates an effective potential $\boldsymbol{V(R)}$ which in turn regulates the slower motion of heavy quarks – and can be used to calculate the spectrum.

The same picture might work for the $\mathcal{T}_{cc}$ and $\mathcal{T}_{bb}$ states, and for the pentaquarks!

Maiani, ADP, Riquer, *Phys.Rev.D* 100 (2019) 1, 014002; *Phys.Rev.D* 100 (2019) 7, 074002; EPJC83 (2023) 5, 378

Maiani, Pilloni, ADP, Riquer, PLB836 (2023) 137624 (on $\mathcal{T}_{cc}$ in B.O.)

Esposito, Papinutto, Pilloni, ADP, Tantalo, Phys Rev D88 (2013) 5, 054029 (on $\mathcal{T}_{cc}$ prediction)

THE $\mathcal{T}_{QQ}$ CASE

$$T = \left|(QQ)_{\bar{\mathbf{3}}}, (\bar{q}\bar{q})_{\mathbf{3}}\right\rangle_{\mathbf{1}} = \sqrt{\frac{1}{3}}\left|(\bar{q}Q)_{\mathbf{1}}, (\bar{q}Q)_{\mathbf{1}}\right\rangle_{\mathbf{1}} - \sqrt{\frac{2}{3}}\left|(\bar{q}Q)_{\mathbf{8}}, (\bar{q}Q)_{\mathbf{8}}\right\rangle_{\mathbf{1}}$$

The potential inside a single orbital is given by

$$V(r) = \frac{\lambda_{Q\bar{q}}}{r} + k_{Q\bar{q}}\, r + V_0 = -\frac{1}{3}\frac{\alpha_s}{r} + \frac{1}{4}k\, r + V_0$$

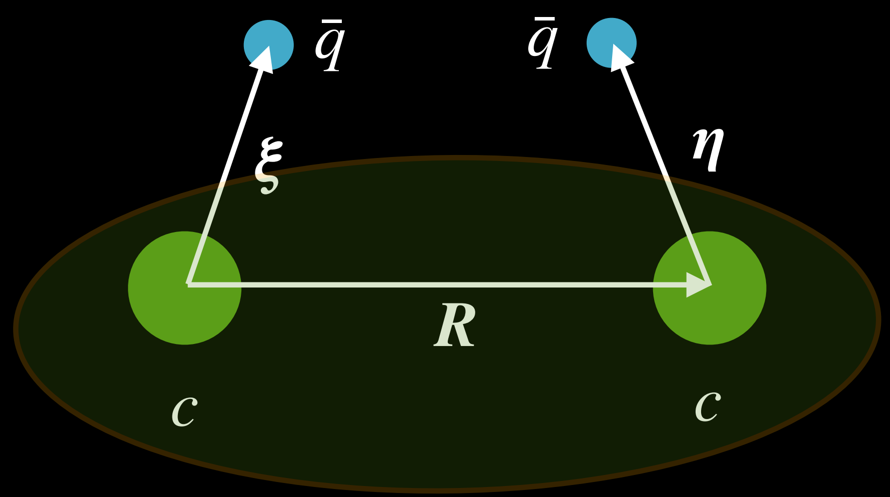
$$\lambda_{Q\bar{q}} = \left[\frac{1}{3}\times\frac{1}{2}\left(-\frac{8}{3}\right) + \frac{2}{3}\times\frac{1}{2}\left(3-\frac{8}{3}\right)\right]\alpha_s = -\frac{1}{3}\alpha_s$$

using the diagonalization formula ($R_1 \otimes R_2 = S_1 \oplus S_2 \oplus \dots$)

$$R_1 \otimes R_2 = \bigoplus_j \frac{1}{2}(C_{S_j} - C_{R_1} - C_{R_2})\mathbf{1}_{S_j}$$

Maiani, Pilloni, ADP, Riquer, PLB836 (2023) 137624 (on $\mathcal{T}_{cc}$ in B.O.)

THE BORN-OPPENHEIMER POTENTIAL



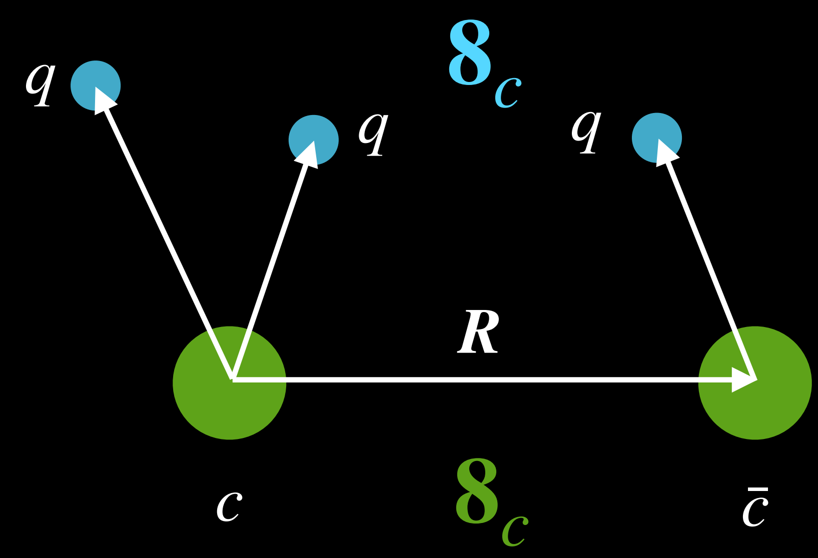
$$\delta V = \lambda_{Q\bar{q}} \left(\frac{1}{|\boldsymbol{\xi} - \boldsymbol{R}|} + \frac{1}{|\boldsymbol{\eta} + \boldsymbol{R}|} \right) + \frac{\lambda_{q\bar{q}}}{|\boldsymbol{\xi} - \boldsymbol{R} - \boldsymbol{\eta}|}$$

$$V_{BO}(R) = -\frac{2}{3}\alpha_s \frac{1}{R} + (\Psi(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{R}), \delta V\, \Psi(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{R}))$$

$$M(\mathcal{T}^{+}_{cc})_{\text{th.}} = 3871 \text{ MeV} \quad M(\mathcal{T}^{+}_{cc})_{\text{exp.}} = 3875 \text{ MeV}$$

$$M(\mathcal{T}_{bb})_{\text{th.}} = 10552 \text{ MeV}$$

PENTAQUARKS AND FERMI STATISTICS



The three light quarks in the pentaquark have to be in a color-octet configuration (a mixed representation).

We show that Fermi statistics applied to the complex of the three light quarks requires three SU(3)$_f$ octets, two with spin 1/2 and one with spin 3/2. Additional lines corresponding to decays into $J/\psi + \Sigma$ and $J/\psi + \Xi$ are predicted.

Maiani, ADP, Riquer, Eur. Phys. J. C 83 (2023) 5, 378

THE EQUAL SPACING RULE

In the vector mesons octet

$$K^* \approx (\phi + \rho)/2$$

The analog of ϕ in the hidden charm tetraquarks is

$$X(1^{++}) = [cs][\bar{c}\bar{s}] \qquad X(4140) \text{ seen in } J/\psi\phi$$

To first order in SU(3) flavor symmetry breaking we might predict

$$Z_{cs} \stackrel{!}{=} (X(4140) + X(3872))/2 = 4009 \text{ MeV}$$

A Z_{cs} has been observed at 4003 MeV.

Maiani, ADP, Riquer, Sci. Bulletin 66, 1616 (2021)

Z_{cs} AND NEGATIVE CHARGE CONJUGATION

Observed by LHCb in the decay

$$B^+ \to \phi + Z_{cs}^+(4003) \to \phi + K^+ + J/\psi$$

In the diquark-antidiquark model we predict that $M(X(1^{++})) = M(Z(1^{+-}))$. Using the same spacing rules, given the $Z(3900)$ and the recently discovered $Z_{cs}(3985)$ we predict a $Z_{ss}(\simeq 4076)$

Maiani, ADP, Riquer, Sci. Bulletin 66, 1616 (2021)

CONCLUSIONS

- It would be useful to have new comparative studies on the r_0 of the X(3872) and of the $\mathcal{T}_{QQ}$ particles, and to agree on the way to extract information from data (not easy).
- It would be of great relevance to learn more, on the experimental side, about deuteron production at high p_T .
- Some states are produced promptly in pp collisions, some are not. There is no clear reason why.
- Are there loosely bound molecules $B\bar{B}*$? Can we formulate more stringient bounds on $X^{\pm}$ particles?
- More basically: are we on the right questions?

BACKUP

THE VICINITY TO THRESHOLD

$X(3872)$	$Z_c^{0\pm}(3900)$	$Z_c^{0\pm}(4020)$	$Z_b^{0\pm}(10610)$	$Z_b^{0\pm}(10650)$
$D^0\bar{D}^{*0}$	$D^0\bar{D}^{*0\pm}$	$D^{*0}\bar{D}^{*0\pm}$	$B^0\bar{B}^{*0\pm}$	$B^{*0}\bar{B}^{*0\pm}$
$\delta \approx 0$	$+7.8$	$+6.7$	$+2.7$	$+1.8$

(MeV)

ONE WAY TO COMPUTE f_w

- Use $e^{-\mu r}$ in place of $e^{i\mu r}$ – in the final expression $\mu \to -i\mu$
- Use the regularized $\chi_s(r) = 2kr\left(\frac{e^{i\delta}\sin(kr+\delta)}{kr} - \frac{e^{i\delta}\sin\delta}{kr}\right)$

  for $r \in [0,\lambda]$ and $\chi_s(r) = 2kr\left(\frac{e^{i\delta}\sin(kr+\delta)}{kr}\right)$ for $r \in [\lambda, \infty]$
- The integral is finite. Substitute $\delta = \cot^{-1}\left(-\frac{1}{ka_s}\right)$
- Double-expand the result around $k = 0$ and $\alpha = 0$.
- Take the limit $\lambda \to 0$
- Set $\mu \to -i\mu$

R. Jackiw, `Delta Function Potentials in two- and three- dimensional quantum mechanics` in Diverse Topics in Theoretical and Mathematical Physics, World Scientific.
See also Gosdzynsky, Tarrach (https://doi.org/10.1119/1.16691) — suggested by Adam Szczepaniak.

SUM RULE IN KÄLLÉN-LEHMAN

An elementary deuteron would not correspond to $Z = 1$ but to whatever $0 < Z < 1$. Strictly speaking, only the case $Z = 0$ corresponds to the exclusively composite state.

Indeed it can be shown that the following sum rule holds

$$\int_0^\infty \rho(\mu^2)\, d\mu^2 = 1$$

which corresponds to

$$Z + \int_0^\infty \sigma(\mu^2)\, d\mu^2 = 1$$

A DERIVATION OF THE DWBA FORMULA

$$f = \frac{e^{i\delta_s}\sin\delta_s}{k} + \frac{e^{i\delta_w}\sin\delta_w}{k}$$

$$f_{\mathrm{Born}} = -\frac{m}{2\pi}\int V(r)\, e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} d^3 r$$

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_{\ell=0}^{\infty} i^{\ell} j_{\ell}(kr)(2\ell+1)P_{\ell}(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{r}})$$

$$e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}} = \sum_{\ell=0}^{\infty} i^{\ell} j_{\ell}(k'r)(2\ell+1)(-1)^{\ell}P_{\ell}(\hat{\boldsymbol{k}}'\cdot\hat{\boldsymbol{r}})$$

$$\int P_{\ell}(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)P_{\ell'}(\boldsymbol{n}_1\cdot\boldsymbol{n}_3)d\Omega_1 = \delta_{\ell\ell'}\frac{4\pi}{(2\ell+1)}P_{\ell}(\boldsymbol{n}_2\cdot\boldsymbol{n}_3)$$

$(-1)^{\ell} i^{2\ell} = +1$ for every ℓ and $k = k'$ for elastic collisions

A DERIVATION OF THE DWBA FORMULA

So we get

$$f = -2m \sum_{\ell=0}^{\infty} (2\ell + 1) P_\ell(\cos\theta) \int V(r)(j_\ell(kr))^2 r^2 dr$$

To be compared with Holtsmark formula

$$f = \sum_{\ell=0}^{\infty} (2\ell + 1) P_\ell(\cos\theta) \frac{e^{i\delta}\sin\delta}{k}$$

giving

$$\frac{e^{i\delta}\sin\delta}{k} = -2m \int V(r)(j_\ell(kr))^2 r^2 dr$$

A DERIVATION OF THE DWBA FORMULA

$$\chi^{(0)}(r) = 2kr\, j_\ell(kr)$$

$$\frac{e^{i\delta}\sin\delta}{k} = -\frac{2m}{4k^2}\int V(r)\,(\chi^{(0)}(r))^2\,dr$$

DWBA consists in computing $\boldsymbol{T}$ with the in/out states of $\boldsymbol{V_s}$

$$T_{\beta\alpha} = \left(\Psi^{-}_{s\,\beta}, V_w \Psi^{+}_{s\,\alpha}\right)$$

Therefore we substitute $\chi^{(0)} \to \chi_s$

$$\frac{e^{i\delta_w}\sin\delta_w}{k} = -\frac{2m}{4k^2}\int_0^\infty V(r)\,\chi_s^2(r)\,dr$$