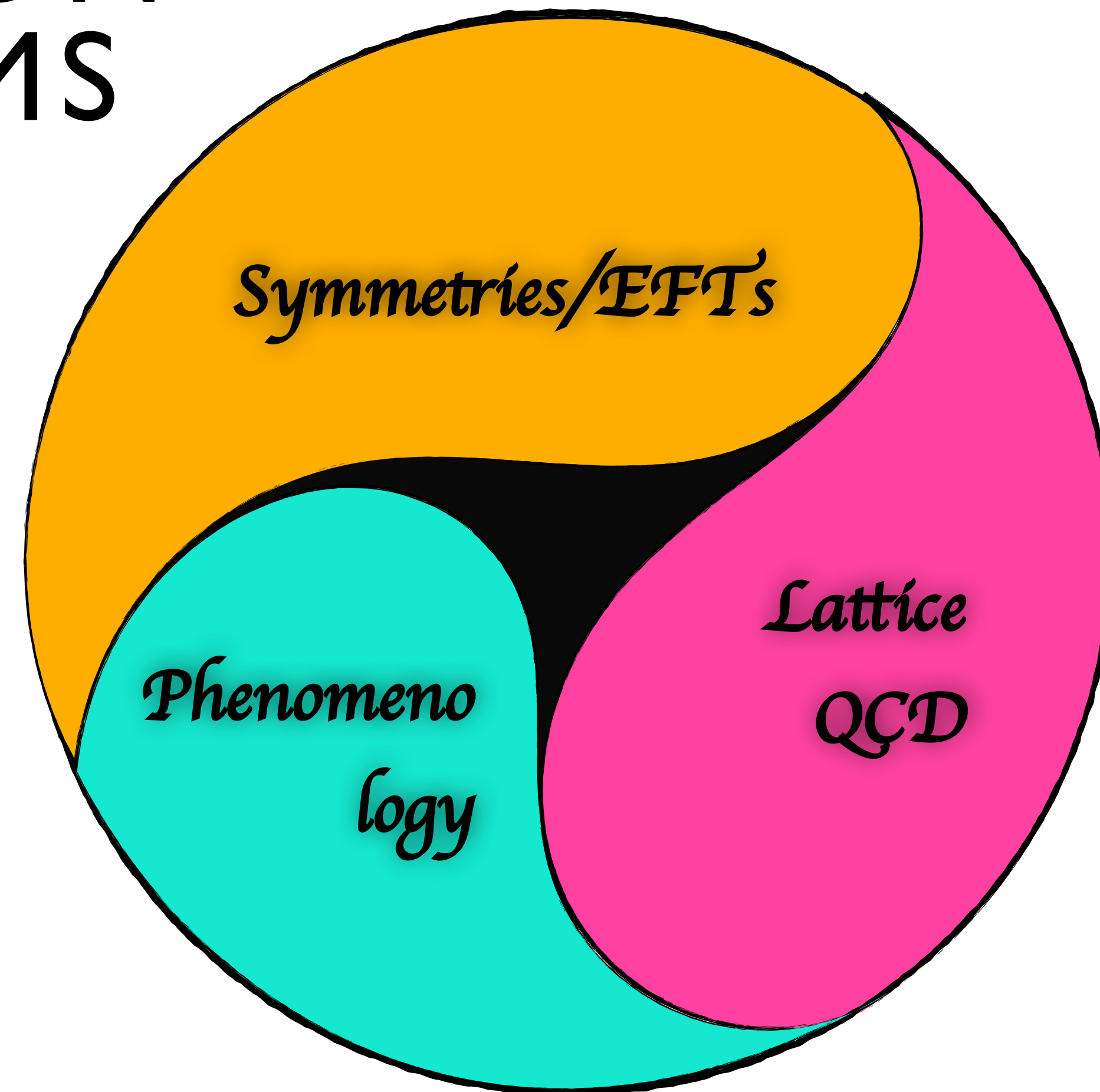


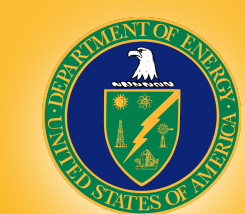
RESONANT HADRON SYSTEMS FROM:



Maxim Mai

University of Bonn

The George Washington University



DE-SC0016582

DE-SC0016583



PHY

2012289

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MA 7156/I



Deutsche
Forschungsgemeinschaft

DFG

Effective field theory

Ulf-G Meißner (U. Bonn, FZ Jülich)

Bastian Kubis (U. Bonn)

Evgeny Epelbaum (U. Bochum)

...

3-body scattering and quantisation condition [FVU]

Michael Döring (GWU/JLAB)

Adam P. Szczepaniak (U. Indiana/JLAB)

Alessandro Pilloni (U. Messina)

Finite-volume effects

Akaki Rusetsky (U. Bonn)

Ulf-G. Meißner (U. Bonn)

Michael Döring (GWU/JLAB)

Daniel Severt (U. Bonn)

2/3-body lattice spectroscopy [GWQCD collaboration]

Michael Döring (GWU/JLAB)

Chris Culver (U. Liverpool)

Ruairi Brett (GWU)

Andrei Alexandru (GWU)

Frank X. Lee (GWU)

[Extended Twisted Mass collaboration]

Carsten Urbach (U. Bonn)

Ferenc Pittler (U. Cyprus)

Marco Garofalo (U. Bonn)

...

Chiral Unitary approaches

Ulf-G Meißner (U. Bonn, FZ Jülich)

Daniel Sadasivan (Ave Maria U.)

LiSheng Geng (Beihang U.)

Aleš Cieply (Rež)

Michael Döring (GWU/JLAB)

...

Data driven tools

Justin Landay (GWU)

Dave Ireland (U. Glasgow)

...

Meson electroproduction [Jülich-Bonn-Washington Collaboration]

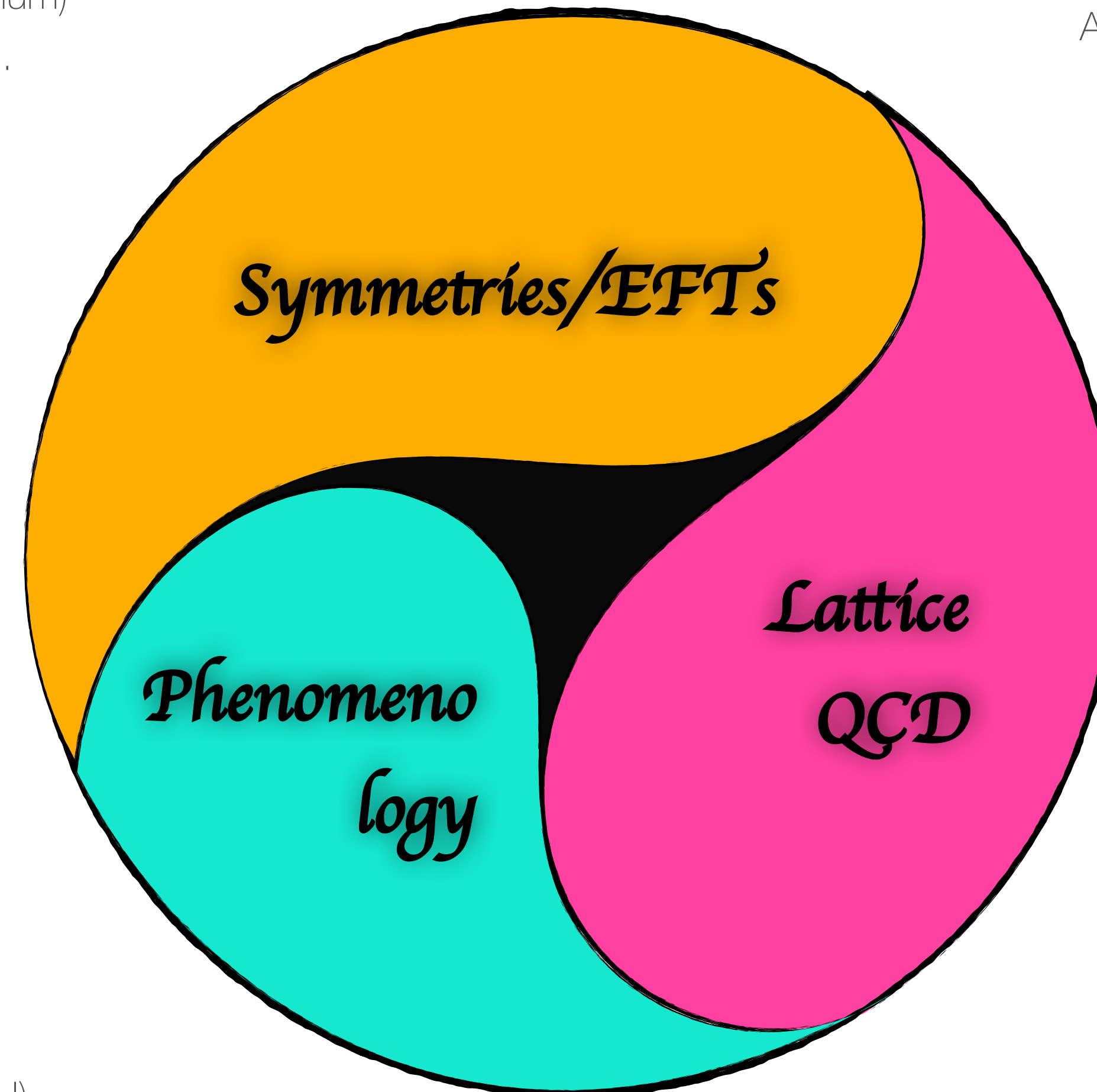
Ron Workman (JBW, GWU)

Michael Döring (GWU/JLAB)

Deborah Rönchen (FZ Jülich)

Helmut Haberzettl (GWU)

...



HADRON SPECTROSCOPY

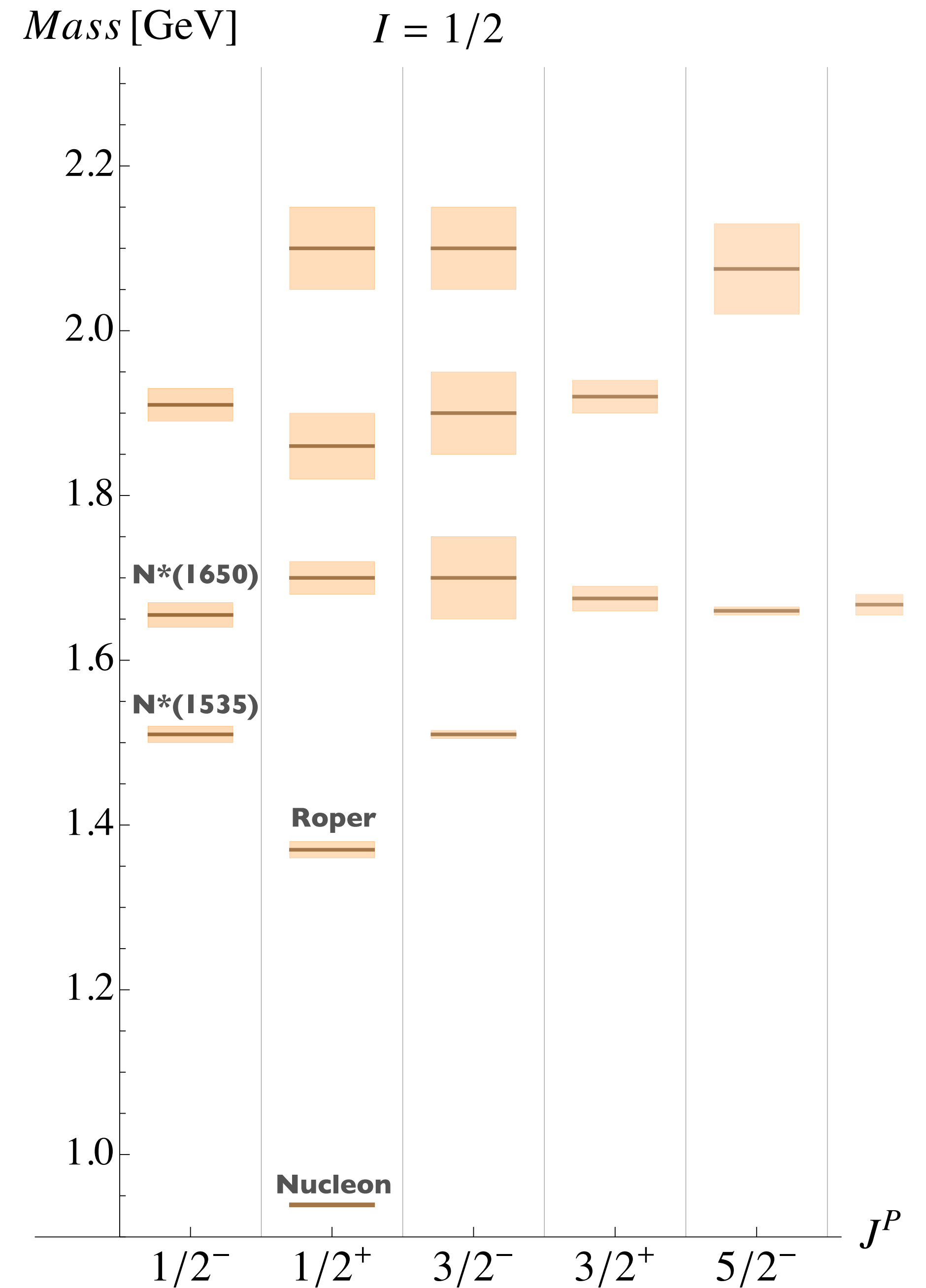
Mostly excited states

≈ 100 mesons & ≈ 50 baryons (****)

Key questions

“what is the pattern of these states?”

“how they are formed?”



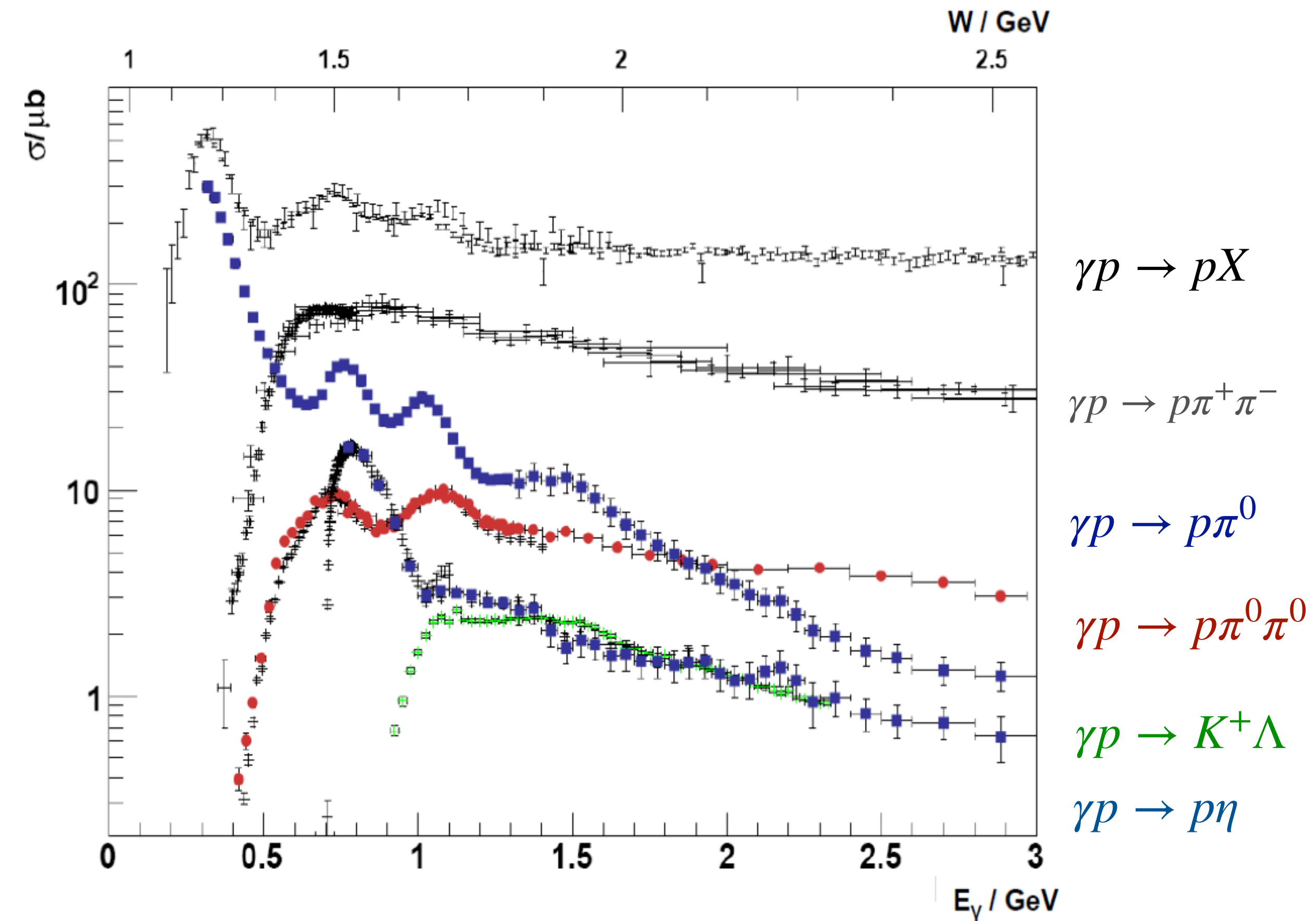
HADRON SPECTROSCOPY

Physical input

- many experimental data
- ongoing experiments^[1]

Resonance parameter

“Not every bump is a resonance, not every resonance is a bump” - R. G. Moorhouse (1960s)

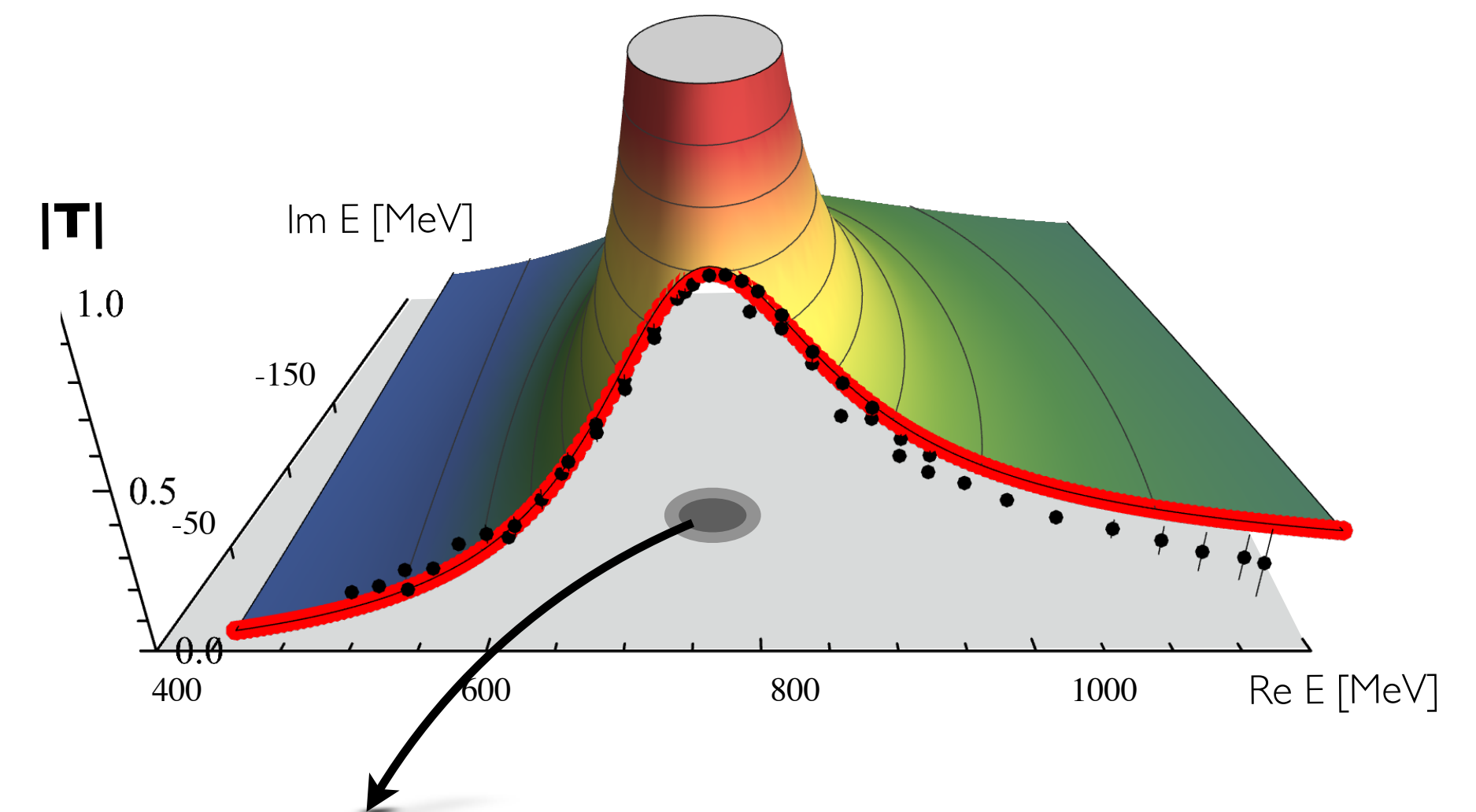


[1] **Talks:** Lucilla Lanza, Victor Mokeev, ...

UNIVERSAL PARAMETERS

Reaction-independent parameters

- pole positions on unphysical Riemann Sheets
- central quantity: **transition amplitudes**



$$M^* = (750 - i60) \text{ MeV}$$

Universal property of the ρ – meson

Review: [“Towards a theory of hadron resonances”](#) Phys. Rept. 1001 (2023) –

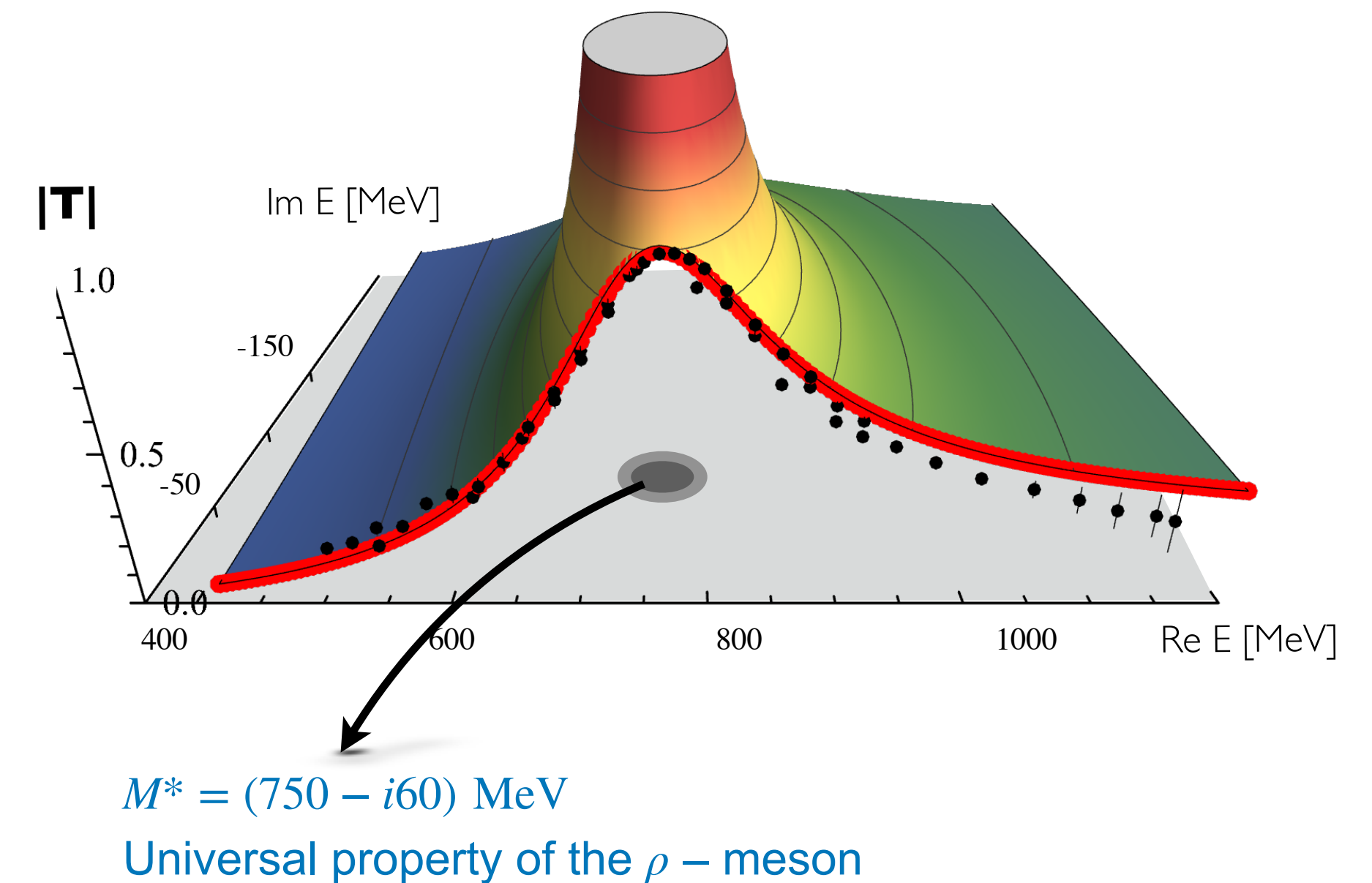
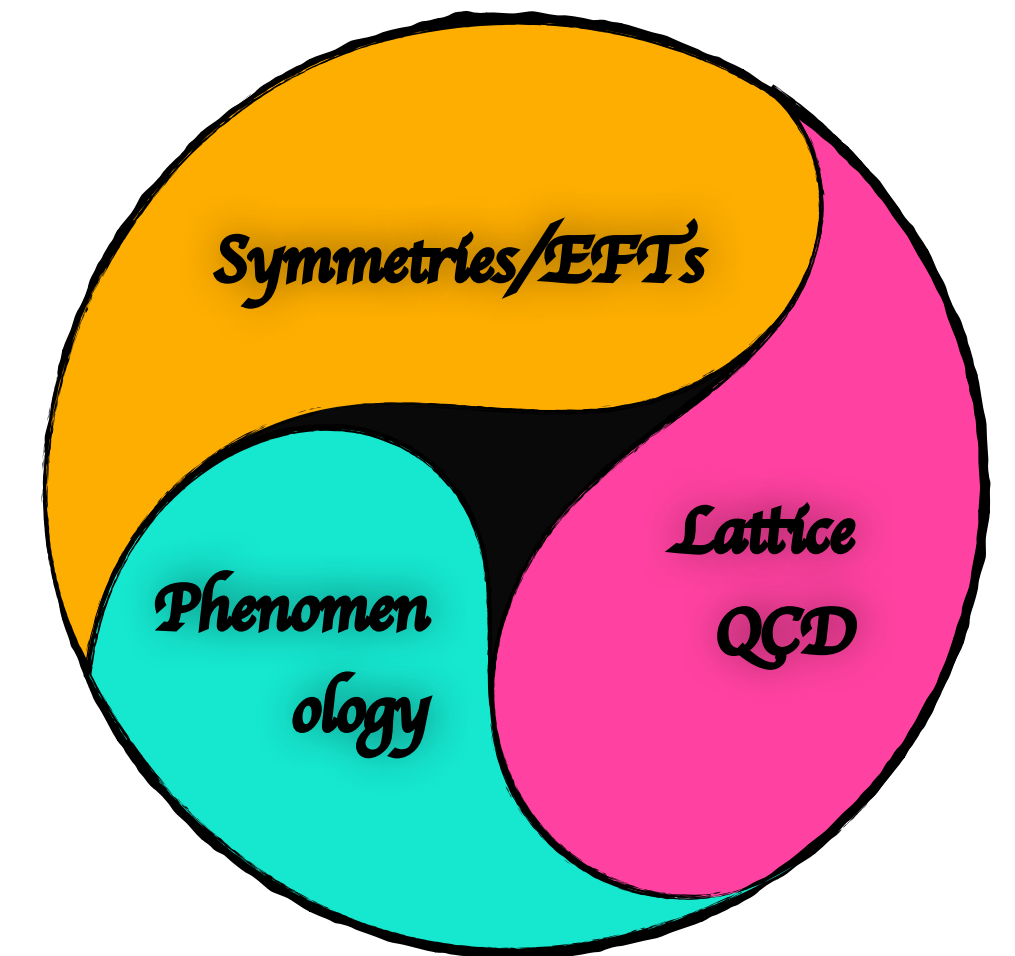
MM/Meißner/Urbach

Talks: Szczepaniak, Guo, Rodas, Pelaez, Döring, ...

UNIVERSAL PARAMETERS

Reaction-independent parameters

- pole positions on unphysical Riemann Sheets
- central quantity: **transition amplitudes**
 - Constraints from S-matrix (Unitarity/Analyticity/Crossing)
 - Constrains from QCD (CHPT/LatticeQCD)



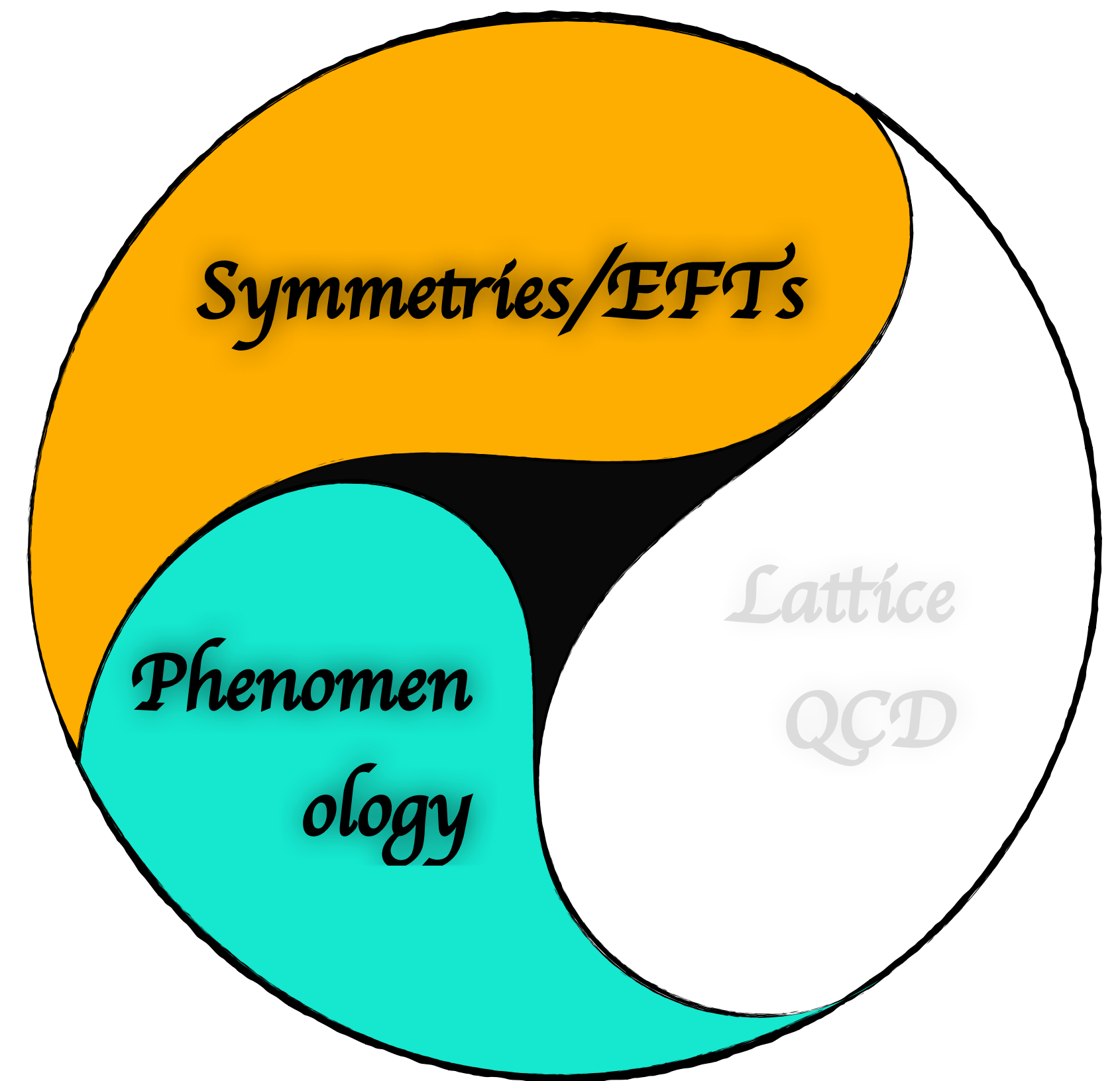
Review: ``Towards a theory of hadron resonances'' Phys. Rept. 1001 (2023) –

MM/Meißner/Urbach

Talks: Szczepaniak, Guo, Rodas, Pelaez, Döring, ...

INDIVIDUAL STATES

MESON-BARYON RESONANCES
FROM CHPT AND UNITARITY



RESONANT MESON-BARYON SYSTEMS

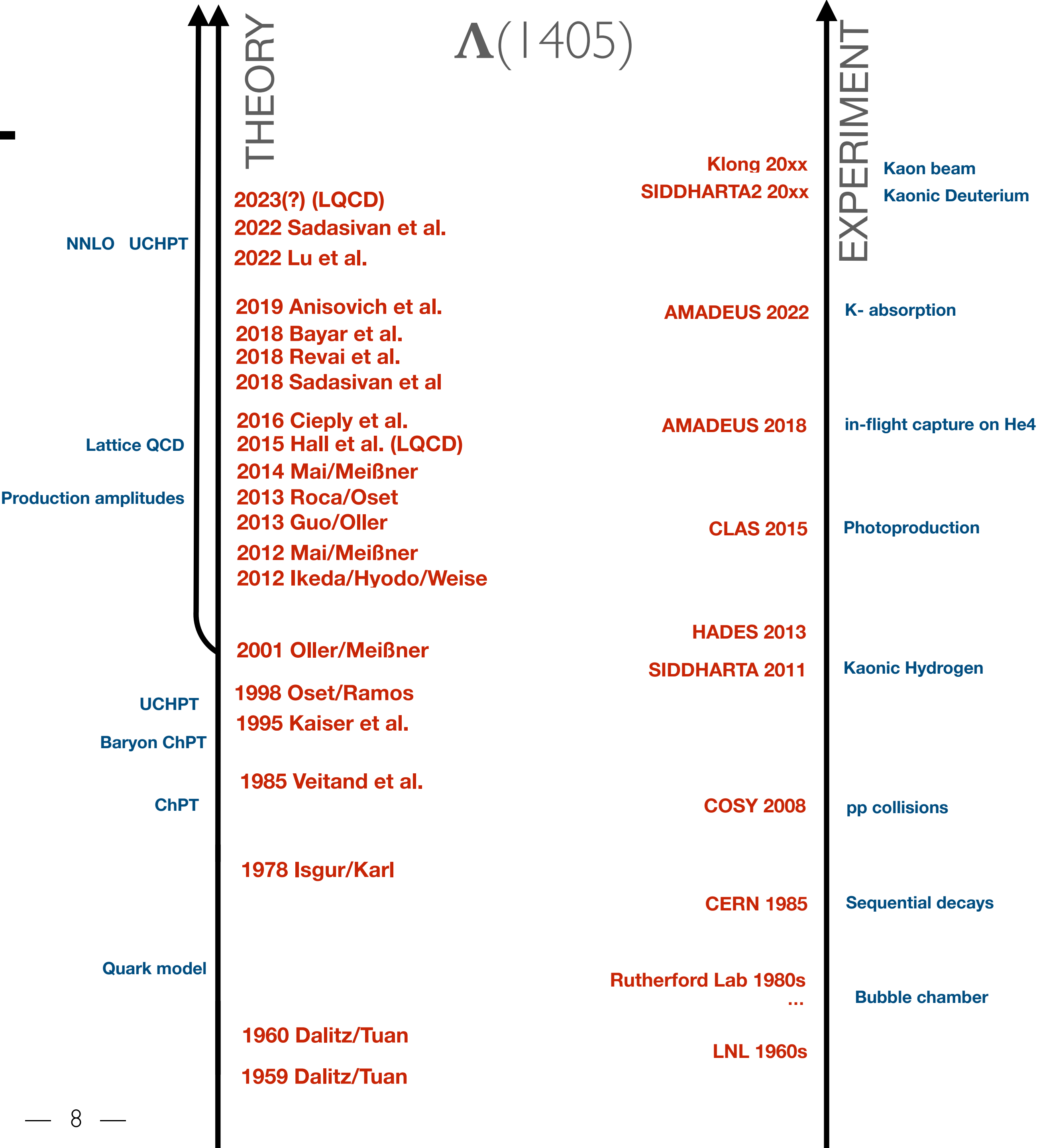


Many examples:

- $N^*(1535)$, $N^*(1650)$, ...
- $\Lambda(1405)$, $\Lambda(1380)$

→ Long history of experimental and theoretical efforts^[1]

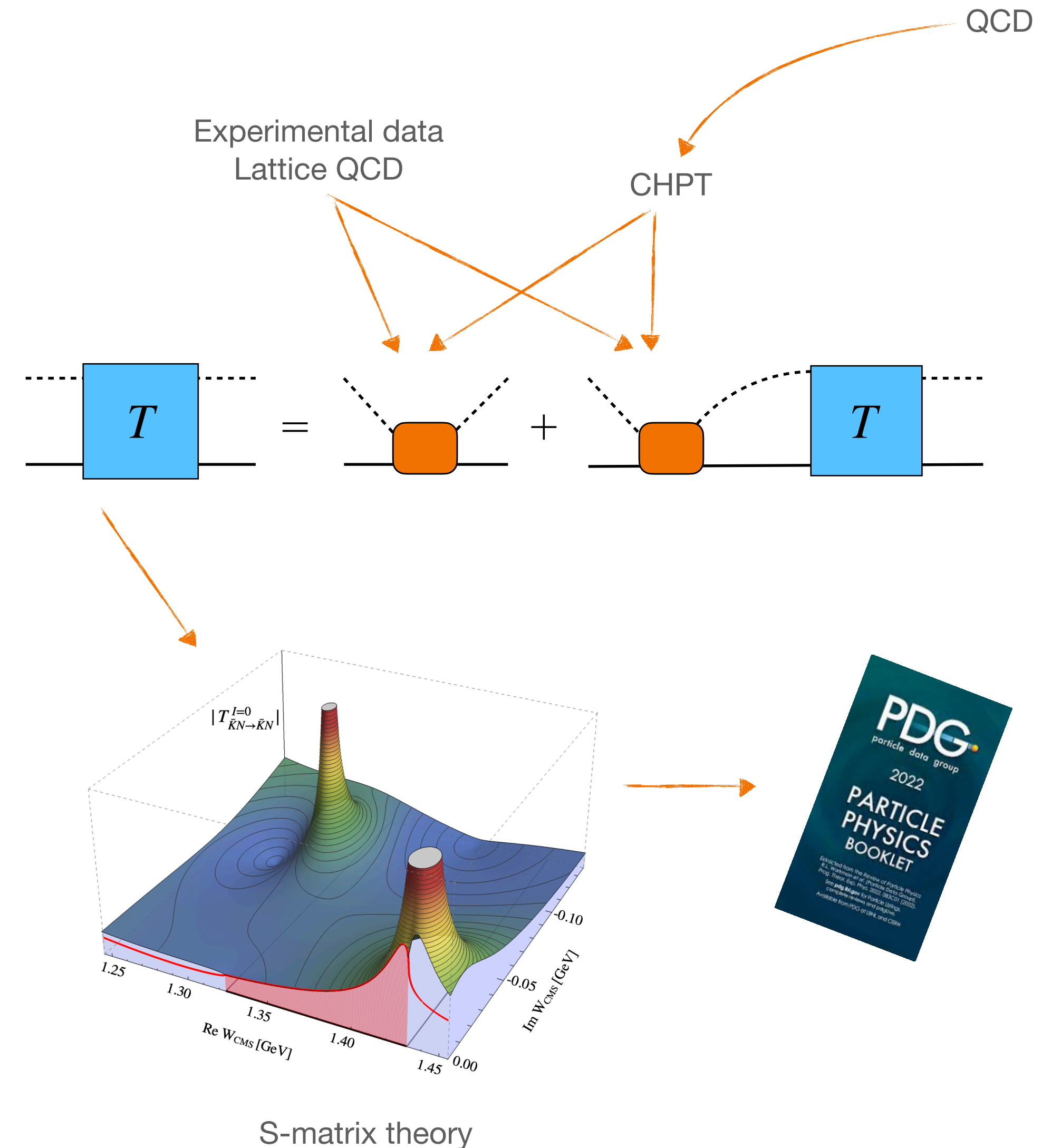
[1] **Reviews:** MM Eur.Phys.J.ST 230 (2021); Hyodo/Niiyama Prog.Part.Nucl.Phys. 120 (2021)



TRANSITION AMPLITUDE

One way:

- Chiral Perturbation Theory (#QCD#EFT) dictates the form of the interaction at low energies
- Unitary scattering amplitude from the Bethe-Salpeter equation
 - Fit free parameters to experimental data / LQCD
 - Extract complex pole positions for complex energies

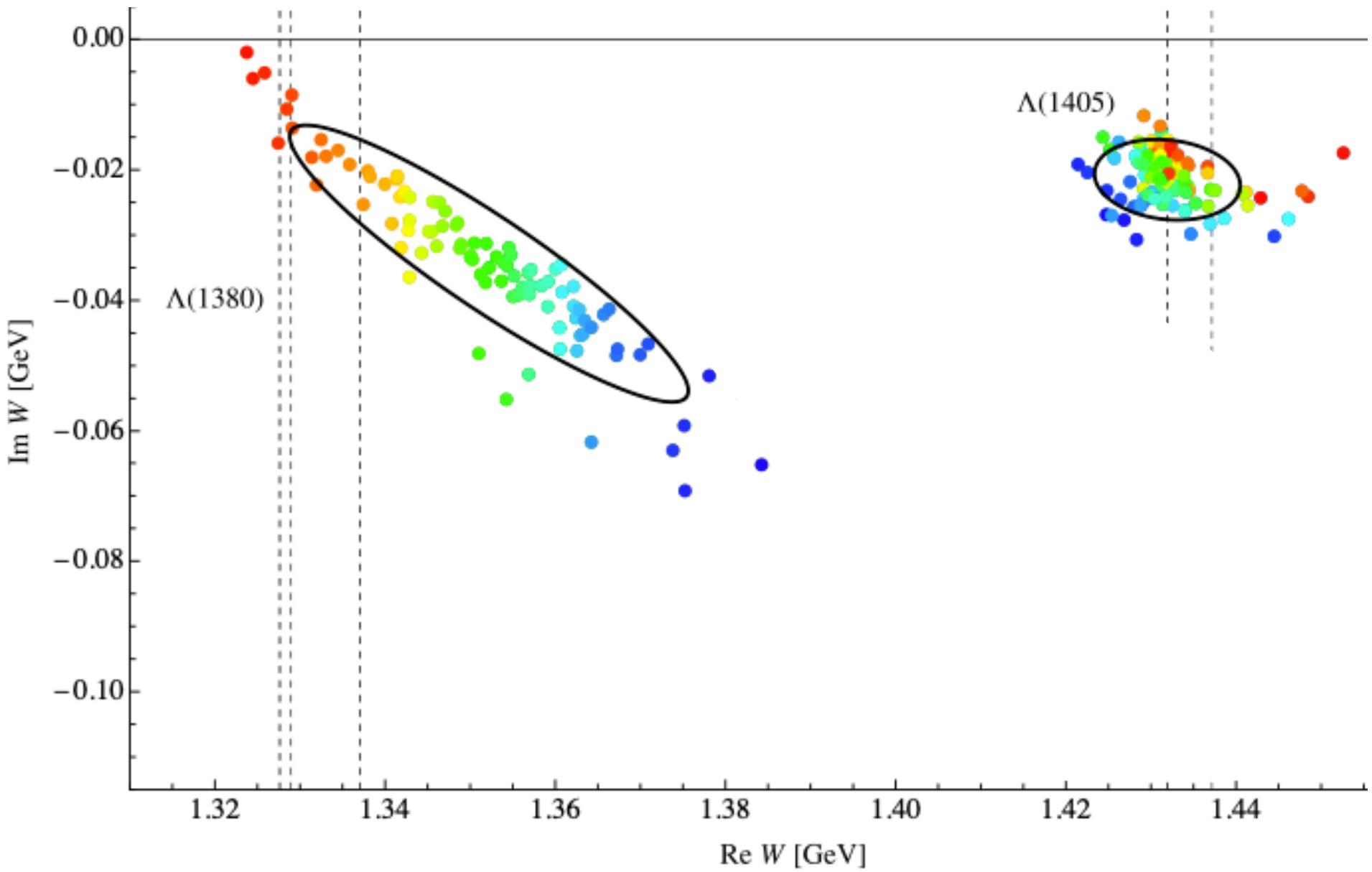


CURRENT FRONTIER

Statistical tests and new data sources

AMADEUS Phys. Lett. B 782 (2018)

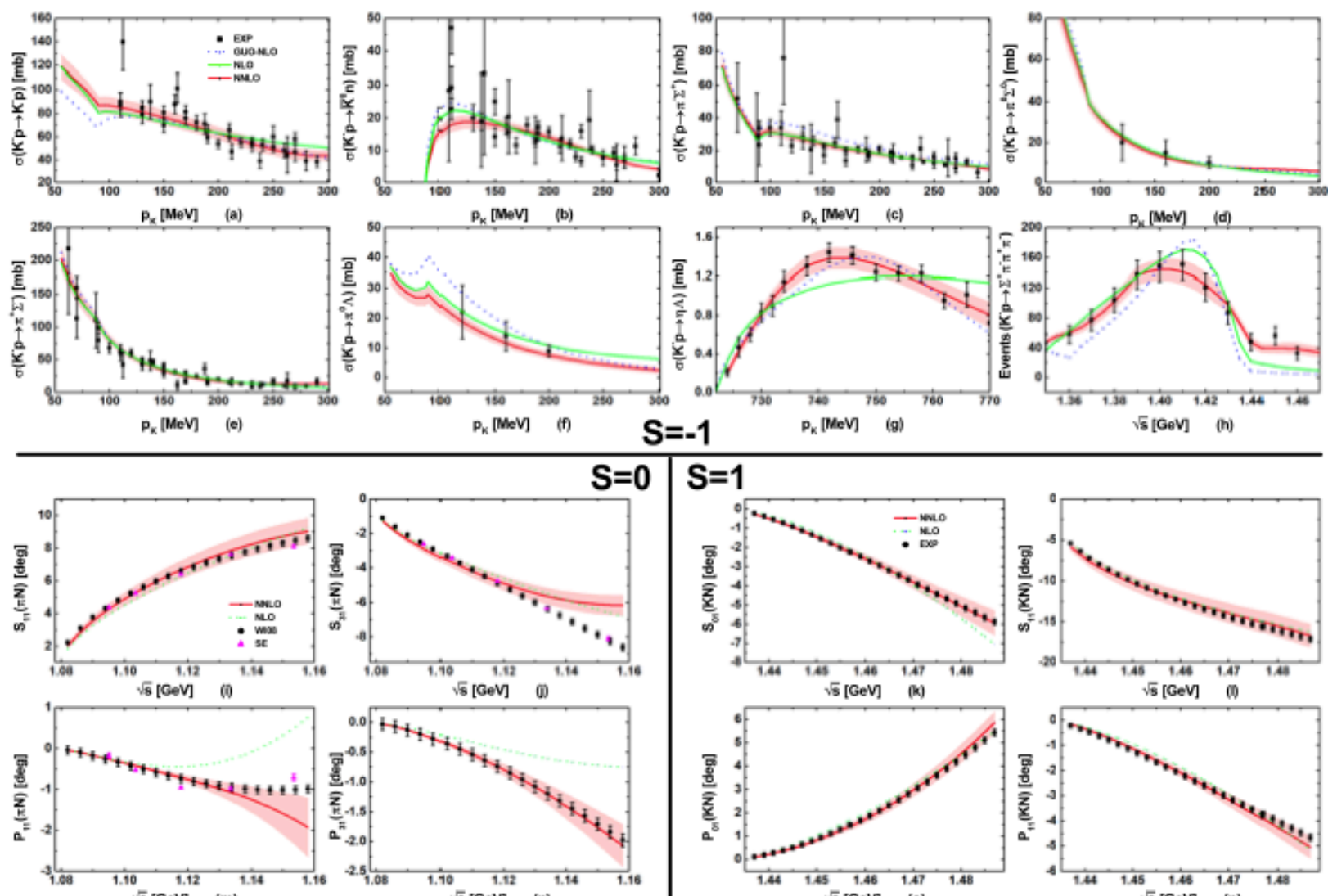
Sadasivan et al Front.Phys. 11 (2023)



- New correlations $\Lambda(1405)/\Lambda(1380)$

Theory update: NNLO UCHPT

Lu/Geng/Döring/MM Phys.Rev.Lett. 130 (2023)



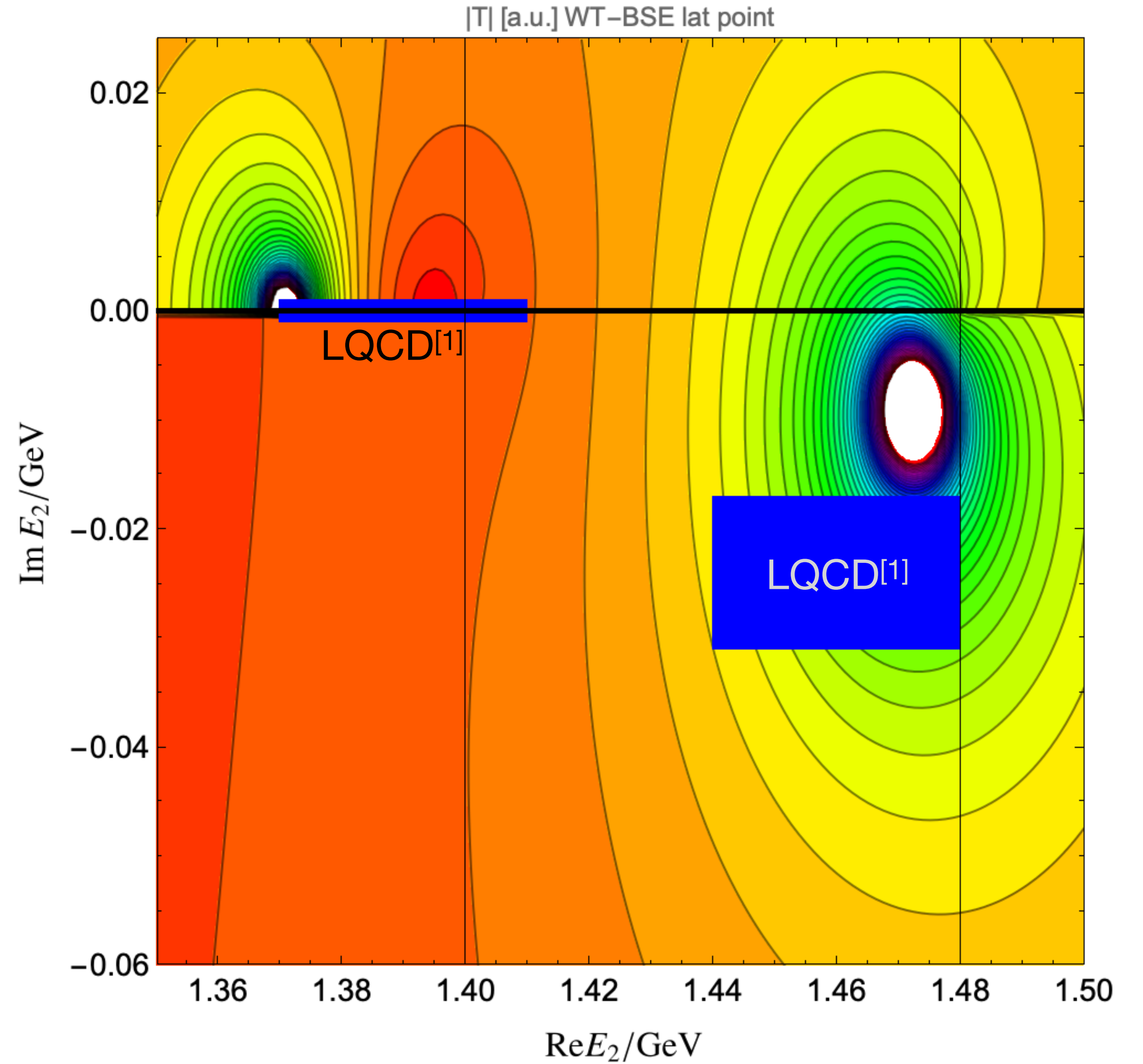
- Simultaneous description of $\pi N/KN/K\bar{N}N$ scattering
- Two-pole structure $\Lambda(1405)/\Lambda(1380)$

- New experimental facilities (Talks: Francesco Sgaramella/...)
- Lattice QCD [2 poles] (Talks: Morningstar/Leinweber)

CURRENT FRONTIER

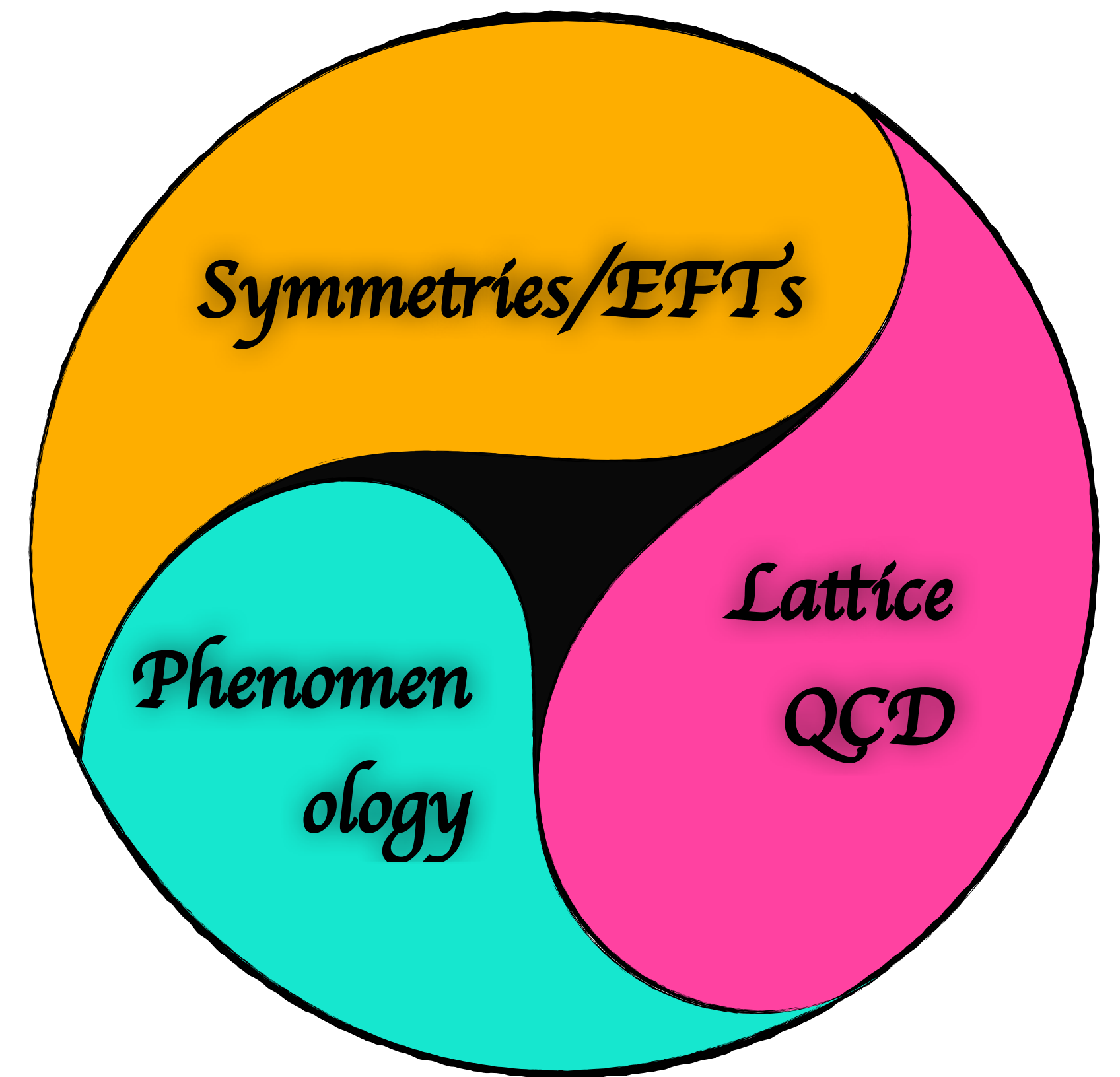
VERY VERY PRELIMINARY

- UCHPT Chiral trajectory



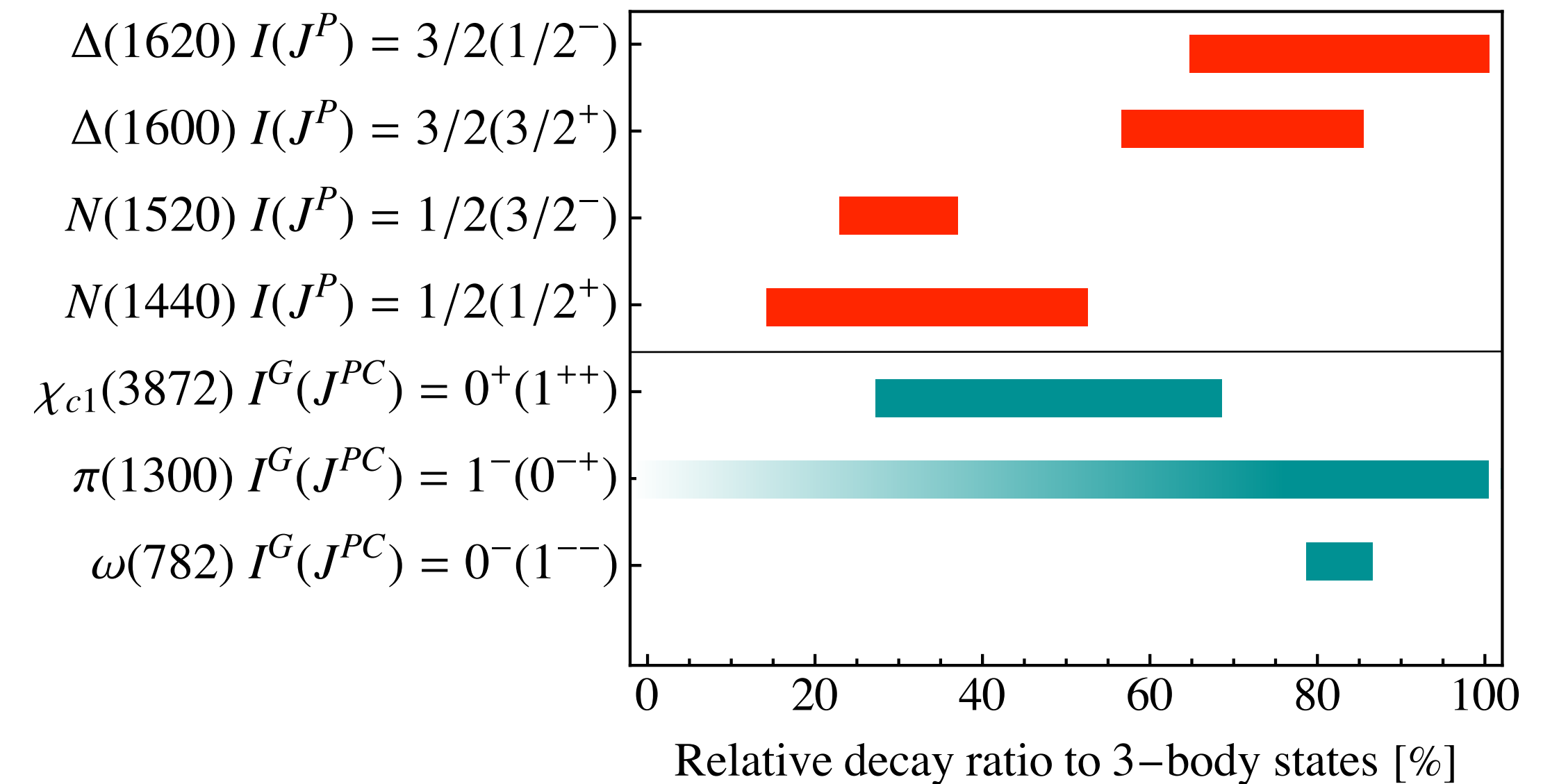
INDIVIDUAL STATES

THREE-BODY SYSTEMS



HADRONIC 3-BODY PROBLEM

- Many known states have large 3-body content:
→ Roper(1440)/X(3872)/a₁(1260)/...
- Beyond Standard Model searches (τ -EDM/...)
- Exotic states of matter^[1]

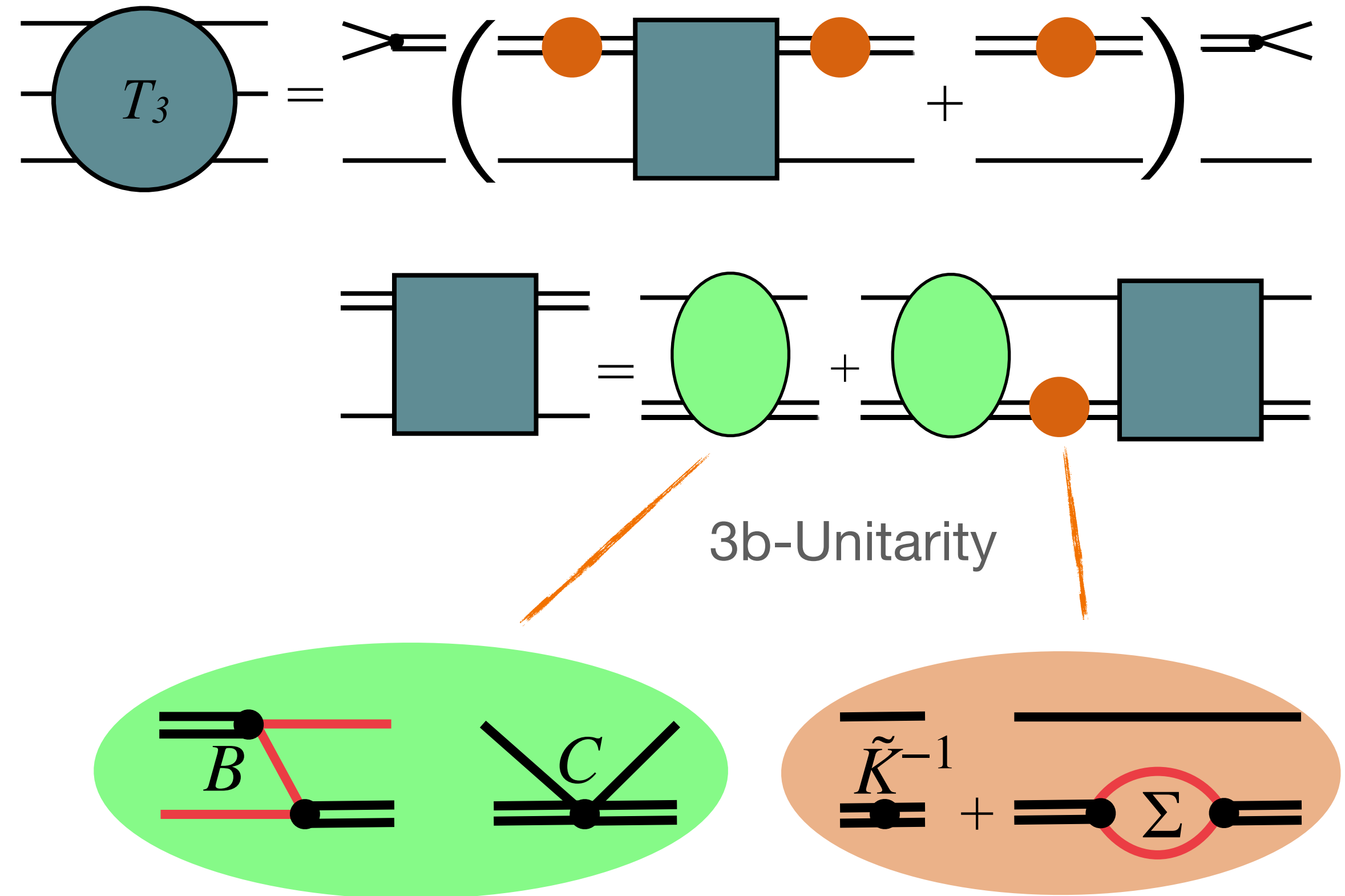


[1] Experimental programs: GlueX@JLAB; COMPASS@CERN;

TRANSITION AMPLITUDE

Three-body scattering amplitude^{[1][2]}

- constructed from unitarity
- novel result from the S-matrix theory

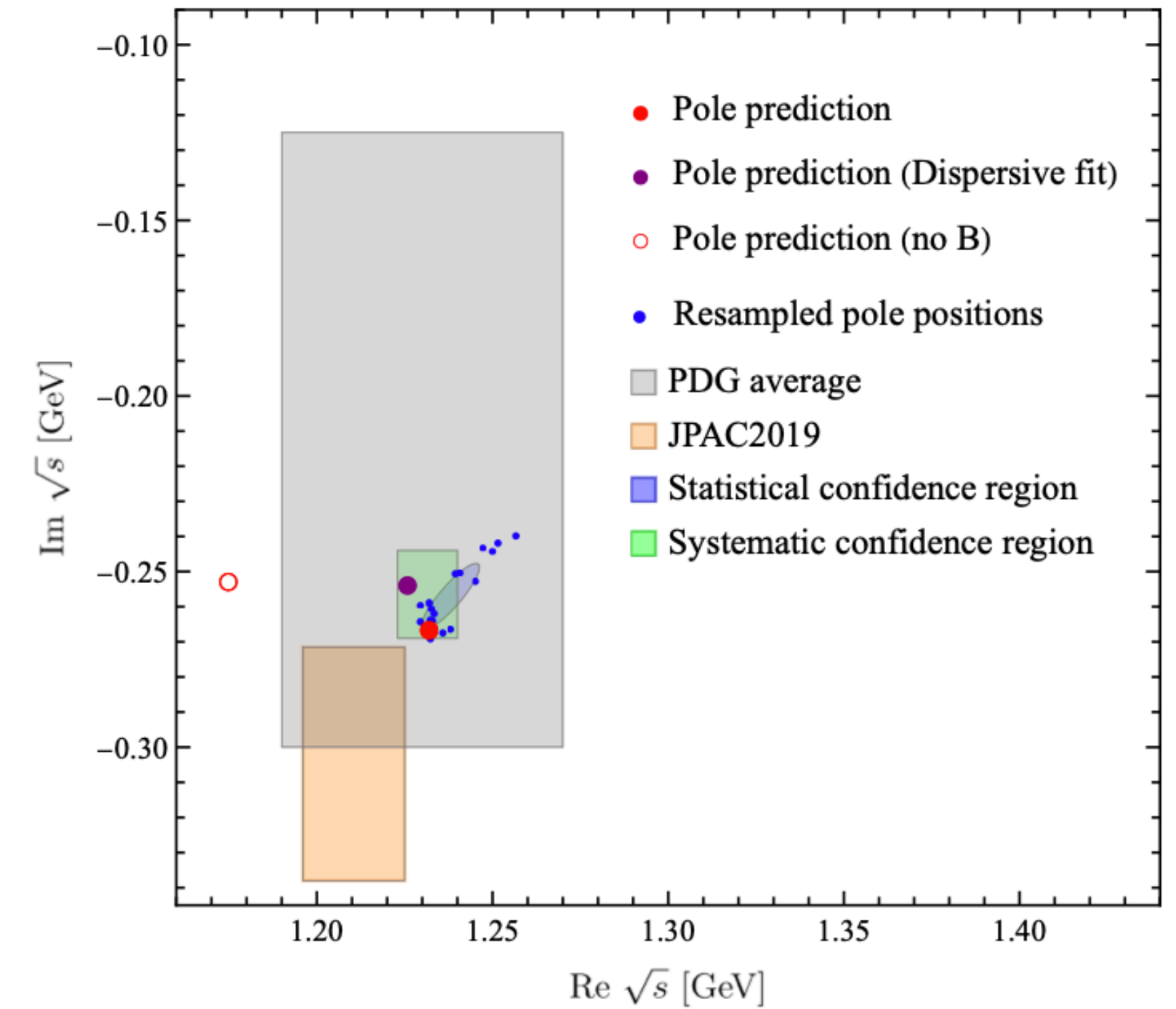
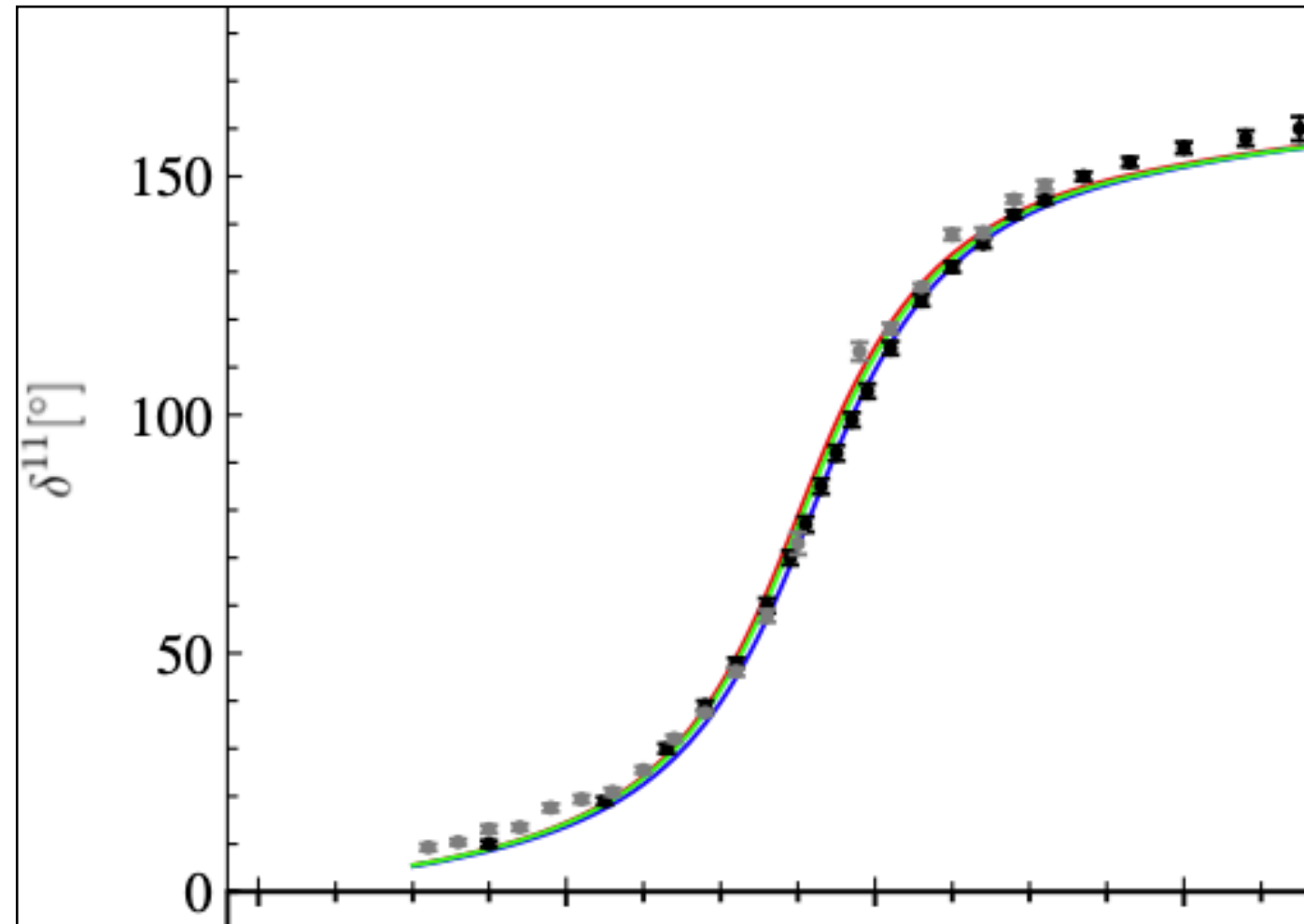
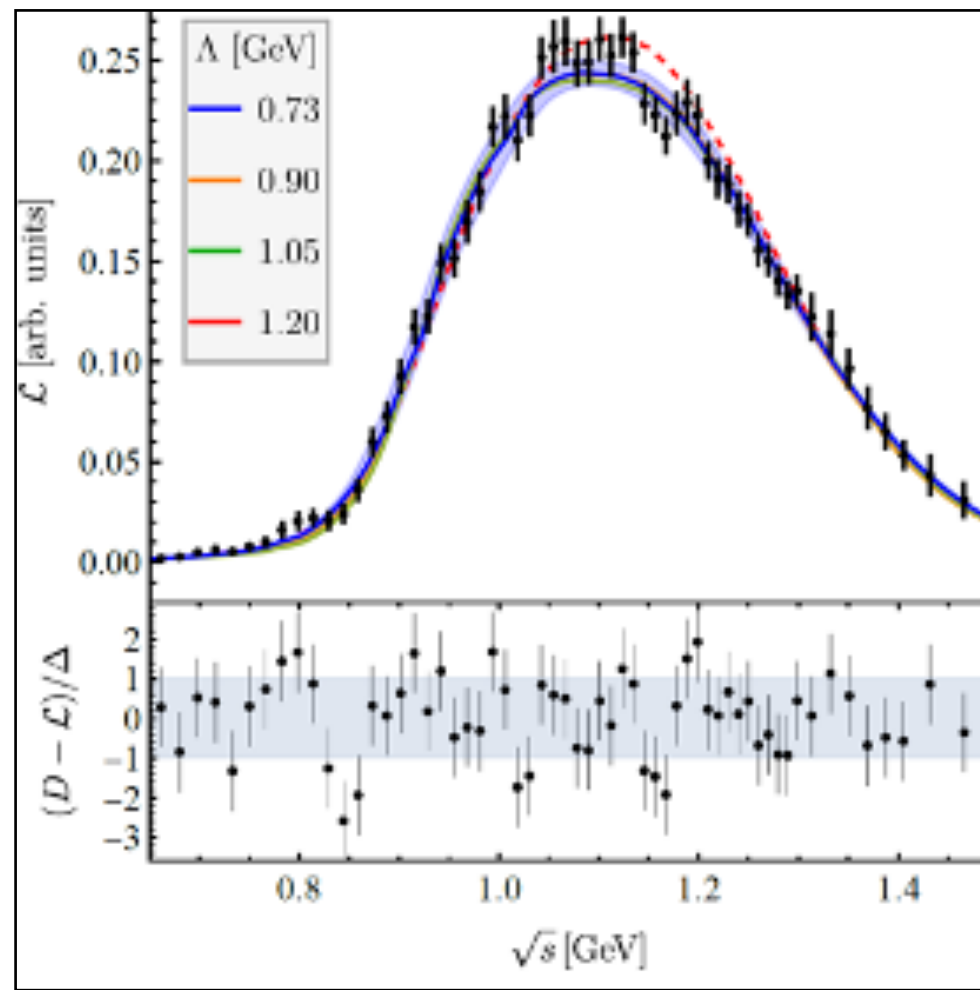
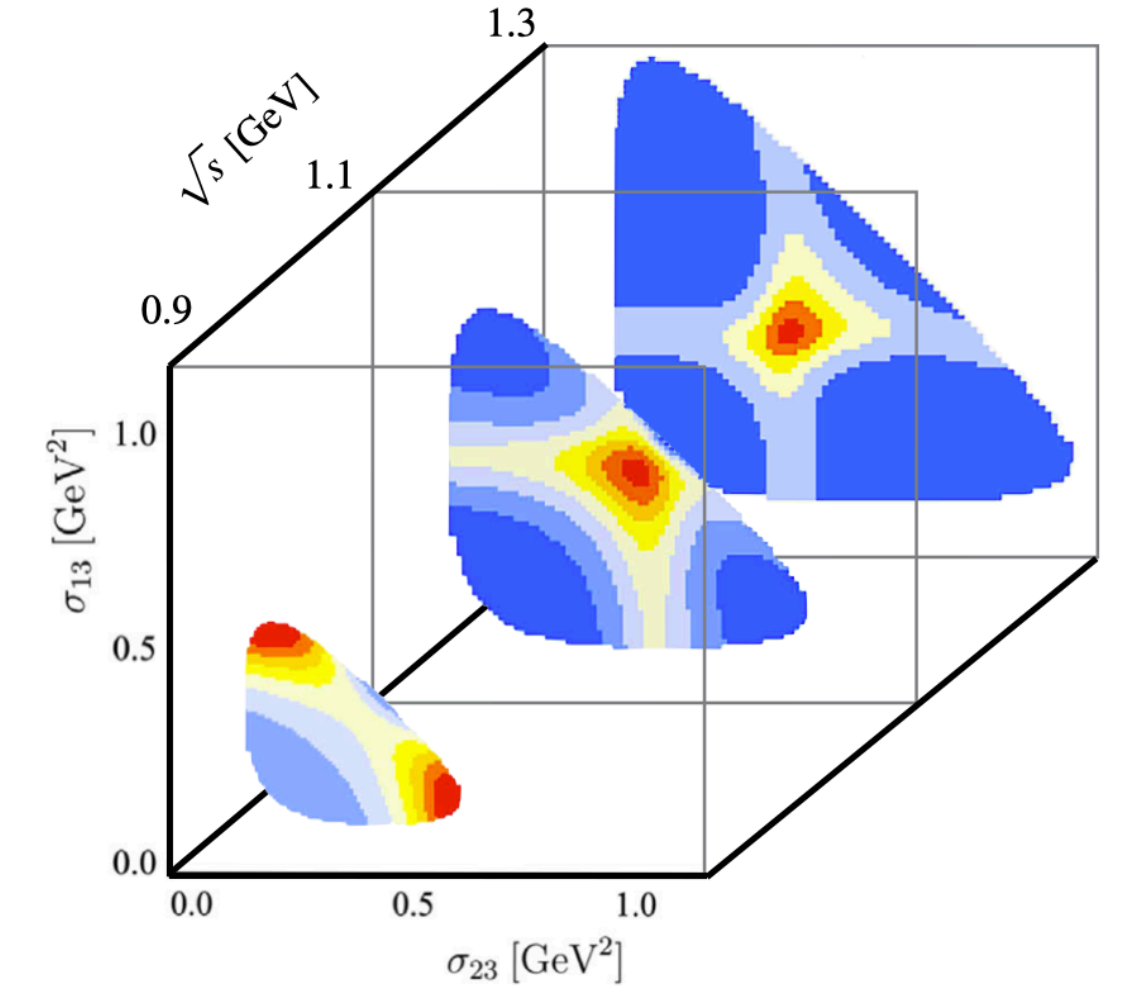


[1] MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)

[2] Related approaches: Wunderlich et al. JHEP 08 (2019); Jackura et al. Eur.Phys.J.C 79 (2019);

a1(1260) Our Universe

$$T^c = B + \textcolor{red}{C} + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B + \textcolor{red}{C})}{2E_l} \frac{1}{\textcolor{green}{\tilde{K}}_n^{-1} - \Sigma_n} T^c$$

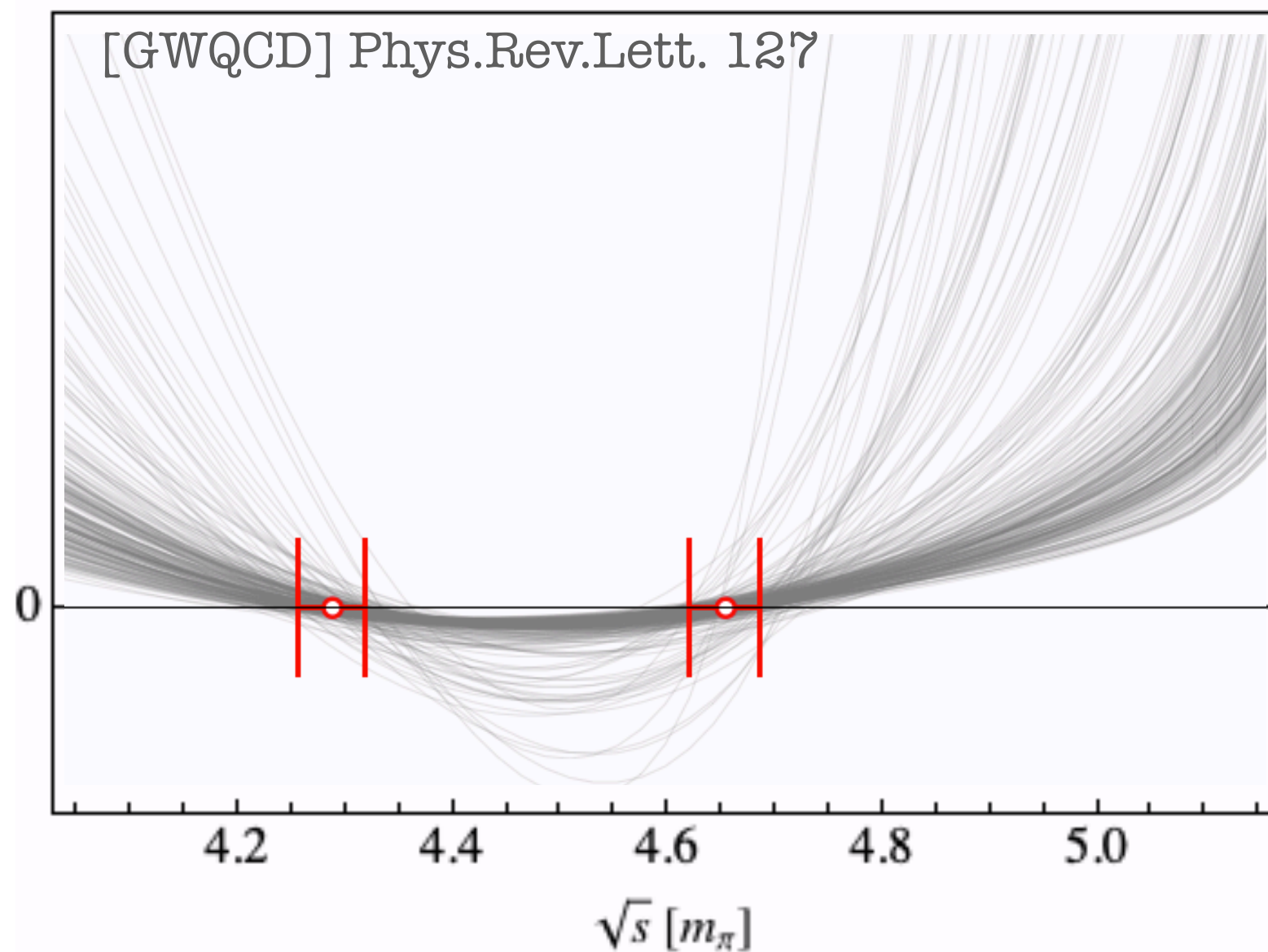
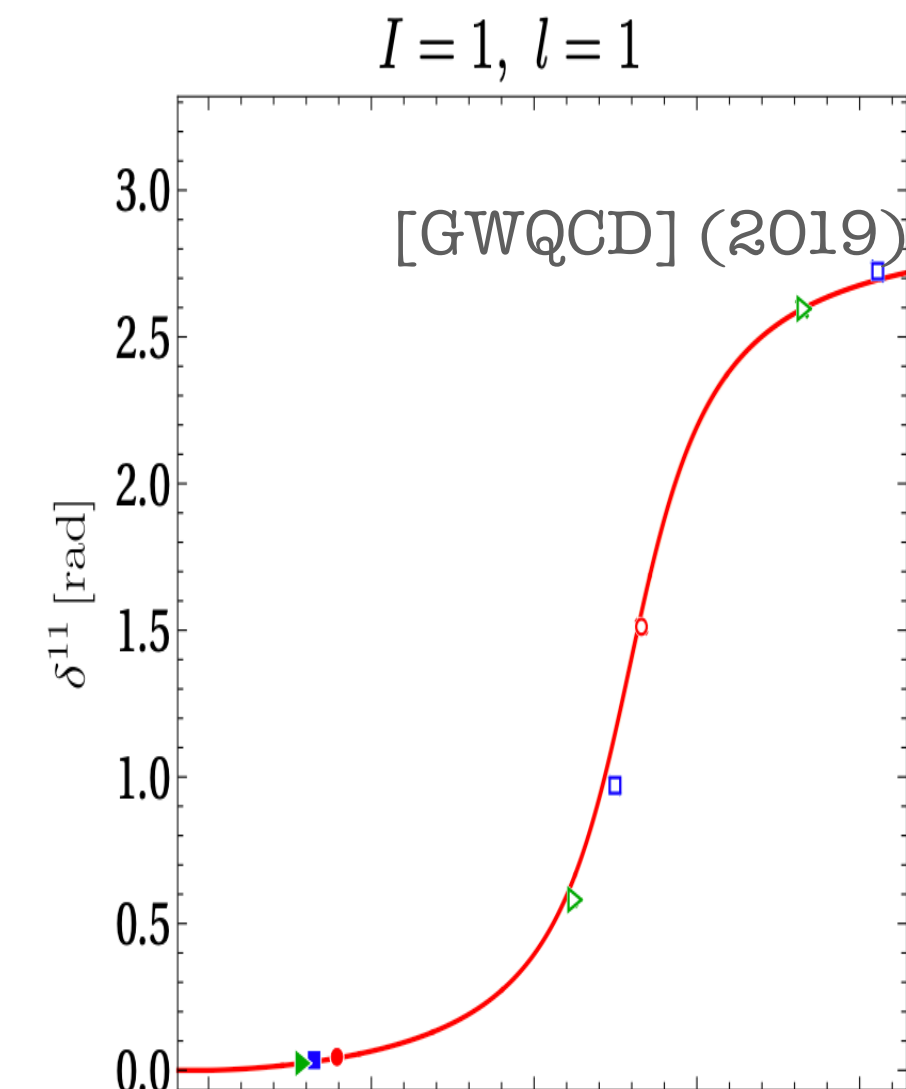


a1(1260)
“Heavy Universe” ($M_\pi : 138 \text{ MeV} \rightarrow 224 \text{ MeV}$)

3-body quantisation condition — FVU [1][2]

$$\det \left[2L^3 E \left(\tilde{K}_n^{-1} - \Sigma \right) - B - \mathbf{C} \right]_{\mathbf{p}'\mathbf{p}} = 0$$

$$T^c = B + \mathbf{C} + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B + \mathbf{C})}{2E_l} \frac{1}{\tilde{K}_n^{-1} - \Sigma_n} T^c$$

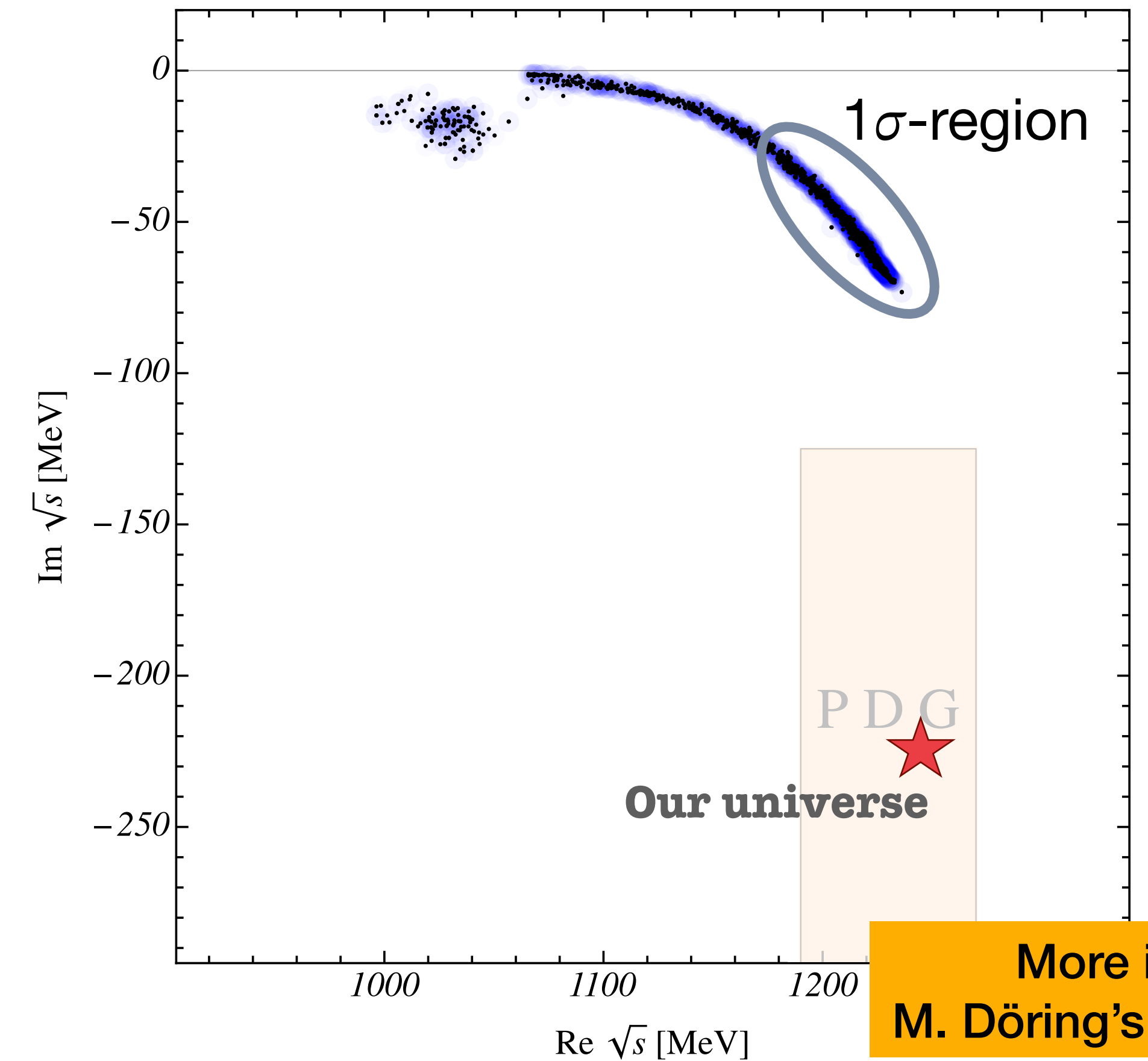
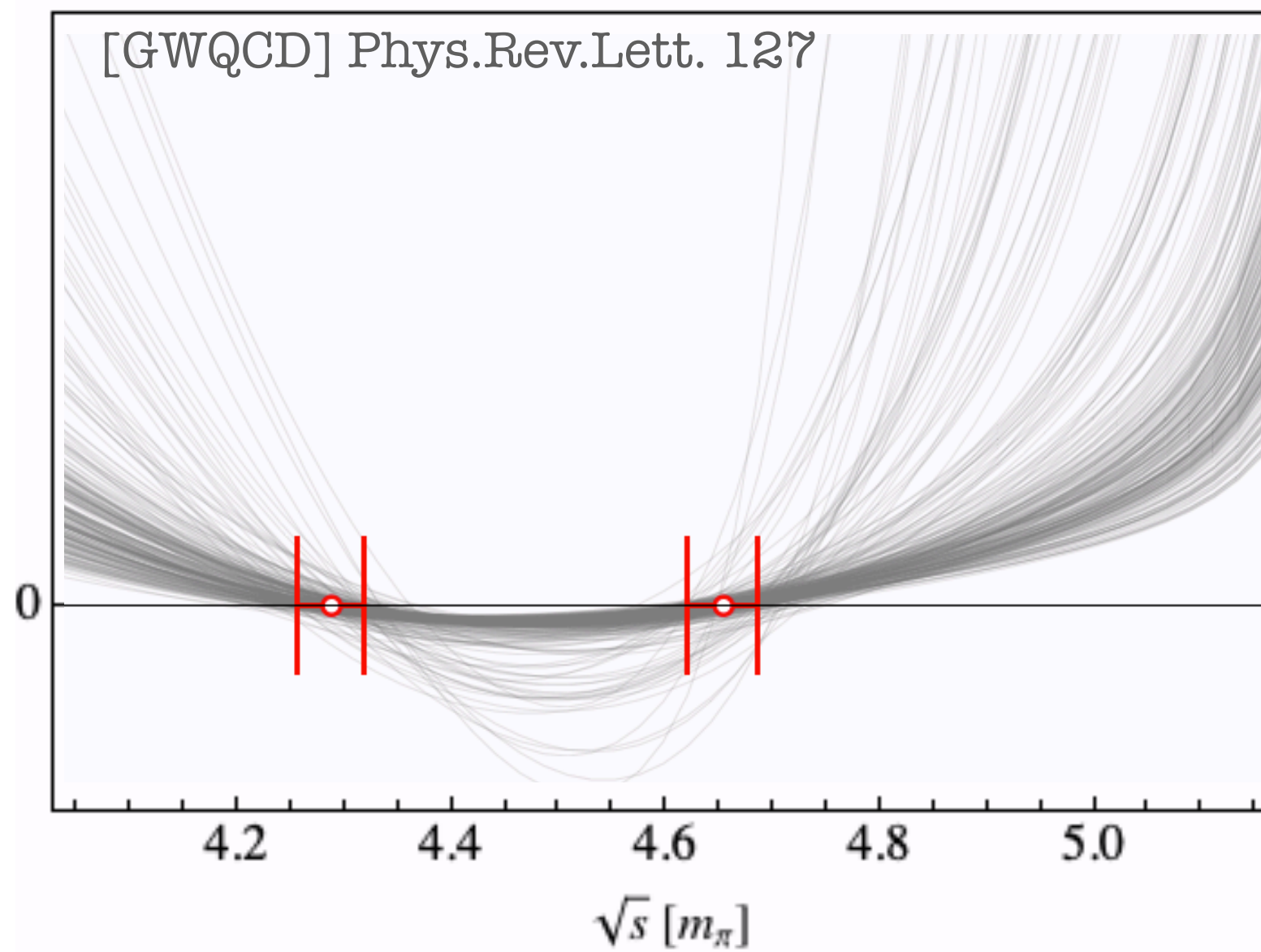
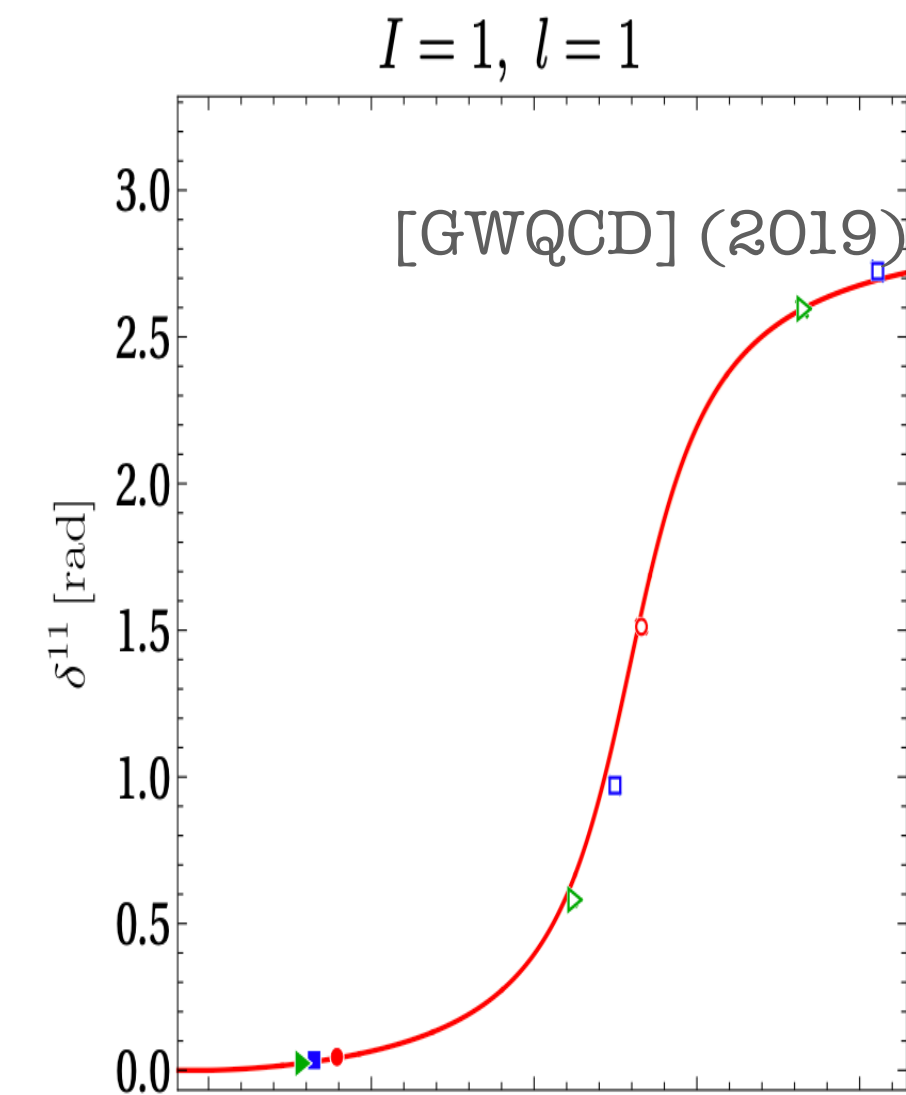


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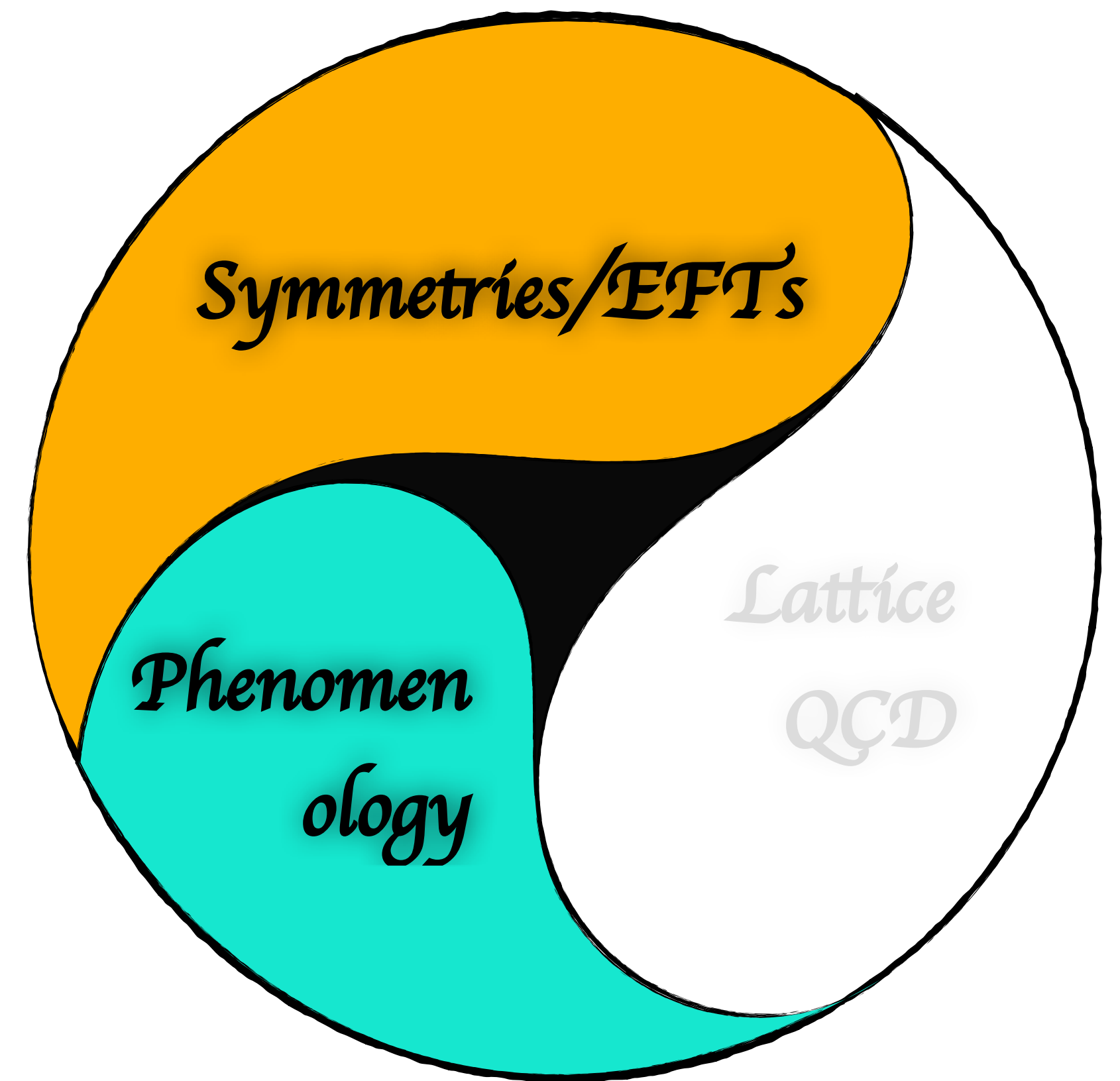
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- [1] MM/Culver/Döring/Alexandru/Lee/Brett/Sadasivan [GWQCD] Phys.Rev.Lett. 127
 [2] **Reviews:** Hansen/Sharpe Ann.Rev.Nucl.Part.Sci. 69 (2019); MM/Döring/Rusetsky Eur.Phys.J.ST 230 (2021);

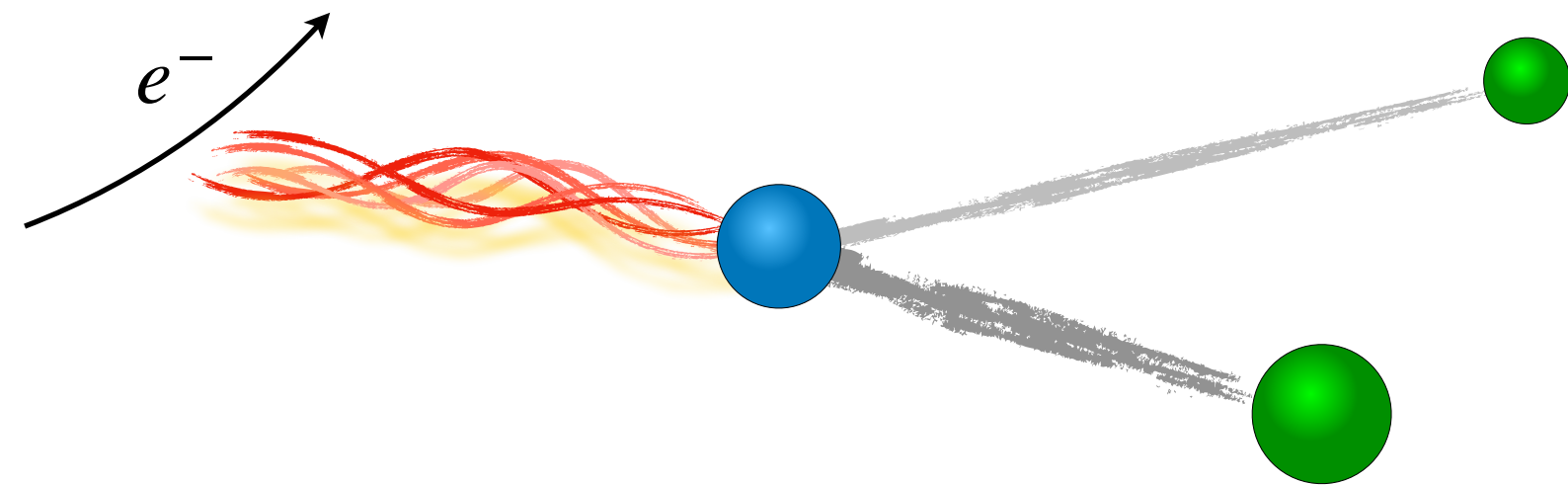
GLOBAL PROPERTIES

MESON-ELECTROPRODUCTION



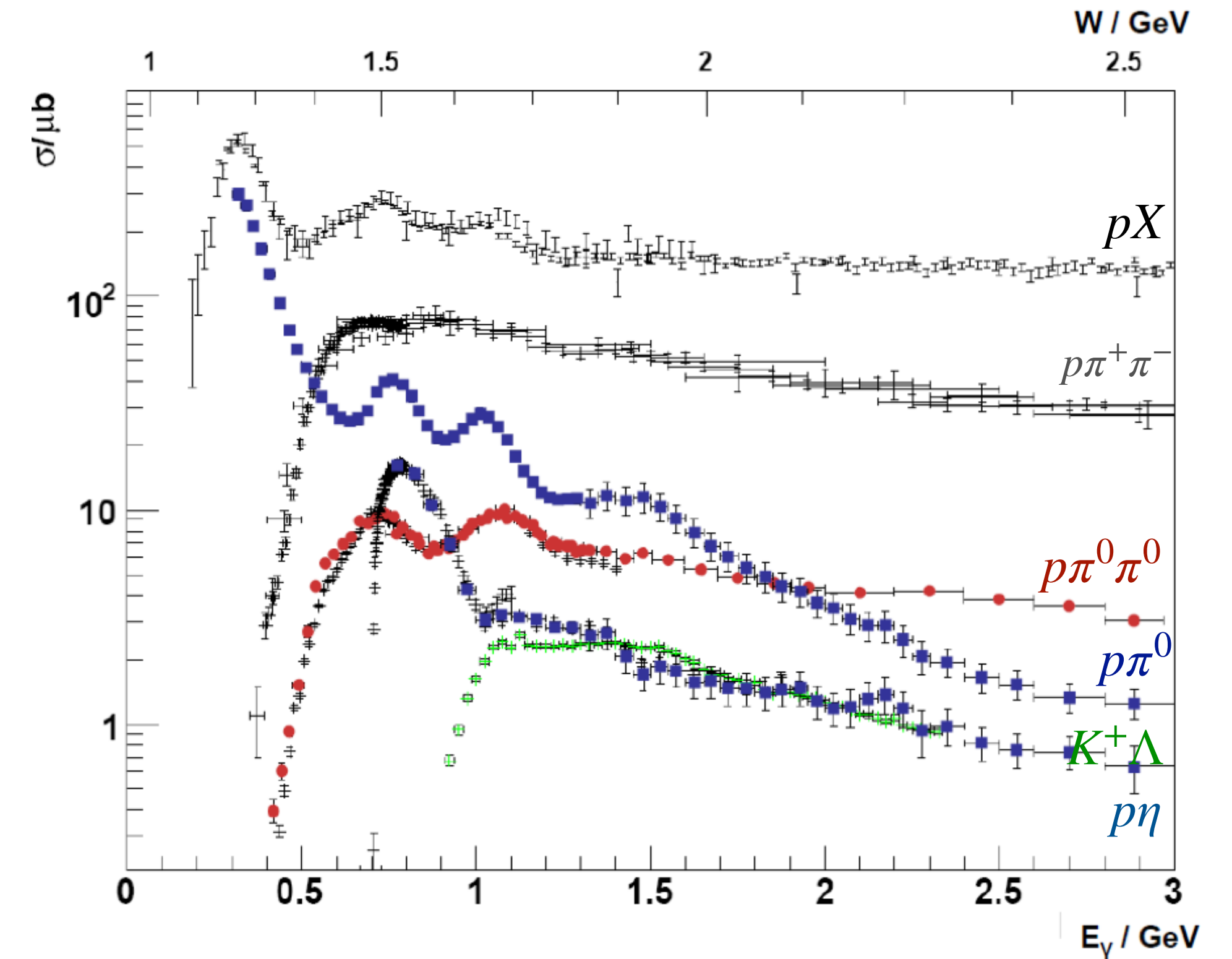
PHOTON-INDUCED EXCITATION

New probe



- 5 independent variables
- Momentum transfer dependence
- State compositeness^[1] (?)

Experimental accessibility

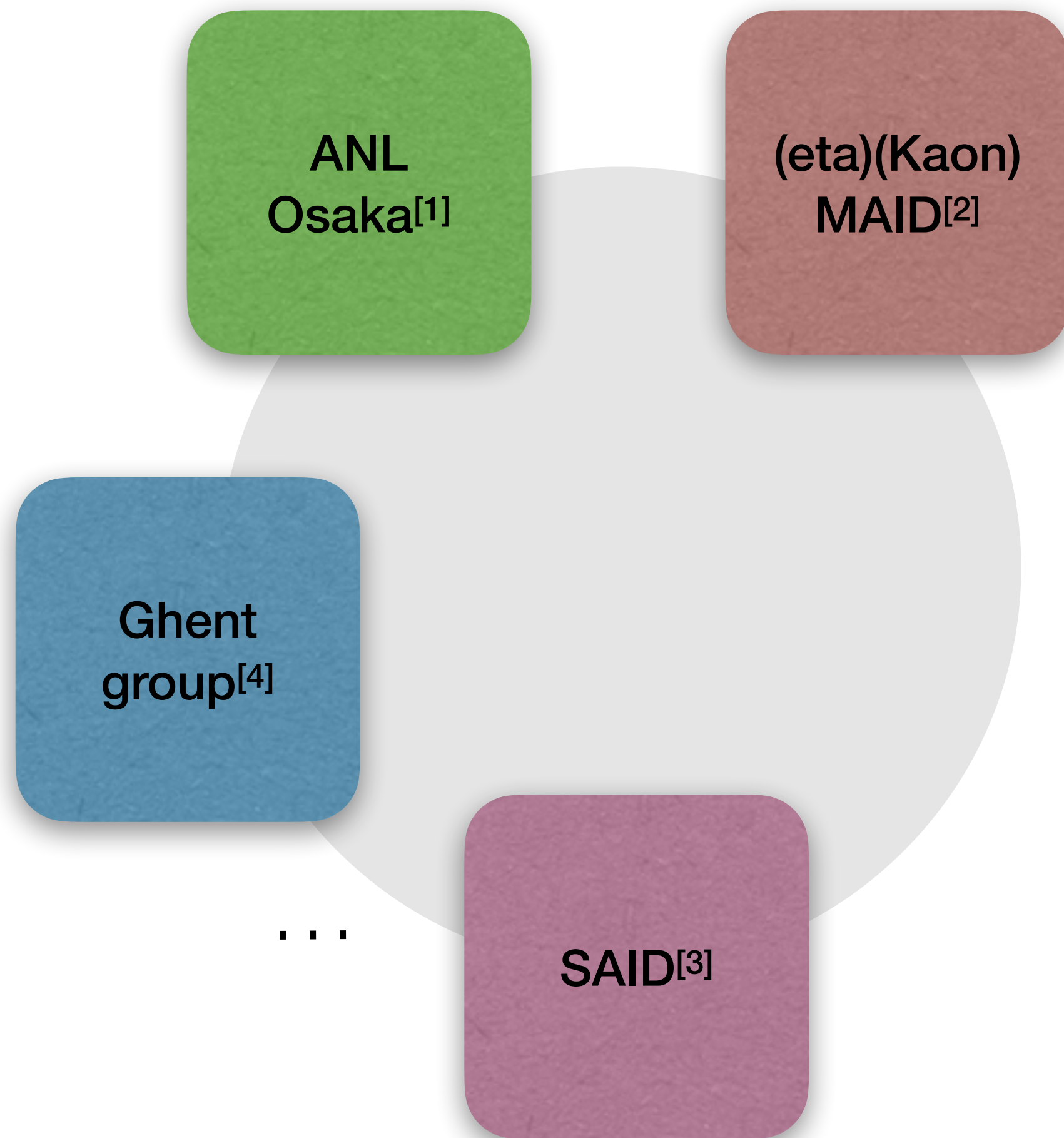


- Large amount of data (10^5)
- more data coming up^[1] ($Q^2=5-12 \text{ GeV}^2$)

[1] e.g., **Review:** Burkert, Roberts, Rev.Mod.Phys. 91 (2019)

[2] Carman, Joo, Mokeev, Few Body Syst. 61, 29 (2020) ... ; [CLAS] Phys.Rev.C 105

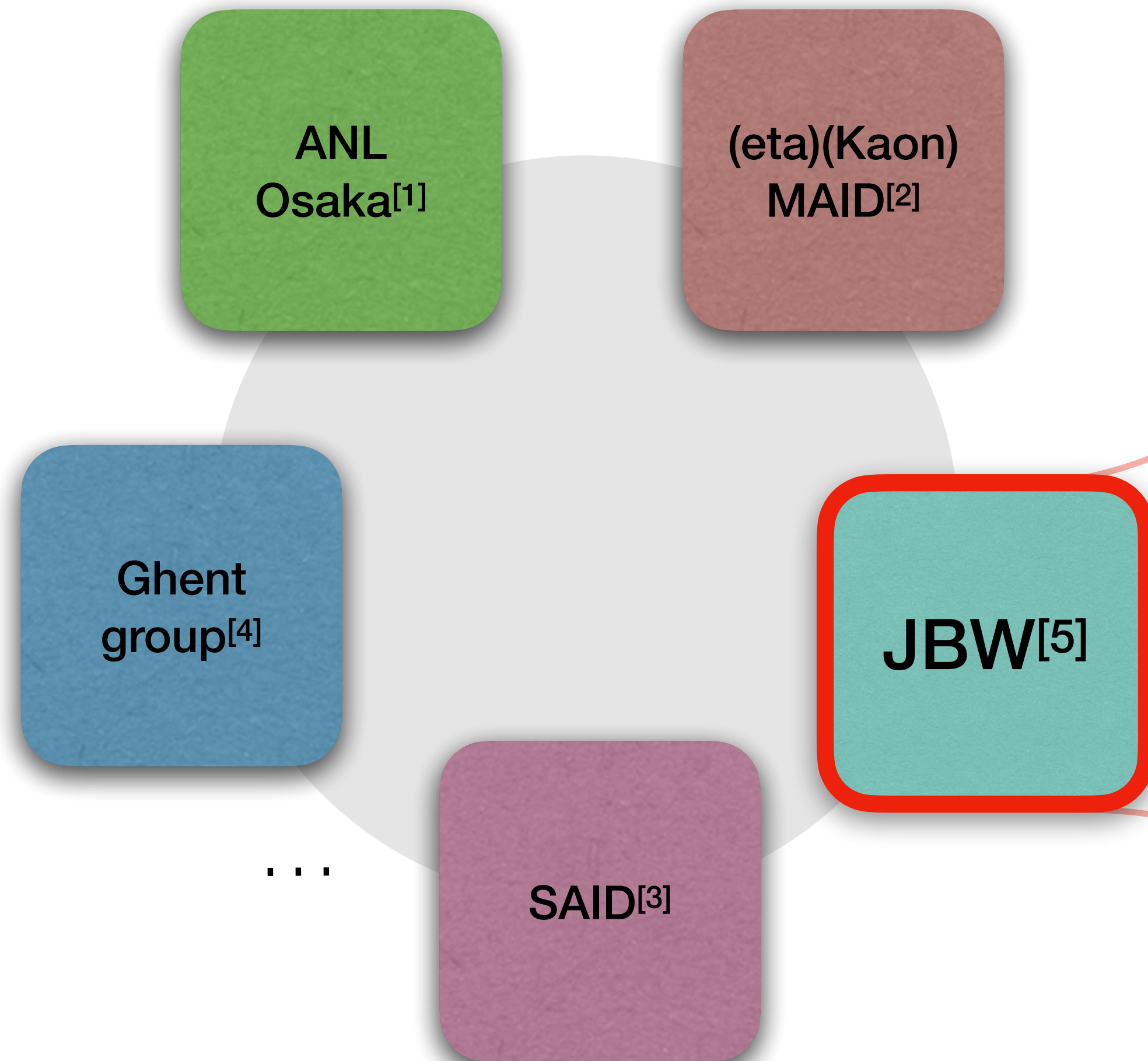
TRANSITION AMPLITUDES: PREVIOUS APPROACHES



Some highlights

- simultaneous description of pion photo-/
electroproduction (MAID)
- low-energy constraints from CHPT (chiral MAID)
- ...

TRANSITION AMPLITUDES: NEW APPROACH



Jülich-Bonn-Washington

- coupled-channel approach (πN , ηN , $K\Lambda$, $K\Sigma$, ...)
- constraints from scattering data and theory:
 - gauge invariance
 - (pseudo)threshold behaviour
 - Final-state unitarity
- web-interface: <https://jbw.phys.gwu.edu/>

More in

M. Döring's talk #2 [6 Jun 2023, 17:20 DAD/5L]

[1] ANL-Osaka PRC 80(2009), Few-Body Syst. 59(2018),... [2] MAID2007, EPJA 34(2007) EtaMAID2018, EPJA 54(2018) [3] SAID, PiN Newsletter 16(2002) [4] Gent group PRC

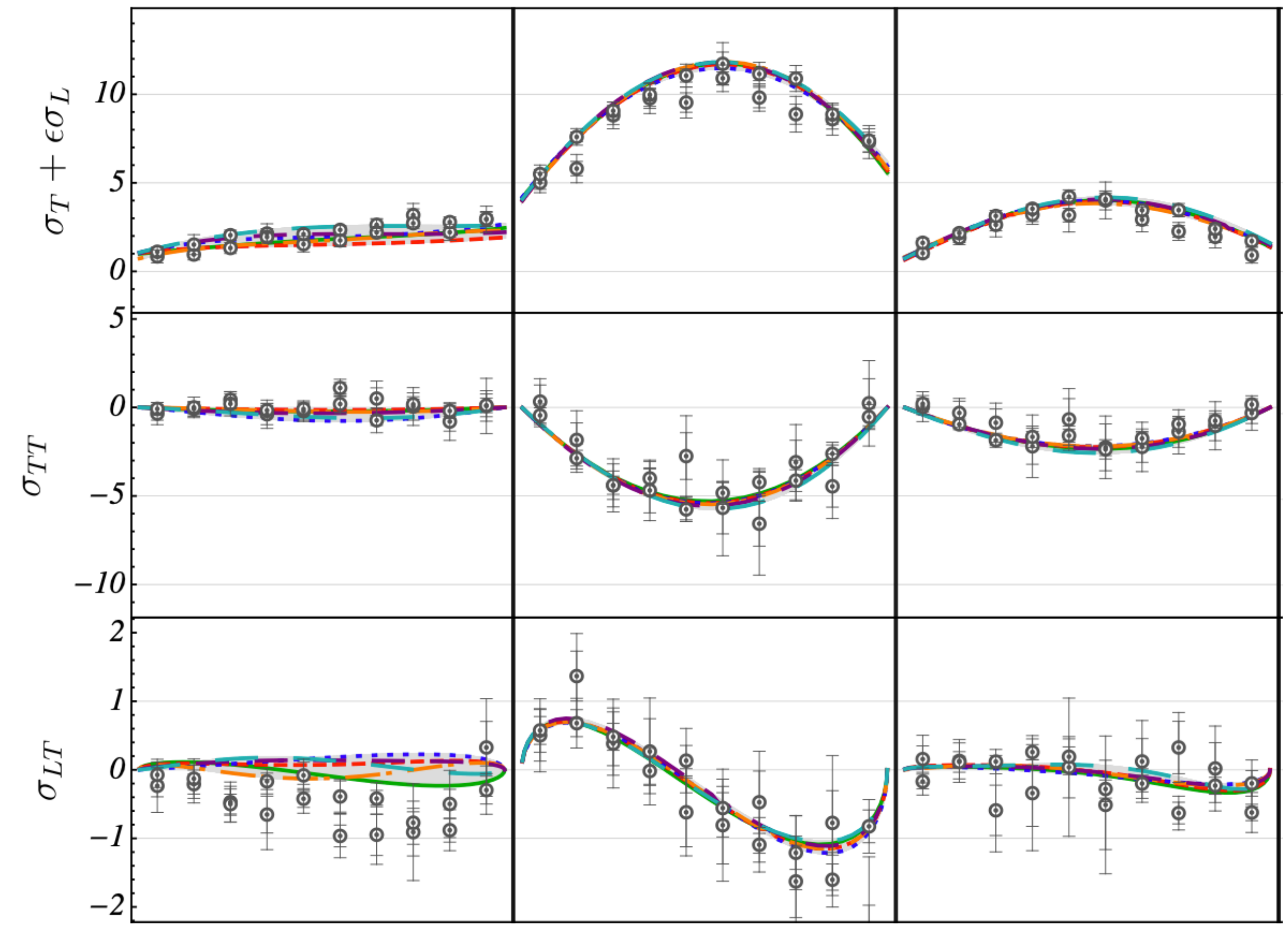
89(2014),... Aznauryan et al., PRC 80(2009), IJMP(2013),...

INTERPOLATIONS AND PREDICTIONS

Fits accomplished: $\pi^0 p / \pi^+ n / \eta p / (+K\Lambda \text{ upcoming})$ ^[1]

Example: Joo data^[2]

- not measured quantities can be estimated
- **interpolator** over observable types and kinematics



[1] [JBW] MM et al. Phys.Rev.C 103 (2021) 6 / Phys.Rev.C 106 (2022) 015201

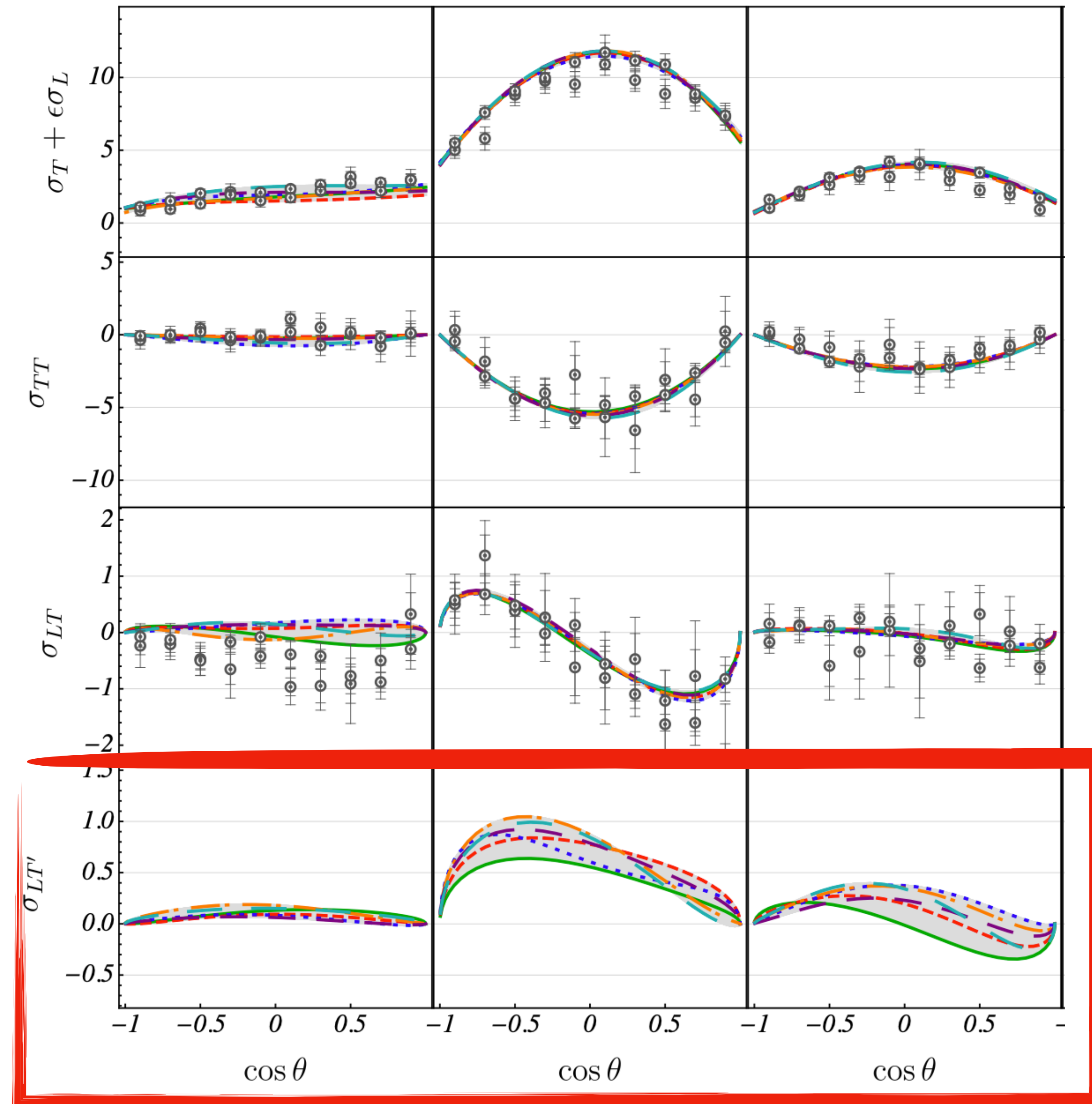
[2] Joo et al. [CLAS] PRC (2003), PRL (2002)

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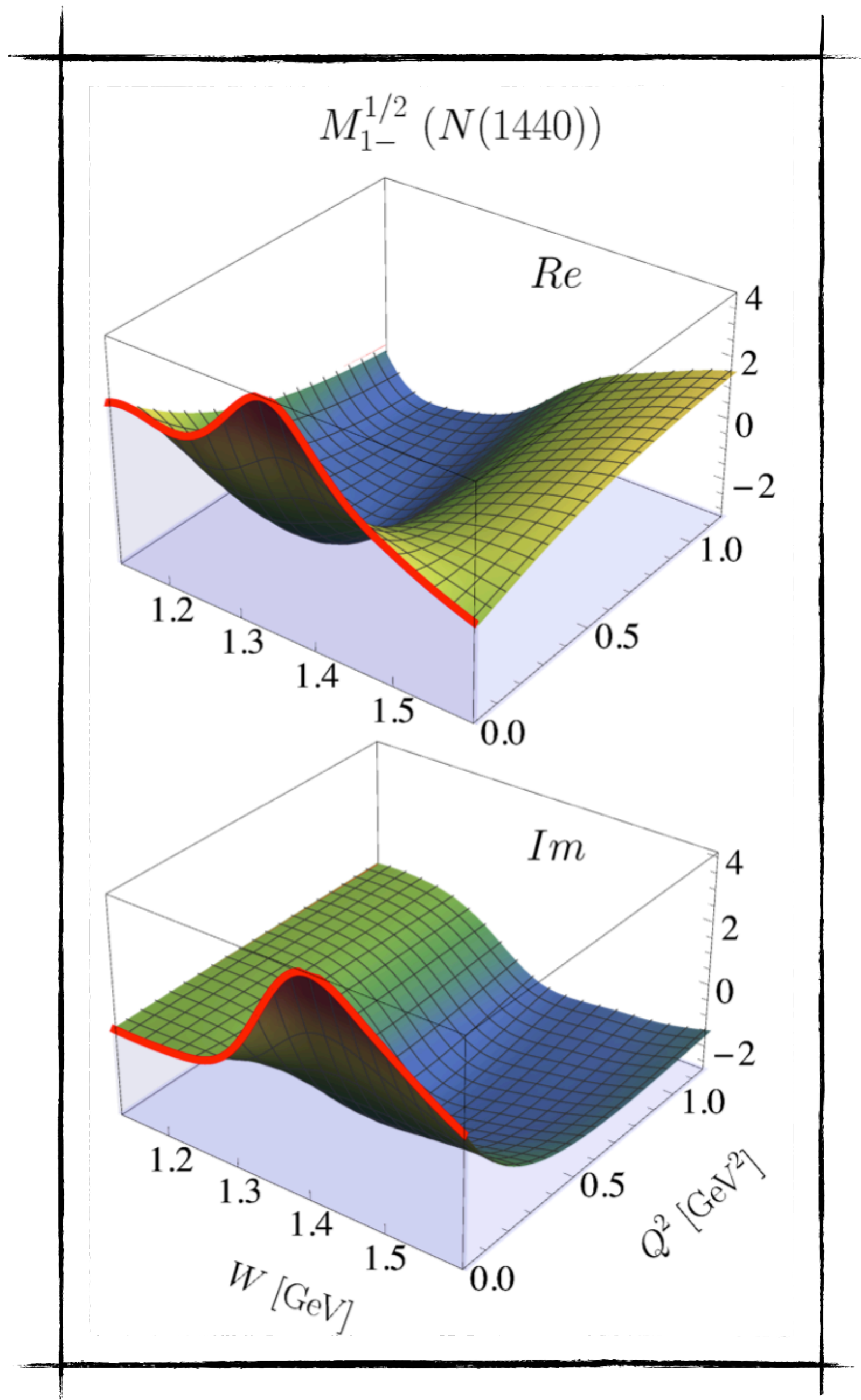
[1] [JBW] MM et al. Phys.Rev.C 103 (2021) 6 / Phys.Rev.C 106 (2022) 015201

[2] Joo et al. [CLAS] PRC (2003), PRL (2002)

INTERPOLATIONS AND PREDICTIONS

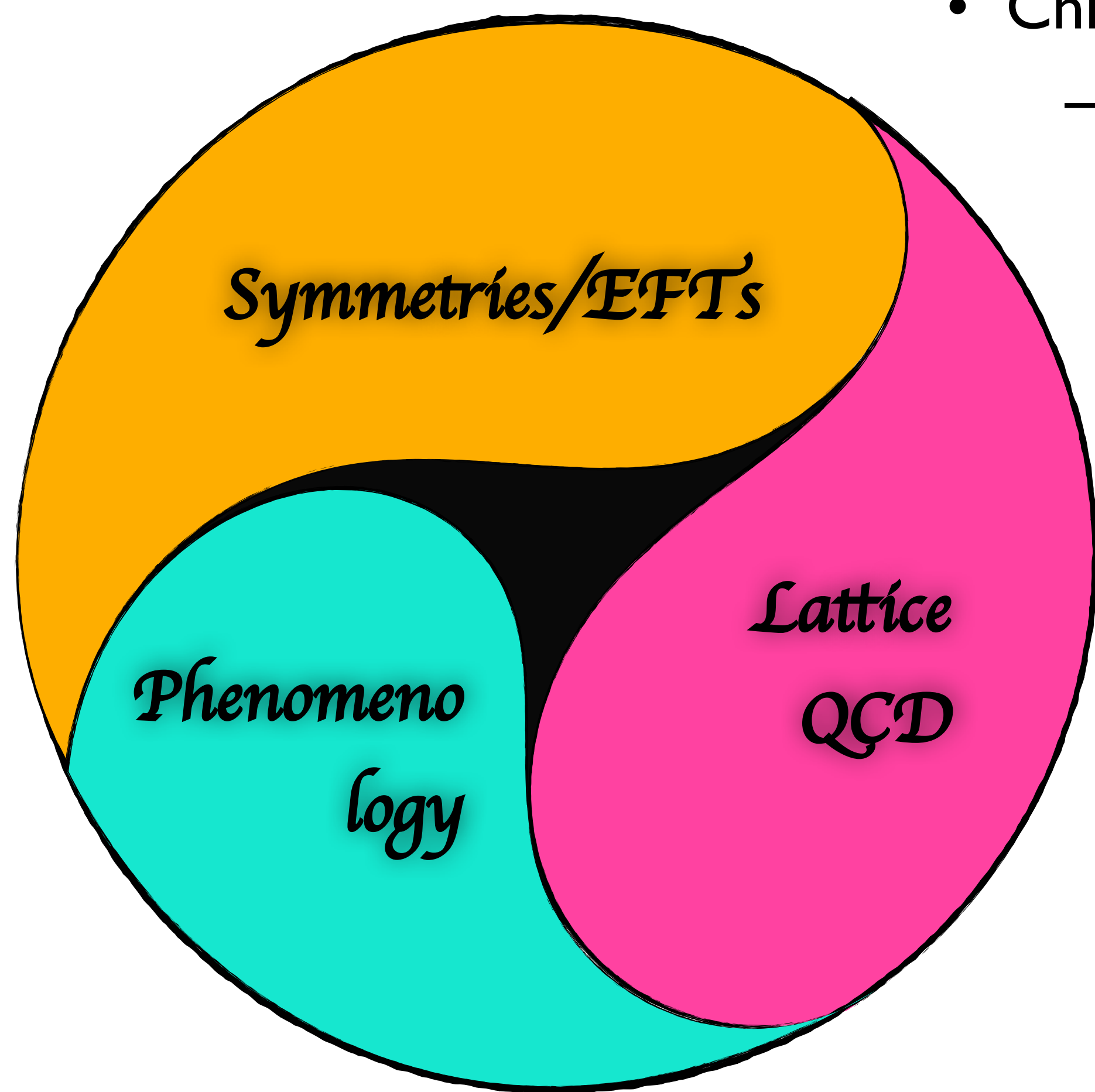
Example: Roper(1440)

- Non-trivial Q^2 behaviour
 - complex structure (**3-body effects, ...**)
 - Helicity couplings (upcoming)



SUMMARY

New synergetic approaches to universal parameters of resonance become available



- Chiral unitary models
 - QCD symmetries constraints to hadron-hadron dynamics
 - Strong predictive power
- Lattice hadron spectroscopy
 - Novel 3-body methodology has matured
 - EFTs and S-matrix theory: bridge to real world physics
- Phenomenological models
 - Link between observables and transition amplitudes

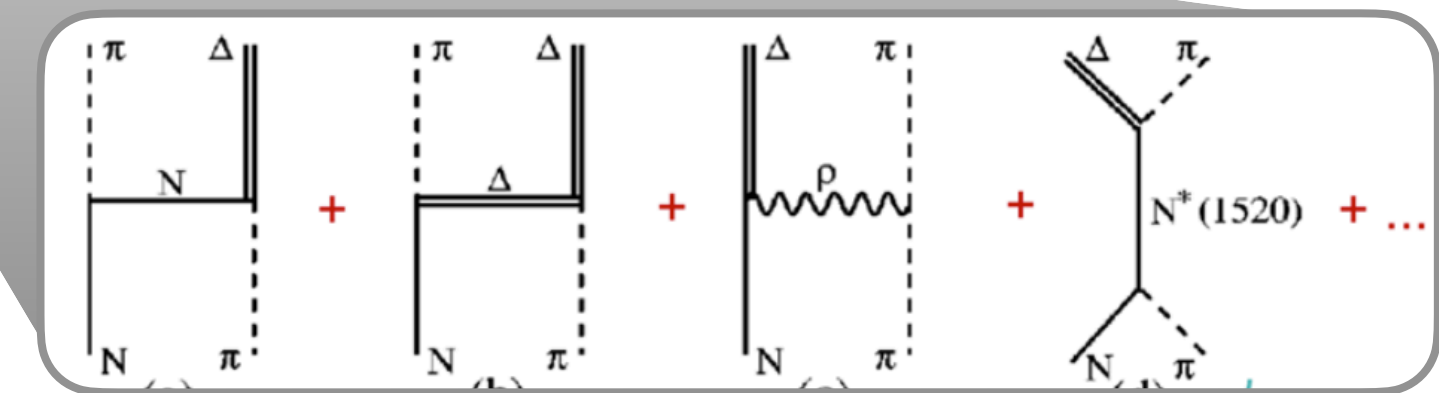
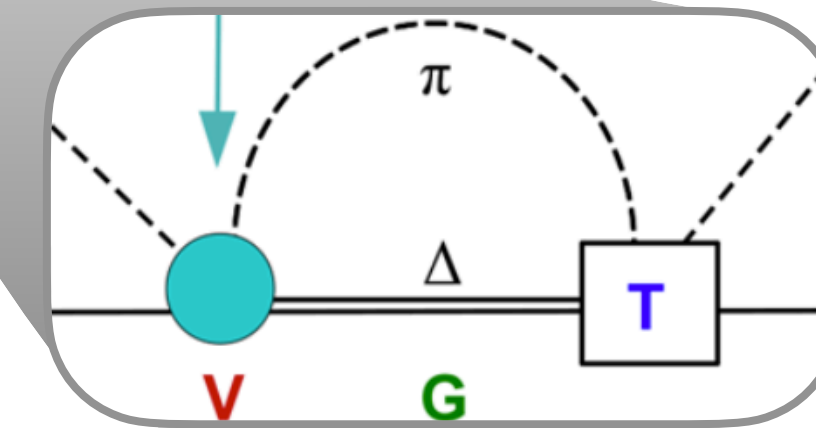
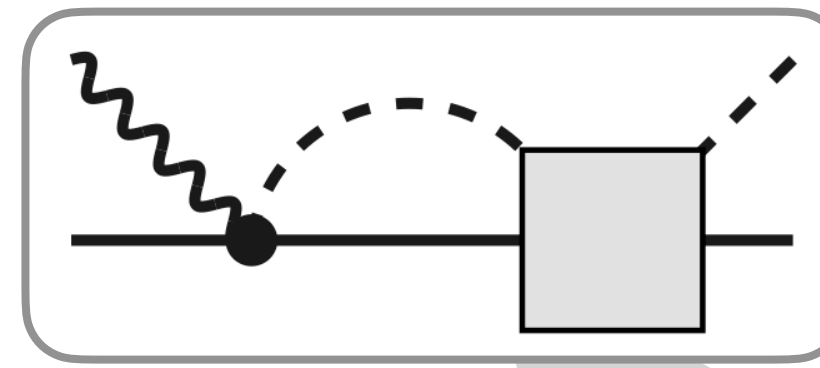
THANK YOU

TRANSITION AMPLITUDE

Final-state unitarity

- Jülich-Bonn dynamical coupled-channel model^I
- Amplitudes fixed from scattering and photoproduction data

$$\pi N \rightarrow xX \text{ and } \gamma N \rightarrow xX \text{ } (\sim 60\text{k data})$$



SYMMETRIES

Five kinematical variables ($3 \cdot (2+3) - 10 = 5$)

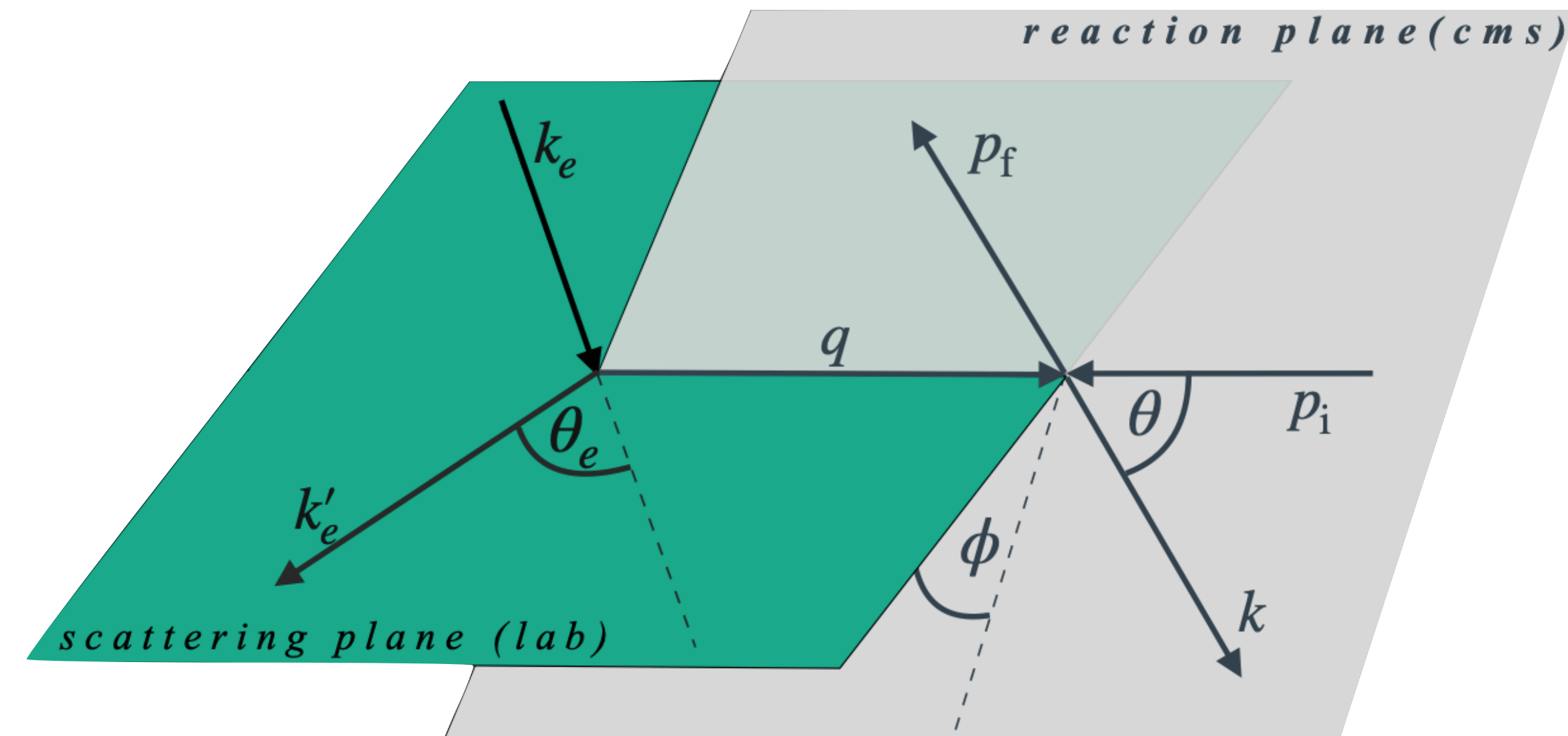
- total energy: **W**
- photon virtuality: **Q^2**
- angles: $\theta_e, \theta, \varphi$

Underlying objects^{1,2}:

- Helicity amplitudes:
- Multipoles:

$$\{H_i(W, Q^2, \theta) \mid i = 1..8\}$$

$$\mathcal{M}_{\ell\pm}(W, Q^2) \in \{E_{\ell\pm}(W, Q^2), L_{\ell\pm}(W, Q^2), M_{\ell\pm}(W, Q^2)\}$$



¹) Chew et al. Phys.Rev. 106 (1957); Denner Phys.Rev. 124 (1961); Berends et al. Nucl.Phys.B 4 (1967);

²) (for explicit formulas) MM et al. Phys.Rev.C 103 (2021)

THEORETICAL CONSTRAINTS

Gauge invariance

- manifest implementation¹ too costly
- instead Ward-Takahashi identity by construction

$$\longrightarrow H_7 = \sum_{i=1}^6 a_i H_i \quad H_8 = \sum_{i=1}^6 b_i H_i$$

Pseudo/threshold constraints:

Siegert's theorem²

- Long-wavelength limit: electric and magnetic multipoles are related

$$\longrightarrow \lim_{k \rightarrow 0} E_{\ell+} = k^{\ell} \dots$$

... fewer parameters needed 👍

$$\longrightarrow L_{\ell\pm} \sim E_{\ell\pm} \quad \textbf{for} \quad q = 0$$

1) Afnan et al.(1995); Kvinikhidze et al.(1999); Haberzettl(19xx-2021); Borasoy et al.
(2007); Ruic et al.(2011); MM et al. (2012); Bruns, Cieplý, MM 2206.08767 [nucl-th]

2) Siegert(1973) Amaldi et al.(1979) Tiator(2016)

MULTIPOLES

$$\mathcal{M}_{\mu\gamma^*}(W, Q^2) = R_{\ell'}(\lambda, q/q_\gamma) \left(V_{\mu\gamma^*}(W, Q^2) + \sum_{\kappa} \int_0^\infty dp p^2 T_{\mu\kappa}^{\text{JUBO}}(p, W) G_\kappa(p, W) V_{\kappa\gamma^*}(p, W, Q^2) \right)$$

$$V_{\mu\gamma}^{\text{JUBO}}(W) e^{-\beta_\mu^0 Q^2/m_p^2} \left(1 + Q^2/m_p^2 \beta_\mu^1 + (Q^2/m_p^2)^2 \beta_\mu^2 \right)$$

👍 Fulfils:

- Final state unitarity / Gauge invariance / Siegert's theorem / Threshold behaviour

👍 Describes:

- Scattering and photoproduction data -- parameters (λ, β) from fits to electroproduction data

🤔 Parametrization dependence due to incomplete data

... even for a truncated complete electroproduction experiment

... in future: Bias-variance tradeoff with statistical criteria

Tiator et al.(2017)

Landay et al., Phys.Rev.D (2019), 1810.00075 [nucl-th]
(2019)

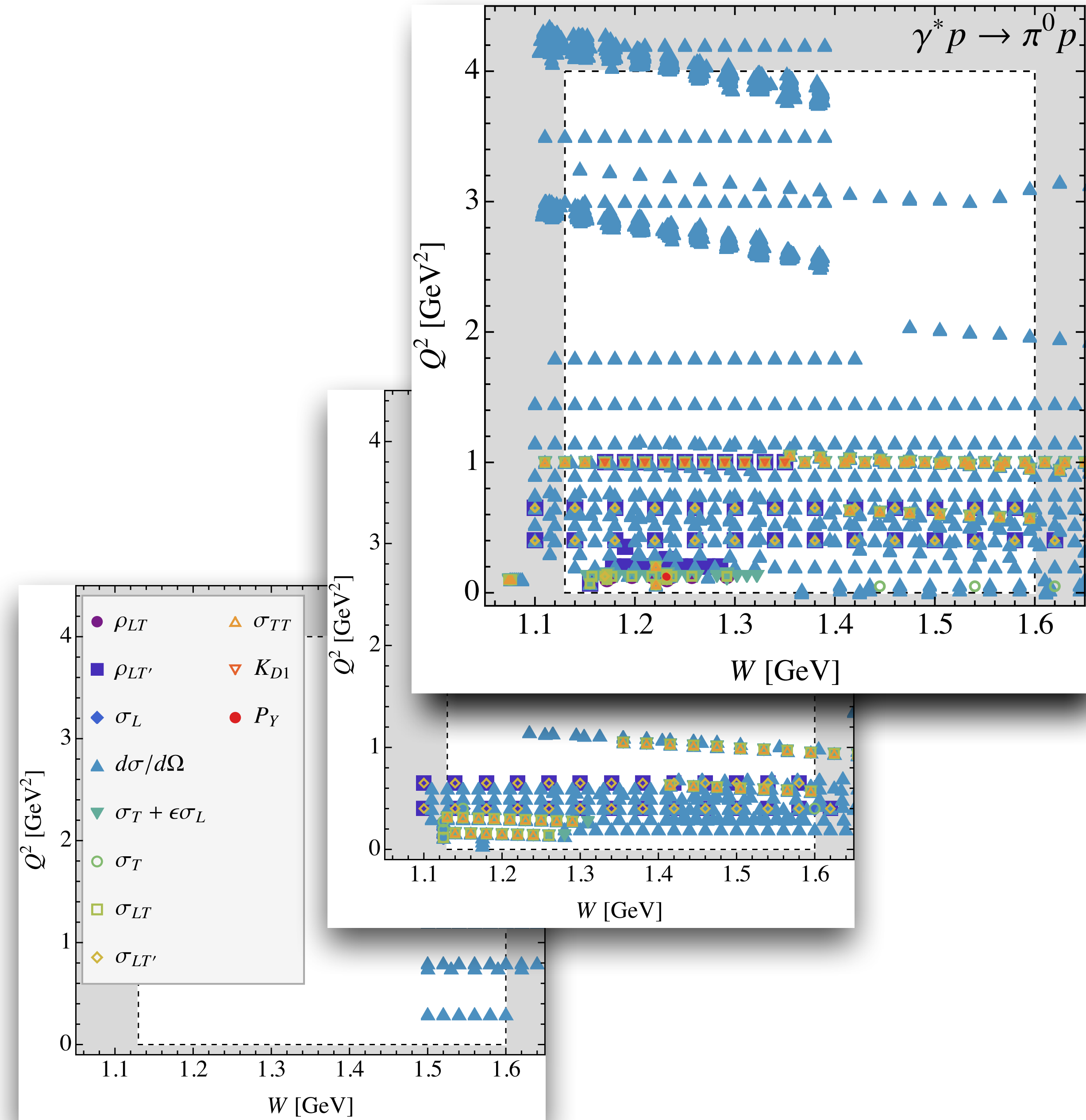
DATA AND PARAMETERS

Experimental data

- 45k($\pi^0 p$) + 37k($\pi^+ n$) + 2k(ηp) = 84k data

Parameters $\{\lambda, \beta\}$

- S/P/D waves
- 26 multipoles $\times$ (10..13 pars) = 257 pars



DATA AND PARAMETERS

Fits¹:

- multiple solutions obtained
- systematic uncertainties studied

	χ^2/dof	$\chi^2_{\pi^0 p/\text{data}}$	$\chi^2_{\pi^+ n/\text{data}}$	$\chi^2_{\eta p/\text{data}}$
$\mathfrak{F}_1^{\text{reg}}$	1.66	1.68	1.61	1.77
$\mathfrak{F}_2^{\text{reg}}$	1.73	1.71	1.71	2.29
$\mathfrak{F}_3^{\text{reg}}$	1.69	1.69	1.66	1.89
$\mathfrak{F}_4^{\text{reg}}$	1.69	1.7	1.64	2.05
$\mathfrak{F}_1^{\text{wt}}$	1.54	1.74	1.63	1.25
$\mathfrak{F}_2^{\text{wt}}$	1.63	1.82	1.79	1.27

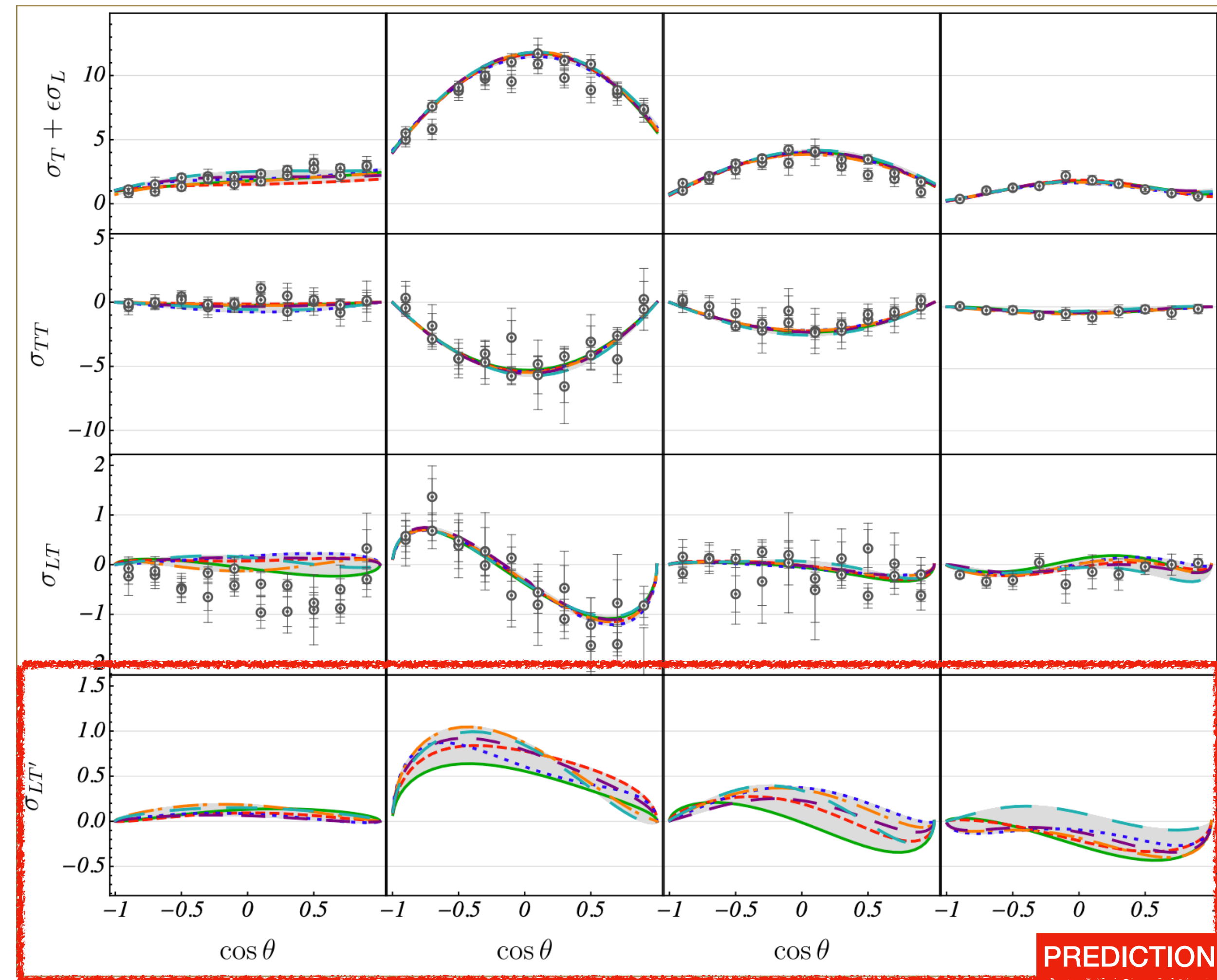
1) [JBW] MM et al. Phys.Rev.C 103 (2021) 6; [Phys.Rev.C 106 \(2022\) 015201](#)

GLOBAL FEATURES

Predictive power

- example: Joo data²
- not measured quantities can be estimated:
 - **interpolator** over observable types and kinematics
 - already usable through web-interface

<https://jbw.phys.gwu.edu/>



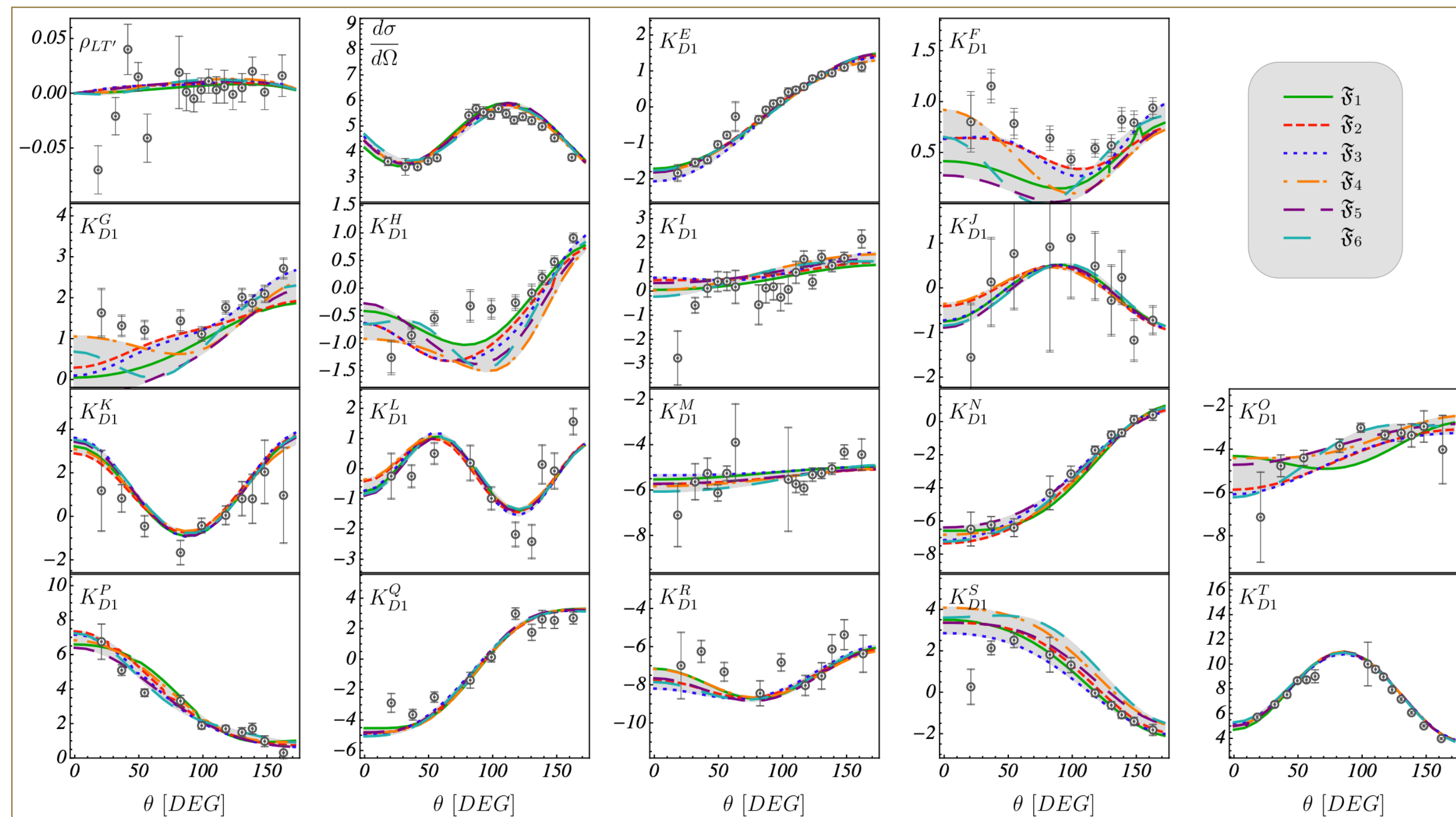
1) [JBW] MM et al. Phys.Rev.C 103 (2021) 6; Phys.Rev.C 106 (2022) 015201

2) Joo et al. [CLAS] PRC (2003), PRL (2002)

RESULTS

πN data fits¹:

- all strategies converge
- different minima (systematic uncertainties)
- Kelly data²



1) [JBW] MM et al. Phys.Rev.C 103 (2021) 6; Phys.Rev.C 106 (2022) 015201

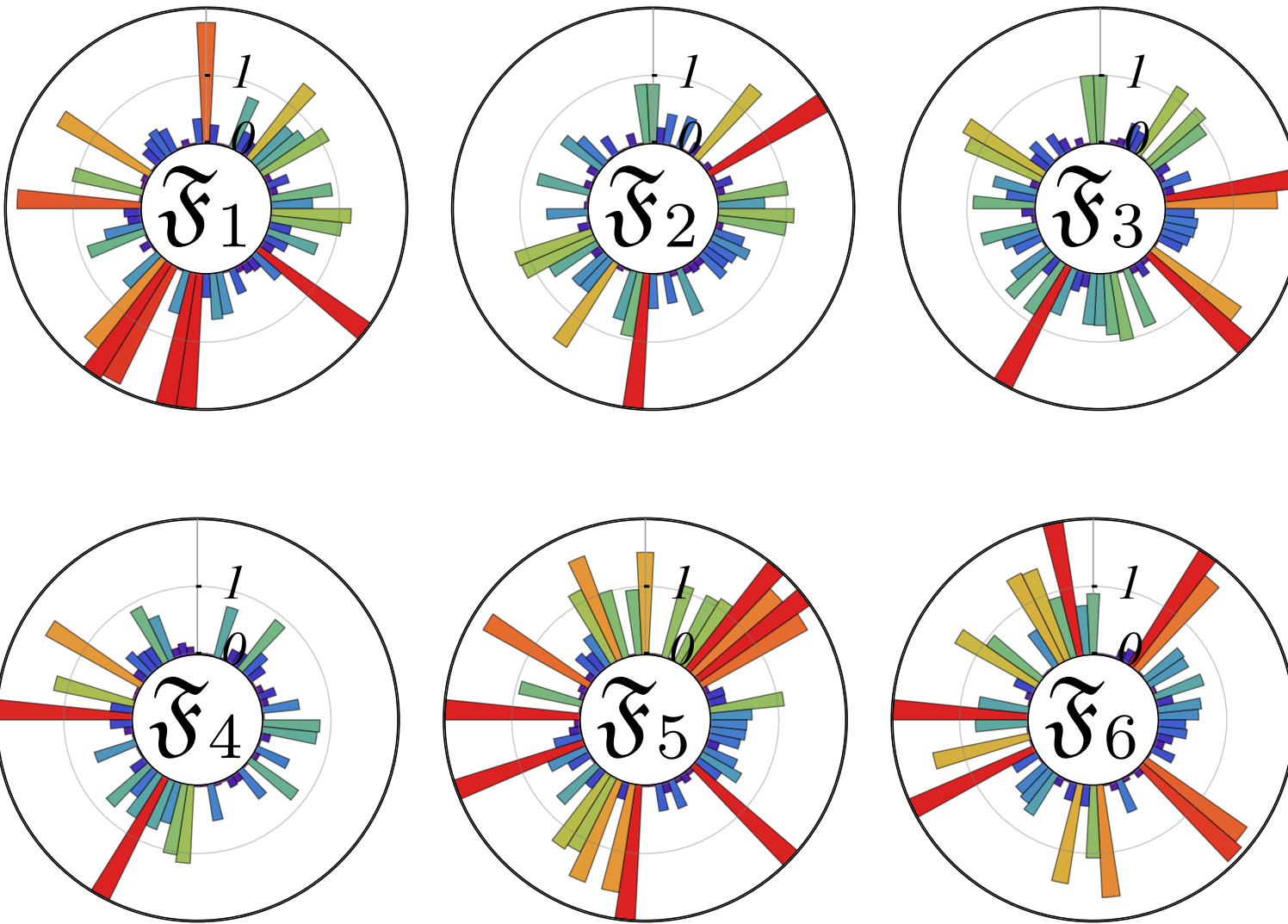
2) Jefferson Lab Hall A Collaboration Phys.Rev.Lett. 95 (2005) 102001

RESULTS

Fits¹:

- all strategies converge
- different minima (systematic uncertainties)

Fit	σ_L		$d\sigma/d\Omega$		$\sigma_T + \epsilon\sigma_L$		σ_T		σ_{LT}		$\sigma_{LT'}$		σ_{TT}		K_{D1}		P_Y		ρ_{LT}		$\rho_{LT'}$		χ^2_{dof}
	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	$\pi^0 p$	$\pi^+ n$	
$\mathfrak{F}_1$	–	9	65355	53229	870	418	87	88	1212	133	862	762	4400	251	4493	–	234	–	525	–	3300	10294	1.77
$\mathfrak{F}_2$	–	4	69472	55889	1081	619	65	78	1780	150	1225	822	4274	237	4518	–	325	–	590	–	3545	10629	1.69
$\mathfrak{F}_3$	–	8	66981	54979	568	388	84	95	1863	181	1201	437	3934	339	4296	–	686	–	687	–	3556	9377	1.81
$\mathfrak{F}_4$	–	22	63113	52616	562	378	153	107	1270	146	1198	1015	4385	218	5929	–	699	–	604	–	3548	11028	1.78
$\mathfrak{F}_5$	–	20	65724	53340	536	528	125	81	1507	219	1075	756	4134	230	5236	–	692	–	554	–	3580	11254	1.81
$\mathfrak{F}_6$	–	18	71982	58434	1075	501	29	68	1353	135	1600	1810	3935	291	5364	–	421	–	587	–	3932	11475	1.78



1) [JBW] MM et al. Phys.Rev.C 103 (2021) 6; Phys.Rev.C 106 (2022) 015201

MULTIPOLES

Observable (e.g. cross section)

$$\frac{d\sigma^\nu}{d\Omega}(W, Q^2, \epsilon, \theta, \phi) = \sigma_T + \epsilon\sigma_L + \sqrt{2\epsilon(1+\epsilon)}\sigma_{LT}\cos\phi + \dots$$

Structure functions

$$\sigma_T(W, Q^2, \theta) = k/q_\gamma \left(|H_1|^2 + |H_2|^2 + |H_3|^2 + |H_4|^2 \right) / 2, \dots$$

Helicity amplitudes

$$H_1(W, Q^2, \theta) = \sin\theta \cos\theta / 2 (-\mathcal{F}_3 - \mathcal{F}_4) / \sqrt{2}, \dots$$

CGLN amplitudes

$$\mathcal{F}_1(W, Q^2, \theta) = \sum_{\ell \geq 0} \ell M_{\ell+}(W, Q^2) P'_{\ell+1}(\cos\theta) + \dots$$

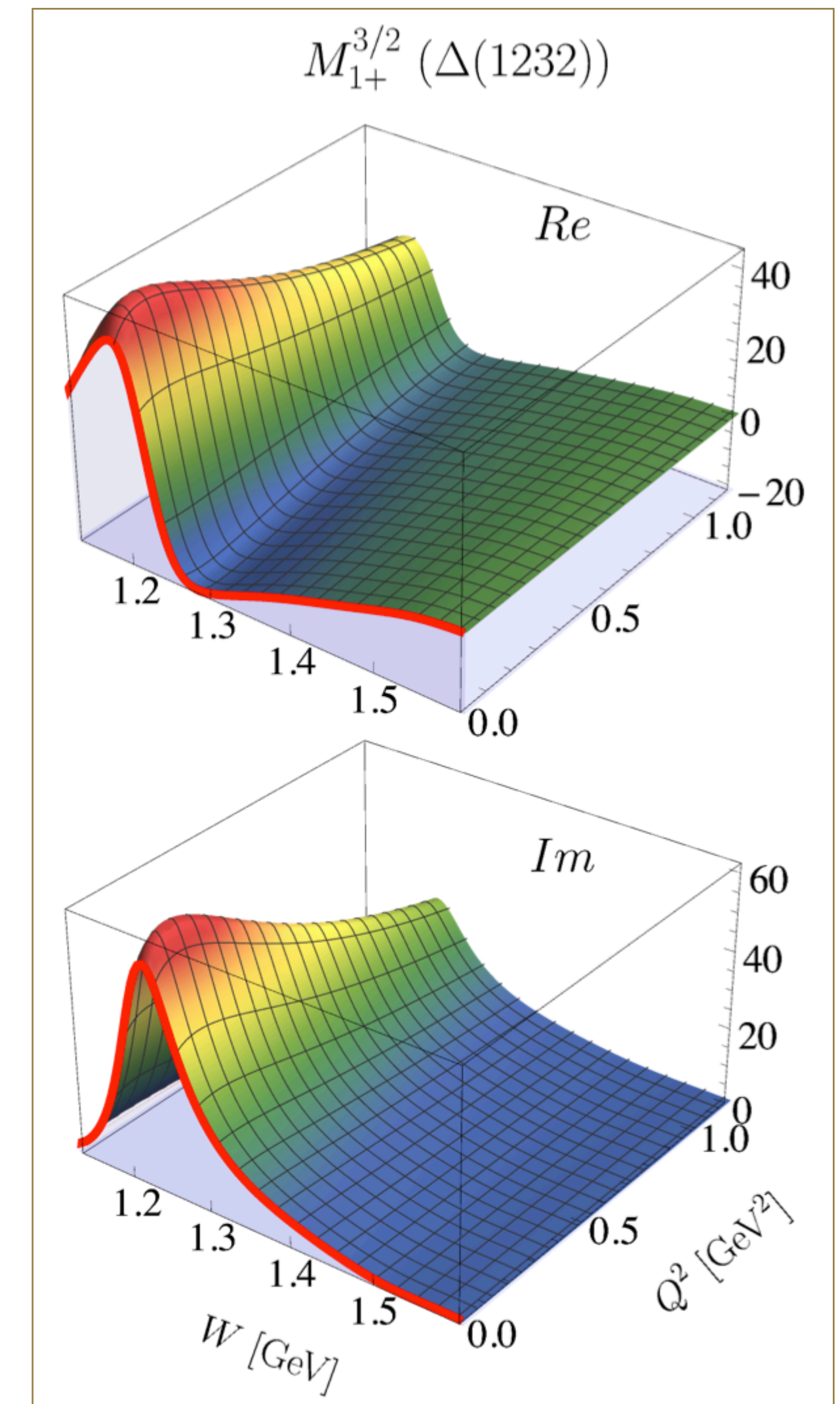
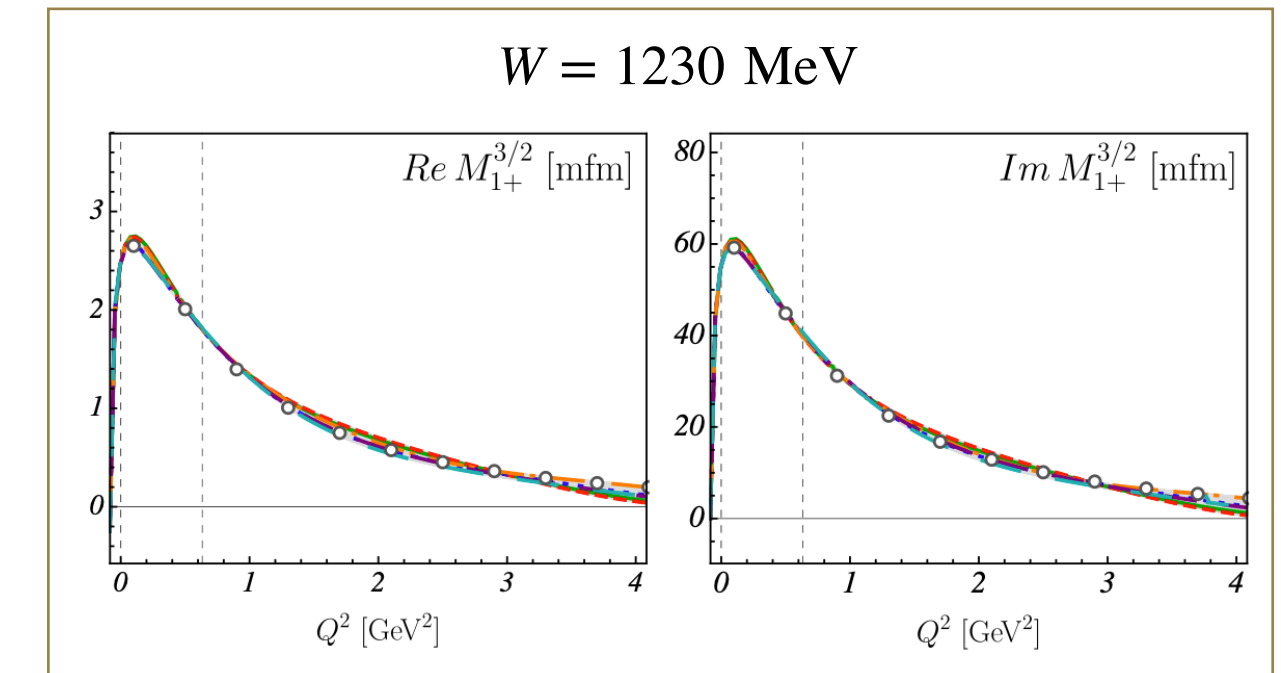
Multipoles

$$\{E_{\ell\pm}(W, Q^2), L_{\ell\pm}(W, Q^2), M_{\ell\pm}(W, Q^2)\}$$

RESULTS

Delta(1232):

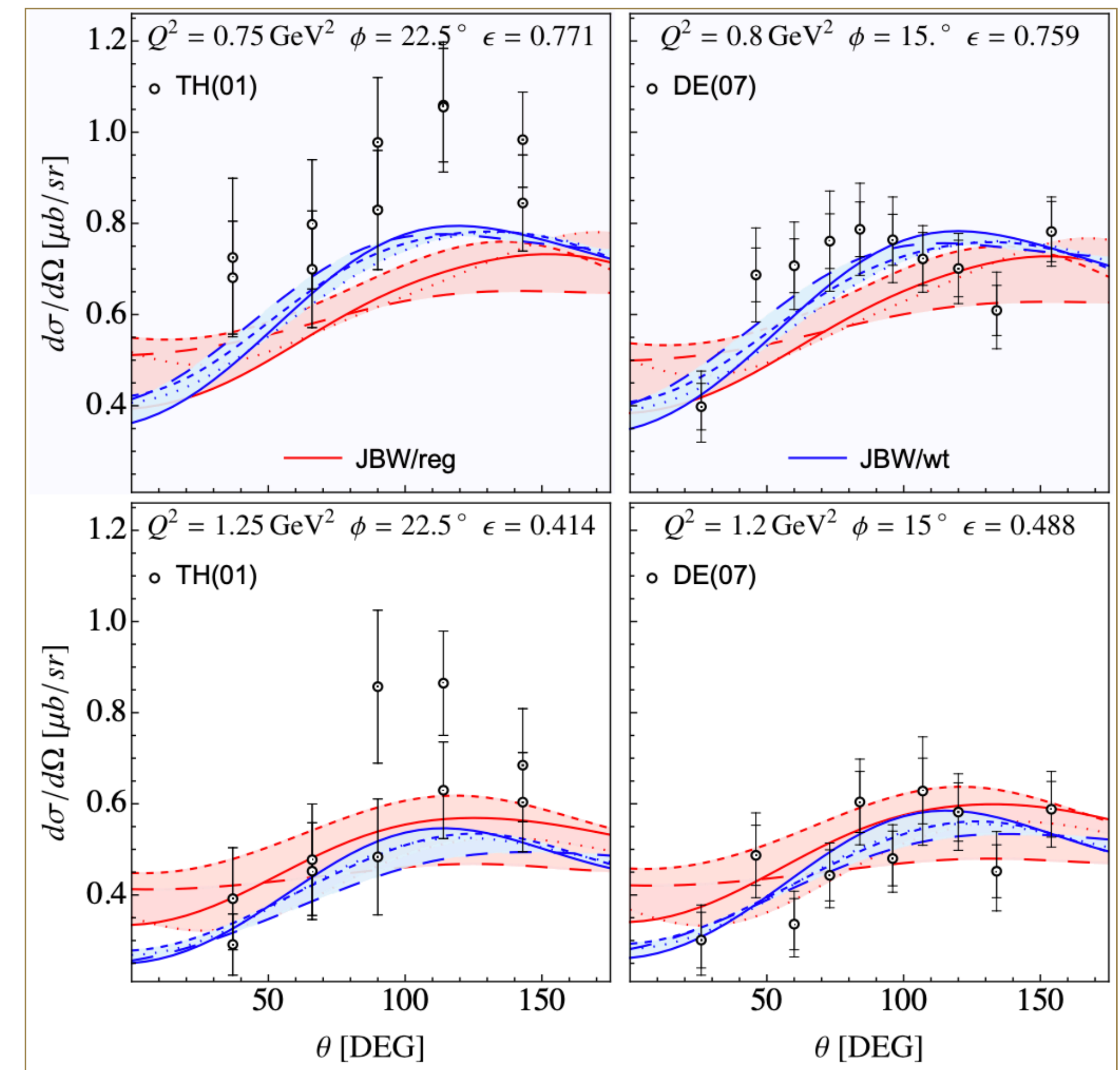
- Large multipoles well determined
- simple Q^2 dependence



RESULTS

Ambiguities in the data:

- example η N data²



1) [JBW] MM et al. Phys.Rev.C 103 (2021) 6; Phys.Rev.C 106 (2022) 015201

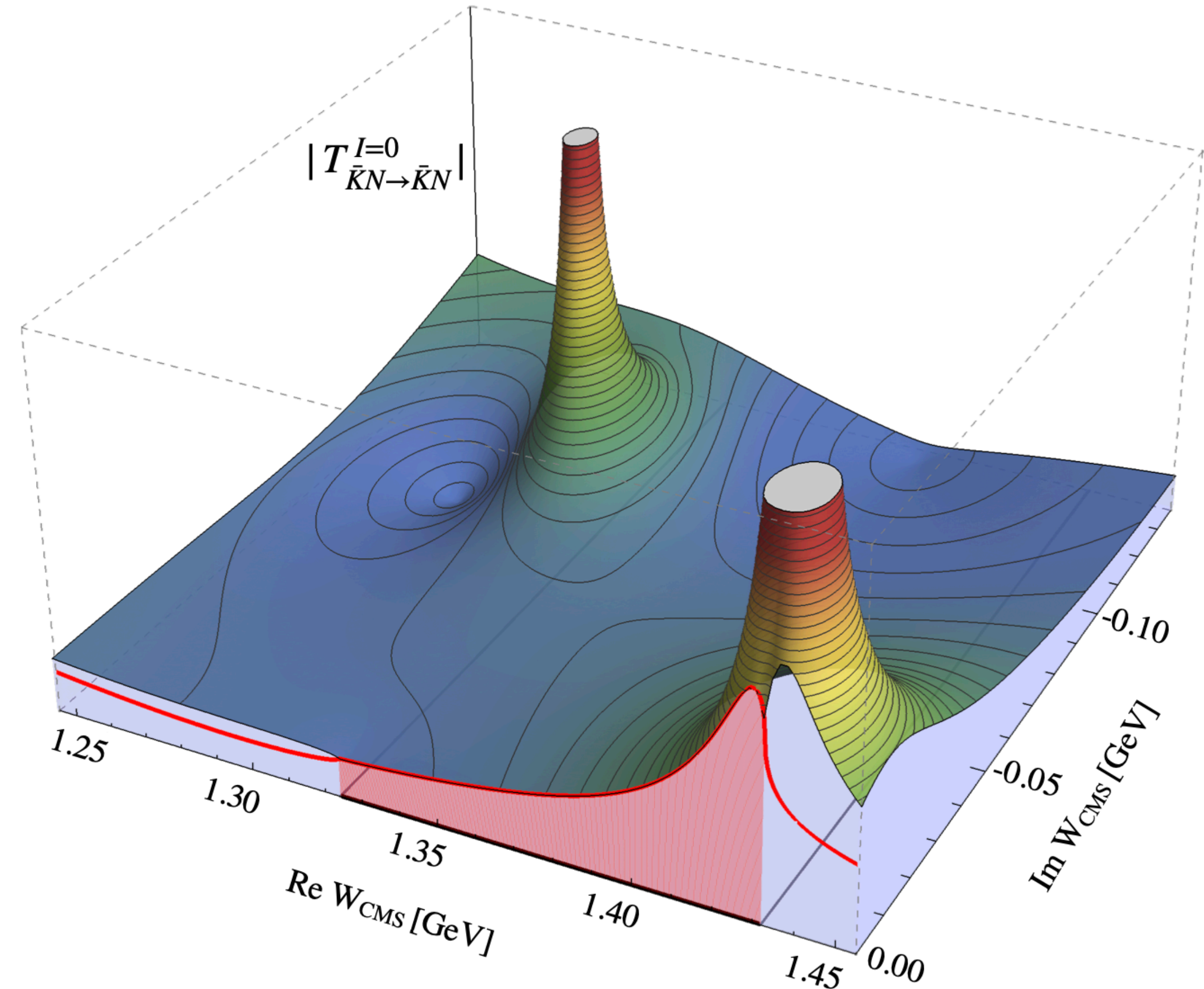
2) H. Denizli et al. (CLAS) PRC 76, 015204 (2007); Thompson et al. (CLAS), PRL86, 1702-1706 (2001); ...

RESONANCE POLE(S)

- Inclusion of chiral symmetry constants demands a second state¹:

$$W^* = (1325 \dots 1381) - i(56 \dots 114) \text{ MeV}$$

→ Common phenomenon in hadron physics²



¹)Oller/Meißner (2001); Ikeda/Hyodo/Weise(2011); MM/Meißner(2012); Guo/Oller(2012),...

²)Meißner, Symmetry 12 (2020) 6, 981

HADRONIC 3-BODY PROBLEM: IMPACT

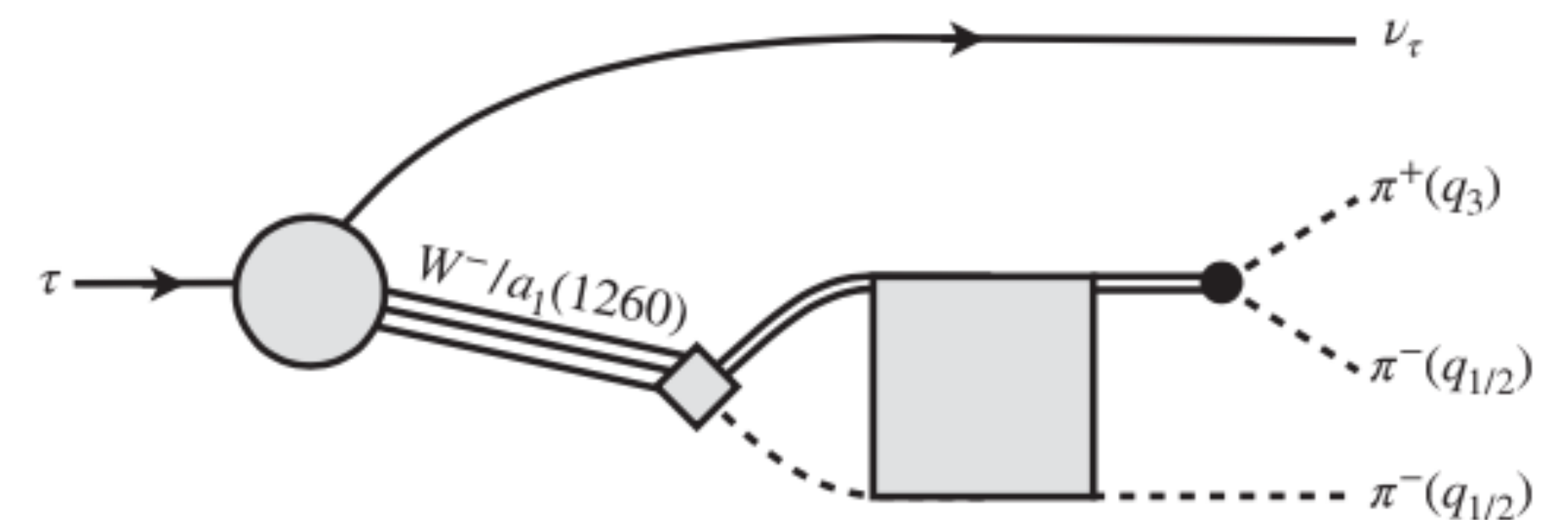
Intricate kinematics/dynamics

- 8 variables
- 2-body sub-channel dynamics

Hadron spectroscopy riddles

- Roper(1440) $\rightarrow \pi\pi N$ [first FV evaluations¹]
- X(3872) $\rightarrow D\bar{D}\pi$
- $a_1(1260) \rightarrow \pi\pi\pi$
- ...

- Beyond Standard Model: τ -EDM



- Precision physics: rare hadronic W-decays²
- Exotic states of matter³

¹) Severt/MM/Meißner JHEP04(2023) >>> PHD talk on Friday

²) Sirunyan et al. [CMS@CERN] PRL122

³) Experimental programs: GlueX@JLAB; COMPASS@CERN;

BROADER IMPACT

Twice non-perturbative regime of QCD

- too low for perturbative QCD
- too high for low-energy EFT

$\bar{K}$ NN & $\bar{K}$ NNN bound states

- dominated by $\bar{K}$ N interaction
- $\bar{K}$ N input is critical for interpretation

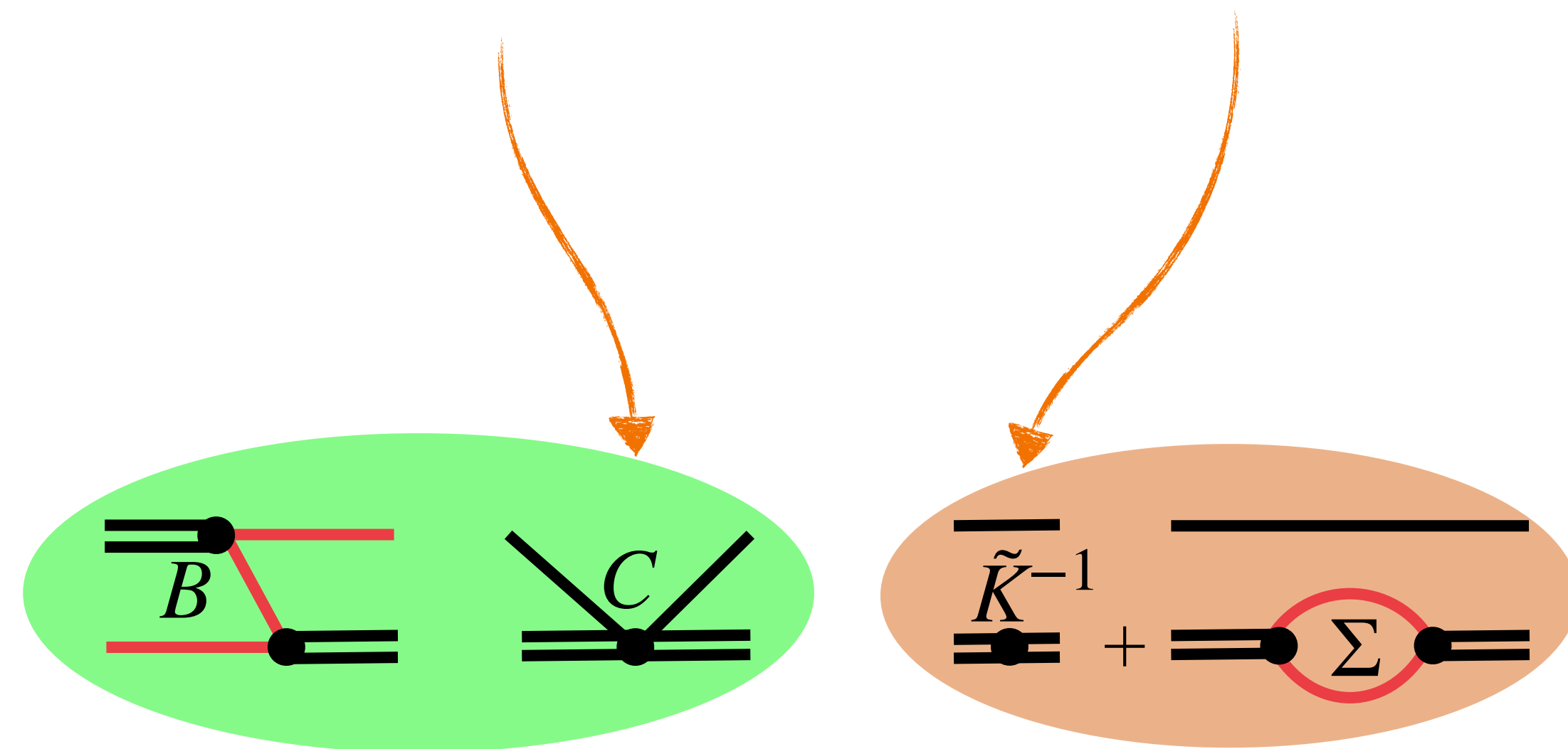
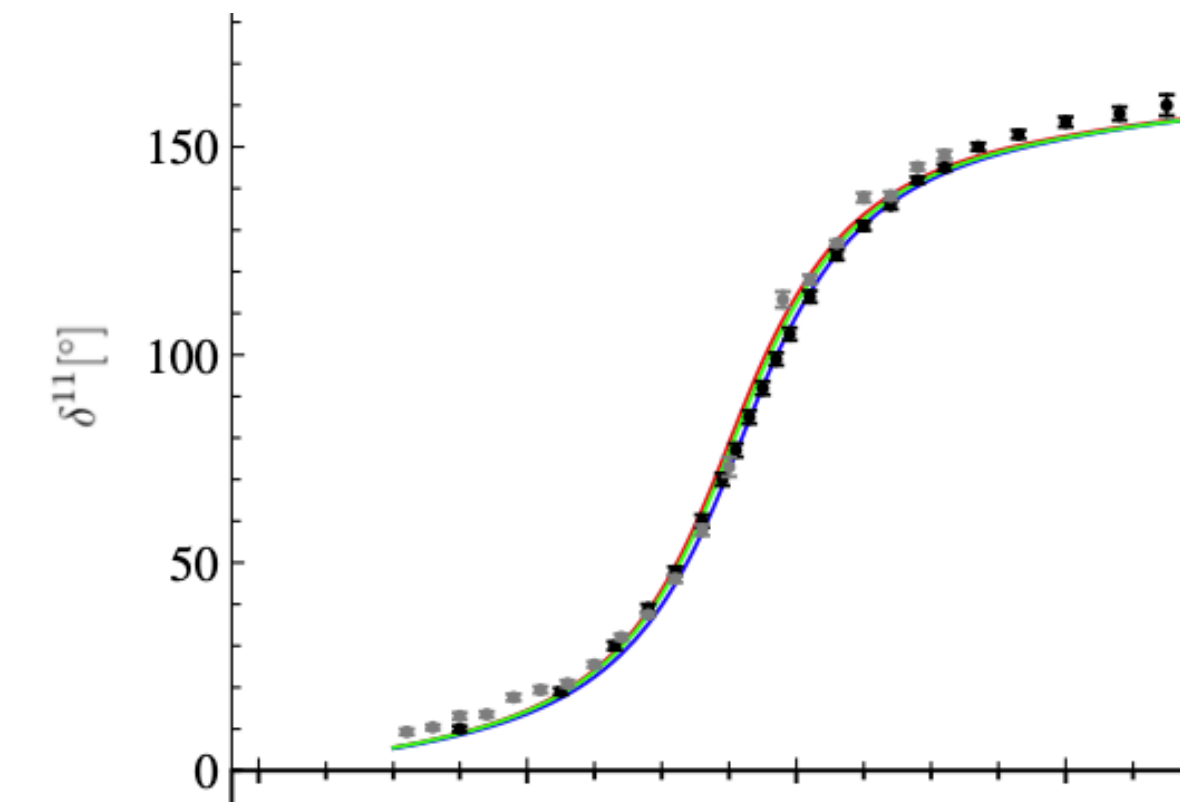
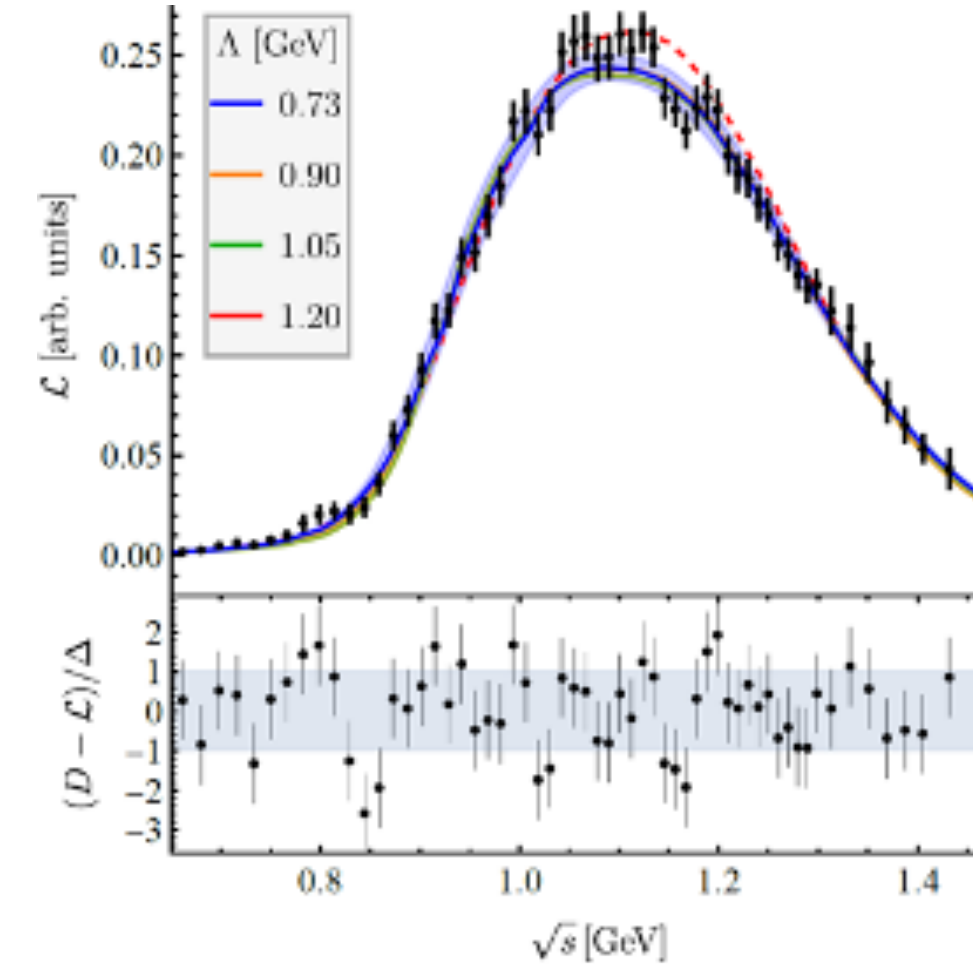
Antikaons in nuclear medium

- Strangeness in the EoS of neutron stars
- K -condensate can change EoS-stiffness



$a_1(1260)$ PHENOMENOLOGY

- Fix quantum numbers to $a_1(1260) \rightarrow \pi\pi\pi$
- solution via complex spectator momentum
- unknown parameter from fits¹ to data

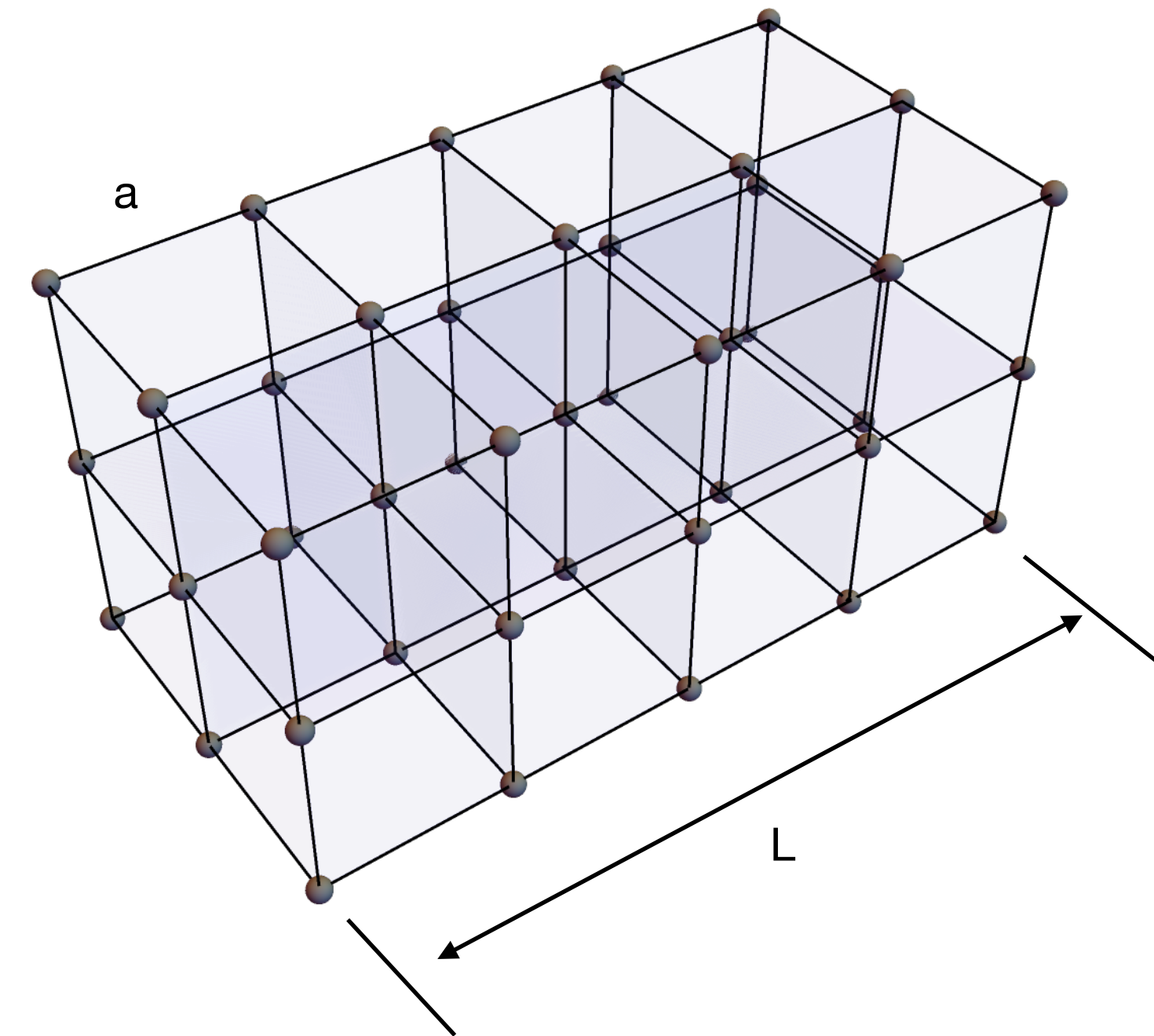


1) Sadasivan/MM/Döring/Alexandru/Culver/Lee Phys.Rev.D

LATTICE HADRON SPECTROSCOPY

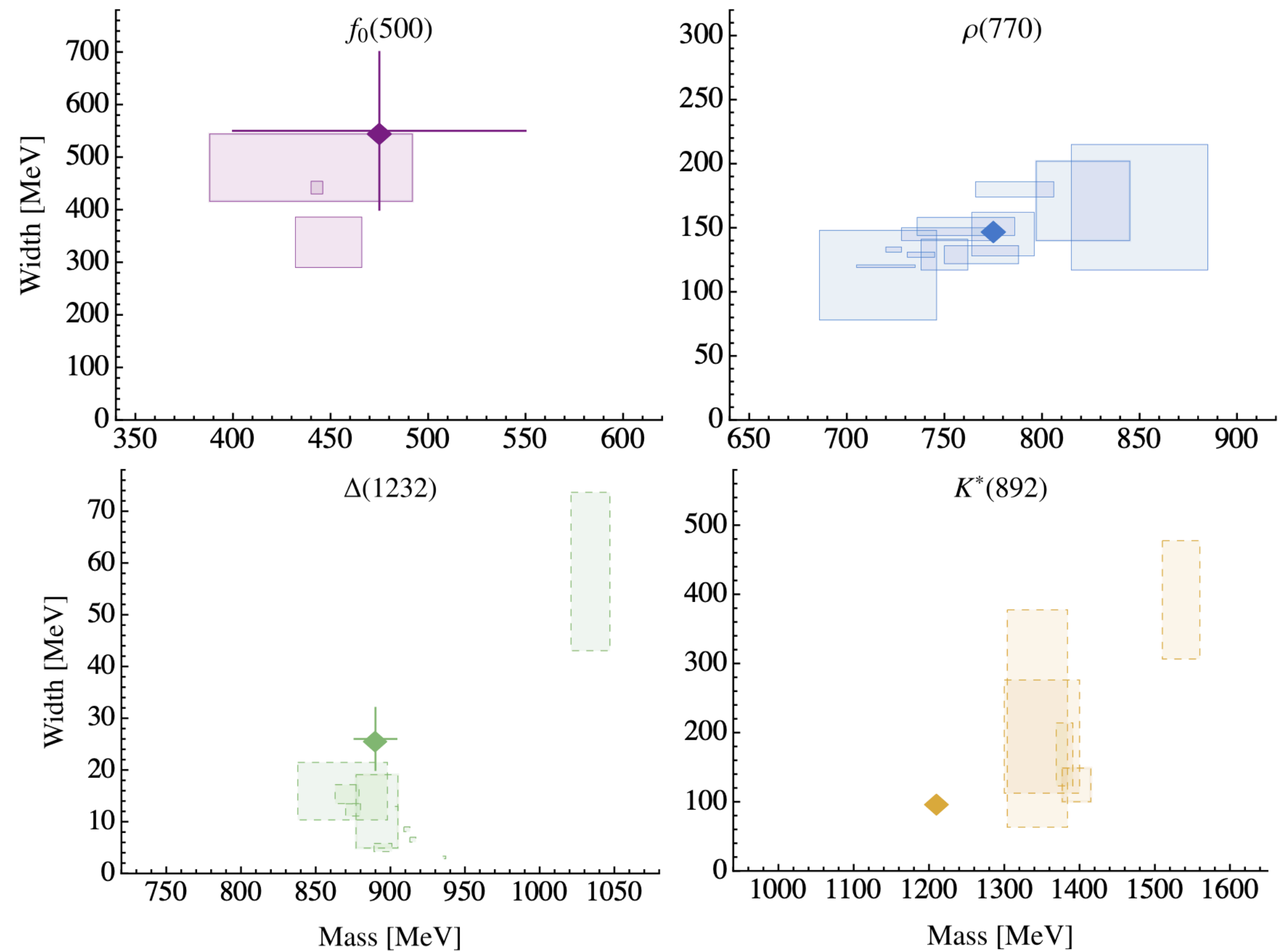
$$\mathcal{L}_{\text{QCD}} = \sum \bar{q}_f^a (i \not{D}_{ab} - m_f \delta_{ab}) q_f^b - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu}$$

- Numerical evaluation of QCD Green's functions



LATTICE HADRON SPECTROSCOPY

- Many studies of 2-body systems¹



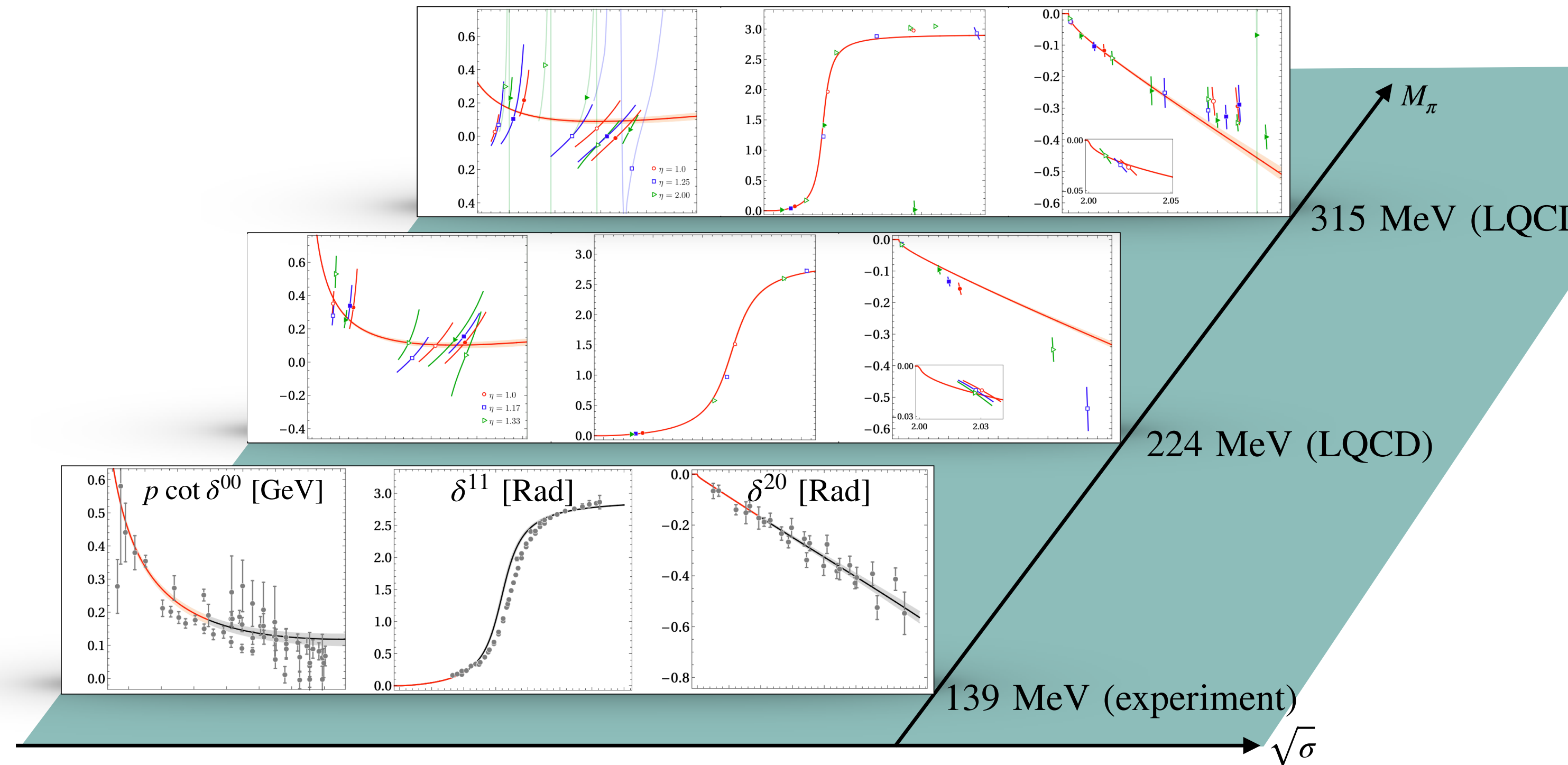
MM/Meißner/Urbach 2206.01477 Phys.Rept.
1001 (2023)

1)[NPLQCD], [RQCD], [ETMC], [HadSpec], ...

2)Reviews: Briceño/Dudek/Young Rev.Mod.Phys. 90 (2018); MM/Meißner/Urbach Phys.Rept. 1001 (2023)

LATTICE HADRON SPECTROSCOPY

- Experimentally inaccessible scenarios:
 - Unconventional quantum numbers
 - Three-body scattering
 - Unphysical pion mass (chiral trajectories)
 - ...



MM/Culver/Brett/Alexandru/Döring/Lee Phys.Rev.D

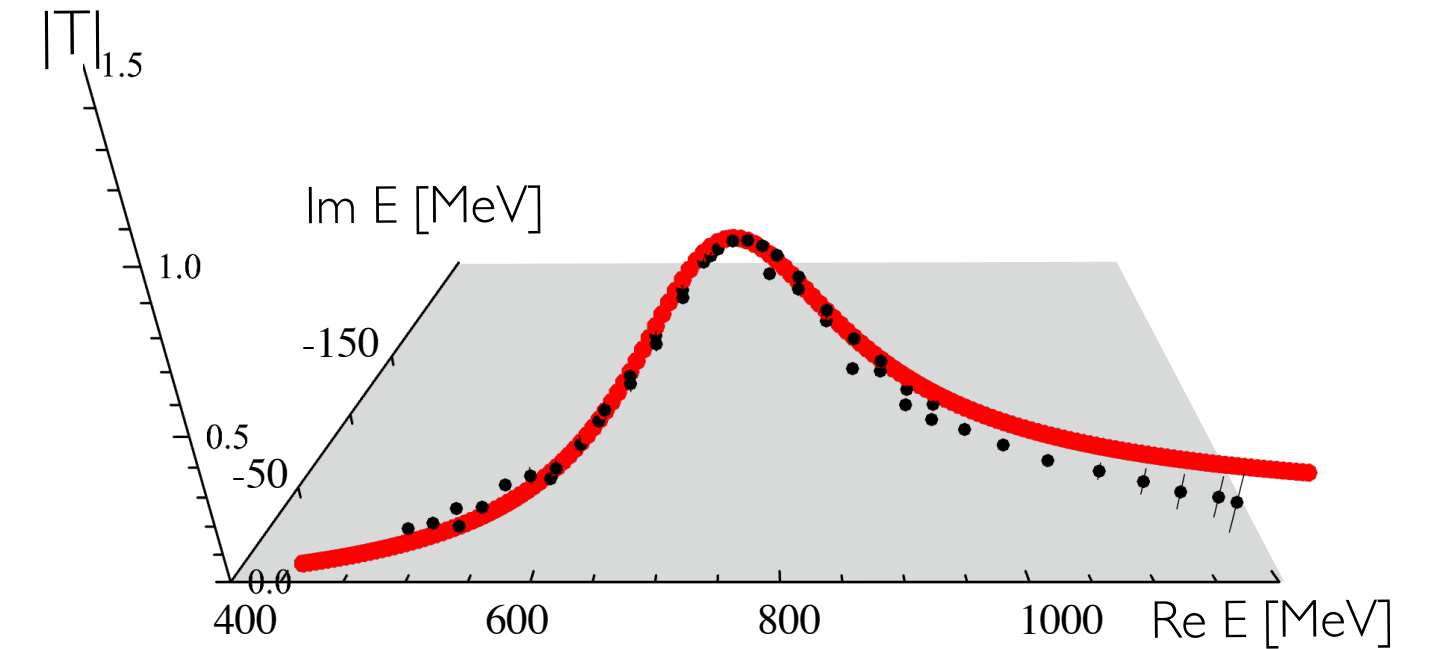
Review: MM/Döring/Rusetsky EPJ ST (2021)

HADRONS IN A BOX

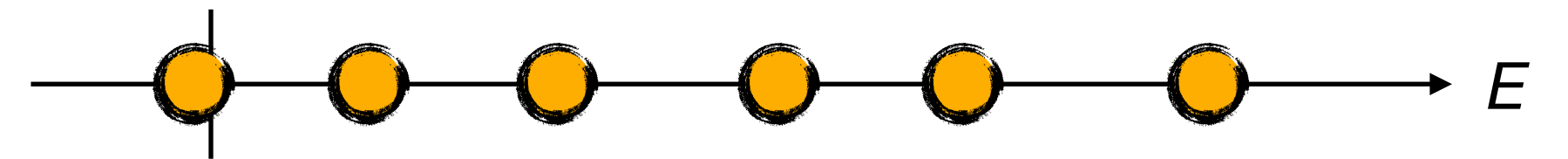
🤗 Heavily simplified:

on-shell particle-configurations: $\Delta E \sim mL$

off-shell particle-configurations: $\Delta E \sim e^{-mL}$



🤗 A unitary "T-matrix" accounts for all $O(mL)$ effects!



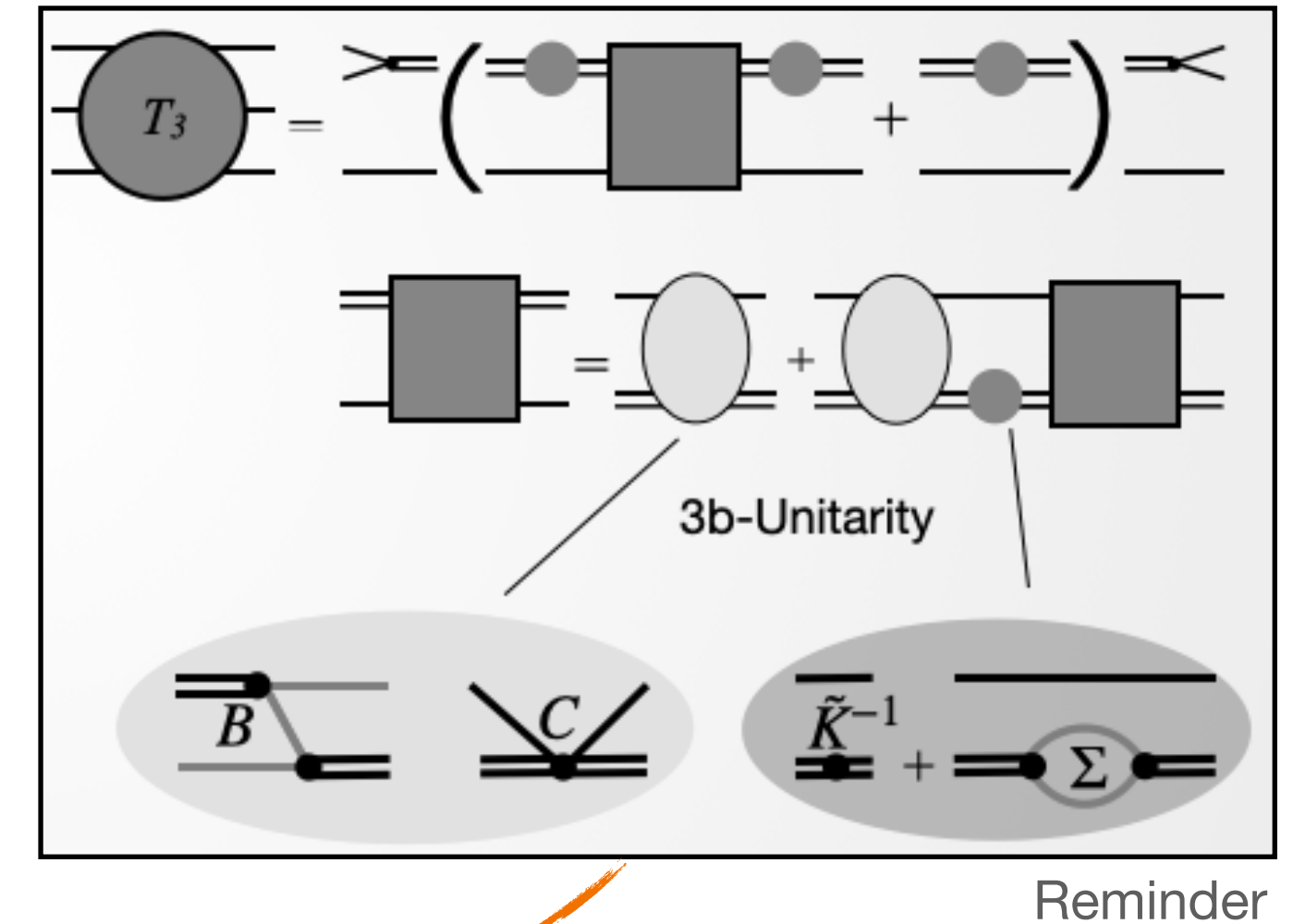
1) Lüscher, Gottlieb, Rummukainen, Feng, Li, Döring, Briceño, Meißner, Rusetsky, Hansen, MM, Blanton, ...

2) Reviews: Hansen/Sharpe Ann.Rev.Nucl.Part.Sci. 69 (2019); MM/Döring/Rusetsky Eur.Phys.J.ST 230 (2021);

3-BODY QUANTISATION CONDITION

Finite-volume unitarity (FVU)^{1,2}

- separates volume dependent terms
- volume independent terms connect infinite/finite-volume spectra

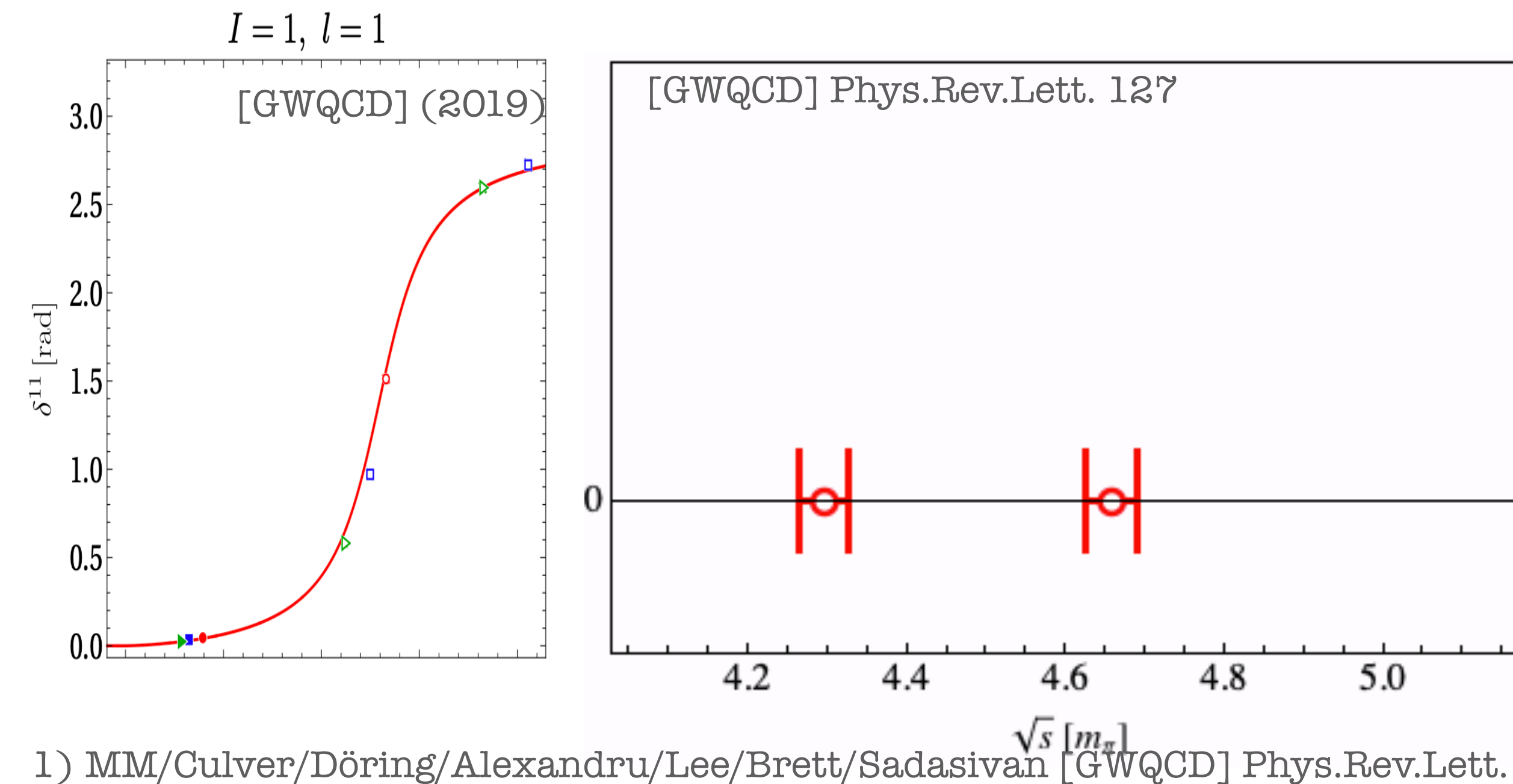


$$0 = \det \left[2L^3 E \left(\tilde{K}_n^{-1} - \Sigma \right) - B - C \right]_{\mathbf{p}'\mathbf{p}}$$

1) Lüscher, Gottlieb, Rummukainen, Feng, Li, Döring, Briceño, Meißner, Rusetsky, Hansen, MM, Blanton, ...

2) Reviews: Hansen/Sharpe Ann.Rev.Nucl.Part.Sci. 69 (2019); MM/Döring/Rusetsky Eur.Phys.J.ST 230 (2021);

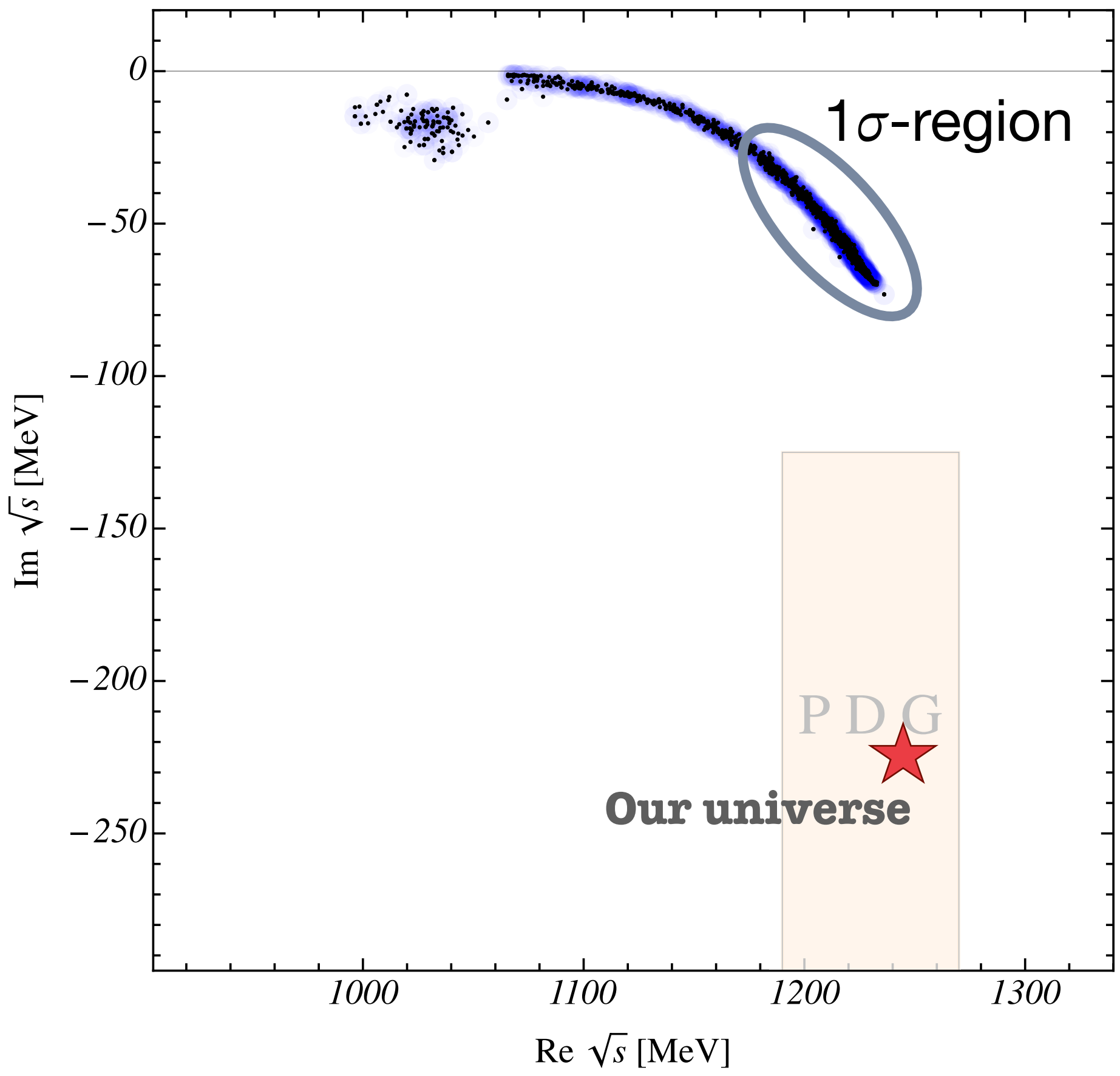
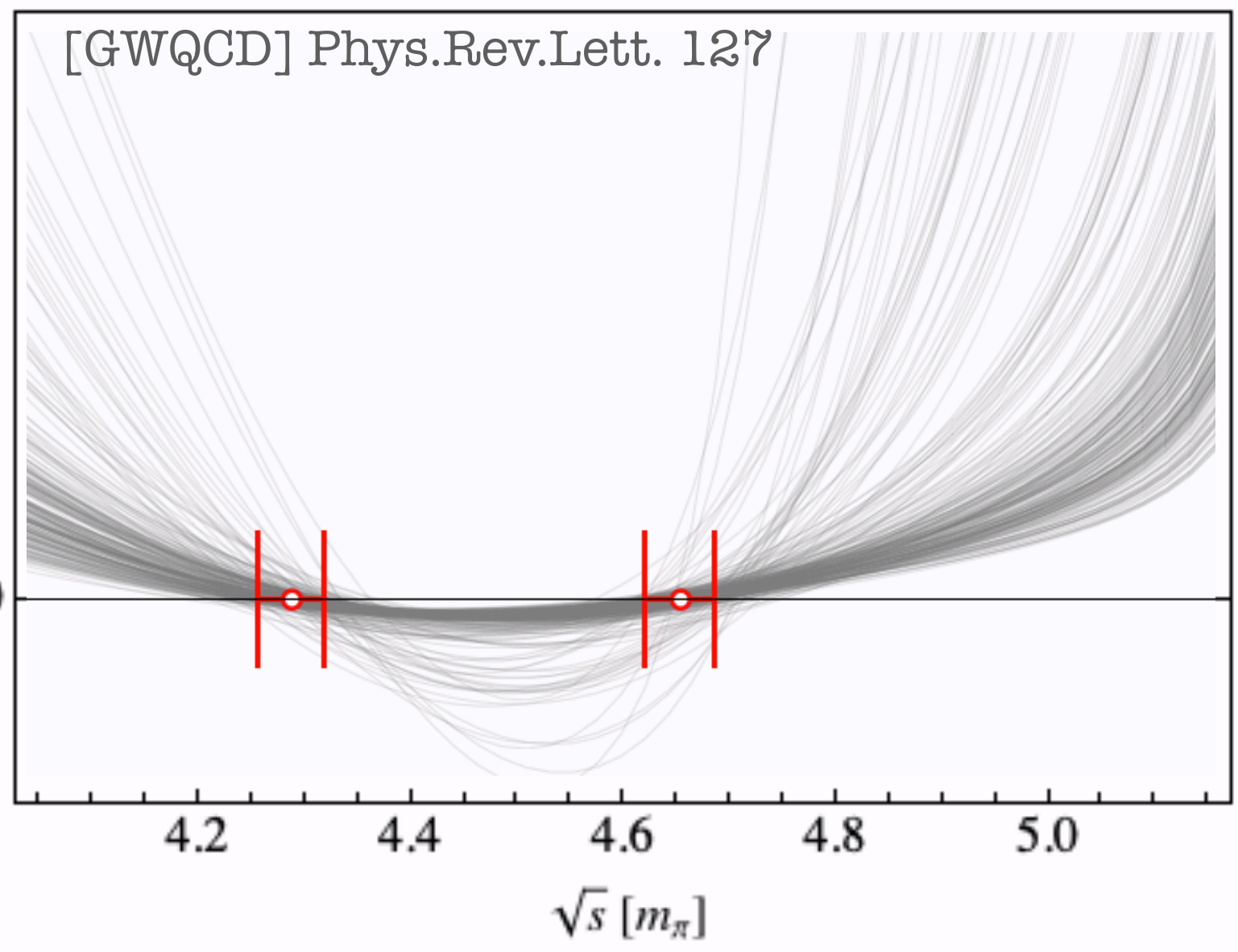
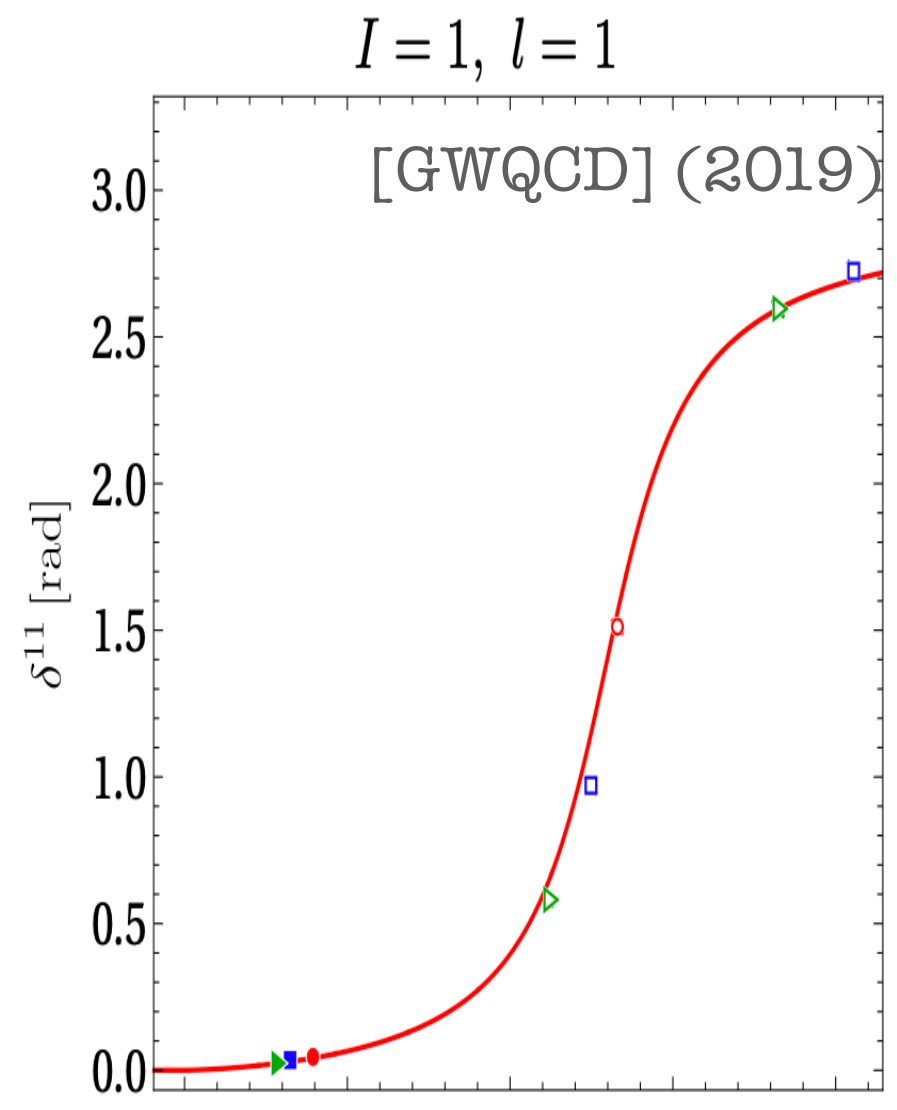
- First LQCD calculation¹ of a resonant 3b system
 - $N_f = 2$ dynamical fermions
 - LapH smearing
 - $\mathbf{P}=(0,0,0)$
 - $m_\pi=\mathbf{224}$ MeV, $m_\pi L=3.3$
 - GEVP with one-/two-/three-meson operators



heavier
Universe"

$$0 = \det \left[2L^3 E \left(\tilde{K}_n^{-1} - \Sigma \right) - B - \textcolor{red}{C} \right]_{\mathbf{p}'\mathbf{p}}$$

$$T^c = B + \textcolor{red}{C} + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B + \textcolor{red}{C})}{2E_l} \frac{1}{\tilde{K}_n^{-1} - \Sigma_n} T^c$$



1) MM/Culver/Döring/Alexander et al. [1807.01600] [GWQCD] Phys.Rev.Lett.

HADRONS IN A BOX

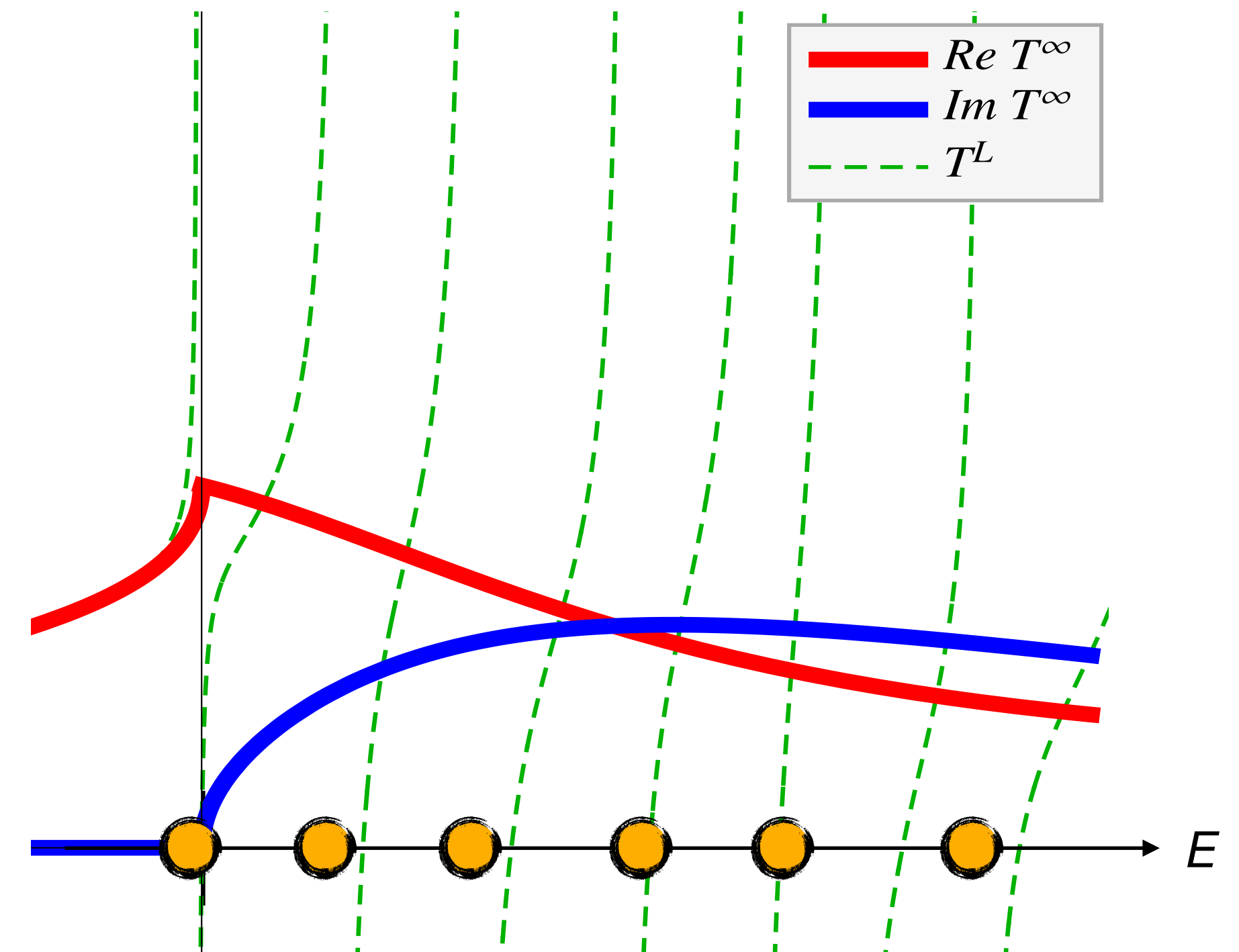
Finite-volume spectrum is real and discrete!
... requires mapping: Quantization condition^{1,2}

🤗 Heavily simplified:

on-shell particle-configurations: $\Delta E \sim mL$

off-shell particle-configurations: $\Delta E \sim e^{-mL}$

🌟 A unitary "T-matrix" accounts for all $O(mL)$ effects!



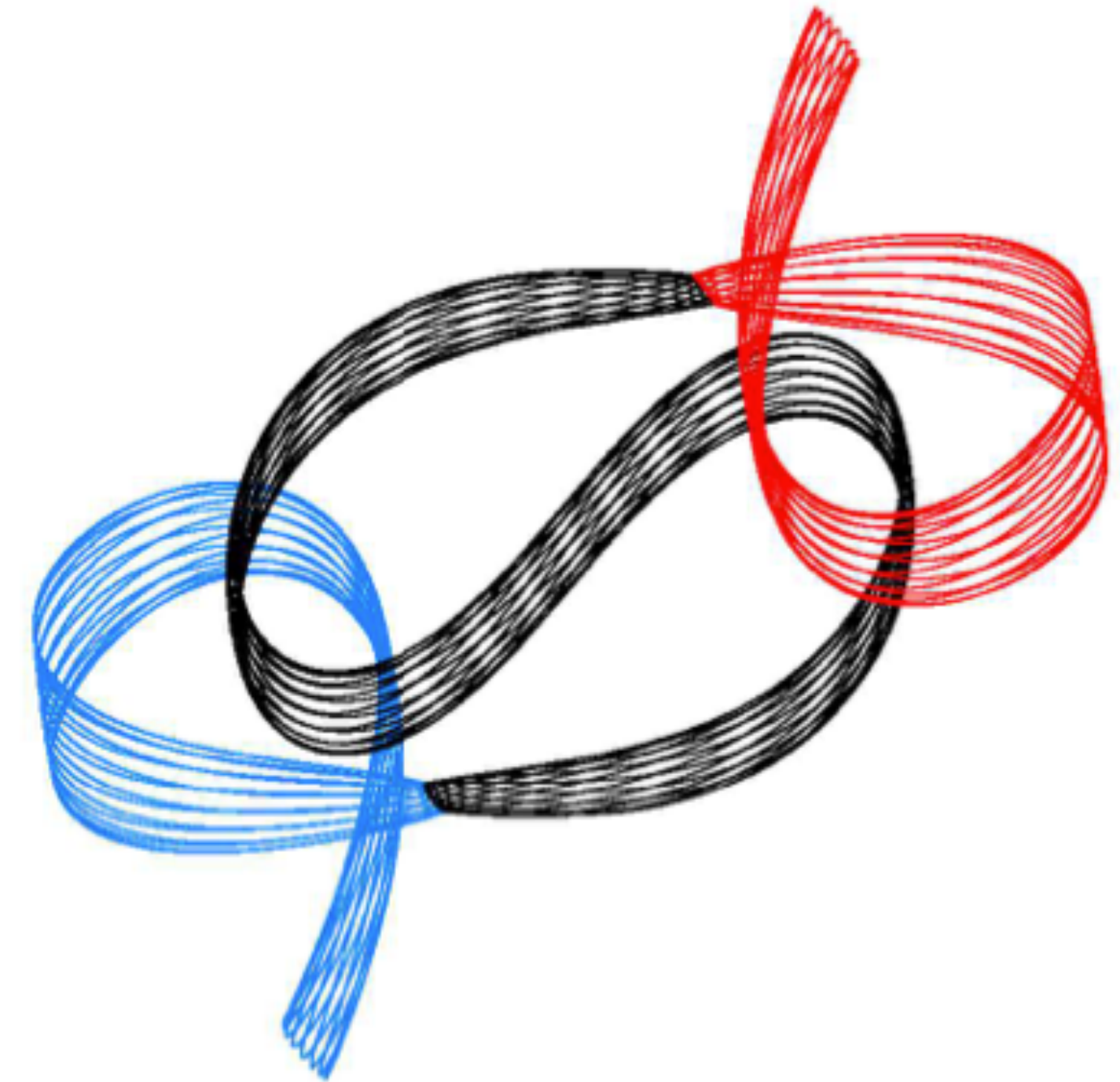
1) Lüscher, Gottlieb, Rummukainen, Feng, Li, Döring, Briceño, Meißner, Rusetsky, Hansen, MM, Blanton, ...

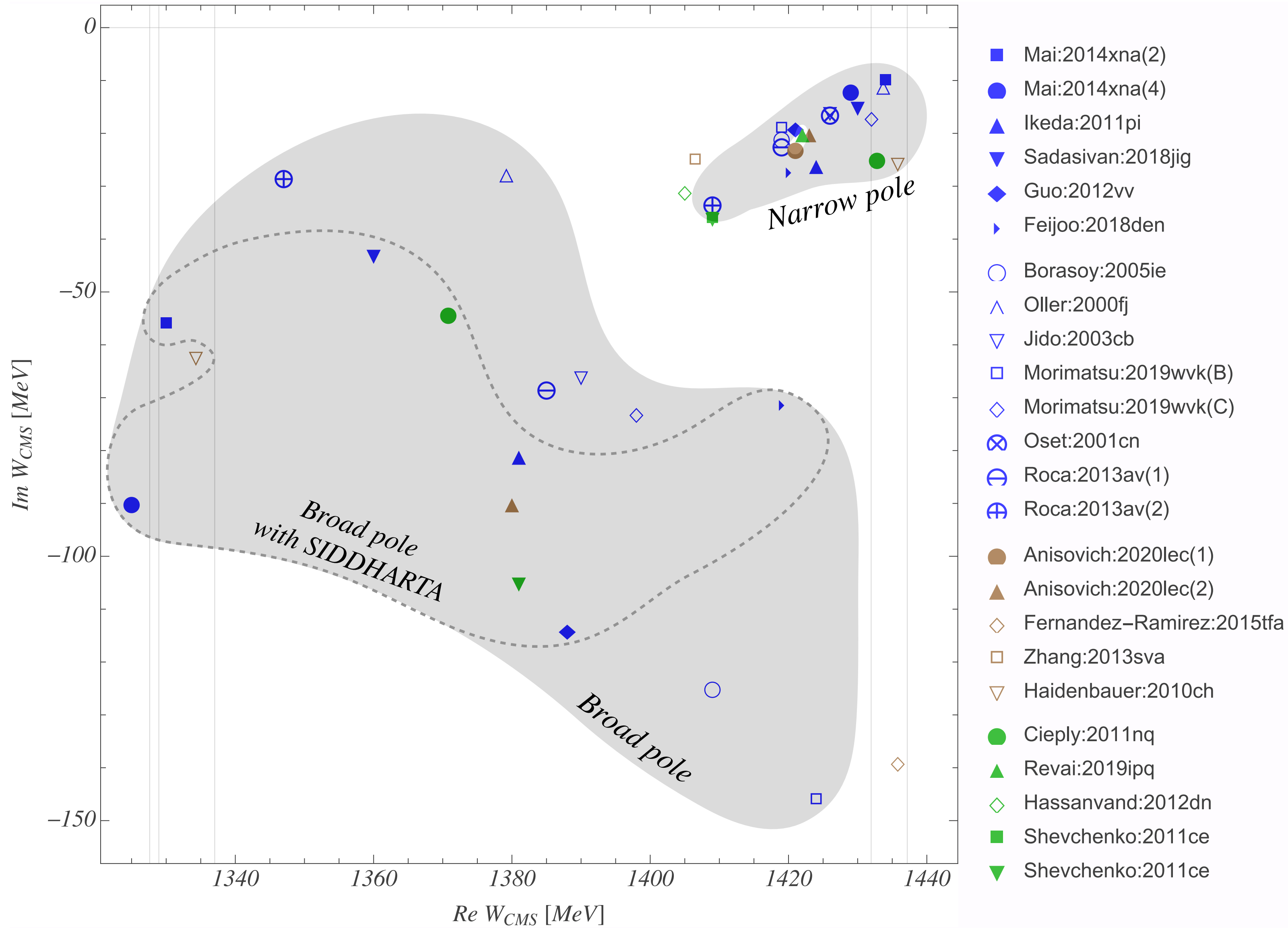
2) Reviews: Hansen/Sharpe Ann.Rev.Nucl.Part.Sci. 69 (2019); MM/Döring/

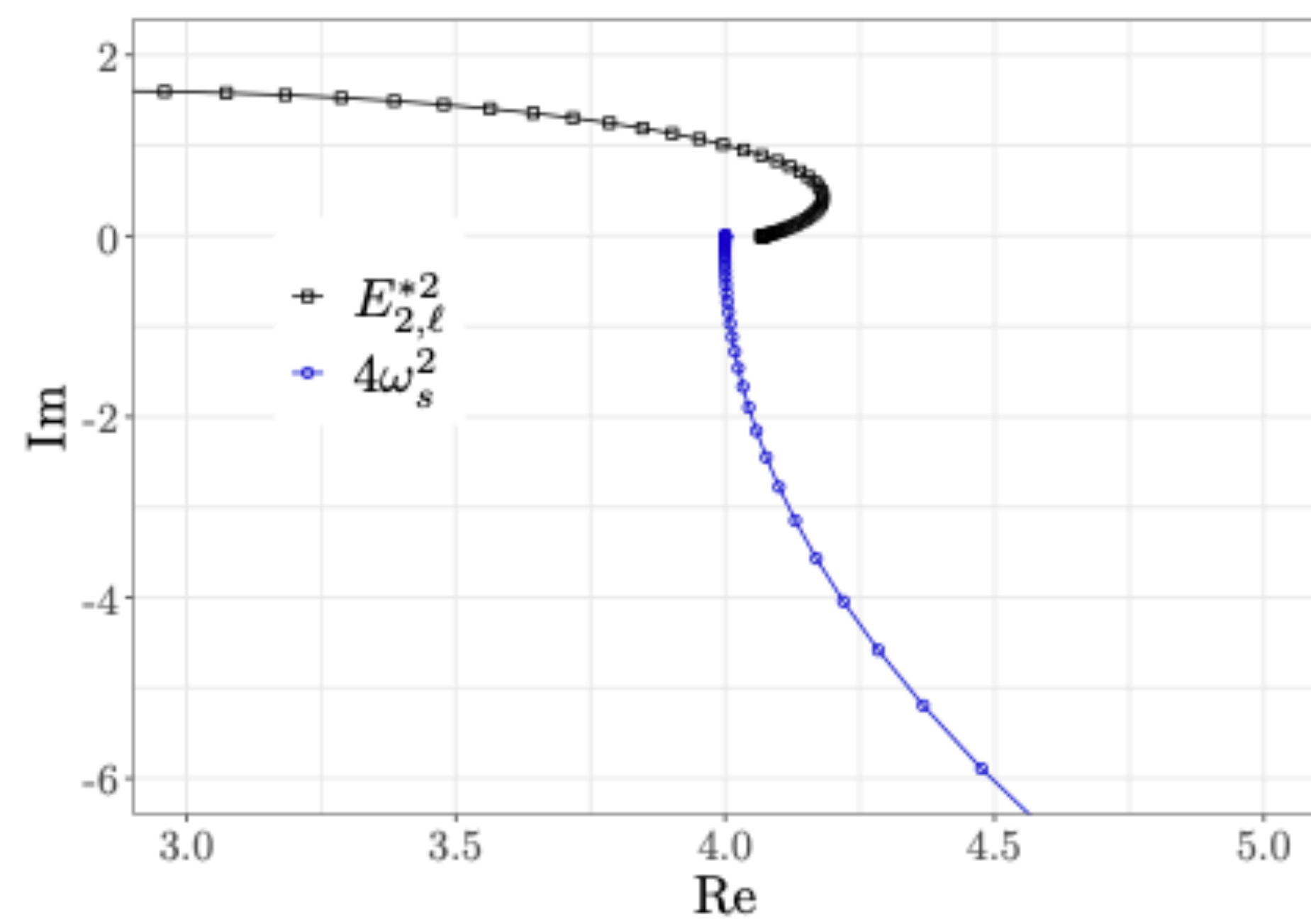
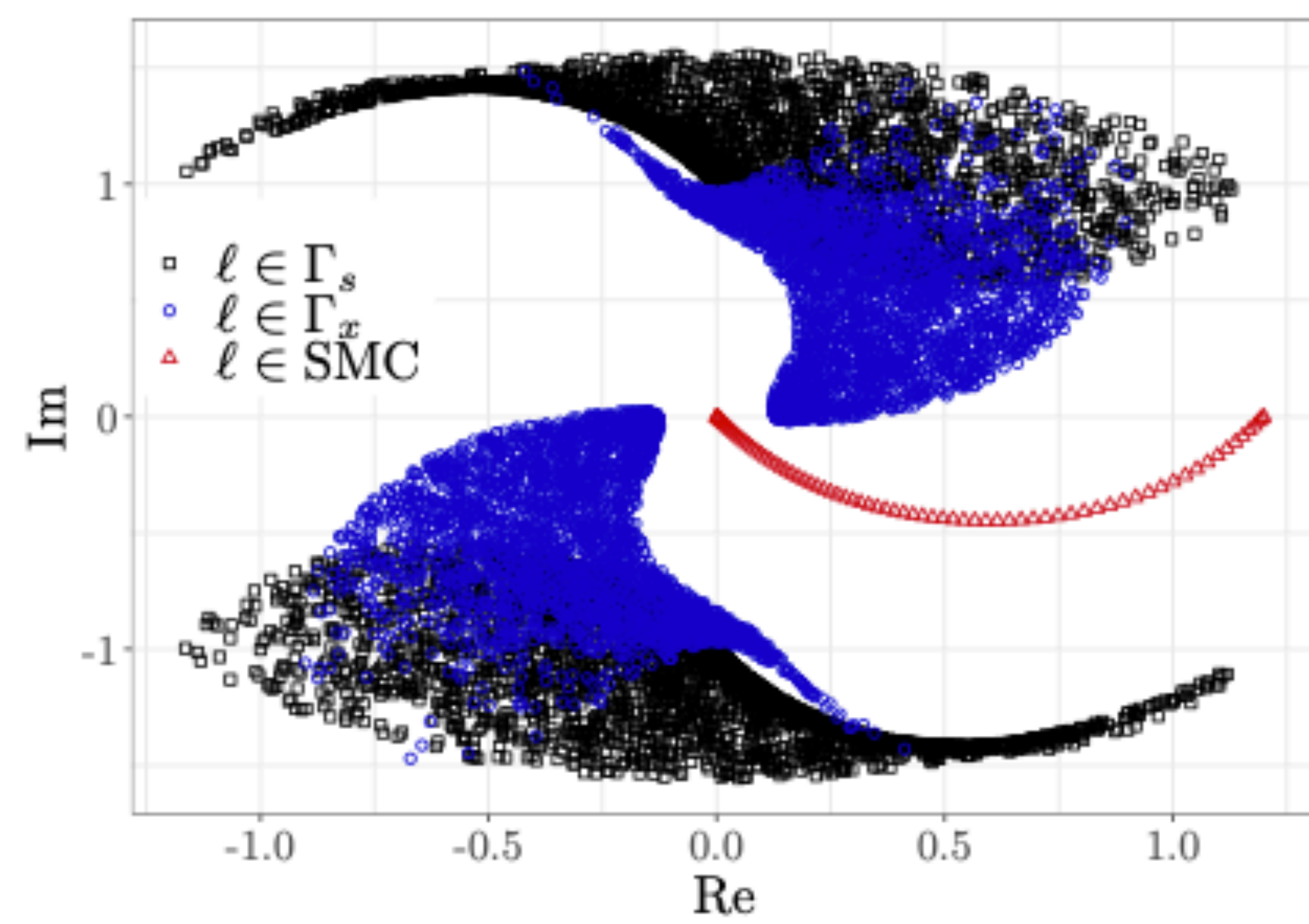
GRAVITATIONAL 3-BODY PROBLEM

... not that we talk about

- birth of mathematical chaos¹
- no closed solution
- in general non-repeating trajectories







Current frontier: 3-body dynamics from LQCD

➡ 3-body Quantization Conditions¹

➡ RFT / FVU / NREFT

➡ many perturbatively interacting systems are studied²

$$0 = \det \left(L^3 \left(\tilde{F}/3 - \tilde{F}(\tilde{K}_2^{-1} + \tilde{F} + \tilde{G})^{-1} \tilde{F} \right)^{-1} + \tilde{K}_{\text{df},3} \right) \quad \text{RFT}$$

$$0 = \det \left(B_0 + C_0 - E_L \left(K^{-1}/(32\pi) + \Sigma_L \right) \right) \quad \text{FVU}$$

1) Rusetsky, Bedaque, Grißhammer, Sharpe, Meißner, Döring, Hansen, Davoudi, Guo....

Reviews:

Hansen/Sharpe Ann.Rev.Nucl.Part.Sci. 69 (2019);

MM/Döring/Rusetsky Eur.Phys.J.ST 230 (2021);

2) MM/Döring PRL122(2019); Blanton et al. PRL 124 (2020); Hansen et al. PRL



3-body force



2-body interaction



one-particle exchange



2-body self-energy

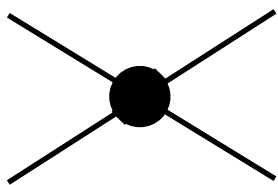
AVOIDED LEVEL CROSS

Variate $g(\varphi_1 \rightarrow \varphi_0 \varphi_0 \varphi_0)$ coupling:

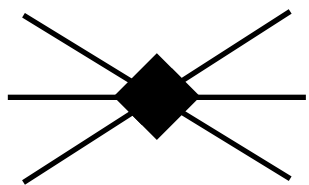
- avoided level crossing becomes wider
- RFT and FVU

g		a	m_1	c_0	c_1	m'_1	c'_0	c'_1	χ^2_{dof}
5	FVU	-0.1512(9)	3.0229(1)	-0.0188(35)	—	—	—	—	2.9
	RFT	-0.1522(12)	—	—	—	3.0232(2)	31.6(8.4)	—	2.5
	FVU	-0.1569(12)	3.0233(2)	-0.0297(57)	2.29(38)	—	—	—	1.5
	RFT	-0.1571(10)	—	—	—	3.0237(2)	37.6(9.0)	2789(540)	1.5
10	FVU	-0.1521(11)	3.0205(2)	-0.0475(66)	—	—	—	—	1.7
	RFT	-0.1531(13)	—	—	—	3.0212(3)	80(14)	—	1.6
	FVU	-0.1549(16)	3.0205(2)	-0.0595(99)	0.93(41)	—	—	—	1.5
	RFT	-0.1563(27)	—	—	—	3.0213(3)	97(16)	1773(980)	1.4
20	FVU	-0.1444(11)	3.0184(2)	-0.1136(77)	—	—	—	—	1.6
	RFT	-0.1450(17)	—	—	—	3.0199(2)	178(17)	—	1.6
	FVU	-0.1464(14)	3.0183(2)	-0.1363(148)	0.84(39)	—	—	—	1.3
	RFT	-0.1484(16)	—	—	—	3.0200(2)	210(23)	2227(600)	1.2

$$q^* \cot \delta = \frac{1}{aM_0}$$



$$C = \frac{c_0}{E_3^3 - m_1^2} + c_1$$

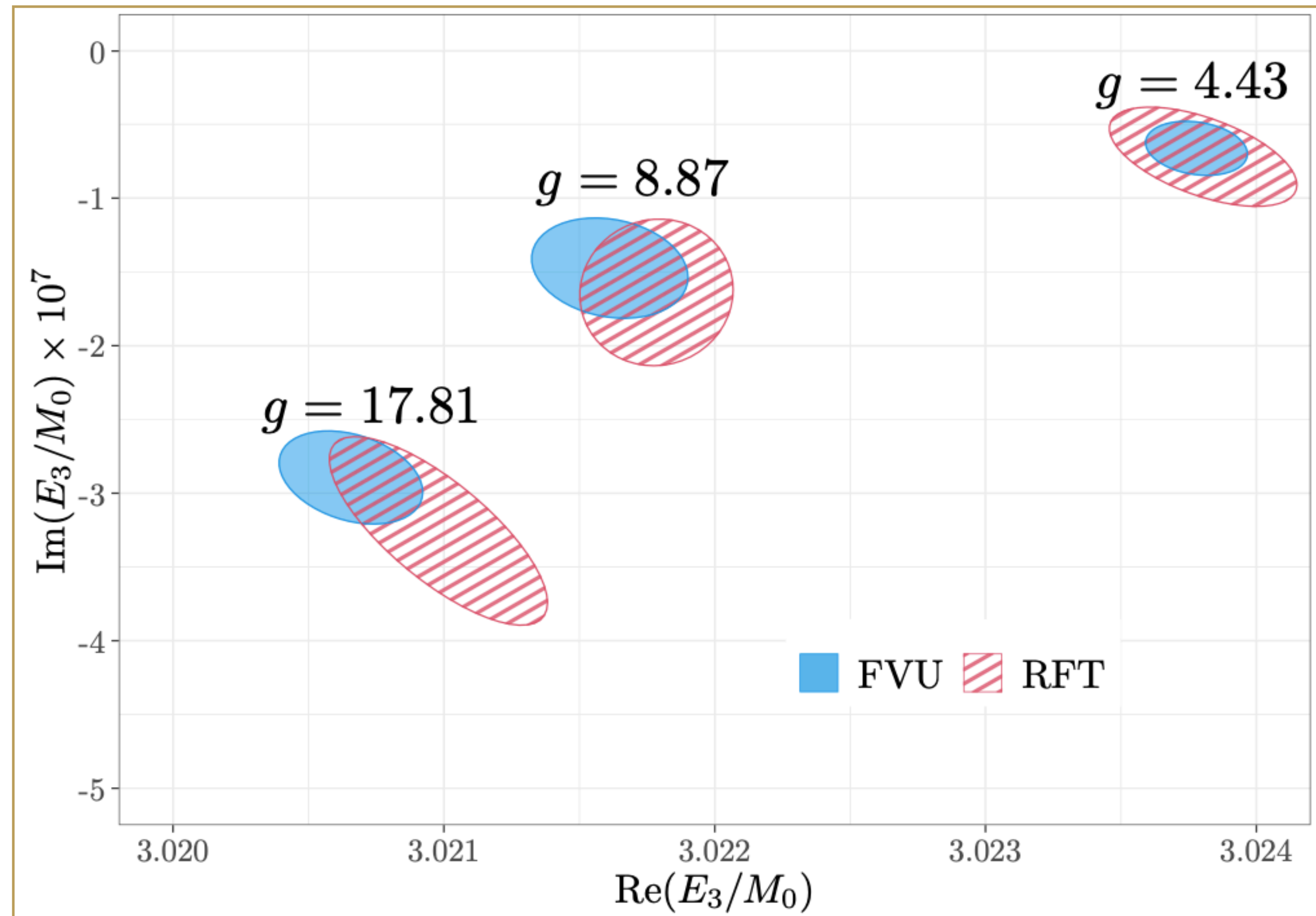


... same fit quality

... observables determined consistently

Pole positions

- FVU: complex energy-plane analysis¹
 - resonance width grows $\sim g^2$
 - avoided level crossing gap $\gg$ width
- Similarly from RFT with Breit-Wigner like approximation



1) Sadasivan/MM/.. Phys.Rev.D 101 (2020)

LATTICE QCD

$$\mathcal{L}_{\text{QCD}} = \sum \bar{q}_f^a (i \not{D}_{ab} - m_f \delta_{ab}) q_f^b - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu}$$

Lattice QCD: numerical evaluation of QCD Green's functions. But...

- discretized Euclidean space time ($a > 0$)
- in finite volume ($L < \infty$)

