Complete experiments, truncated partial-wave analyses and Bayesian inference

some slides based on work done in collaboration with:

Philipp Kroenert, Farah Afzal and Annika Thiel

cf.: [P. Kroenert *et al.*, arXiv:2305.10367 [nucl-th].]

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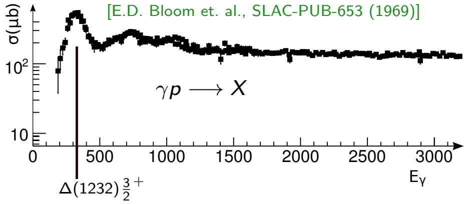
Introduction: Baryon Spectroscopy

$*$) Photoproduction is a generic reaction used to study baryon resonances:



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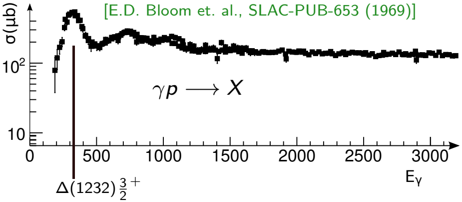

- $\ast$) Baryon resonances $\left(\Delta(1232)\frac{3}{2}^+, N(1440)\frac{1}{2}^+, \ldots\right)$ are Fermions
  - $\hookrightarrow$ Scatter particles with spin to excite systems with half-integer J
- $\ast$) 'T-matrix' $\mathcal{T}_{fi}$ parameterized by N spin-amplitudes $\{b_i, i = 1, \ldots, N\}$
- $\ast$) The usual reactions under study are:
  - Pion-Nucleon (πN-) scattering: $\pi N \longrightarrow \pi N$ (2 spin-amplitudes)
  - Pion photoproduction: $\gamma N \longrightarrow \pi N$ (4 spin-amplitudes)
  - Pion electroproduction: $eN \longrightarrow e'\pi N$ (6 spin-amplitudes)
  - 2-Pion photoproduction: $\gamma N \longrightarrow \pi\pi N$ (8 spin-amplitudes)
  - ...

Spin amplitudes

Generic case:

*) General meson-production reaction:

$$\mathcal{P} N \rightarrow \{\varphi_i\}\, B, \text{ with:}$$

- $\mathcal{P}$: 'probe'-particle $(\pi, \gamma, \gamma^{*}, \ldots)$,
- N: target-nucleon,
- $\{\varphi_i\}$: one or multiple meson(s),
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$$\mathcal{T}_{fi} = \chi_B^{\dagger} \left[\sum_{k=1}^{N_{\mathcal{A}}} \boldsymbol{\kappa}_k b_k \left(\Omega_2^{n_f} \right) \right] \chi_N ,$$

- $\kappa_i \left(\{p_j\}, \{\sigma_i\} \right)$: spin-kinem. operators,
- $b_i \left(\Omega_2^{n_f} \right)$: spin ('transversity') ampl.'s,
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with additional factor of (1/2) in case of a $2 \rightarrow 2$ reaction (parity!)

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$$\gamma N \rightarrow \varphi B.$$

$*)$ Then: $N_{\mathcal{A}} = \frac{1}{2} \left(\boldsymbol{n}_\gamma \boldsymbol{n}_N \boldsymbol{n}_\varphi \boldsymbol{n}_B \right) = \frac{1}{2} \left(2 \times 2 \times 1 \times 2 \right) = \underline{\underline{4}}.$

$*)$ We have:

$$\mathcal{T}_{fi} = \chi_B^\dagger [\boldsymbol{\kappa}_1 b_1 + \boldsymbol{\kappa}_2 b_2 + \boldsymbol{\kappa}_3 b_3 + \boldsymbol{\kappa}_4 b_4] \chi_N, \text{ where:}$$

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*) $\{\boldsymbol{\kappa}_1, \ldots, \boldsymbol{\kappa}_4\}$ are complicated, e.g.:

$$\boldsymbol{\kappa}_1 = \frac{(\hat{q} \cdot \hat{\epsilon})}{\sqrt{2} \sin^2 \theta} \left[e^{-i \frac{\theta}{2}} \left(\hat{k} \cdot \hat{\boldsymbol{\sigma}} \right) - e^{i \frac{\theta}{2}} \left(\hat{q} \cdot \hat{\boldsymbol{\sigma}} \right) \right]$$

$\boldsymbol{\kappa}_2 = \ldots$, where:

- $\hat{k}, \hat{q}$: photon- and meson momentum,
- $\hat{\epsilon}$: γ-polarization,
- $\hat{\boldsymbol{\sigma}} = (\sigma_x, \sigma_y, \sigma_z)^T$: Pauli-matrices,
- W: CMS-energy,
- θ: CMS scattering-angle.

Polarization observables

<u>Generic case:</u>
Observables defined as *bilinear forms*:

$\mathcal{O}^{\alpha} = \boldsymbol{c}^{\alpha} \sum_{i,j=1}^{N_{\mathcal{A}}} b_i^{*} \tilde{\Gamma}_{ij}^{\alpha} b_j,$

for $\alpha = 1, \ldots, N_{\mathcal{A}}^2$.
($\boldsymbol{c}^{\alpha}$: normaliz. factors)

*) The $\tilde{\Gamma}^{\alpha}$ are a set of complete, orthogonal complex $N_{\mathcal{A}} \times N_{\mathcal{A}}$-matrices ('Clifford algebra')

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Photoproduction observable	Class
$\sigma_0 = \frac{1}{2}\left(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	
$-\check{\Sigma} = \frac{1}{2}\left(\lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 - \lvert b_4\rvert^2\right)$	$\mathcal{S}$
$-\check{T} = \frac{1}{2}\left(-\lvert b_1\rvert^2 + \lvert b_2\rvert^2 + \lvert b_3\rvert^2 - \lvert b_4\rvert^2\right)$	
$\check{P} = \frac{1}{2}\left(-\lvert b_1\rvert^2 + \lvert b_2\rvert^2 - \lvert b_3\rvert^2 + \lvert b_4\rvert^2\right)$	
$\mathcal{O}_{1+}^{a} = \lvert b_1\rvert\lvert b_3\rvert \sin\phi_{13} + \lvert b_2\rvert\lvert b_4\rvert \sin\phi_{24} = \mathrm{Im}\left[b_3^* b_1 + b_4^* b_2\right] = -\check{G}$	
$\mathcal{O}_{1-}^{a} = \lvert b_1\rvert\lvert b_3\rvert \sin\phi_{13} - \lvert b_2\rvert\lvert b_4\rvert \sin\phi_{24} = \mathrm{Im}\left[b_3^* b_1 - b_4^* b_2\right] = \check{F}$	$a = \mathcal{BT}$
$\mathcal{O}_{2+}^{a} = \lvert b_1\rvert\lvert b_3\rvert \cos\phi_{13} + \lvert b_2\rvert\lvert b_4\rvert \cos\phi_{24} = \mathrm{Re}\left[b_3^* b_1 + b_4^* b_2\right] = -\check{E}$	
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$\mathcal{O}_{1+}^{b} = \lvert b_1\rvert\lvert b_4\rvert \sin\phi_{14} + \lvert b_2\rvert\lvert b_3\rvert \sin\phi_{23} = \mathrm{Im}\left[b_4^* b_1 + b_3^* b_2\right] = \check{O}_{z'}$	
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Truncated partial-wave analysis (TPWA)

Generic case: partial-wave exp. for $2 \rightarrow 2$ spin-reaction in helicity-formalism:

$$\mathcal{T}_{\mu_1 \mu_2, \lambda_1 \lambda_2}(s, t) = e^{i(\lambda - \mu)\phi} \sum_{j=\max(|\lambda|,|\mu|)}^{\infty} (2j+1) \mathcal{T}^{j}_{\mu,\lambda}(s)\, d^{j}_{\mu,\lambda}(\theta),$$

where $\lambda := \lambda_1 - \lambda_2$, $\mu := \mu_1 - \mu_2$ and $\{b_i\} \Leftrightarrow \{H_i\} = \{\mathcal{T}_{\pm\pm,\pm\pm}\}$.

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E.g.: for photoproduction ($N_{\mathcal{A}} = 4$), one has the multipole-series ($x = \cos\theta$):

$$F_1(W,\theta) = \sum_{\ell=0}^{\cancel{\infty}\,\ell_{\max}} \left[\ell M_{\ell+}(W) + E_{\ell+}(W)\right] P^{'}_{\ell+1}(x) + \left[(\ell+1) M_{\ell-}(W) + E_{\ell-}(W)\right] P^{'}_{\ell-1}(x),$$

$$\vdots$$

$$F_4(W,\theta) = \sum_{\ell=2}^{\cancel{\infty}\,\ell_{\max}} \left[M_{\ell+}(W) - E_{\ell+}(W) - M_{\ell-}(W) - E_{\ell-}(W)\right] P^{''}_{\ell}(x), \text{ where } \{F_i\} \Leftrightarrow \{b_i\},$$

$*$) $4\ell_{\max}$ complex multipoles present in every truncation-order $\ell_{\max} \geq 1$:

$$\mathcal{M}_\ell = \{E_{0+}, E_{1+}, M_{1+}, M_{1-}, E_{2+}, E_{2-}, \ldots, M_{\ell_{\max}-}\}.$$

(Generic case: $N_{\mathcal{A}} * \ell_{\max}$ waves for every order $\ell_{\max} \geq 1$.)

$*$) $\mathcal{M}_\ell$ determined up to 1 overall phase $\Rightarrow$ $\underline{8\ell_{\max} - 1}$ real par.'s in TPWA.

Complete experiments

Complete experiments: 'amplitude formulation'

A *complete experiment* is a minimum subset selected from entire set of $N_{\mathcal{A}}^2$ polarization observables that allows for an unambiguous extraction of the complex amplitudes describing the process (either b_i or $\mathcal{M}_\ell$), up to one unknown overall phase ($\phi(W, \theta)$ for the CEA, $\phi(W)$ for the TPWA).

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A *complete experiment* is a set of measurements that is sufficient to predict all other possible experiments. For polarization experiments, this means a subset of all existing polarization observables that is capable of determining all the remaining observables. cf.: [L. Tiator, AIP Conf. Proc. **1432**, no.1, 162-167 (2012)]

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Predict all remaining observables from determined amplitudes ✓

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$\hookrightarrow \exists$ solution-methods for both CEA and TPWA

CEA-solution using graphs

Standard assumption: $N_{\mathcal{A}}$ moduli $|b_1|, \ldots, |b_{N_{\mathcal{A}}}|$ *fixed* from $N_{\mathcal{A}}$ 'diagonal' obs.'s

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Construct graphs: $*$) Every node (or vertex) $\leftrightarrow$ one amplitude b_i,
$*$) Every edge '$i \rightarrow j$' $\leftrightarrow$ bil. product $b_j^* b_i$ (or rel. phase ϕ_{ij}),

Criteria for *completeness*:

(i) Connectedness of the graph removes continuous ambiguities,

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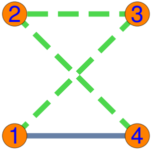
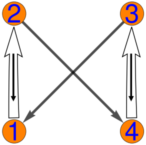
E.g.: photoproduction ($N_{\mathcal{A}} = 4$)

$*$) Moravcsik-graphs (# dashed lines *odd*) [YW *et al.*, PRC **102**, 034605 (2020)]



$\Leftrightarrow$ (over-) complete set: $\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}, \check{F}, \check{G}, \check{C}_{x'}, \check{C}_{z'}, \check{O}_{x'}, \check{O}_{z'}\}$.

$*$) New 'directional' graphs: [YW, PRC **104**, 045203 (2021)]



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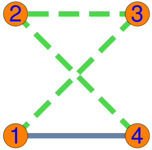

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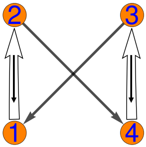

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$\hookrightarrow$ Quite powerful solution-method; generalizes to cases of higher $N_{\mathcal{A}}$ ($N_{\mathcal{A}} > 4$)!

$\hookrightarrow$ Empirically: at least 2 $N_{\mathcal{A}}$ observables in a complete set!

CEA-solution using Fierz identities

Clifford algebra $\left\{ \tilde{\Gamma}^{\alpha} \right\}$ implies so-called 'Fierz-identities':

$$\check{\Omega}^{\alpha} \check{\Omega}^{\beta} = \sum_{\delta,\eta} C^{\alpha\beta}_{\delta\eta} \check{\Omega}^{\delta} \check{\Omega}^{\eta}, \text{ with } C^{\alpha\beta}_{\delta\eta} := \frac{1}{N_{\mathcal{A}}^{2}} \mathrm{Tr} \left[\tilde{\Gamma}^{\delta} \tilde{\Gamma}^{\alpha} \tilde{\Gamma}^{\eta} \tilde{\Gamma}^{\beta} \right].$$

$\Rightarrow$ Use these to solve for complete experiments! ('measurement formulation')

cf.: [Chiang & Tabakin, Phys. Rev. C **55**, 2054-2066 (1997)]

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E.g.: πN-scattering, i.e. $N_{\mathcal{A}} = 2$, $\{b_1, b_2\}$ vs. $N_{\mathcal{A}}^2 = 4$ obs.'s $\left\{\sigma_0, \check{P}, \check{R}, \check{A}\right\}$;
$\sigma_0 = |b_1|^2 + |b_2|^2$, $\check{P} = |b_1|^2 - |b_2|^2$, $\check{R} = -|b_1||b_2| \sin\phi_{21}$, $\check{A} = |b_1||b_2| \cos\phi_{21}$

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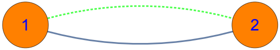
E.g.: πN-scattering, i.e. $N_{\mathcal{A}} = 2$, $\{b_1, b_2\}$ vs. $N_{\mathcal{A}}^2 = 4$ obs.'s $\left\{ \sigma_0, \check{P}, \check{R}, \check{A} \right\}$;

$\sigma_0 = \left| b_1 \right|^2 + \left| b_2 \right|^2$, $\check{P} = \left| b_1 \right|^2 - \left| b_2 \right|^2$, $\check{R} = - \left| b_1 \right| \left| b_2 \right| \sin \phi_{21}$, $\check{A} = \left| b_1 \right| \left| b_2 \right| \cos \phi_{21}$

$*$) Graphical solution (Moravcsik):



$\Leftrightarrow \{\sin \phi_{21}, \cos \phi_{21}\} \Leftrightarrow$ all 4 obs.'s needed ✓

CEA-solution using Fierz identities

Clifford algebra $\left\{\tilde{\Gamma}^{\alpha}\right\}$ implies so-called 'Fierz-identities':

$$\check{\Omega}^{\alpha}\check{\Omega}^{\beta} = \sum_{\delta,\eta} C_{\delta\eta}^{\alpha\beta}\check{\Omega}^{\delta}\check{\Omega}^{\eta}, \text{ with } C_{\delta\eta}^{\alpha\beta} := \frac{1}{N_{\mathcal{A}}^{2}}\mathrm{Tr}\left[\tilde{\Gamma}^{\delta}\tilde{\Gamma}^{\alpha}\tilde{\Gamma}^{\eta}\tilde{\Gamma}^{\beta}\right].$$

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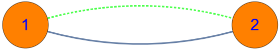

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*) There exists one relevant Fierz-identity:

$$\sigma_0^2 - \check{P}^2 - \check{R}^2 - \check{A}^2 = 0$$

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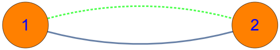

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$\hookrightarrow$ Consistent, but cumbersome, alternative solution-method

$\Rightarrow$ Maybe easier to automate in the future ... (?)

Complete experiments for the TPWA

$*)$ Example: photoproduction ($N_{\mathcal{A}} = 4$). Consider group $\mathcal{S}$ $\left\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\right\}$, i.e. 'diagonal' observables: $\check{\Omega}^{\alpha_S} \propto \pm \left|b_1\right|^2 \pm \left|b_2\right|^2 \pm \left|b_3\right|^2 + \left|b_4\right|^2$.

$\Rightarrow$ Use $t := \tan\left(\frac{\theta}{2}\right)$ and write linear factorizations (for finite ℓ_{max}):

$$b_1\left(\theta\right) \propto \frac{\exp\left(-i\frac{\theta}{2}\right)}{\left(1+t^2\right)^{\ell_{\mathrm{max}}}} \prod_{j=1}^{2\ell_{\mathrm{max}}} \left(t + \beta_j\right), \;\; b_2\left(\theta\right) \propto \frac{\exp\left(i\frac{\theta}{2}\right)}{\left(1+t^2\right)^{\ell_{\mathrm{max}}}} \prod_{j=1}^{2\ell_{\mathrm{max}}} \left(t - \beta_j\right), \; \ldots$$

with $4\ell_{\mathrm{max}}$ roots $\left\{\alpha_k, \beta_j\right\} \in \mathbb{C}$ equivalent to multipoles: $\left\{E_{\ell\pm}, M_{\ell\pm}\right\}$.

[A. S. Omelaenko, Sov. J. Nucl. Phys. **34**, 406 (1981)]

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$\hookrightarrow$ Study discrete ambiguities of group $\mathcal{S}$, generated by:

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$\Rightarrow$ Surprise: *all* ambiguites can be resolved using less than $2N_{\mathcal{A}} = 8$ observables!

[YW, R. Beck and L. Tiator PRC **89**, no.5, 055203 (2014)]

[R. L. Workman, *et al.*, PRC **95**, no.1, 015206 (2017)]

[YW, arXiv:2008.00514 [nucl-th]]

Ideal tool: Bayesian inference

$\ast$) Use *parametric version* of Bayes' Theorem:

$$\underbrace{p(\boldsymbol{\theta}|\boldsymbol{y})}_{\text{'posterior'}} = \frac{\overbrace{p(\boldsymbol{y}|\boldsymbol{\theta})}^{\text{'likelihood'}}\,\overbrace{p(\boldsymbol{\theta})}^{\text{'prior'}}}{\underbrace{\int d^D\theta p(\boldsymbol{y}|\boldsymbol{\theta})\,p(\boldsymbol{\theta})}_{\text{'Bayesian evidence'}}}.$$

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$\ast)$ We have, for either the CEA or the TPWA:

- Parameters: either $\boldsymbol{\theta} = \{b_i\}$, or $\boldsymbol{\theta} = \{E_{\ell\pm}, M_{\ell\pm}\}$,
- $\boldsymbol{y} = [\boldsymbol{y}^{\sigma_0}, \boldsymbol{y}^{\check{G}}, \ldots, \boldsymbol{y}^{\check{F}}]^T$: values of measured observables,
- Likelihood has form 'exp$[-\frac{1}{2}\chi^2]$ (in most cases),
- Prior $p(\boldsymbol{\theta})$ is chosen *flat*, within the 'physically allowed' region for the amplitudes; this region is *constrained* by:
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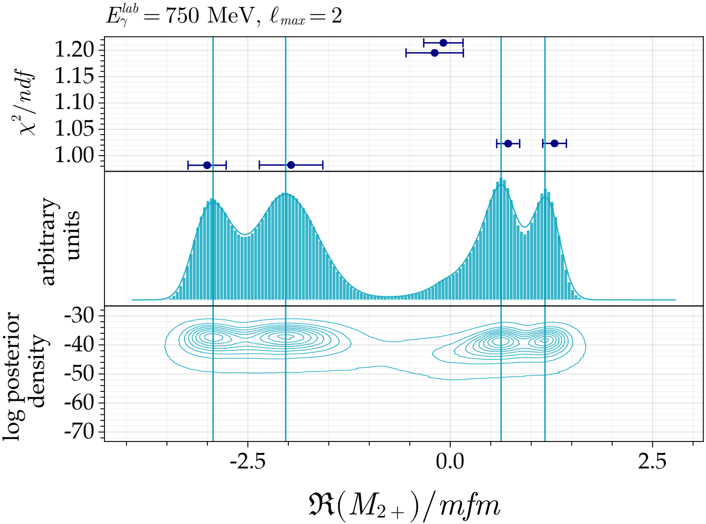
$\Rightarrow$ Obtain (for instance) marginalized probability-distribution for each parameter θ_i, i.e.: $p(\theta_i|y) = \int \ldots \int d\theta_1 \ldots d\theta_{i-1} d\theta_{i+1} \ldots d\theta_D\, p(\theta_1 \ldots \theta_D|\boldsymbol{y})$, using state-of-the-art (Markov chain) Monte Carlo methods.

Some results for a TPWA - I

Example: TPWA for $\gamma p \longrightarrow \eta p$, using observables $\{\sigma_0, \Sigma, T, E, G, F\}$, and $\ell_{\mathrm{max}} = 2$ cf.: [P. Kroenert, YW, F. Afzal and A. Thiel, arXiv:2305.10367 [nucl-th].]



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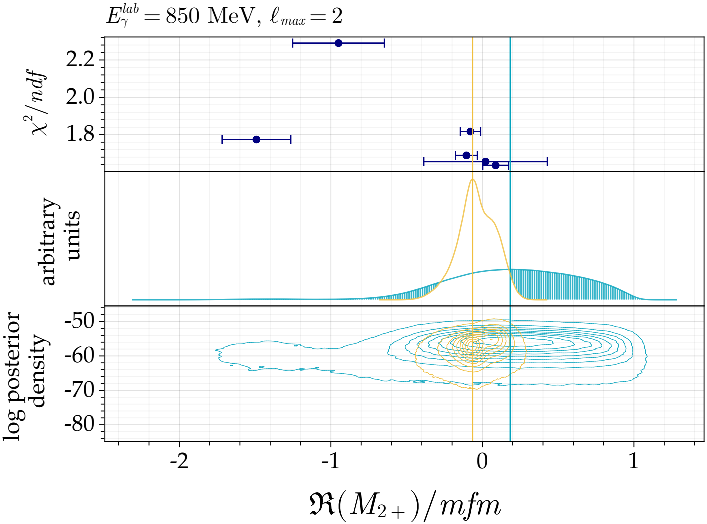
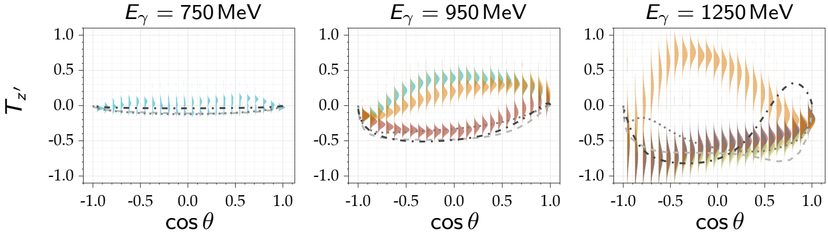


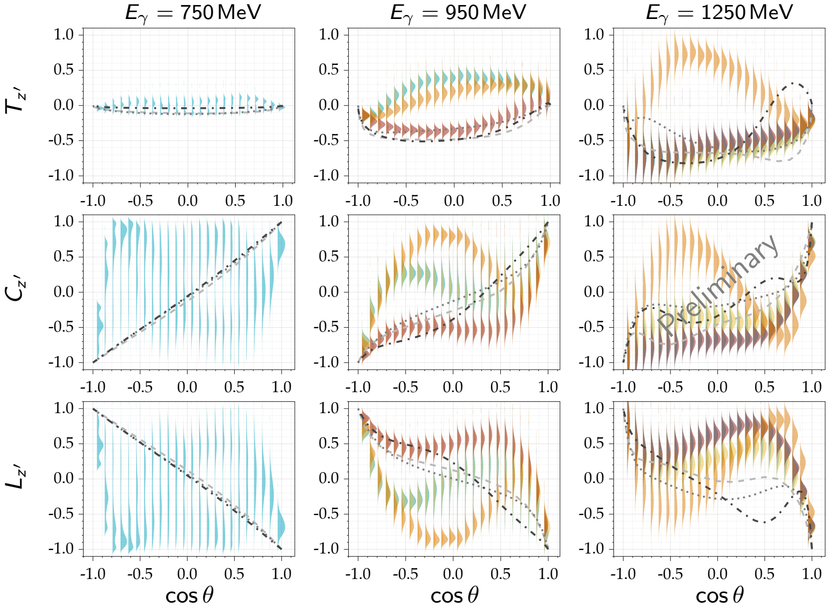
Some results for a TPWA - II



PWA-curves:

EtaMAID2018 (dashed); BnGa-2019 (dotted); Jülich-Bonn-2022 (dash-dotted).

Some results for a TPWA - II



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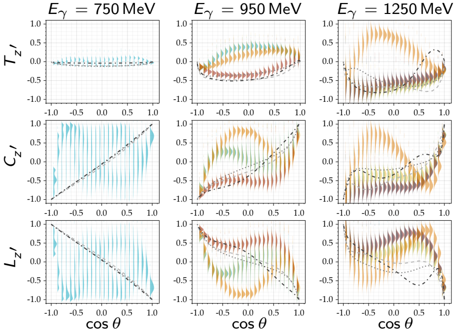

$\hookrightarrow$ Predict promising *candidate observables* for resolving discrete ambiguities:

E_γ^{lab} / MeV	Observables
750	$C_{x'}, C_{z'}, L_{x'}, L_{z'}$
850	$C_{x'}, C_{z'}, L_{x'}, L_{z'}, T_{x'}, T_{z'}$
950	$C_{x'}, C_{z'}, L_{x'}, L_{z'}, T_{z'}$
1050	$C_{x'}, C_{z'}, L_{x'}, O_{z'}, T_{z'}$
1150	$C_{z'}, O_{x'}, T_{x'}, T_{z'}$
1250	$C_{z'}$

cf.: [P. Kroenert, YW, F. Afzal and A. Thiel, arXiv:2305.10367 [nucl-th].]

Thank You!

Additional Slides

Measuring the overall phase

Measure the overall phase in scattering experiments using vortex beams

$\equiv$ beams of particles with intrinsic orbital angular momentum $\langle L_z \rangle = \hbar \ell$ along the axis of propagation (i.e. z-axis) cf. [Ivanov, Phys. Rev. D **85**, 076001 (2012)], [Ivanov, arXiv:2205.00412 [hep-ph] (2022)]

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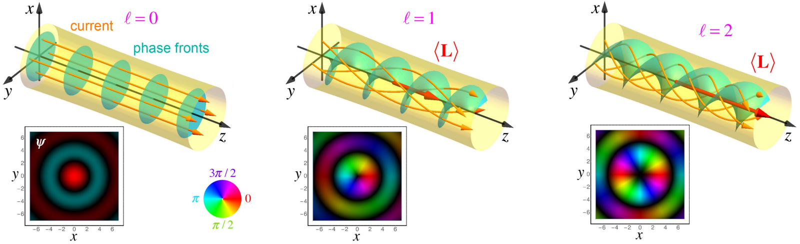

[arXiv:2205.00412, Fig.1]

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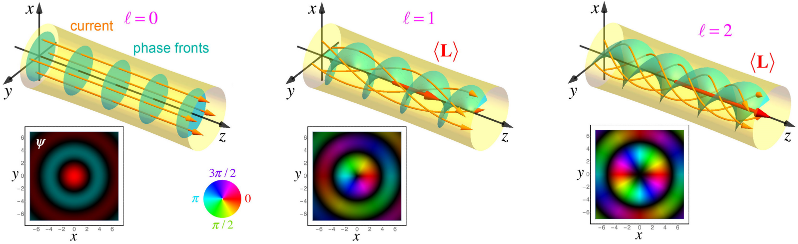


[arXiv:2205.00412, Fig.1]

Proposal [Ivanov, Phys. Rev. D **85**, 076001 (2012)]: for the example $\gamma p \to \pi p$, consider double-twisted γp-collision (i.e. both γ and p are in a [Bessel] vortex-state)

$\Rightarrow$ Measure *azimuthal asymmetry* $A = \frac{\Delta\sigma}{\sigma}$ ('sine-weighted' c.s. $\Delta\sigma$; non-weighted σ)

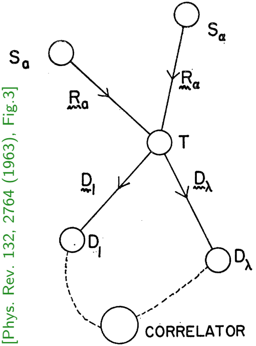
$\Rightarrow$ Then, one has $\boxed{A = \frac{d\phi\left(\theta_{\gamma\pi}^{\mathrm{LAB}}\right)}{d\theta_{\gamma\pi}^{\mathrm{LAB}}} \cdot P}$, with an 'analyzing power' P.

$\Rightarrow$ For insanely good accuracy and statistics, integration yields: $\phi\left(\theta_{\gamma\pi}^{\mathrm{LAB}}\right) + \mathcal{C}$.

Vortex-beams at the GeV-scale maybe feasible within 10-20 years [Ivanov, priv. comm. (2022)]

The Hanbury-Brown and Twiss experiment

Measure the overall phase via intensity correlations in a *Hanbury-Brown and Twiss-type* experiment [Goldberger, Lewis & Watson, Phys. Rev. 132, 2764 (1963)]



[Phys. Rev. 132, 2764 (1963), Fig.3]

- $\ast$) Two sources, S_a and S_α, emitting beam-particles
- $\ast$) One single irradiated target T
- $\ast$) Two spatially separated detectors, D_l and D_λ
- $\ast$) A CORRELATOR, which registers only in case D_l and D_λ count in *coincidence*

$\hookrightarrow$ The correlator counting-rate contains an isolatable term, which is proportional to:

$$\mathrm{Re}\,[\mathcal{T}_{\lambda\leftarrow\alpha}\mathcal{T}^*_{l\leftarrow\alpha}\mathcal{T}_{l\leftarrow a}\mathcal{T}^*_{\lambda\leftarrow a}].$$

$\ast$) Assuming $|\mathcal{T}_{f\leftarrow i}|$ as known, one can measure $\cos(\Gamma)$, where:

$$\Gamma := \phi(\boldsymbol{g}_{la}) - \phi(\boldsymbol{g}_{l\alpha}) + \phi(\boldsymbol{g}_{\lambda\alpha}) - \phi(\boldsymbol{g}_{\lambda a}), \text{ with } \boldsymbol{g}_{la} \equiv \hat{R}_a - \hat{D}_l, \ \ldots .$$

$\ast$) *Varying positions* of detectors and sources, do measurements for *many* angles:

$$\left\{\boldsymbol{g}^{(1)}_{la}, \boldsymbol{g}^{(2)}_{la}, \ldots\right\}, \left\{\boldsymbol{g}^{(1)}_{l\alpha}, \boldsymbol{g}^{(2)}_{l\alpha}, \ldots\right\}, \left\{\boldsymbol{g}^{(1)}_{\lambda\alpha}, \boldsymbol{g}^{(2)}_{\lambda\alpha}, \ldots\right\}, \left\{\boldsymbol{g}^{(1)}_{\lambda a}, \boldsymbol{g}^{(2)}_{\lambda a}, \ldots\right\}.$$

$\hookrightarrow$ Extract: $\phi\left(g^{(\nu+1)}_{la}\right) - \phi\left(g^{(\nu)}_{la}\right) \equiv \delta\phi(\nu) \longrightarrow$ Overall phase: $\phi(\boldsymbol{g})$.