On the prediction of spectral densities from Lattice QCD: theoretical aspects

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Motivations

Predictions of hadronic amplitudes, decay rates, spectral densities
- important tests of the Standard Model
  - $(g-2)_\mu$ based e.g. on $\gamma \to \pi^+\pi^-$, $\pi^0 \to \gamma\gamma$
  - test of CP violation in K, D decays
- improve our understanding of strong interactions
  - properties of resonances like ρ^0

Lattice QCD: non-perturbative formulation of QCD
- first-principle predictions from 3-4 input parameters
- systematically improvable w/ better algorithms and HPC

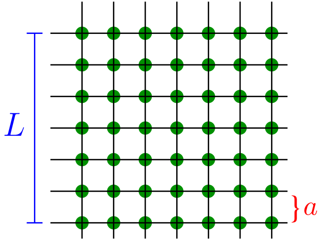
Lattice field theories

lattice spacing $a \to$ regulate UV divergences

finite size $L \to$ infrared regulator

Continuum theory $a \to 0$, $L \to \infty$

Euclidean metric $\to$ Boltzman interpretation of path integral



$$\langle O \rangle = \mathcal{Z}^{-1} \int [DU] e^{-S[U]} O(U) \approx \frac{1}{N} \sum_{i=1}^{N} O[U_i]$$

Very high dimensional integral $\to$ Monte-Carlo methods

Markov Chain of gauge field configs $U_0 \to U_1 \to \cdots \to U_N$

Analytic continuation - I

Time-momentum representation very natural for Lattice QCD
  project operator $\mathcal{O}$ to definite spatial momentum
  evaluate $C(t) = \langle \mathcal{O}(t)\mathcal{O}(0) \rangle$

$+$ Physical observables as integrals of spectral densities
  $P = \int d\omega\, \kappa(\omega)\, \rho(\omega)$
  e.g. inclusive diff. decay rate semileptonic [Gambino, Hashimoto '20]

$+$ Correlator is integral of spectral density $C(t) = \int d\omega e^{-\omega|t|} \rho(\omega)$

$=$ Solve $P = \int dt\, f(t)\, C(t)$ for unknown f?
  $\kappa(i\omega)$: study analytic continuation of kernel

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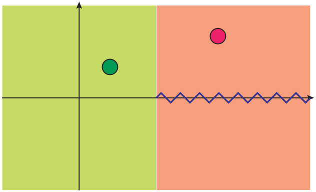
Analytic continuation - II



ρ has branch cuts starting at multi-particle thresholds $E_{\rm thr}$

Kernel κ can have poles s_i in complex plane

if $\mathrm{Re} s_i \leq E_{\rm thr}$ direct analytic continuation from Euclidean $C(t)$

$\forall t \; \exists M > 0 \mid f(t) C(t) < e^{-Mt}$

e.g. HVP contribution to $(g-2)_\mu$ [Blum '02][Bernecker-Meyer '11]

if $\mathrm{Re} s_i > E_{\rm thr}$ direct analytic continuation not possible

$\sum_t f(t)\, G(t)$ diverges exponentially

Inverse Problem

Lattice QCD: $C(t)$ for finite Euclidean t + non-zero errors

inverse Laplace ill-defined problem $\to$ approximate num. solutions

[M. Saccardi's talk] [Backus, Gilbert '68][HLT '19][Bailas et al. '20 ...]

smooth Dirac δ_σ

$C(t)$ finite discrete Euclidean times

cannot extract continuous $\rho(\omega)$

$\rho_\sigma(\omega) = \int d\omega' \, \rho(\omega') \, \delta_\sigma(\omega' - \omega)$

$\delta_\sigma(x) = \frac{\sigma/\pi}{x^2+\sigma^2}$

Smeared ρ needed for 2 reasons

large smearing improves regularized solutions

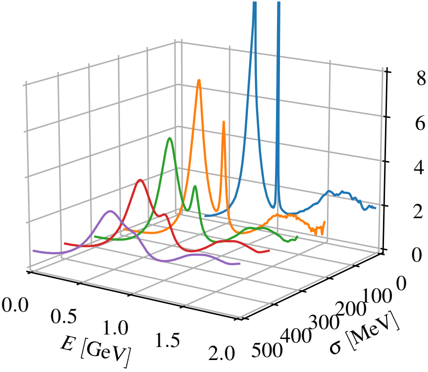
large smearing improves finite-volume errors (this talk)

Cutoff effects of smeared ρ under investigation following [Sommer Lat22]

Smeared spectral densities

ρ_σ analityc continuation in complex plane of ρ [Poggio, Quinn, Weinberg '76]

$$\rho_\sigma(\omega) = \int dx \rho(x)\delta_\sigma(x,\omega) = \frac{\sigma}{\pi}\int dx \frac{\rho(x)}{(x-\omega)^2+\sigma^2}$$
$$= \frac{1}{2\pi i}\Big[\int_{\mathbb{R}+i\sigma} - \int_{\mathbb{R}-i\sigma}\Big] dz \frac{\rho(z)}{z-\omega}$$



ρ_σ has physical meaning

Calculable in PT deep in complex plane ($\sigma \gg 2$ GeV)

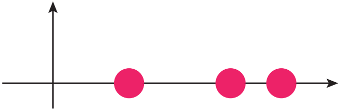
Lattice calculations $L \simeq 1/\sigma$

R-ratio data [F. Jegelehner]

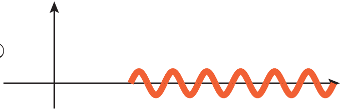
Lattice Simulations performed in finite box $L^3 \times T$ (T large)

periodic BC $\vec{p} = \frac{2\pi}{L}\vec{n}$, $\vec{n} \in \mathbb{Z}^3$

$\rightarrow$ spectrum quantized



$L \rightarrow \infty$



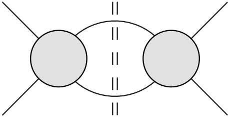
Hamiltonian $\hat{H}_L$ (on slice L^3), momentum operator $\hat{P}_i$

Hilbert space $\hat{H}_L|n,\vec{p}\rangle_L = E_n(\vec{p},L)|n,\vec{p}\rangle_L$

scattering? decay rates?

what is meaning of $|n,\vec{p}\rangle_L$ and $E_n(\vec{p},L)$?

1. s-channel one-loop diagram



$$\int \frac{dk_0}{2\pi}\int \frac{d\vec{k}}{(2\pi)^3}\frac{iz(k)}{k^2-m^2+i\epsilon}\,\frac{iz(P-k)}{(P-k)^2-m^2+i\epsilon}$$
$$\int \frac{dk_0}{2\pi}\,\frac{1}{L^3}\sum_{\vec{k}}\frac{iz(k)}{k^2-m^2+i\epsilon}\,\frac{iz(P-k)}{(P-k)^2-m^2+i\epsilon}$$

2. evaluate integral-sum difference w/ Poisson's formula
   non-analytic function $\rightarrow 1/L^n$ corrections, i.e. loop legs on-shell
   analytic function $\rightarrow e^{-mL}$ corrections, i.e. loop legs off-shell

3. re-sum all $2 \rightarrow 2$ diagrams
   $2 \rightarrow 4$ diagrams $1/L^k$ correction if $\sqrt{s} > 4m$
   quantization condition $Q(E_n) = n\pi$

Smeared spectral densities

Finite volume effects

Scalar current J projected to zero-momentum

$$\rho(\omega) = \langle 0|\hat{J}\,\delta(\hat{H}-\omega)\,\delta^3(\vec{P})\,\hat{J}|0\rangle \quad \rightarrow \quad \rho(\omega|L) = \sum_n |\langle 0|\hat{J}|n\rangle_L|^2$$

Goal: finite volume effects of $\int d\omega\, \rho(\omega|L)\,\kappa(\omega)$ [MB, Hansen in prep]

$\kappa(\omega) = e^{-\omega t}$: correlator (checks w/ literature)
$\kappa(\omega) = \delta_\sigma(\omega - E)$: smeared ρ_σ

Setup of our derivation:

1. lowest partial wave Lüscher quantization condition $Q(E_n) = n\pi$
2. applicable to $I = 1$ vector-vector channel

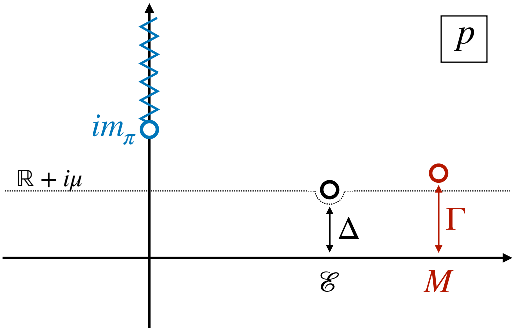
Our work builds upon [Lellouch-Lüscher '00][Hansen-Sharpe '12][...]
[Bulava, Hansen, Hansen, Patella, Tantalo '21]

Finite volume effects

Preliminary

[MB, Hansen in prep]

$$\rho_\kappa(\omega|L) - \rho_\kappa(\omega) \to \int_{\mathbb{R}+i\mu} dp \frac{p^2}{p^2+m_\pi^2}\, e^{2iQ(\omega)}\, |F_\pi(\omega)|^2\, \kappa(\omega)$$



$$\omega(p) = 2\sqrt{m_\pi^2 + p^2}$$

$$e^{2iQ(\omega)} \simeq e^{ipL} \to e^{-\mu L}$$

how large μ?

Finite-vol effects driven by analytic structure of

$\omega(p)$

$|F_\pi(\omega(p))|^2$

$\kappa(\omega(p))$

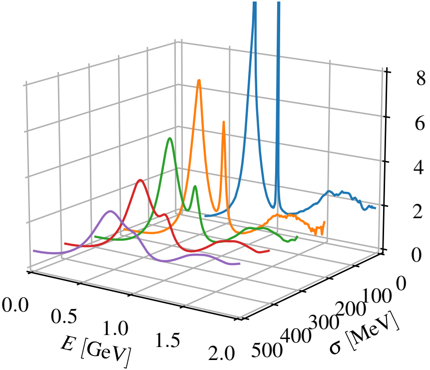
Conclusions

Smeared spectral densities ρ_σ have physical meaning

$\sigma \gg 0$ needed to control finite-vol effects

is $\sigma \simeq 0$ really needed for physics?

e.g. CMD3 vs BaBar vs KLOE $\sigma \simeq m_\pi$ likely sufficient!



[MB, Hansen in prep]

(a plausible) recipe

1. take lattice w/ given $m_\pi L$
2. take smeared ρ w/ $\sigma = m_\pi$
3. calculate ρ w/ stat. errs 1%
4. move to larger L (same m_π)

for point 3. [M. Saccardi's talk]

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Thanks for the attention