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The role of convergence methods as fitting functions in the context of the MUonE experiment

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Content

- Muon anomalous magnetic moment (deep introduction: last monday talk by Mattia Bruno)
- MUonE Experiment
- Padé Approximants Introduction
- Method & simulations
- Results with realistic errors using Padé Approximants
- D-Log Padé Approximants (work in progress)
- Outlook & Conclusions

Muon Anomalous magnetic moment

The anomalous magnetic moment of the muon

- The magnetic moment (for charged leptons)

$$\vec{\mu_l} = g_l \frac{e}{2m_l} \cdot \vec{S}$$

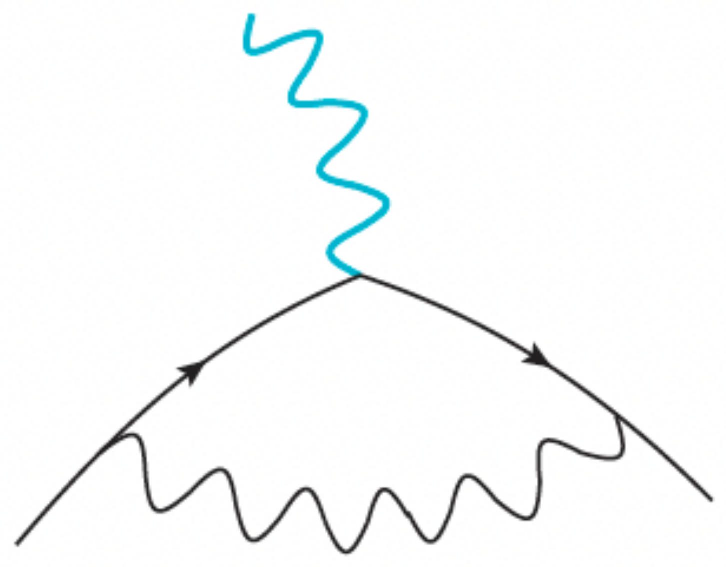
g_l is gyromagnetic ratio

$\xrightarrow[\text{Spin = 1/2}]{\text{Dirac Theory}}$ $g_l = 2$ $\xrightarrow{\text{Radiative correction of QFT}}$ $g_l \neq 2$

- The magnetic anomaly (deviation from Dirac value)

$$a_l = (g_l - 2)/2$$

This observable can be both precisely measured experimentally and predicted in the Standard Model, providing a stringent test of the SM.



"These quantum fluctuations modify g"

- The first order correction (by J. Schwinger)

$$a_e^{\text{QED,LO}} = \alpha/2\pi \approx 1.16 \times 10^{-3}$$

α is fine structure constant

The anomalous magnetic moment of the muon

- E821 @BNL measurement with an error of 0.54 ppm

$$a_\mu^{\text{exp}} = 116592089(63) \times 10^{-11}$$

Bennet et al, PRD73,072003 (2006)

Merge with FNAL measurement

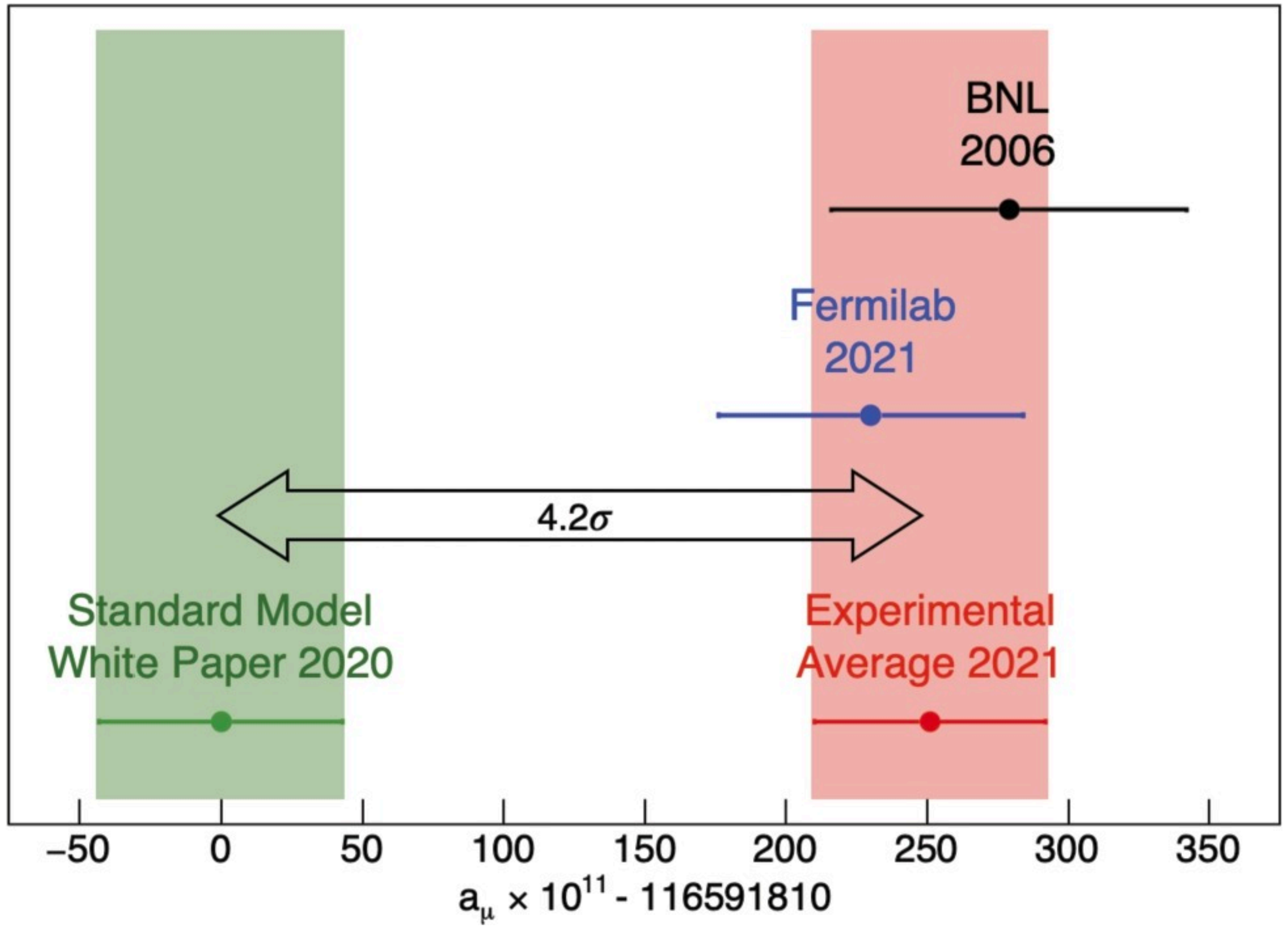
$$a_\mu^{\text{exp}} = 116592089(63) \times 10^{-11}$$ [0.35 ppm]

B. Abi et al, Phys. Rev. Lett.126,141801(2021)

- The theoretical calculation with SM (approved consensus)

$$a_\mu^{\text{SM}} = 116591810(43) \times 10^{-11}$$

By the Muon $g-2$ Theory Initiative
T Aoyama et al, Phys. Rep., 887 (2020)



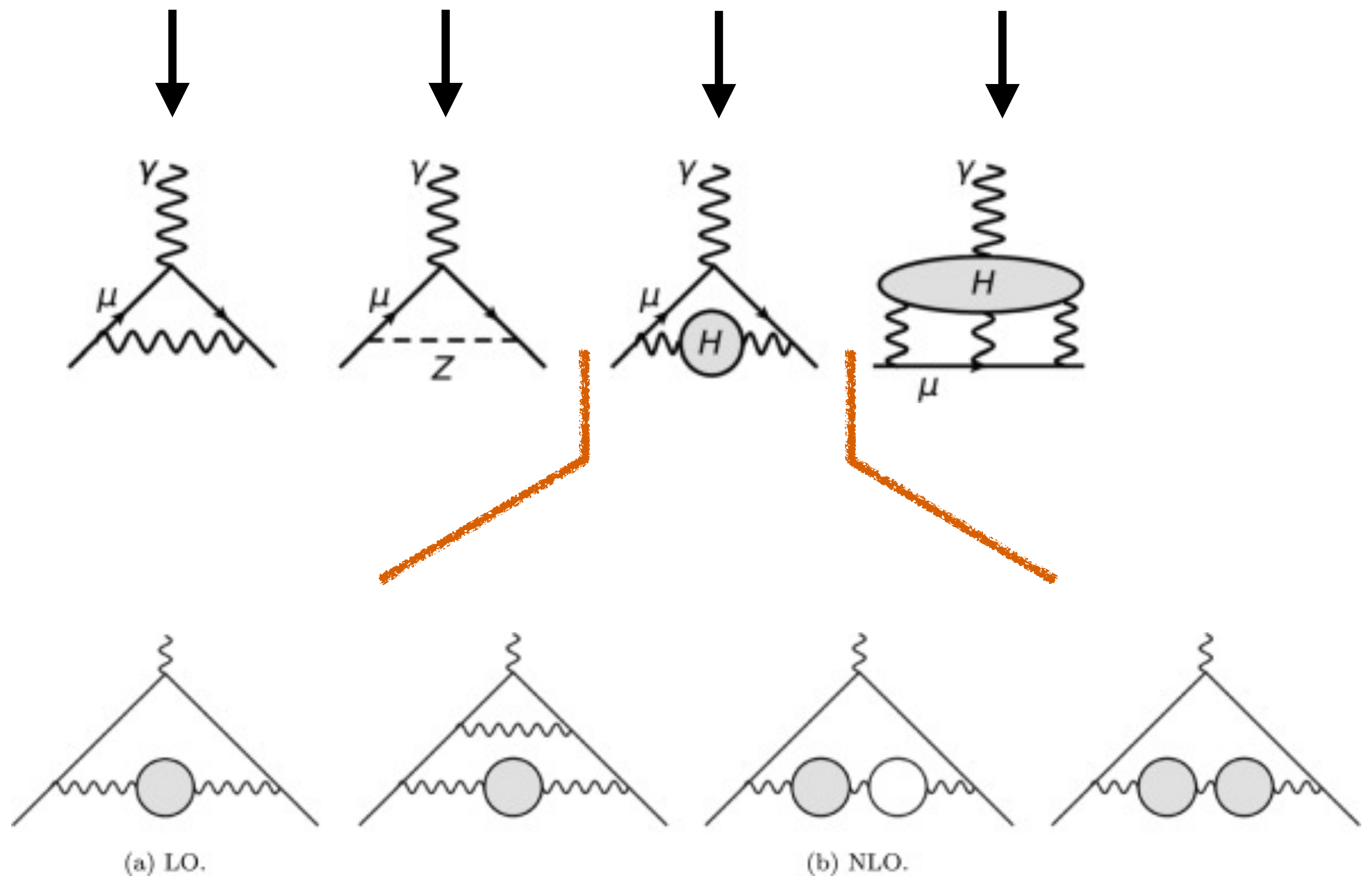
Current discrepancy limited by:

- Experimental uncertainty: last run at FNAL and J-PARC x4 accuracy
- In the theoretical frame: How to calibrate Hadronic Contributions uncertainties.

$$\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{th}} = 251(59) \times 10^{-11}$$

The anomalous magnetic moment of the muon

$$a_\mu^{SM} = a_\mu^{QED} + a_\mu^{EW} + a_\mu^{HVP} + a_\mu^{HLbL}$$



Main contribution: LO Vacuum Polarization estimated
rel. uncertainty 0.35% - 0.6%

$$a_\mu^{HVP} = a_\mu^{LO,HVP} + a_\mu^{NLO,HVP} + a_\mu^{NNLO,HVP}$$

- QED corrections known up to 5 loops with related precision $\sim 7 \times 10^{-10}$

$$a_\mu^{QED} = 116584718.931(104) \times 10^{-11}$$ *Aoyama, Hayakawa, Kinoshita,Nio (2012)*

- EW corrections up to 2 loops with precision $\sim 10^{-9}$ rel. uncertainty $< 1\,\%$

$$a_\mu^{EW} = 153.6(1.0) \times 10^{-11}$$ *Gnendiger, D. Stöckinger, H. Stöckinger-Kim (2013)*

- Hadronic contribution $\sim 7 \times 10^{-8}$; the dominant theoretical uncertainty

$$a_\mu^{HVP} = 6845(40) \times 10^{-11}$$ *T Aoyama et al, Phys. Rep., 887 (2020)*

>> not calculable by pQCD <<

Data-Driven approach (Timelike domain)

Lattice QCD

Experiment MUonE (Spacelike domain)

How to calculate the HVP contribution ?

- Using the optical theorem
- Involving the total hadronic cross section measured experimentally at e +e − machines

Using the dispersive relation integral:

$$a_\mu^{\mathrm{HVP,LO}} = \frac{\alpha}{\pi^2}\int_0^\infty \frac{ds}{s}\, K(s)\, \mathrm{Im}\Pi_{\mathrm{had}}(s+i\epsilon) \quad ; \quad K(s) = \int_0^1 \frac{x^2(1-x)}{x^2+(1-x)\frac{s}{m_\mu^2}}\, dx \sim \frac{1}{s}$$

$\mathrm{Im}\Pi_{\mathrm{had}}(s)$ Is the hadronic contribution to the photon vacuum polarisation function

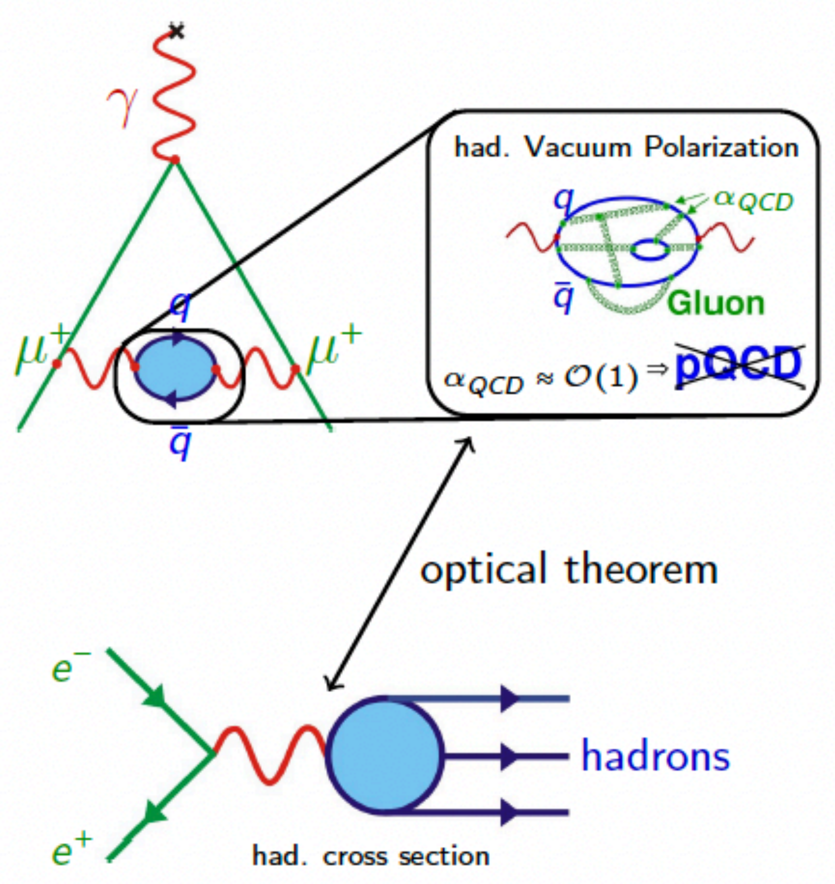
Kernel has the lepton information !

Data-Driven

$$a_\mu^{\mathrm{HVP,LO}} = \left(\frac{\alpha m_\pi}{3\pi}\right)^2 \int_{4m_\pi^2}^\infty ds\, \frac{K(s)R(s)}{s^2}$$

$$R(s) = \left(\frac{3s}{4\pi\alpha^2}\right) \quad \sigma_{e^+e^-\to \mathrm{hadrons}}(s) = 12\pi \mathrm{Im}\Pi_{\mathrm{had}}$$

$$a_\mu^{\mathrm{HVP,LO}} = \left(\frac{\alpha m_\pi}{3\pi}\right)^2 \int_{4m_\pi^2}^\infty ds\, K(s)\, \sigma_{e^+e^-\to \mathrm{hadrons}}(s)$$



Alternative representation usually use in:

Lattice

$$\int ds \int dx \rightarrow \int dx \int ds$$

In terms of the Euclidean space

$\bar{\Pi}(Q^2)\ ;\ Q^2 = -q^2 < 0$

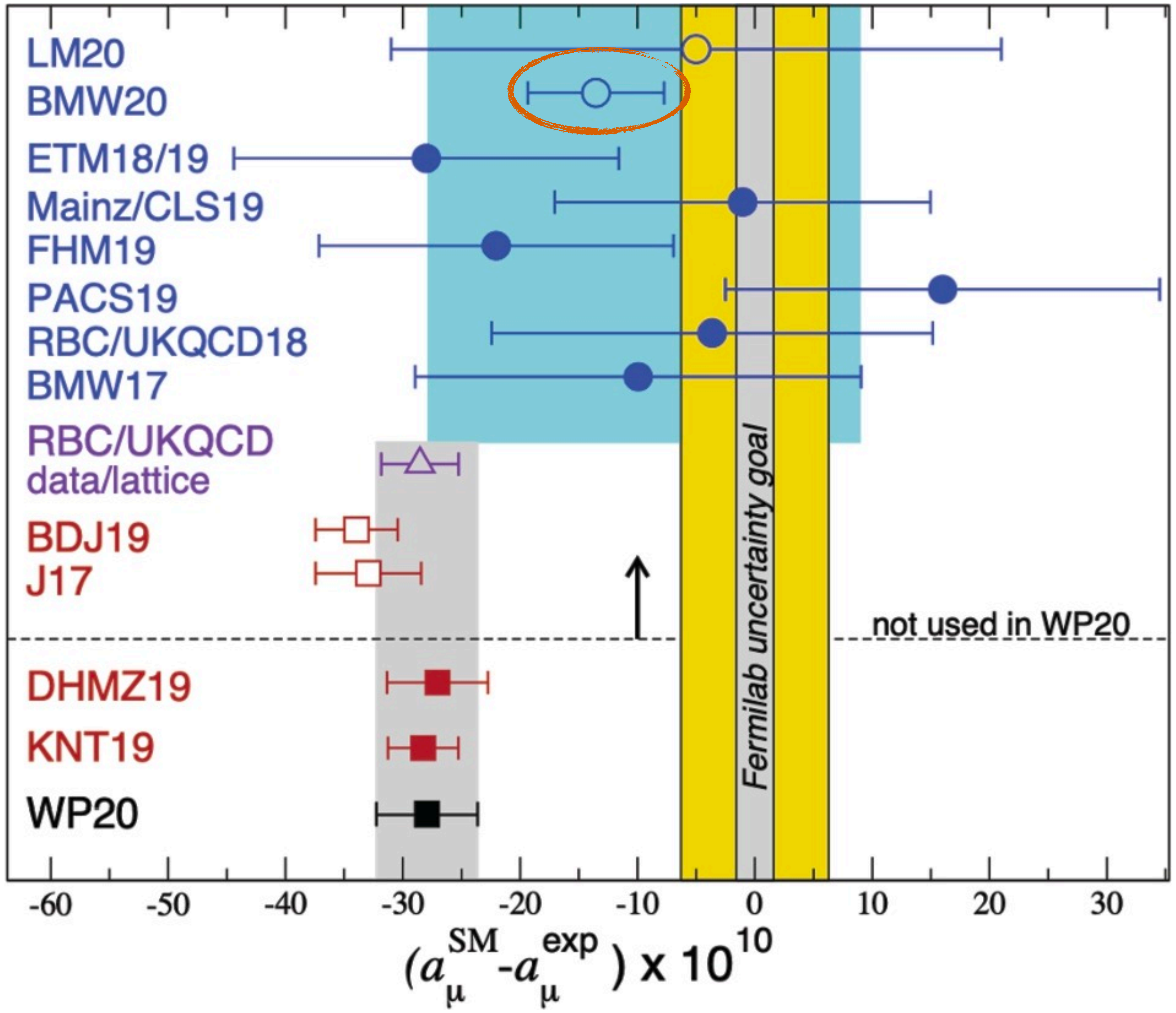
$$a_\mu^{\mathrm{HVP,LO}} = \frac{\alpha}{\pi}\int_0^1 dx\ (x-1)\,\bar{\Pi}^{\mathrm{HVP}}(Q^2)$$

$$\bar{\Pi}^{\mathrm{HVP}}(Q^2) = \int_{s_0}^\infty \frac{ds}{s}\frac{Q^2}{(s+Q^2)}\frac{1}{\pi}\mathrm{Im}\Pi_{\mathrm{had}}(s)$$

This representation would be useful for MUonE experiment since it is in the space-like domain

C. Aubin, T. Blum, Phys. Rev. D, 75 (2007)
P. Boyle, L. Del Debbio, E. Kerrane, J. Zanotti, Phys. Rev. D, 85 (2012)
X. Feng, K. Jansen, M. Petschlies, D.B. Renner, Phys. Rev. Lett., 107 (2011)

Tension between Data-driven Vs Lattice QCD

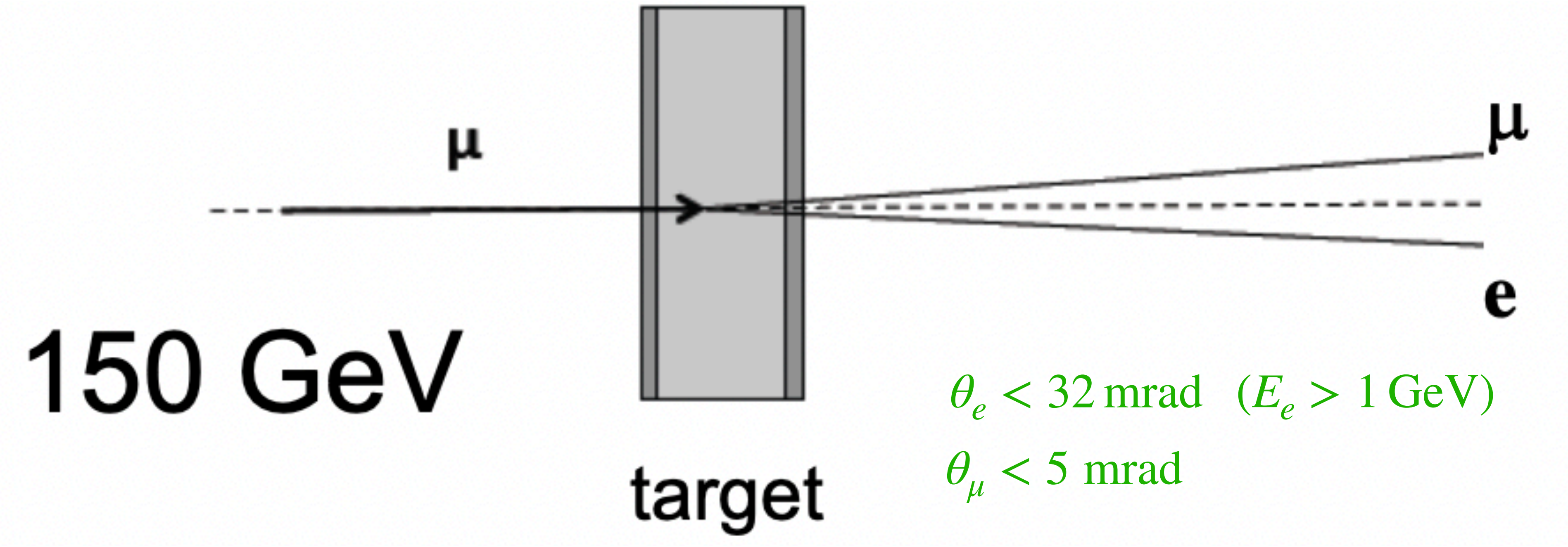


MUonE Experiment

MUonE Experiment ($\mu e \to \mu e$) @CERN

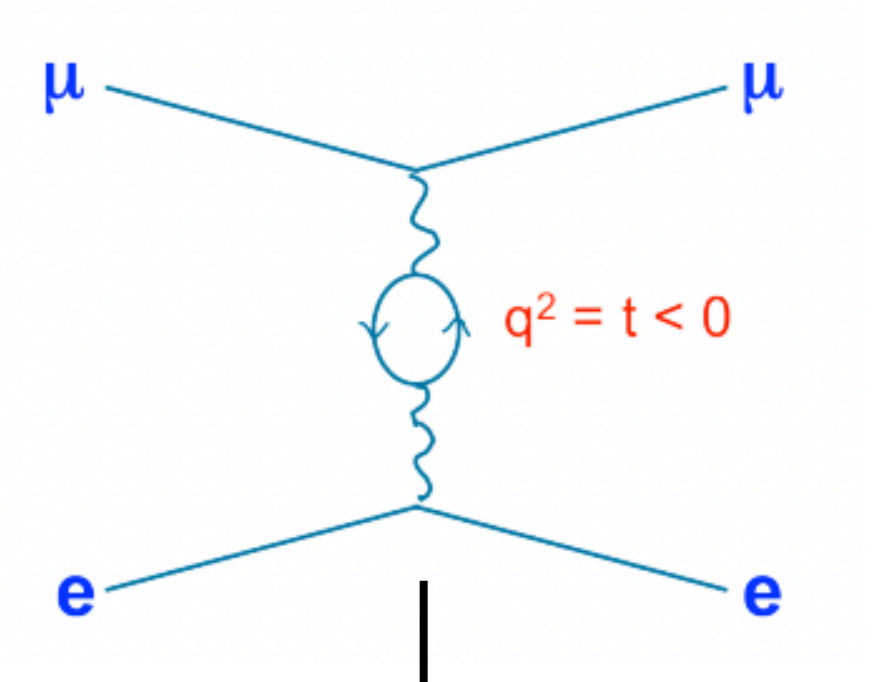
It is a new experimental proposal @ CERN:

- Scattering μ's on e's in a low Z target looks like an ideal process (fixed target experiment)
- It is a pure t-channel process at tree level
- The M2 muon beam-line ($E_\mu \simeq 150\,\mathrm{GeV}$) is available at CERN
- Useful C.M. energy to test dominant region of $a_\mu^{LO,HVP}$ $\sqrt{s} \simeq 0.4\,\mathrm{GeV} \;\to\; q^2 < 0.11\,\mathrm{GeV}^2$
- With ~ 3 years of data taking, a statistical accuracy of 0.35% on $a_\mu^{LO,HVP}$ can be achieved
- Easy selection based on the correlation of the electron and muon scattering angles.



$\theta_e < 32\,\mathrm{mrad} \quad (E_e > 1\,\mathrm{GeV})$

$\theta_\mu < 5\,\mathrm{mrad}$



$$\frac{d\sigma}{dt} \approx \frac{d\sigma_0}{dt}\left|\frac{\alpha(t)}{\alpha_0}\right|^2$$

Running of α

$$\left|\frac{1}{1-\Delta\alpha(t)}\right|^2 = \left|\frac{\alpha(t)}{\alpha_0}\right|^2$$

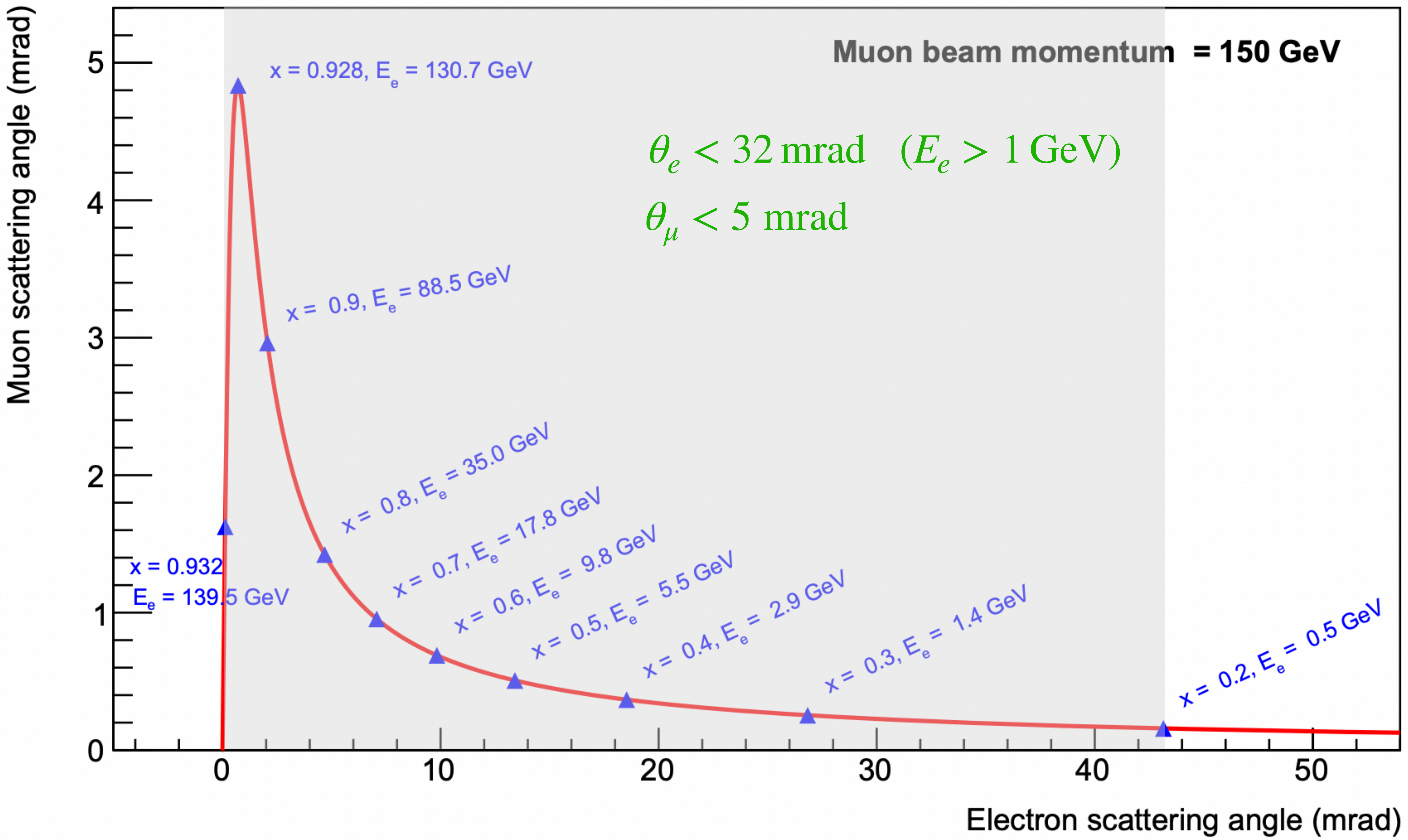
$$\Delta\alpha(t) = \Delta\alpha_{lep}(t) + \Delta\alpha_{had}(t)$$

Known from QED

To be measured

Since it is related with HVP contribution of a_μ

MUonE experiment - event selection



MUonE experiment - measurement in the space-like momentum

$\Delta\alpha(q^2) = -\,\mathrm{Re}\,\bar{\Pi}(q^2)$
$\mathrm{Im}\,\bar{\Pi}(q^2 < 0) = 0$

Recovering the Master Formula

$$a_\mu^{\mathrm{HVP,LO}} = \frac{\alpha^2}{\pi}\int_0^1 \mathrm{d}x\ \ (1-x)\,\Delta\alpha_{\mathrm{had}}[t(x)]$$

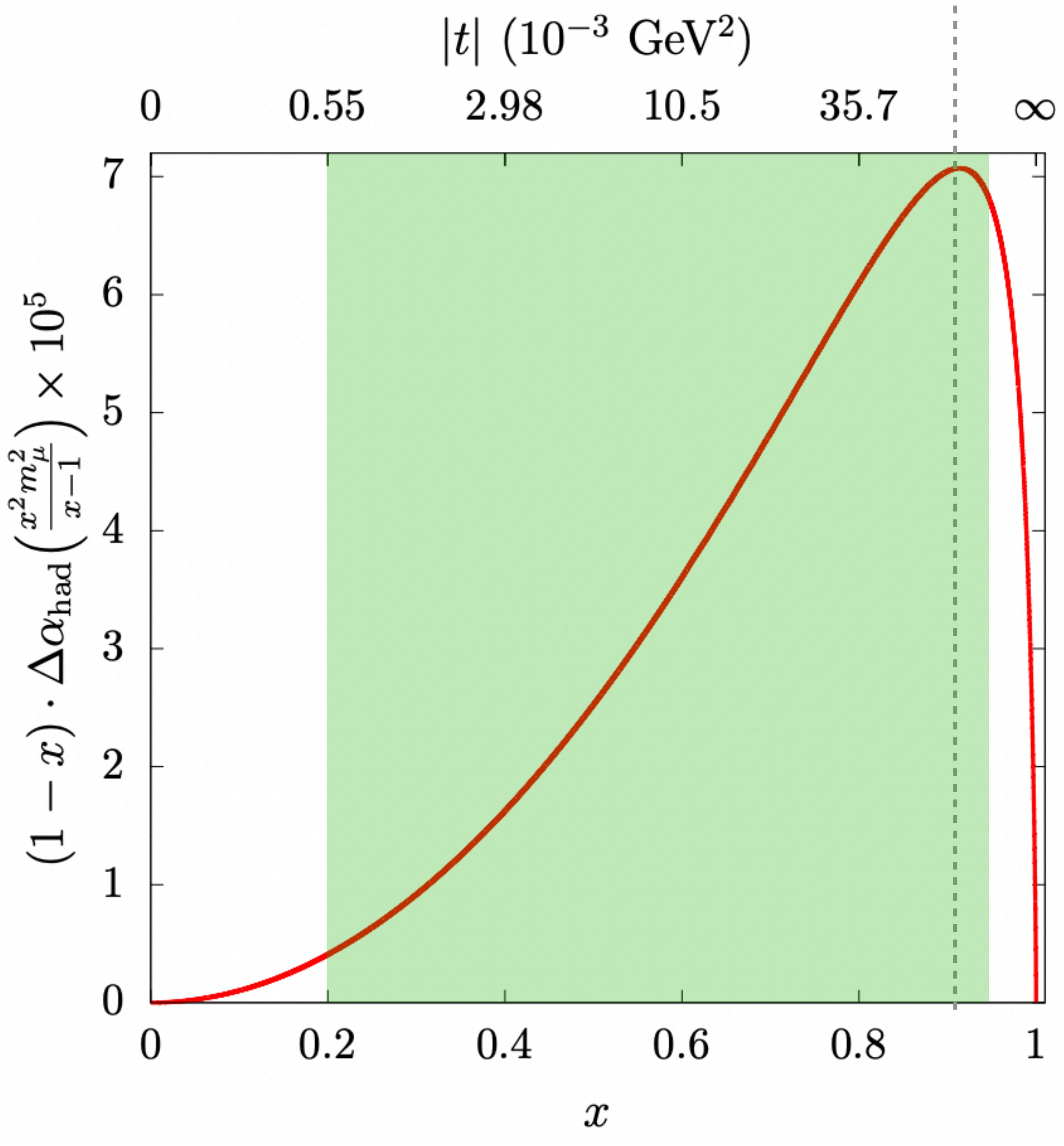
$$t(x) = -\frac{x^2 m_\mu^2}{1-x} \qquad 0 \le -t < \infty \qquad t \leftrightarrow x \qquad x(t) = \frac{t}{2m_\mu^2}\left(1-\sqrt{1-\frac{4m_\mu^2}{t}}\right) \qquad 0 \le x < 1$$

$t < -0.143\,\mathrm{GeV}^2$ <<. Real Experimental Range. >> $0.2 < x < 0.93$

using the output of the routine hadr5n12 (which uses time-like hadron production data and perturbative QCD)

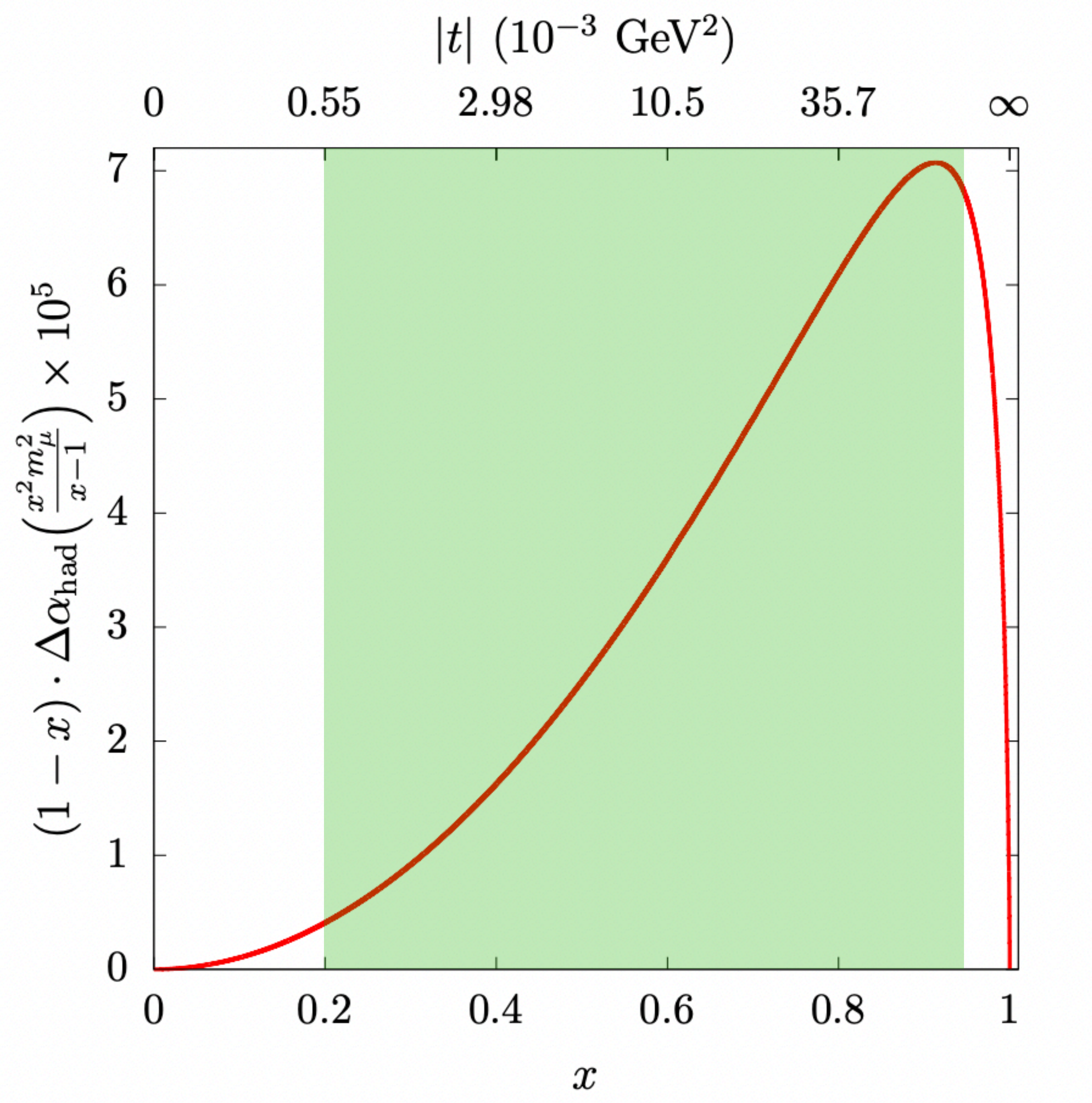
taken from F. Jegerlehner. Nucl Phy proc suppl (2008)

$x_{peak} \simeq 0.914$
$t_{peak} \simeq -0.108\ \mathrm{GeV}^2$



- $a_\mu^{LO,HVP}$ is given by the integral of the curve (smooth behaviour)
- Requires a measurement of $\Delta\alpha_{\mathrm{had}}$ in the space-like region ($t = q^2 < 0$)
- It enhances the contribution from low q^2 region (below 0.11 GeV^2)
- They expect to cover 87% of the $a_\mu^{LO,HVP}$ with the space-like integral
  - But they need to extrapolate: x → 1, (13% missing)
- Its precision is determined by the uncertainty on $\Delta\alpha_{\mathrm{had}}$ in this region

Here is our motivation !



Finding a reliable method to fit the data + good ***extrapolation*** outside data region

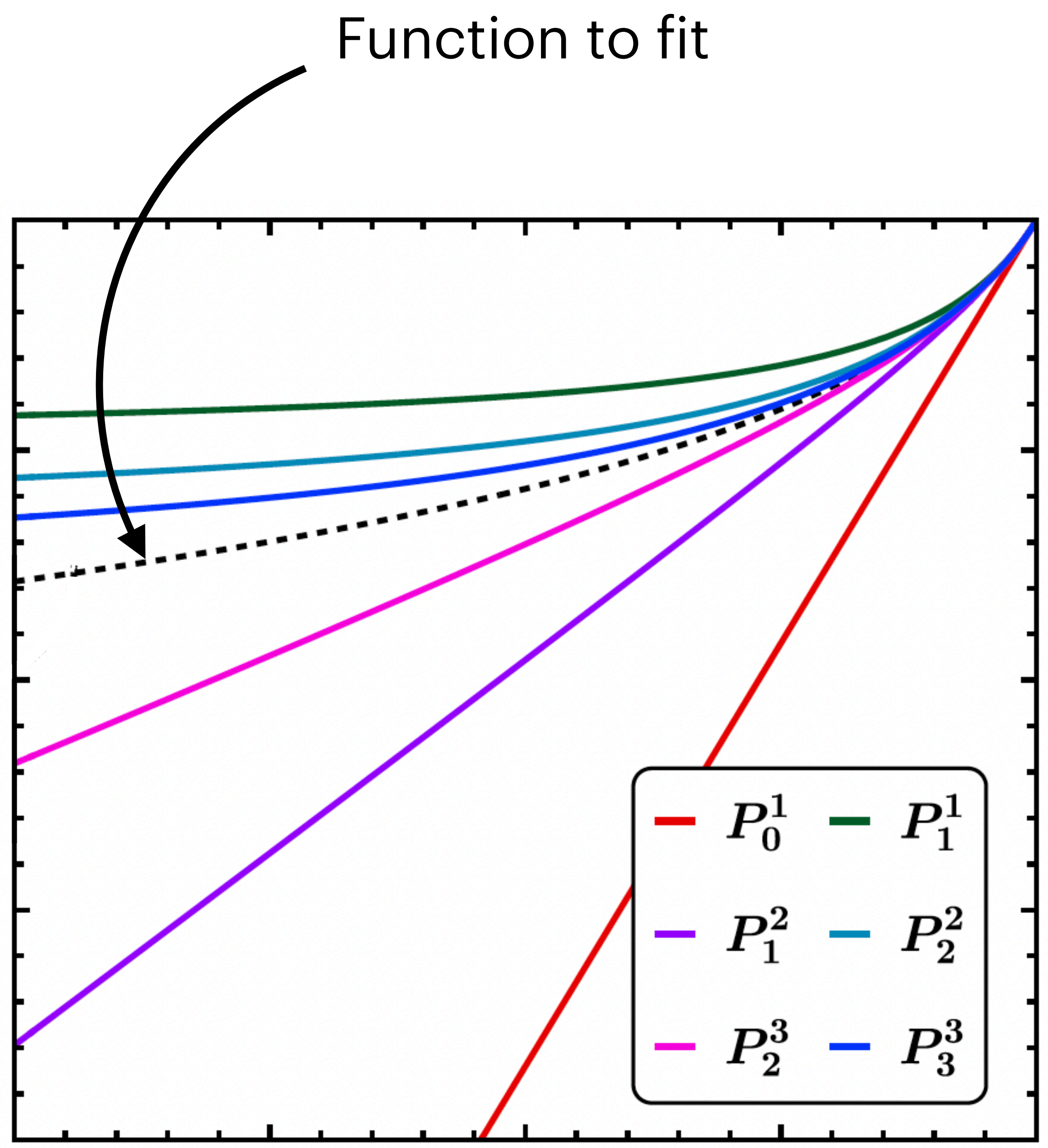
<u>This fitting method must be:</u>

- A very precise fitting
- A fast convergence method to the original function
- It must have the same analytical properties as the original function

Padé Approximants

Pade Approximants

$$P_M^N(z) = \frac{Q_N(z)}{R_M(z)} = \frac{q_0 + q_1 z + \cdots + q_N z^N}{1 + r_1 z + \cdots + r_M z^M}$$



Advantages

- Systematic and model-independent method
- Partial reconstruction of analytical (physical) properties
- Efficient approximation
- Possible to provide a convergence error

This type of approximants are already applied successfully in the context of g-2 and HVP:

- *Masjuan, Peris (2009)* arXiv:0903.0294 [hep-ph]
- *Masjuan (2012)* arXiv:1206.2549v2 [hep-ph]
- *Golterman, Maltman, Peris (2014)* arXiv:1405.2389 [hep-lat]
- *Aubin, Blum, Chau, Golterman, Peris, Tu (2017)* arXiv:1701.05829 [hep-ph]
- *Masjuan, Sanchez-Puertas (2017)* arXiv:1601.03071 [hep-lat]

Padé Approximants

To construct the PAs
(matching with Taylor coef.)

$$f(0) = P(0),$$
$$f'(0) = P'(0),$$
$$f''(0) = P''(0),$$
$$\vdots$$
$$f^{(m+n)}(0) = P^{(m+n)}(0)\,.$$

Coefficients "q_i" and "r_i" of the Padé's approximant can be obtained from the Taylor's coefficients (a_i) using the above relationships

→ Convergency is guaranteed

Stieltjes function

$$f(z) = \int_0^\infty \frac{\mathrm{d}\phi(u)}{1+zu}$$

→ Useful for us

✔ Stieltjes

$$f(z) \equiv \Delta\alpha_{\rm had}(t)$$
$$-\infty < t < 0$$

To ensure that poles are real and t-positive axis

We can use the convergence theorem to bound the function

$$P_1^1(t) \leq P_2^2(t) \leq \ldots \leq \Delta\alpha_{\rm had} \leq \ldots \leq P_1^2(t) \leq P_0^1(t)$$

Method & simulations

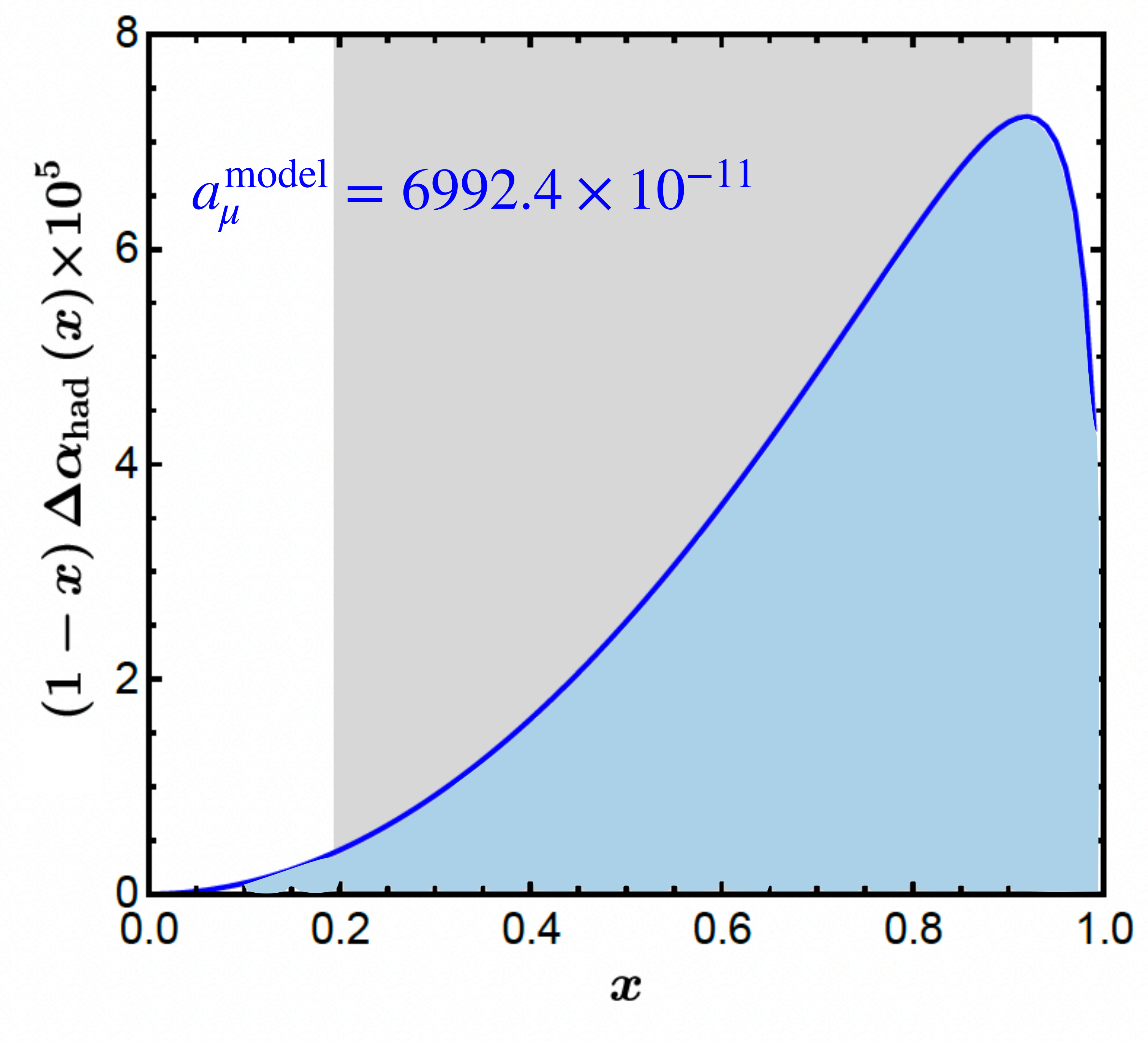
Fitting strategy - overview

Using a toy model

Pseudo Data

A phenomenological and ChPT. model proposed by Greynat and de Rafael was used (*JHEP 05 (2022) 084* - arXiv:2202.10810 [hep-ph])

$$a_\mu^{\mathrm{HVP,LO}} = \frac{\alpha^2}{\pi} \int_0^1 \mathrm{d}x \; (1-x)\, \Delta\alpha_{\mathrm{had}}[t(x)]$$



Fitting method

- x - variable limits are [0.2,0.93] divided in 30 bins
- data points are the mid-value of each bin
- Fitting: χ^2 minimisation
- Conditions in the Padé approximants:
    - Only real positive poles
    - No poles between [0,1]
    - No cancelations between zeros and poles
- Other constraints come from the Taylor series reproduction and change of variable from "t" to "x"

An example of Padé's approximant (dataset without errors)

First we can consider a data point set without uncertainties just to see the behaviour of the Padé approximant

Example: Padé Approximant (2 dofs)

$\mathrm{P}_1^1(t)$

In terms of Taylor Coeffs. (a_i)

$$\frac{a_1\, t}{1 - a_2\, t}$$

$t \rightarrow x$

$$\frac{b_1\, m_\mu^2\, x^2}{1 - x + b_2\, m_\mu^2\, x^2}$$

$a_\mu^{\mathrm{PA}} = 6933 \times 10^{-11}$ (0.9%)

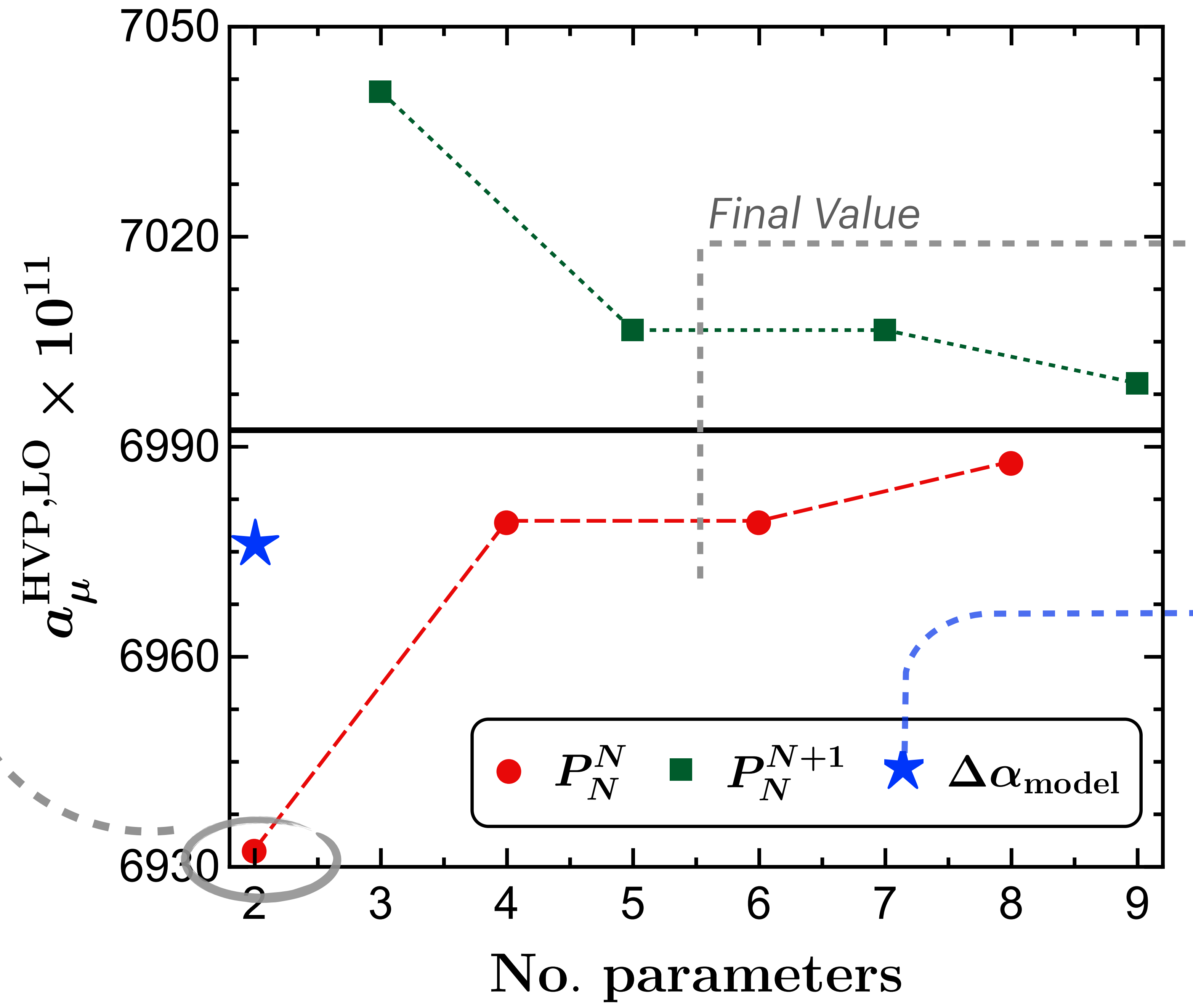
$a_1 = -912\ \mathrm{GeV}^{-2}$ $(0.7\%)^\dagger$

$a_2 = -1489\ \mathrm{GeV}^{-4}$ $(15\%)^\dagger$

$\chi^2_{min}/\mathrm{dof} = 1.18 \times 10^{-3}$

p-value = 1

† Relative error from Taylor coefficients of the model



$\bar{a}_\mu^{\mathrm{PAs}} = 6991 \times 10^{-11}$, (0.02%)

We compute a weighted average for the final value of a_μ^{HLO} (*weights = N*)

$a_\mu^{\mathrm{LO,HVP}}$ computed using the parametrisation used by the MUonE team

$$\Delta\alpha_{had}(t) = k\left\{-\frac{5}{9} - \frac{4M}{3t} + \left(\frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6}\right)\frac{2}{\sqrt{1-\frac{4M}{t}}}\log\left|\frac{1-\sqrt{1-\frac{4M}{t}}}{1+\sqrt{1-\frac{4M}{t}}}\right|\right\}$$

G. Abbiendi, 2022, (arXiv:2201.13177 [physics.ins-det])

- The fitting functions for the two sequences proposed (P_N^N and P_N^{N+1}) can be obtained in the same way (results in the right plot)

Results with realistic errors

Fitting with realistic errors - Data generation

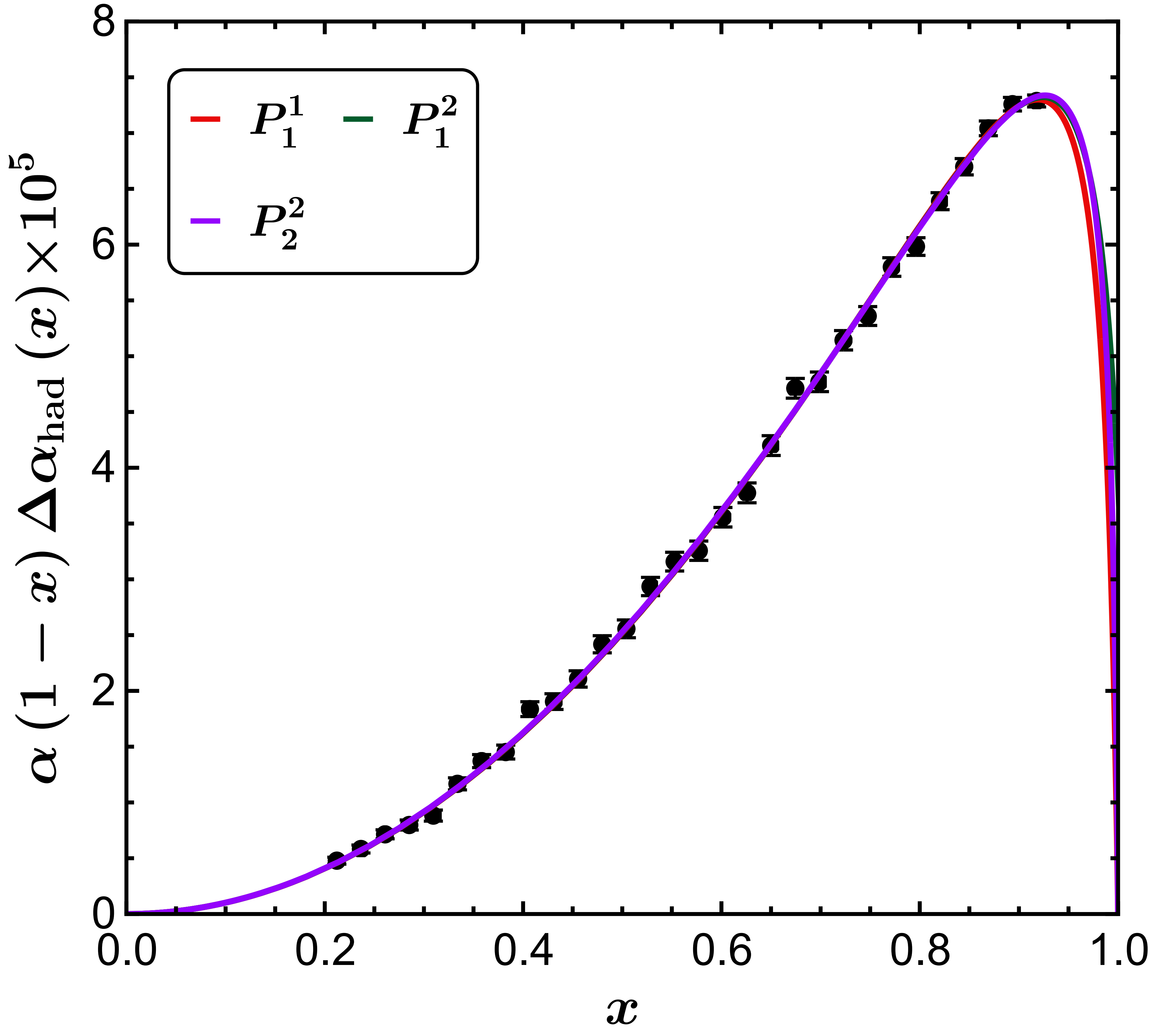
Data Generation

- 1000 toy data sets were generated
- 30 data points equally spaced in the interval [0.2,0.93] in x-variable
- Central value for each data point come from gaussian distribution with error ranging: (0.7% - 6.7%)
- Selecting data sets: $\eta_{\text{err}} < 1\,\%$; $\eta_{\text{err}} = \dfrac{a_\mu^{\text{partial}} - a_\mu^{\text{model}}}{a_\mu^{\text{model}}}$
- $\bar{a}_\mu^{\text{HLO}} = 6991^{+22}_{-20} \times 10^{-11}$ using the partial evaluations, as we expected for our final prediction.

Fitting considerations

- Monte Carlo analysis of the fits for each Padé approximant
- fits with $\chi^2/\text{dof} < 0.5$ or $\chi^2/\text{dof} > 1.5$ and $\text{p} - \text{value} < 10^{-4}$ are not considered
- PAs obtained were used to extrapolate the whole region

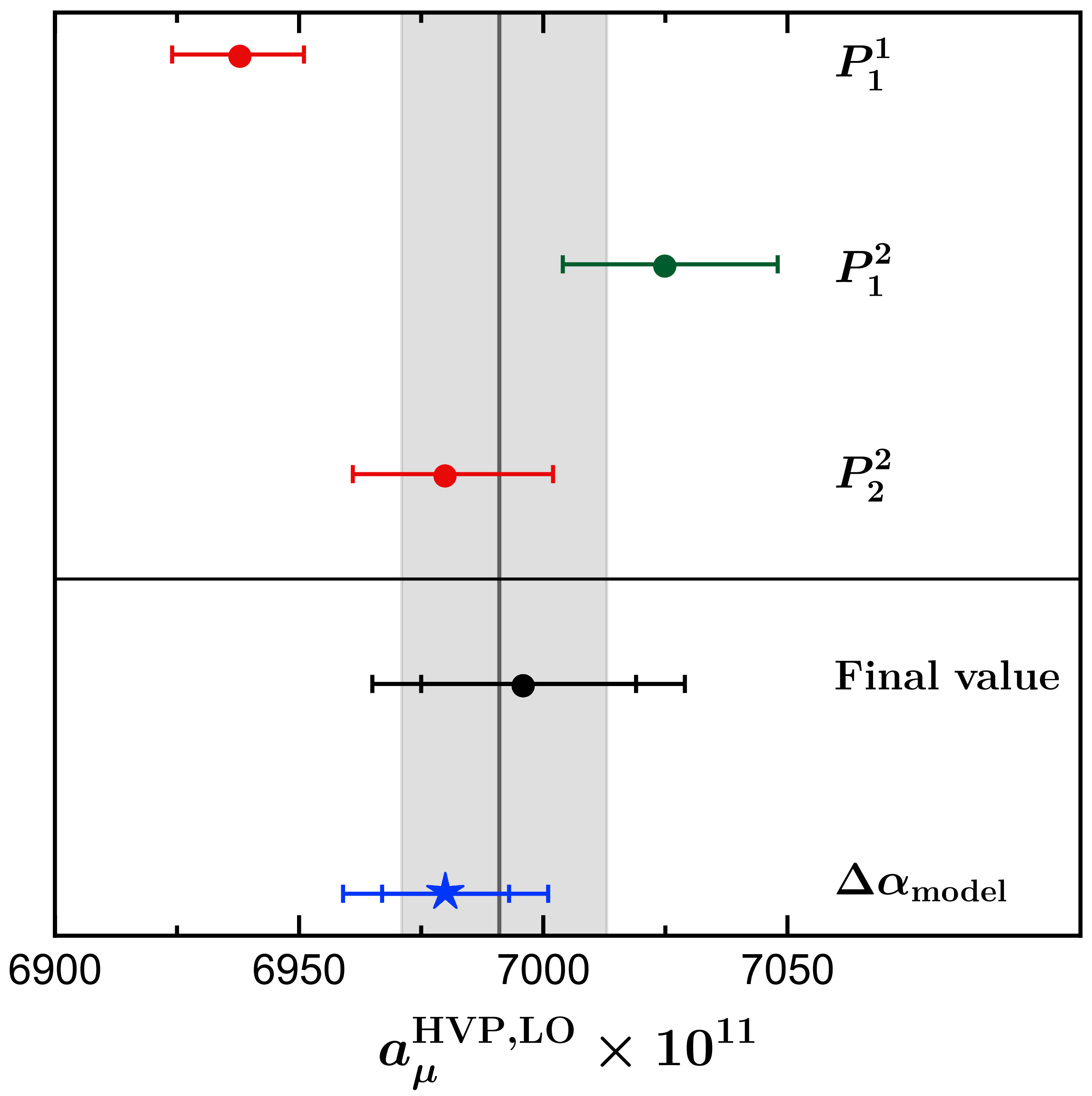
One dataset fitted with different Padé approximants



PAs with realistic errors - Results

	$a_\mu^{\mathrm{HVP,\,LO}} \times 10^{11}$	χ^2/dof	p-value
P_1^1	6938^{+13}_{-14}	$1.00^{+0.16}_{-0.15}$	0.485
P_1^2	7025^{+23}_{-21}	1.02 ± 0.16	0.445
P_2^2	6980^{+22}_{-19}	1.05 ± 0.16	0.394

$$\bar{a}_\mu^{\mathrm{PAs}} \times 10^{11} = 6991\,\left(^{+23}_{-21}\right)_{\mathrm{stat}} \pm (23)_{\mathrm{th}}$$



$$\bar{a}_\mu^{\mathrm{model}} \times 10^{11} = 6980 \pm (13)_{\mathrm{stat}} \pm (17)_{\mathrm{th}}$$

Theoretical uncertainty :

$$\mathrm{Err}_{\mathrm{th}} = \frac{\left| P_N^N|_{max} - P_N^{N+1}|_{max} \right|}{2}$$

They could be reduced by:

- Increasing the number of data sets
- lowering the uncertainties of each data set

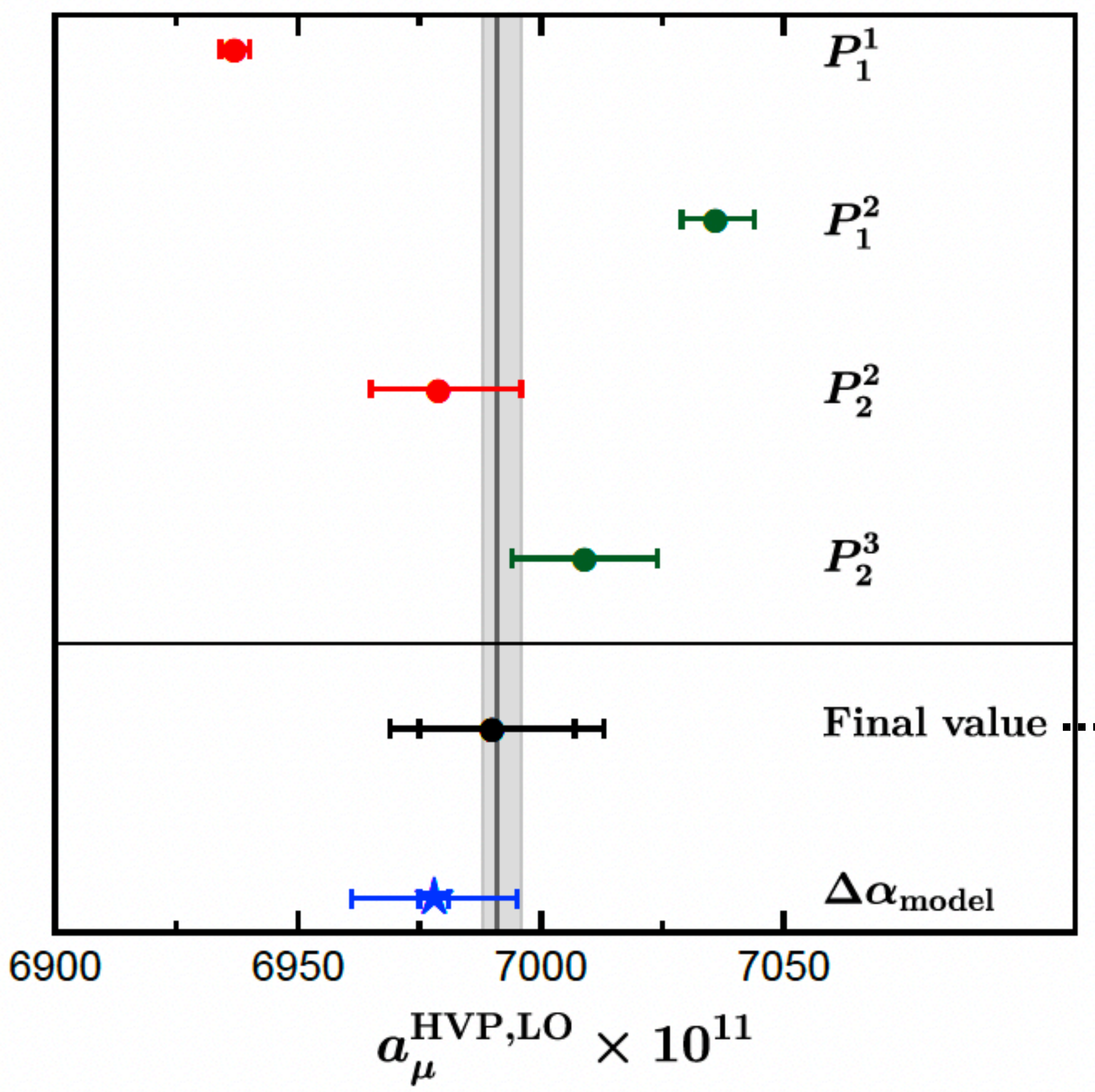
reducing x5 the error

than the ones awaited in the MUonE

considering a gaussian distribution with errors between 0.14% and 1.3% in the data set generation

	$a_\mu^{\mathrm{HVP,\,LO}} \times 10^{11}$	χ^2/dof	p-value
P_1^1	6937 ± 3	$1.23^{+0.18}_{-0.19}$	0.273
P_1^2	7027^{+14}_{-9}	1.00 ± 0.16	0.475
P_2^2	6979^{+17}_{-14}	$1.03^{+0.16}_{-0.17}$	0.450
P_2^3	7009 ± 15	1.06 ± 0.17	0.356

$$\bar{a}_\mu^{\mathrm{PAs}} \times 10^{11} = 6990\,\left(^{+17}_{-15}\right)_{\mathrm{stat}} \pm (15)_{\mathrm{th}}$$



$$\bar{a}_\mu^{\mathrm{model}} \times 10^{11} = 6978 \pm (3)_{\mathrm{stat}} \pm (17)_{\mathrm{th}}$$

Work in Progress

D-Log Padé approximants

D-Log Padé approximants

Advantages respect to Padés

- It is useful to reproduces not only poles but also cuts or branch points of the original function
- Faster convergence
- Also model independent to find the singularity position value and its multiplicity

Consider the following function:

$$f(z) = A(z)\frac{1}{(\mu - z)^{\gamma}} + B(z) \quad \gamma \in \Re$$

Set a new function applying logarithm and derivative

$$F(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln f(z) \approx \frac{\gamma}{(\mu - z)}$$

Get a Padé approximant P[N,N] to this new function and go back to original function

$$\mathrm{Dlog}_N^M(z) = f(0)\,\exp\left\{ \int \mathrm{d}z P_N^M \left[F(z)\right] \right\}$$

Is not longer a rational approximant

D-Logs Padé approximants - construction

Make a Taylor series from the toy model in t-variable → Construct a sequence of PAs in the D-log space → Get the D-log Padé approximants to the original function in t → Analysing the structure of approximants and propose a general function to fit

$f(t)$ → $F(t)$ → $\mathrm{D}_N^M(t)$

$$f(t) \cong \mathrm{D}_N^M(t) = f(0)\frac{t\, e^{\sum_{i=1}^{M-N+1} a_i t^i}}{(\mu_1 - t)^{\gamma_1} \dots (\mu_{N-1} - t)^{\gamma_{N-1}}}$$

Making change of variable

$t \to x$

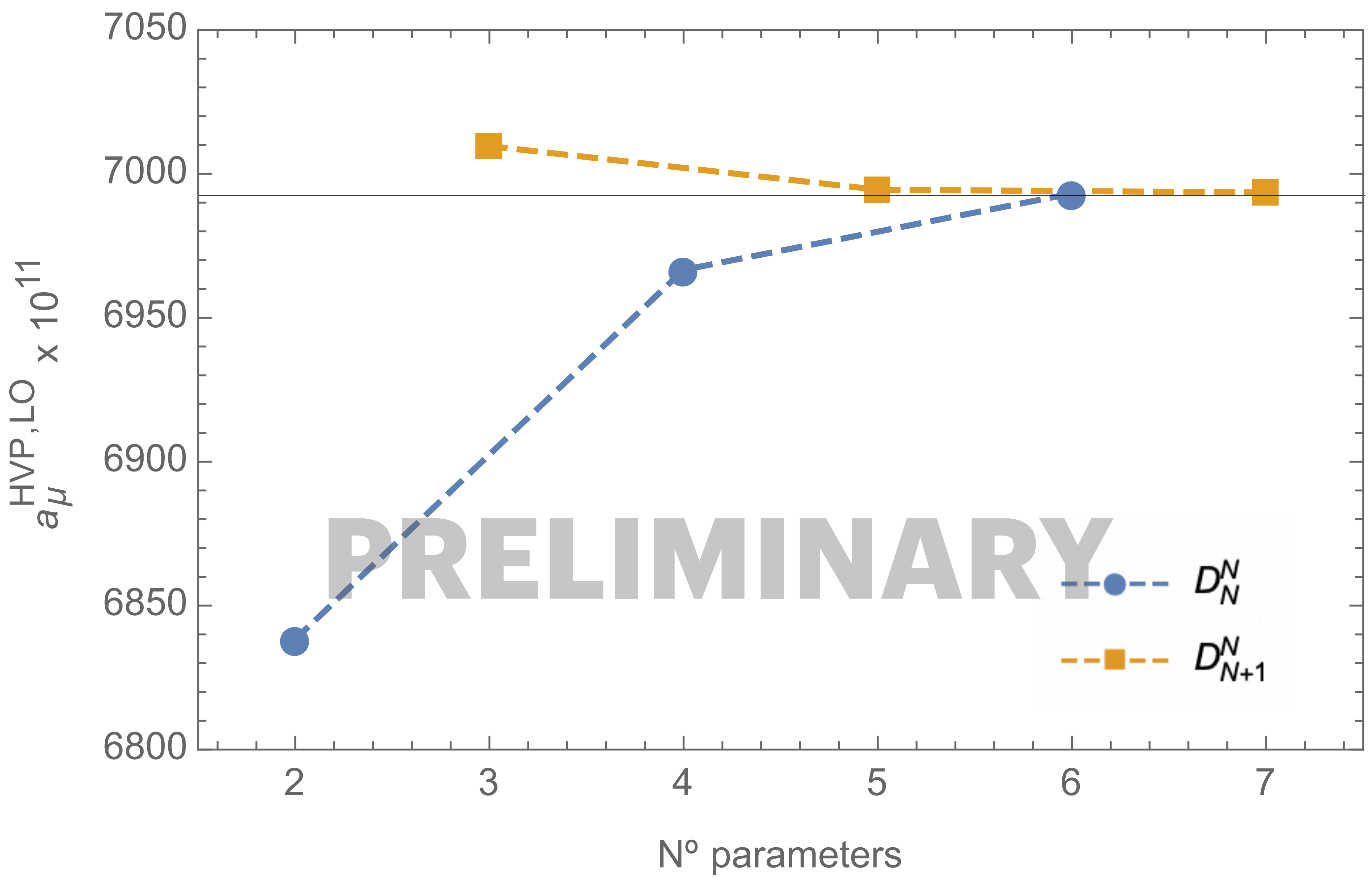
$$\mathrm{D}_N^M(x) = \frac{(x-1)^{-1+\sum \gamma_i}\, m_\mu^2\, x^2\, e^{\sum_{i=1}^{M-N+1} a_i \left(m_\mu^2 x^2/x - 1\right)^i}}{(\mu_1 x - \mu_1 - m_\mu^2 x^2)^{\gamma_1} \dots (\mu_{N-1} x - \mu_{N-1} - m_\mu^2 x^2)^{\gamma_{N-1}}}$$

Fitting function

D-Log Padé approximants - Preliminary Results

- Results and Convergence without error
- Simulations with error (working in progress)

	$a_\mu^{\mathrm{HVP,LO}} \times 10^{11}$	η_{err}	χ^2/dof
D_1^1	6838.31	(2.20%)	4.0×10^{-3}
D_1^2	7009.62	(0.25%)	1.88×10^{-6}
D_2^2	6966.63	(0.37%)	8.66×10^{-9}
D_2^3	6994.49	(0.03%)	1.21×10^{-12}
D_3^3	6993.16	(0.011%)	9.18×10^{-13}
D_3^4	6993.50	(0.016%)	1.56×10^{-16}



D-Log Padé approximants - Preliminary Results

- Results and Convergency without error
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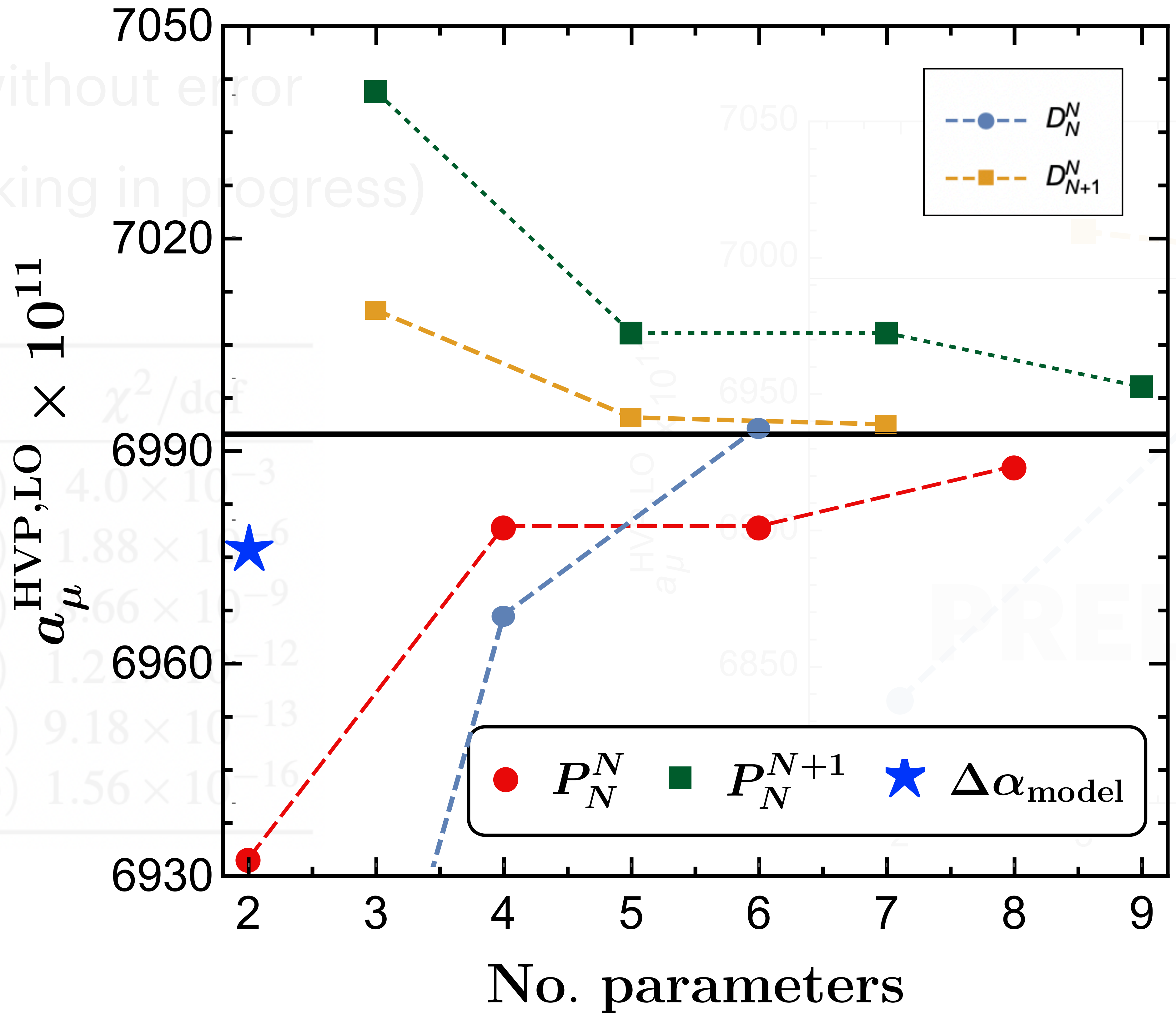
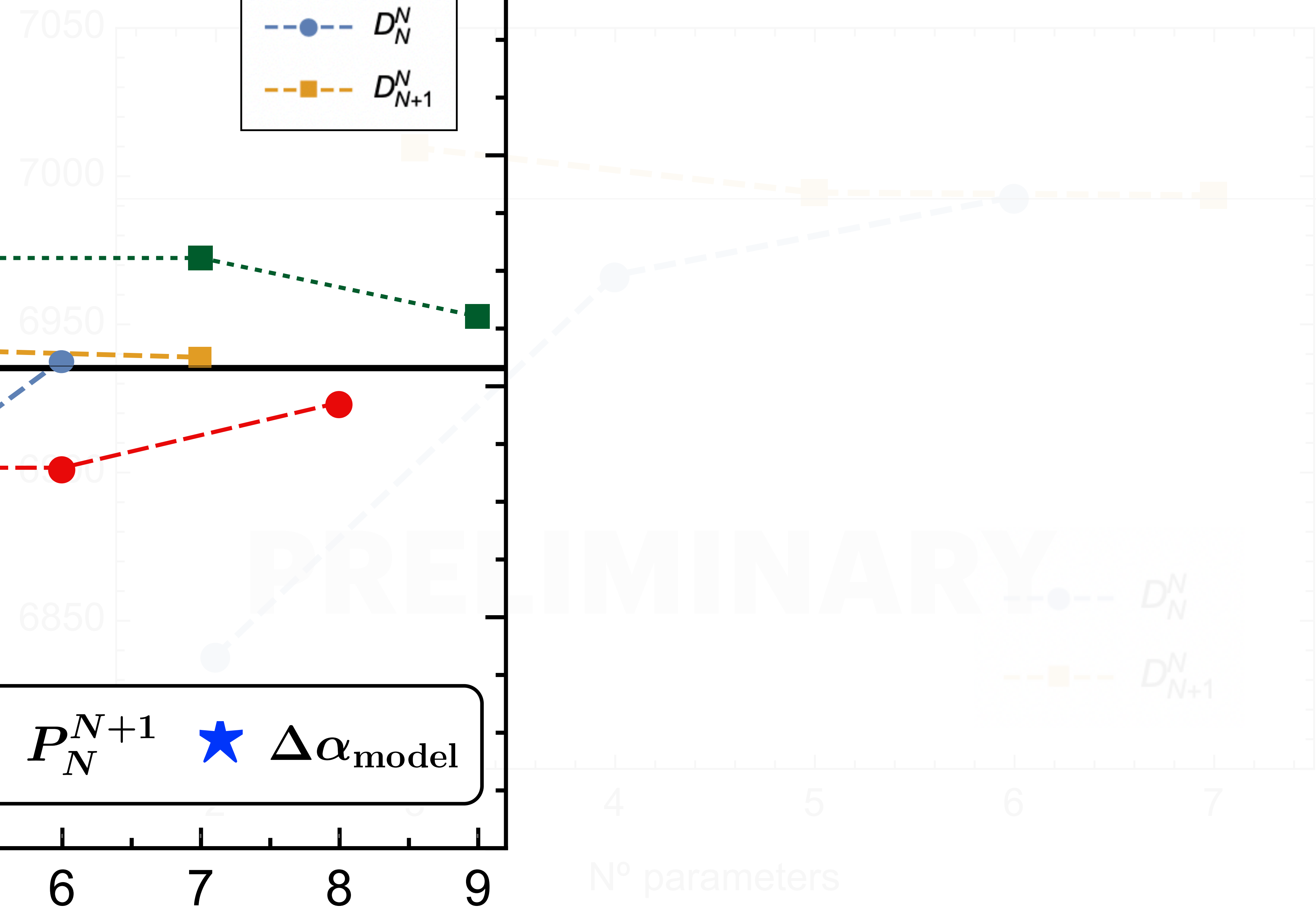
D_1^1	6838.31	(2.20%)	4.0×10^{-3}
D_1^2	7009.62	(0.25%)	1.88×10^{-6}
D_2^2	6966.63	(0.37%)	5.66×10^{-9}
D_2^3	6994.49	(0.03%)	[illegible]
D_3^3	6993.16	(0.011%)	9.18×10^{-13}
D_3^4	6993.50	(0.016%)	1.56×10^{-16}





Conclusions

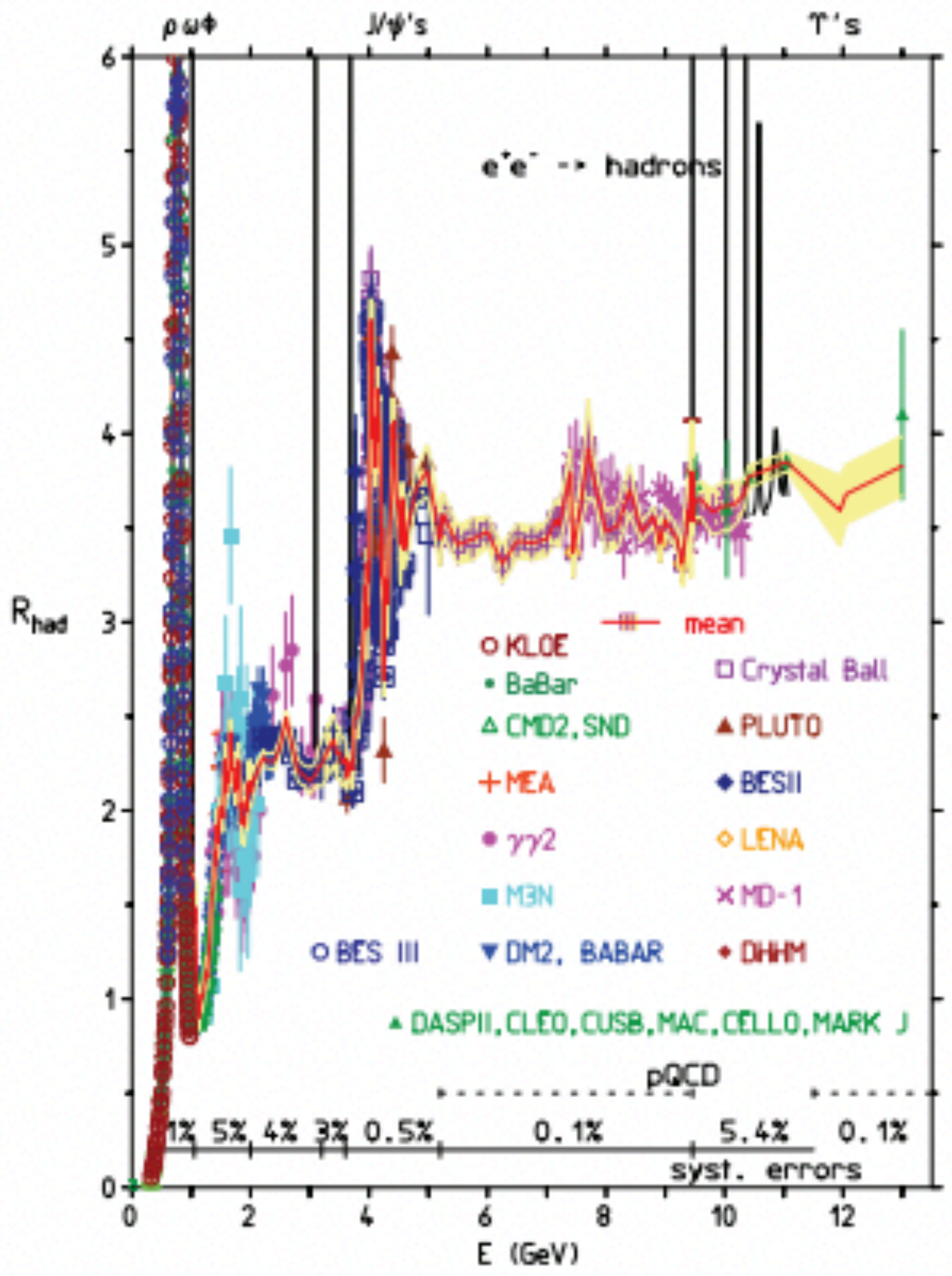
Outlook & Conclusions

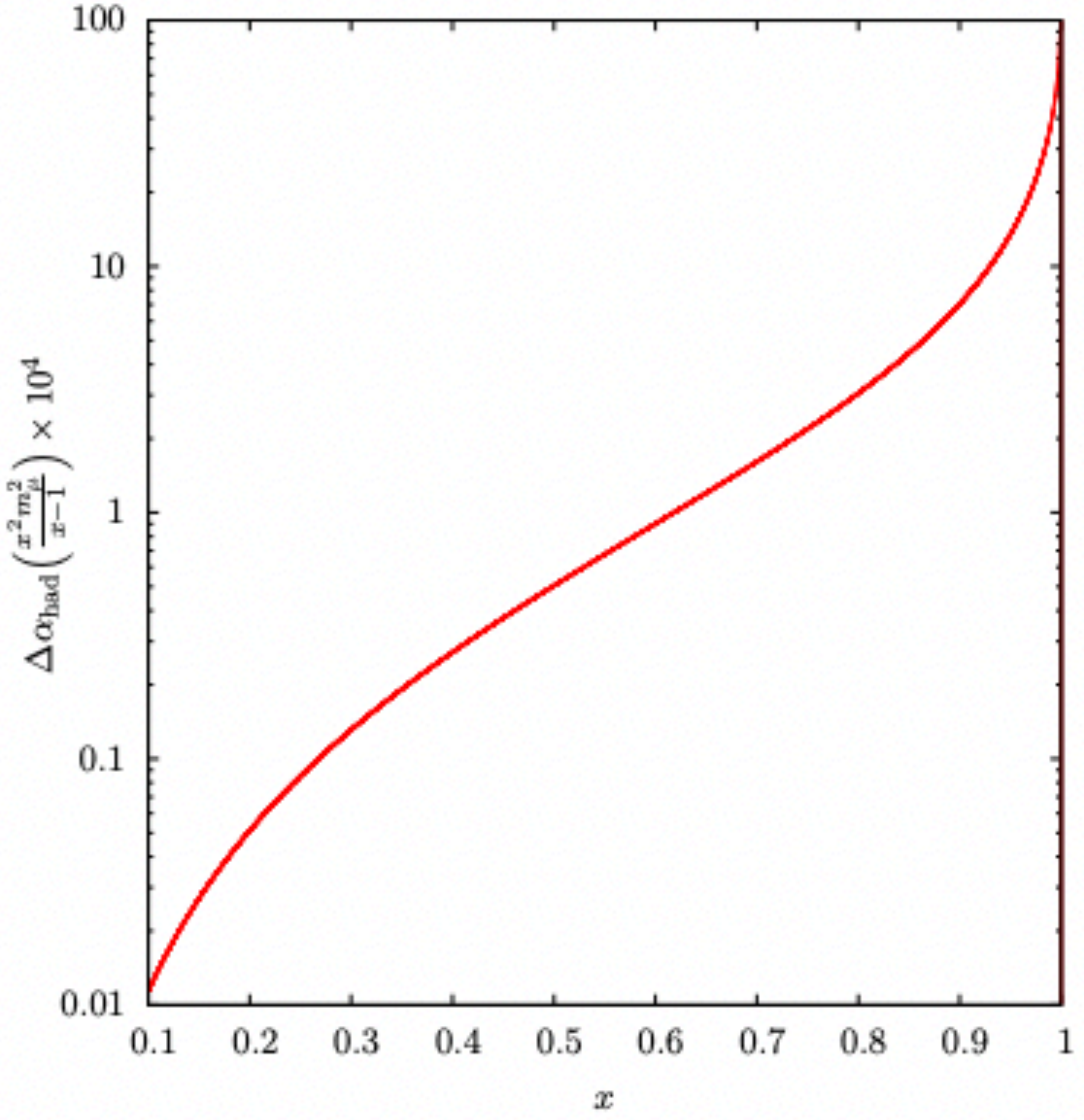
- a_μ^{HVP} discrepancy between E821+FNAL results and SM predictions reached the 4σ level
- Hadron vacuum polarisation contribution is the dominant source of theoretical uncertainty
- different methods required to allow independent cross-checks
    - time-like dispersive approach: the most precise up to now
    - a growing tension between data-driven and lattice is emerging
    - space-like dispersive approach and MUonE experiment proposal: promising, provided theoretical and experimental systematics are kept under control at the level of 10^{-5}
- Different fitting methods also could provide a better accuracy and precision
- Padé approximants or even the Dlogs approximants can be a good alternative to fit the MUonE experiment data in the future.

Thanks

Backup slides

Time-like $\mapsto$ Space-like





Model parametrisation used by MUonE team

Physics-inspired from the calculable contribution of lepton-pairs and top quarks at t<0

$q^2 = t < 0$

$$\Delta\alpha_{had}(t) = k\left\{-\frac{5}{9} - \frac{4M}{3t} + \left(\frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6}\right)\frac{2}{\sqrt{1-\frac{4M}{t}}}\log\left|\frac{1-\sqrt{1-\frac{4M}{t}}}{1+\sqrt{1-\frac{4M}{t}}}\right|\right\}$$

M with dimension of mass squared, related to the mass of the fermion in the vacuum polarization loop
k depending on the coupling $\alpha(0)$, the electric charge and the colour charge of the fermion

Ref: Giovanni Abbiendi arXiv:2201.13177v1 ***[physics.ins-det]***

Model Greynat and De Rafael

$$\Delta\alpha_{\text{had}}[q^2] = \bar{\Pi}(q^2) = q^2 \int_{4m_\pi^2}^{\infty} \mathrm{d}s \frac{\operatorname{Im}\Pi(s)}{s(s - q^2 + i\epsilon)}$$

$$\operatorname{Im}\Pi_{\text{had}}(s) = \frac{1}{4\pi}\left(1 - \frac{4m_\pi^2}{s}\right)^{3/2} \left(\frac{|F(s)|^2}{12} + \sum_i q_i^2\ \Theta(s, s_c, \Delta)\right)\theta(s - 4m_\pi^2)$$

$$|F(s)|^2 = \frac{m_\rho^4}{(m_\rho^2 - s)^2 + m_\rho^2\,\Gamma(s)^2}$$

$$\Gamma(s) = \frac{m_\rho\, s}{96\pi f_\pi^2}\left[\left(1 - \frac{4m_\pi^2}{s}\right)^{3/2}\theta(s - 4m_\pi^2) + \frac{1}{2}\left(1 - \frac{4m_k^2}{s}\right)^{3/2}\theta(s - 4m_k^2)\right]$$

$$\Theta(s, s_c, \Delta) = \frac{2}{\pi}\left[\frac{\arctan\left(\frac{s - s_c}{\Delta}\right) - \arctan\left(\frac{4m_\pi^2 - s_c}{\Delta}\right)}{\frac{\pi}{2} - \arctan\left(\frac{4m_\pi^2 - s_c}{\Delta}\right)}\right]$$