Thermal hadron resonances and Ward Identities: results for the QCD phase diagram

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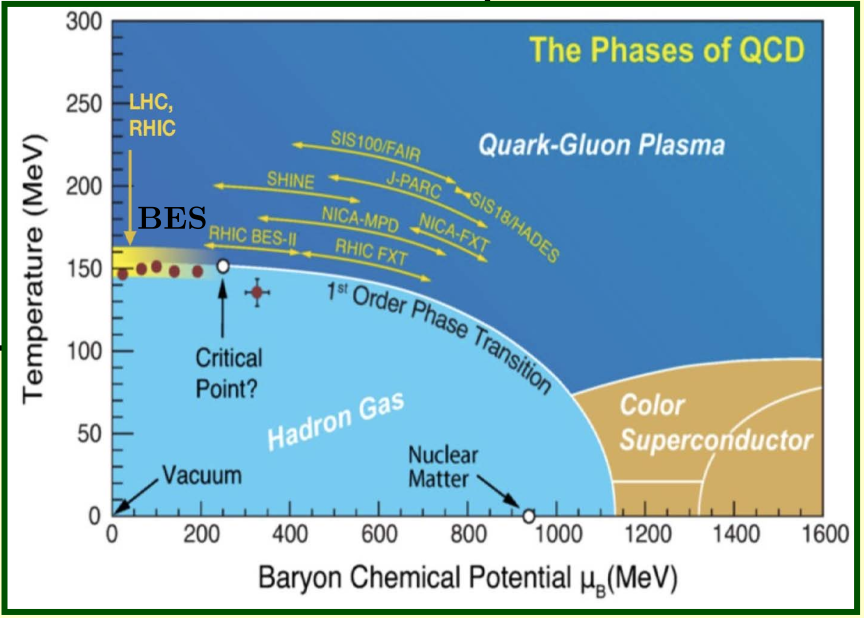
OUTLINE:

- **Chiral and $U(1)_A$ restoration**
- **Ward Identities**
- **Predictions from effective theories**
- **The role of thermal resonances**

QCD phase diagram

SOME OPEN PROBLEMS:

- **Nature of chiral restoration and connection with $U(1)_A$**
- **Properties of Resonances in Hadronic Medium**
- **High density beyond BES programme**
- **Existence and nature of critical point**
- **Other dimensions: μ_I, μ_S, μ_5, eB, ...**

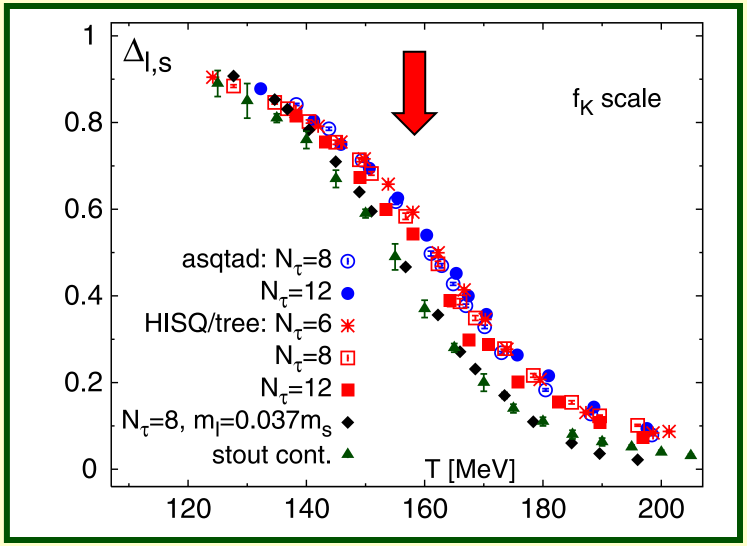


A.Bazavov et al [USQCD coll] EPJA 55, 194 (2019)

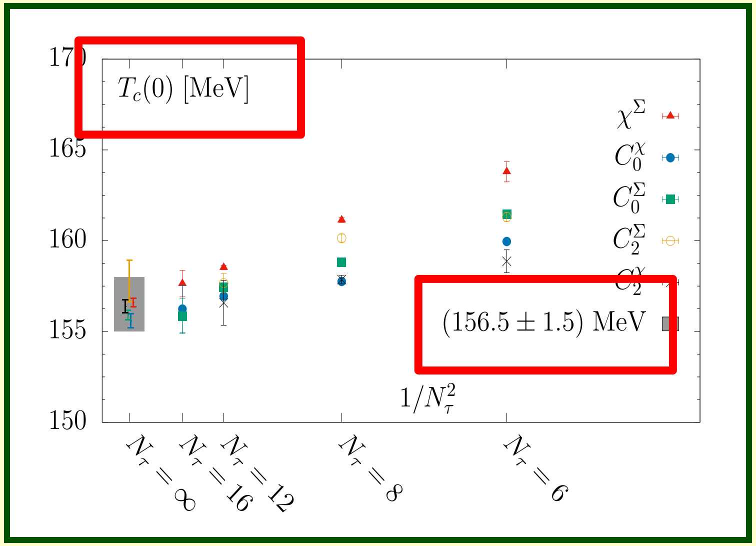
Signals of Chiral Symmetry Restoration

Inflection point for the light quark condensate $\langle \bar{q}q \rangle_l = \langle \bar{u}u + \bar{d}d \rangle$

A.Bazavov et al PRD85, 054503 (2012)



A.Bazavov et al, Phys.Lett.B 795 (2019) 15



$$\Delta_{l,s} = \frac{\langle \bar{q}q \rangle_T - (2m_q/m_s)\,\langle \bar{s}s \rangle_T}{\langle \bar{q}q \rangle_0 - (2m_q/m_s)\,\langle \bar{s}s \rangle_0}$$

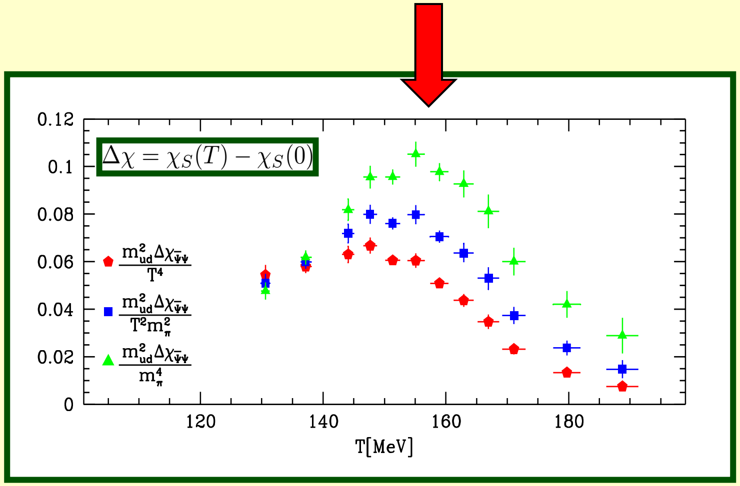
subtracted quark condensate (avoids lattice divergences)

Crossover-like transition in physical case ($N_f = 2+1$, massive quarks)

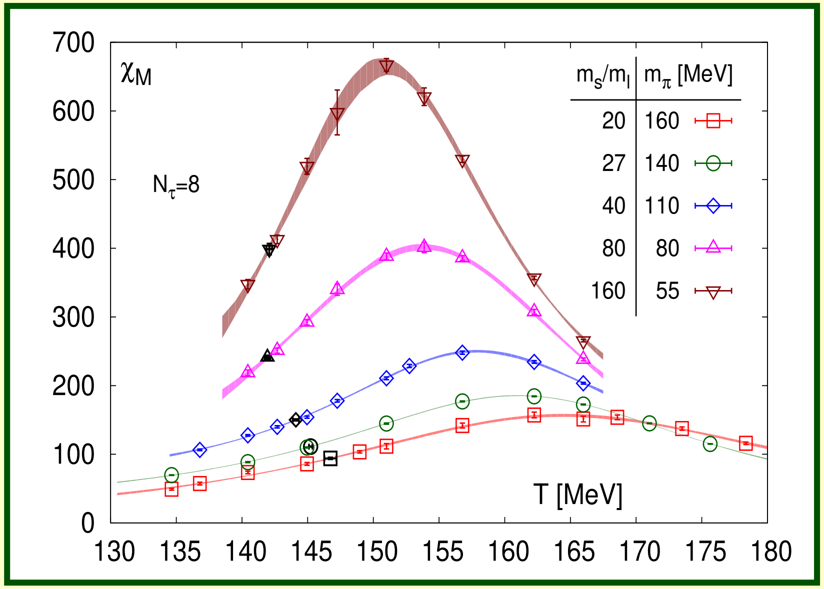
Signals of Chiral Symmetry Restoration

Peak of scalar susceptibility $\chi_S = \int_x \left[\langle \bar{q}q(x)\bar{q}q(0)\rangle - \langle \bar{q}q\rangle_l^2\right]$

H. T. Ding et al PRL123, 062002 (2019)



Y.Aoki et al [BW coll] JHEP 0906, 088 (2009)



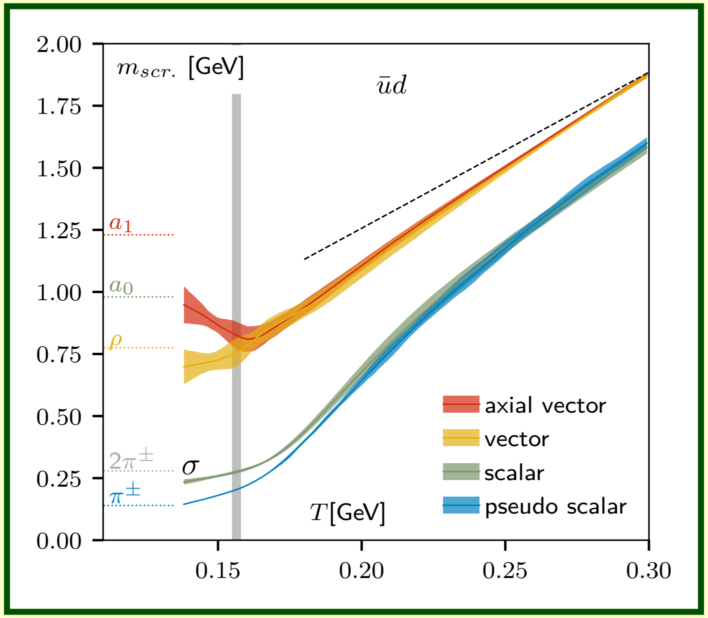
$$\Rightarrow \quad T_c^{m_{ud}=0} = 132^{+3}_{-6} \text{ MeV}$$

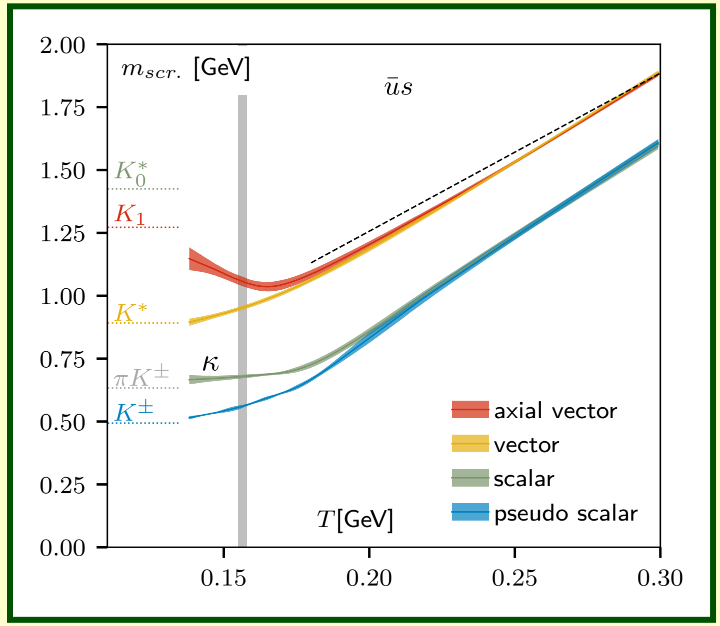
Phase transition in light chiral limit for $N_f = 2$, possibly of second order

R. D. Pisarski and F. Wilczek PRD 29, 338 (1984)

Signals of Chiral Symmetry Restoration

• **Screening masses and susceptibilities of chiral partners degenerate around T_c**





A.Bazavov et al (Hot QCD coll) PRD 100, 094510 (2019)

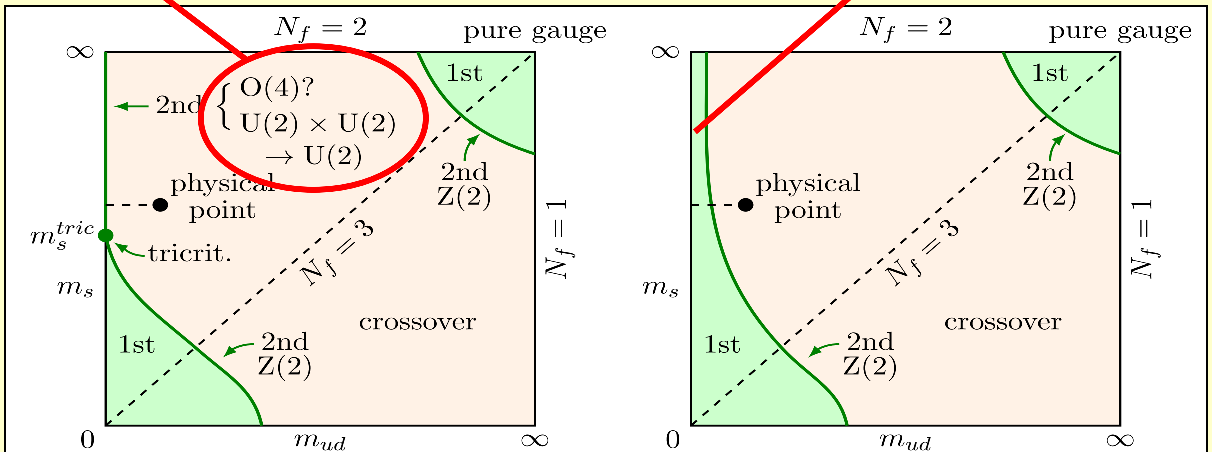
• **Key to understand the nature of chiral transition vs $U(1)_A$**

Chiral Symmetry Restoration vs $U(1)_A$

D.J.Gross, R.D.Pisarski, L.G.Yaffe, RMP53, 43 (1981) → $U(1)_A$ asymptotic restoration via instanton suppression

Universality class depends on the **strenght** of $U(1)_A$ **breaking** @T_c

Transition order can even change if $U(1)_A$ is sufficiently restored



B.B. Brandt et al 2019

E.V.Shuryak, CNPP21, 235 (1994); T.D.Cohen PRD54 R1867 (1996); S.H.Lee, T.Hatsuda PRD54, R1871 (1996)

A. Pelissetto, E. Vicari PRD88, 105018 (2013)

$\Rightarrow M_{\eta'}$ and χ_{top} reduction if $U(1)_A$ efficiently restored around T_c

J. I. Kapusta, D. Kharzeev, L. D. McLerran, PRD53, 5028 (1996)

T. Csorgo et al, PRL105, 182301 (2010)

AGN, J.Ruiz de Elvira, A.Vioque-Rodríguez, JHEP11, 086 (2019)

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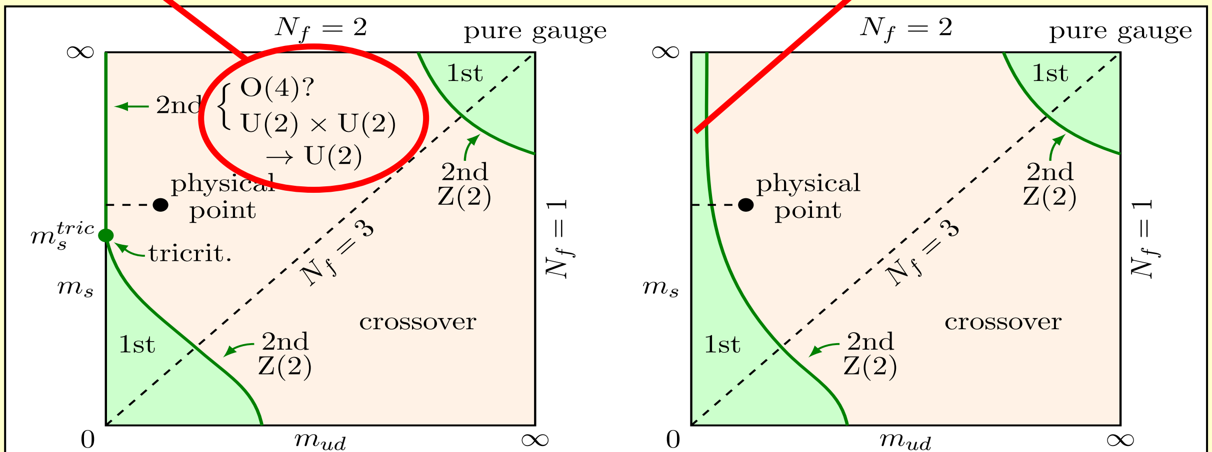


B.B. Brandt et al 2019

⇒ How effective is $U(1)_A$ asymptotic restoration at T_c?
(specially around $N_f = 2$ in light chiral limit)

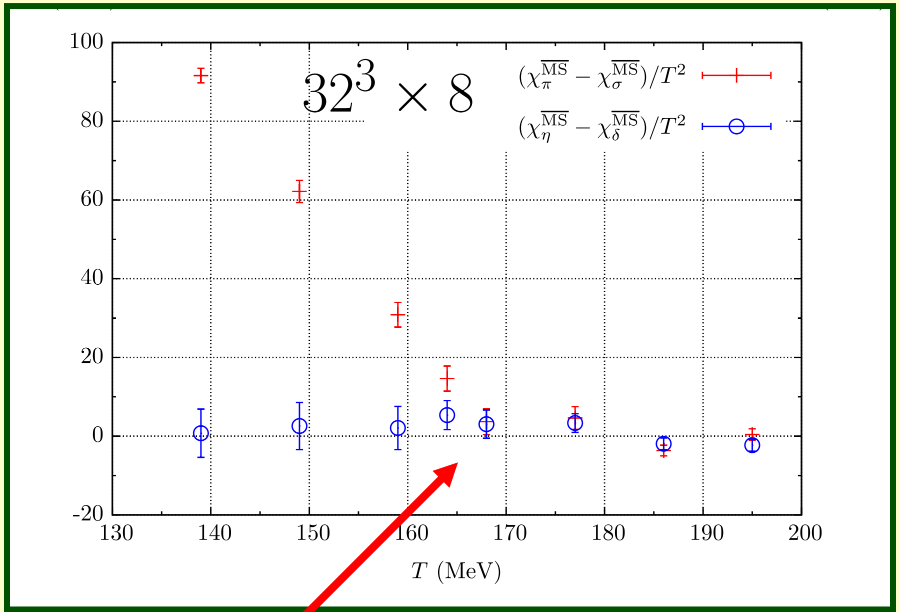
$O(4)$ vs $U(1)_A$: lattice results (light scalar/pseudoscalar sector)

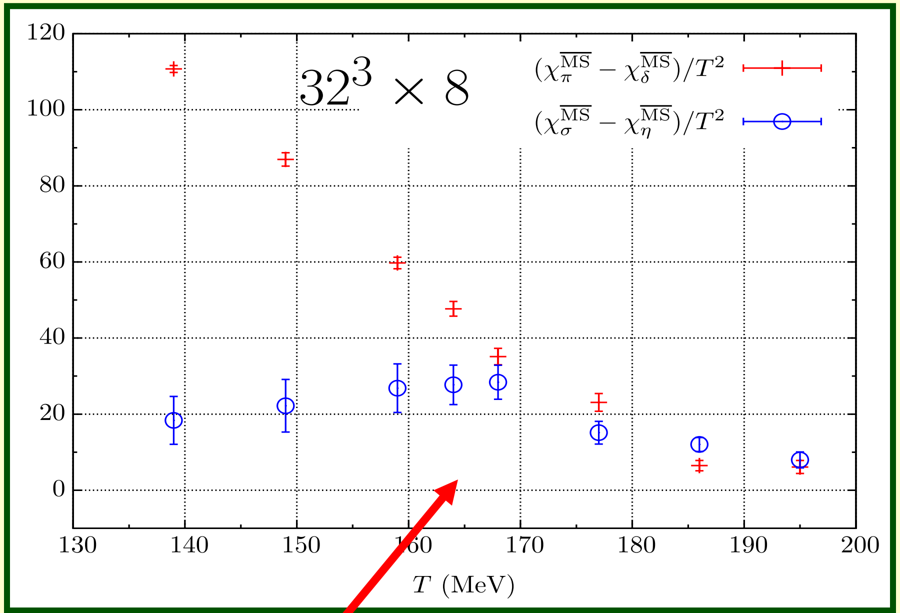
$I=1$		$I=0$
$\pi^a = i\bar{q}_l\gamma_5\tau^a q_l$	$\overset{SU(2)_A}{\longleftrightarrow}$ $O(4)$ partners	$\sigma_l = \bar{q}_l q_l$
$\updownarrow U(1)_A$		$\updownarrow U(1)_A$
$\delta^a(a_0) = \bar{q}_l\tau^a q_l$	$\overset{SU(2)_A}{\longleftrightarrow}$ $O(4)$ partners	$\eta_l = i\bar{q}_l\gamma_5 q_l$

- $N_f = 2+1$ **susceptibilities**

(physical m_{ud}, m_s, domain wall fermions)

Buchoff et al (LLNL/RBC coll) PRD89 (2014)





$O(4)$ **OK** (with large uncertainties in $\chi_\eta - \chi_\delta$)

significant $U(1)_A$ breaking @ T_c for physical masses

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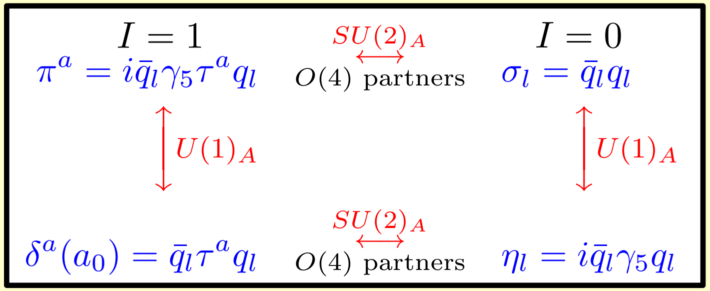


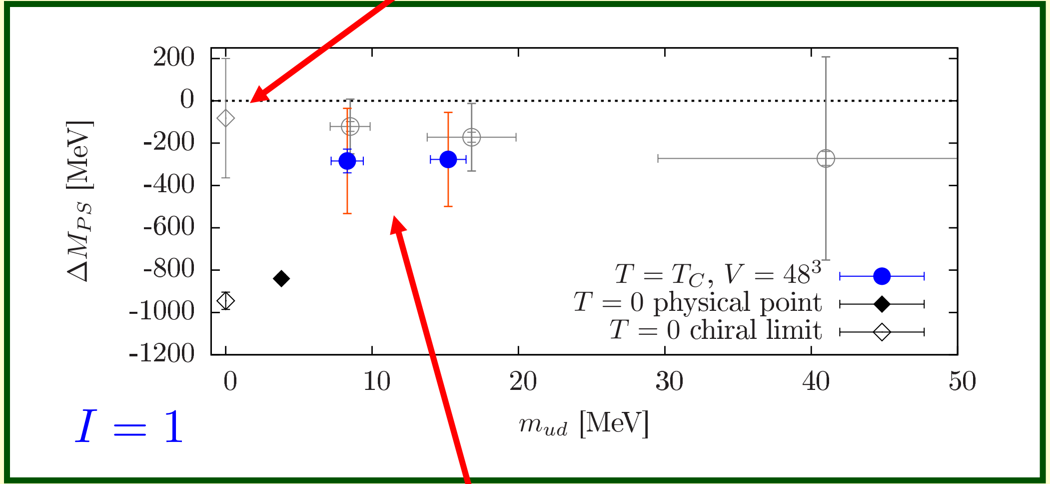
- $N_f = 2$ **screening masses** (dynamical Wilson fermions)

G.Cossu et al PRD87, 114514 (2013)

B.B.Brandt et al JHEP 12, 158 (2016), PoS CD2018, 055 (2019)

A.Tomiya et al PRD96, 034509 (2017)

Compatible with $U(1)_A$ restoration @ T_c in chiral limit



Soft $U(1)_A$ breaking for $m \neq 0$, increasing with V

Same behaviour (domain wall fermions)

S.Aoki et al (JLQCD coll) PRD103, 074506 (2021)

$O(4)$ vs $U(1)_A$: Ward Identities

AGN, J.Ruiz de Elvira, R.Torres, PRD88, 076007 (2013)
AGN, J.Ruiz de Elvira, JHEP1603, 186 (2016), PRD97, 074016 (2018), PRD98, 014020 (2018)

$$\chi_P^{ls} = \int_T dx \langle \mathcal{T} \eta_l(x) \eta_s(0) \rangle$$

$$\chi_{top} \equiv -\frac{1}{36} \int_T dx \langle \mathcal{T} A(x) A(0) \rangle$$

$$A(x) = \frac{3\alpha_s}{4\pi} Tr_c G_{\mu\nu} \tilde{G}^{\mu\nu}$$

Allow to relate susceptibilities and quark condensates from QCD generating functional:

$$\boxed{\chi_P^{ls}(T) = -2\frac{m_l}{m_s}\chi_{5,disc}(T) = -\frac{2}{m_l m_s}\chi_{top}(T)}$$

$\chi_{5,disc} = \frac{1}{4}\left(\chi_P^\pi - \chi_P^{\eta_l}\right)$ **measures** $O(4) \times U(1)_A$ **restoration**

AGN, J.Ruiz de Elvira, R.Torres, PRD88, 076007 (2013)
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- $SU(2)_A$ **transforms** $\langle \eta_l \eta_s \rangle \xrightarrow{(*)} \langle \delta \eta_s \rangle = 0$ **by parity**

$$\boxed{\eta_l \overset{O(4)}{\sim} \delta \Rightarrow \chi_P^{ls} \overset{O(4)}{\sim} 0 \Rightarrow \chi_{5,disc} \overset{O(4)}{\sim} 0, \ \chi_{top} \overset{O(4)}{\sim} 0}$$

$(*)\ \eta_l \to i\bar{q}_l \gamma_5 e^{i\frac{\pi}{2}\gamma_5 \tau^b} q_l = -\delta^b$

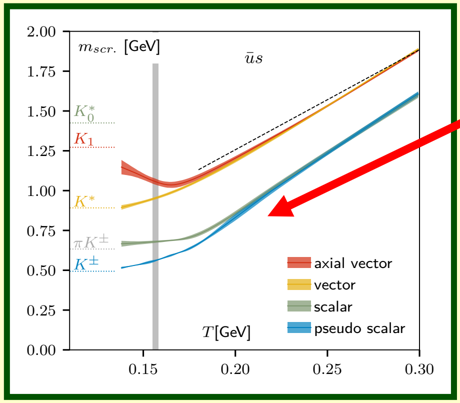
$$\boxed{O(4) \times U(1)_A \text{ pattern for } \textit{exact} \text{ CSR}}$$

Consistent with lattice for $N_f = 2$, $m_l \to 0$

$O(4)$ vs $U(1)_A$: $I=1/2$ sector and role of strangeness

AGN, J.Ruiz de Elvira, A.Vioque-Rodríguez, D. Álvarez-Herrero, EPJC81 (2021) 637

- $K-\kappa$ **expected to degenerate around** $O(4)\times U(1)_A$ **restoration region**



$a=4,\dots 7$

$$K^a = i\bar{q}\gamma_5\lambda^a q \quad \underset{U(1)_A}{\overset{SU(2)_A}{\longleftrightarrow}} \quad \kappa^a = \bar{q}\lambda^a q$$

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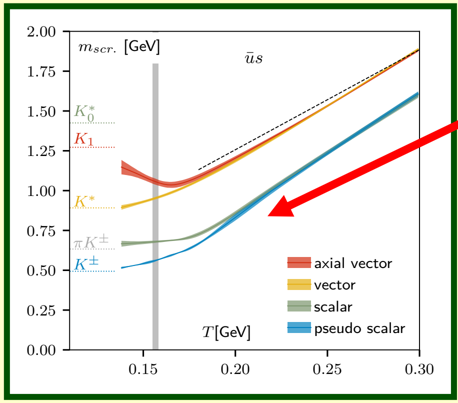

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- **WIs in this sector:**

$$\chi_P^K(T) = -\frac{\langle \bar{q}q \rangle_l (T) + 2 \langle \bar{s}s \rangle (T)}{m_l + m_s}$$

(recall both $\langle \bar{q}q \rangle_l, \langle \bar{s}s \rangle < 0$)

$$\chi_S^\kappa(T) = \frac{\langle \bar{q}q \rangle_l (T) - 2 \langle \bar{s}s \rangle (T)}{m_s - m_l}$$

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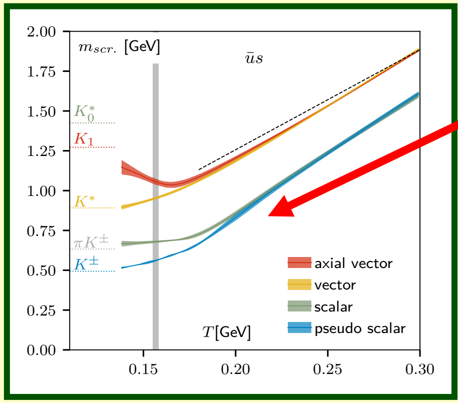

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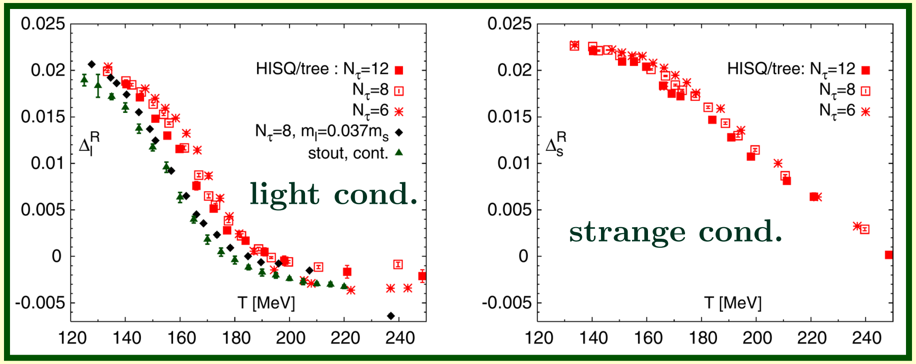


- **WIs in this sector:**

$$\chi_P^K(T) = -\frac{\langle\bar{q}q\rangle_l(T) + 2\langle\bar{s}s\rangle(T)}{m_l+m_s}$$

$\Rightarrow$ **decreases (faster as $T\to T_c$ from below)**

(recall both $\langle\bar{q}q\rangle_l, \langle\bar{s}s\rangle < 0$)

$$\chi_S^\kappa(T) = \frac{\langle\bar{q}q\rangle_l(T) - 2\langle\bar{s}s\rangle(T)}{m_s-m_l}$$

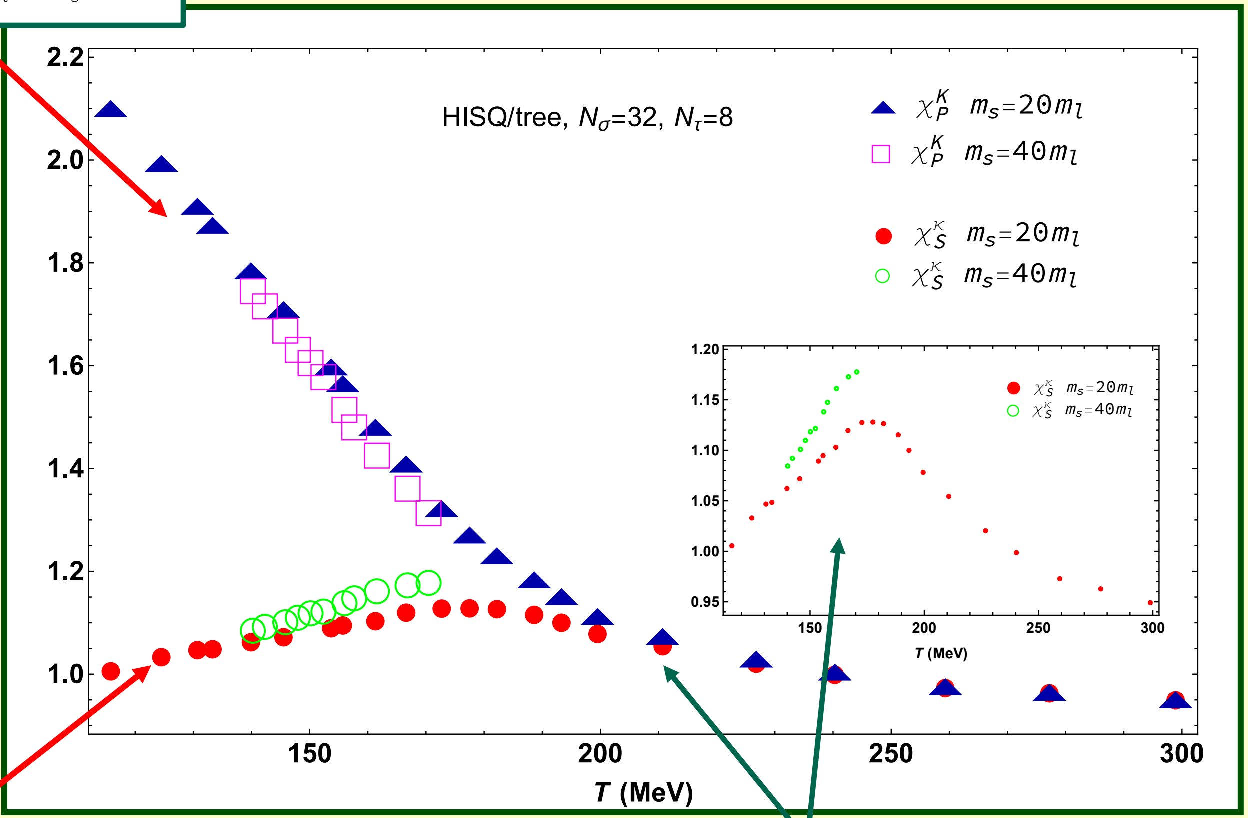
$\Rightarrow$ **increases by $\langle\bar{q}q\rangle_l$ reduction as $T\to T_c$**
but decreases as $\langle\bar{s}s\rangle$ activates
approaching χ^K in $O(4)\times U(1)_A$ region

$\Rightarrow \chi_S^\kappa$ **PEAK above T_c**

Reconstructed susceptibilities from WI and lattice condensate data confirm this behaviour (no direct $\chi^{K,\kappa}$ results yet)

$$\chi_P^K(T) = -\frac{\langle \bar{q}q\rangle_l(T) + 2\langle \bar{s}s\rangle(T)}{m_l + m_s}$$

Lattice points from
A.Bazavov et al (Hot QCD) 2012-14 $(N_f = 2+1)$

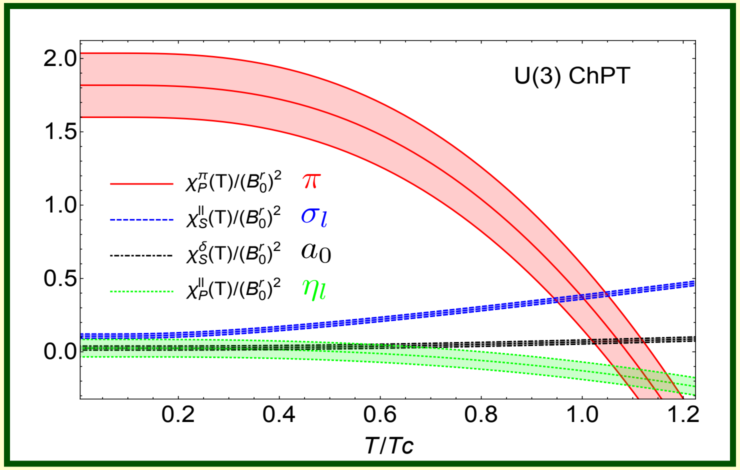


$$\chi_S^\kappa(T) = \frac{\langle \bar{q}q\rangle_l(T) - 2\langle \bar{s}s\rangle(T)}{m_s - m_l}$$

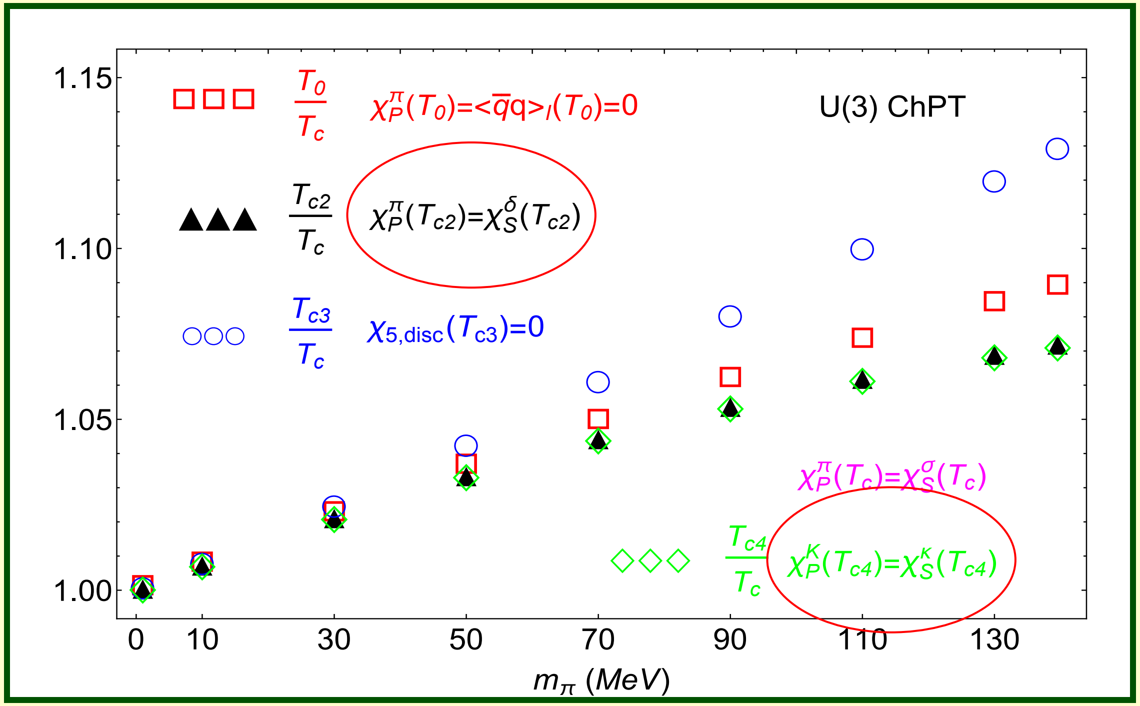
- $\chi_S^\kappa \to \chi_P^K$ **in** $O(4)\times U(1)_A$ **region, above** χ_S^κ **peak**
- **Peak behaviour driven by** m_l/m_s

Effective Theories: chiral and $U(1)_A$ partners within $U(3)$ ChPT

AGN, J.Ruiz de Elvira, PRD98, 014020 (2018)



$1/N_c \sim m_q \sim T^2 \sim p^2$ counting



$T_{U(1)_A} \sim 1.1 T_{chiral}$
(within ChPT uncertainties)

LECs from
X. K. Guo et al 2015

$\rightarrow O(4) \times U_A(1)$ in chiral limit

($K - \kappa$ coincidence with $\pi - \delta$ support need for $U(1)_A$ restor.)

$\Rightarrow$ Compatible with WIs, lattice and models (NJL Ishii et al PRD 2017)

Scattering and Resonances within finite-T Unitarized ChPT

A.Dobado, AGN, F.J.Llanes-Estrada, J.R.Peláez, J.Ruiz de Elvira, A.Vioque-Rodríguez

- **Unitarized** **meson scattering from** **thermal unitarity** **including physical** **thermal-bath processes:**

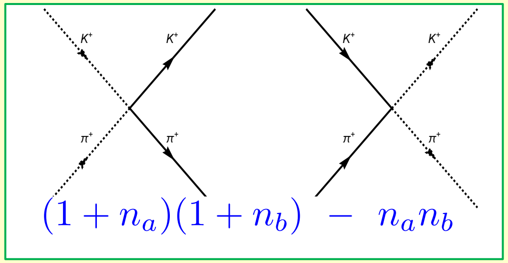
$$\textbf{IAM} \rightarrow t_U(s;T) = \frac{\left[t_2(s)\right]^2}{t_2(s) - t_4(s;T)} \qquad t_2 + t_4 + \cdots \ \text{ChPT PW}$$

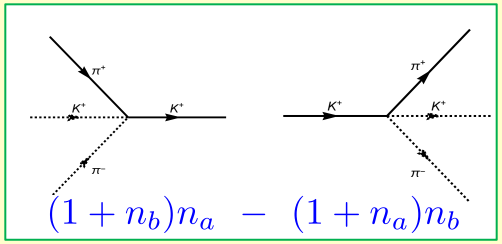
$$\mathbf{Im}\, t_U(s;T) = \begin{cases} \sigma^T(s)\left[t_U(s;T)\right]^2, & s \geq (M_a + M_b)^2 \ \textbf{(unit.cut)} \\ \tilde{\sigma}^T(s)\left[t_U(s;T)\right]^2, & 0 \leq s \leq (M_a - M_b)^2 \ \textbf{(Landau thermal cut)} \end{cases}$$

- **Thermal phase space:**

$$\sigma^T_{ab}(s) = \sigma_{ab}(s)\left[1 + n_B\left(\frac{s+\Delta_{ab}}{2\sqrt{s}}\right) + n_B\left(\frac{s-\Delta_{ab}}{2\sqrt{s}}\right)\right]$$

$$\tilde{\sigma}^T_{ab}(s) = \sigma_{ab}(s)\left[n_B\left(\frac{\Delta_{ab}-s}{2\sqrt{s}}\right) - n_B\left(\frac{s+\Delta_{ab}}{2\sqrt{s}}\right)\right]$$





$\Delta_{ab} = M_a^2 - M_b^2$, σ_{ab} **two-body** $T=0$ **phase space**

$n_i \equiv n_B(E_i) = \frac{1}{e^{E_i/T}-1}$ **BE distrib.**

- **Poles in proper Riemann sheet** $\rightarrow$ **thermal resonances** $s_{pole}(T) = \left[M_p(T) - i\frac{\Gamma_p(T)}{2}\right]^2$

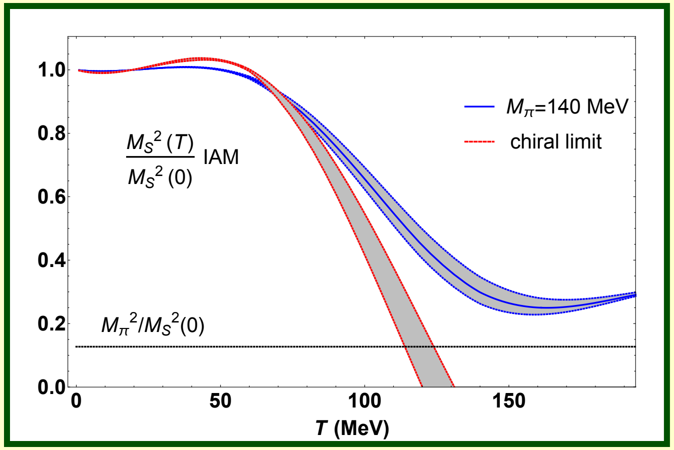
Saturating scalar susceptibilities with light thermal resonances

S.Ferreres-Solé, AGN, A.Vioque-Rodríguez, PRD99, 036018 (2019)

$\Rightarrow$ χ_S **saturated by lightest** $I=J=0$ **state, i.e.** $f_0(500)$ **generated in unitarized finite-**T $\pi\pi$ **scattering**

$$\chi_S(T) \simeq \chi_S(0)\frac{M_S^2(0)}{M_S^2(T)}$$

$M_S^2(T) = M_p^2(T) - \Gamma_p^2(T)/4 = \mathbf{Re}(s_{pole}(T)) \sim \mathbf{Re}\Sigma_\sigma$ **behaves as** $p=0$ **thermal mass in this channel (scaling near** T_c **checked with LSM analysis)**



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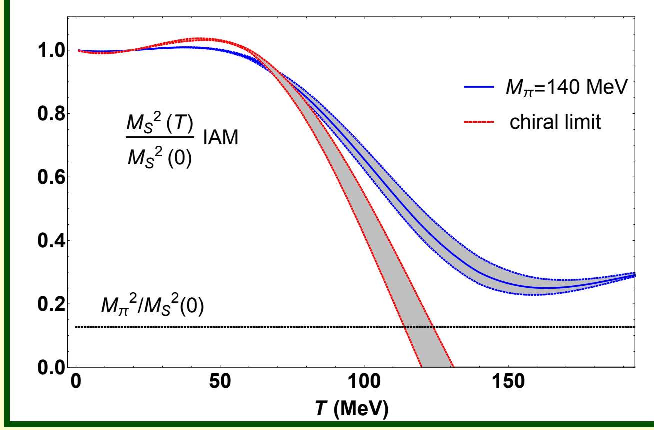
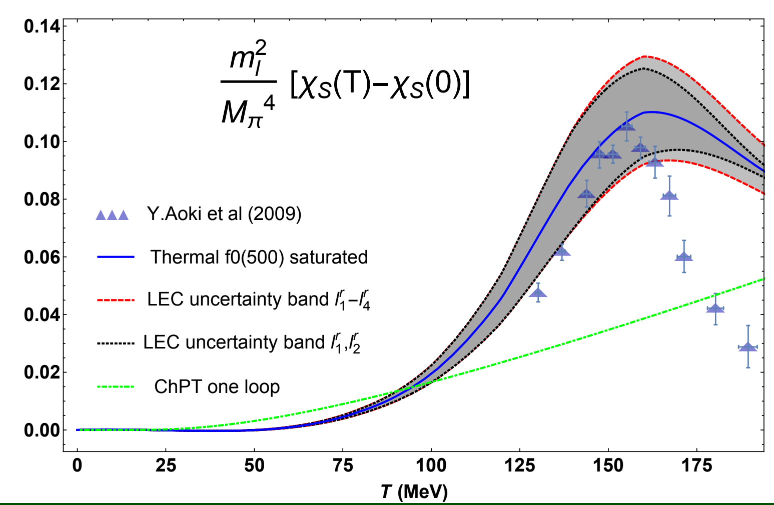


- **Reproduces expected peak $T_c \sim 158$ MeV**
- **Agrees with lattice below the peak within uncertainties**
- **Consistent T_c reduction and χ_S growth near chiral limit**



$\chi_S(0) = \chi_S^{ChPT}(0)$

LECs FLAG coll.& Hanhart, Peláez, Ríos PRL100 (2008)

$s_p = 446.5 - i220.4$ **MeV at** $T = 0$

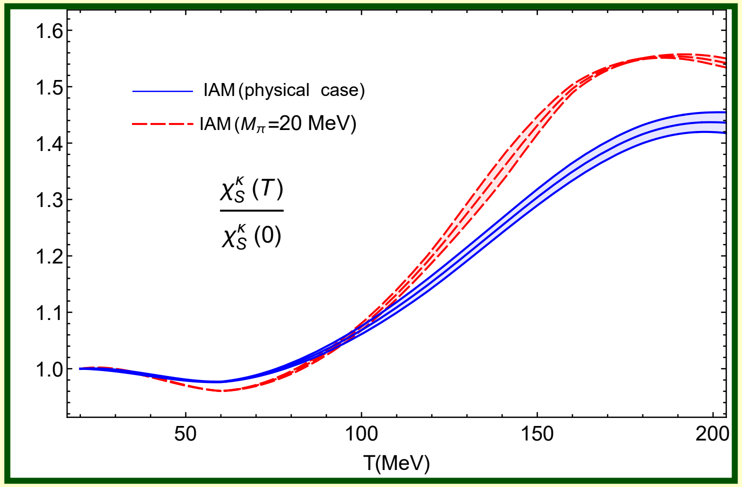
Thermal interactions crucial!

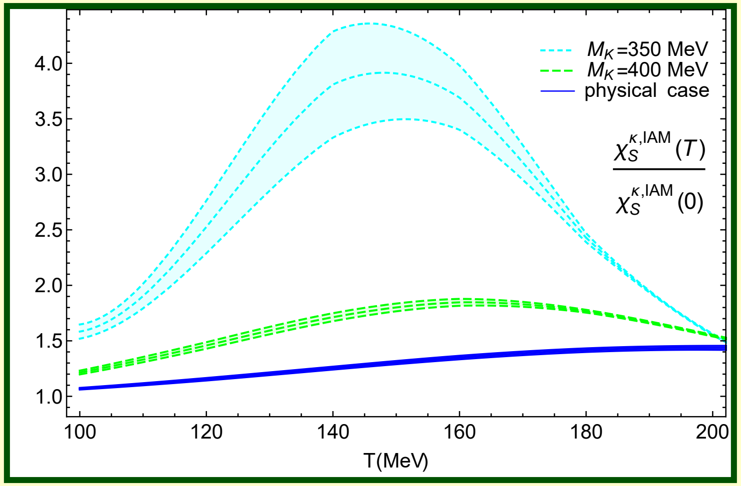
Saturating scalar susceptibilities with light thermal resonances

$\Rightarrow \chi_S^\kappa$ **saturated by $I=1/2$ $K_0^*(700)$ scalar pole from unitarized ChPT thermal $K\pi$ scattering**

AGN, J.Ruiz de Elvira, A.Vioque-Rodríguez, D. Álvarez-Herrero, EPJC81 (2021) 637

AGN, J.Ruiz de Elvira, A.Vioque-Rodríguez, 2304.08786 [hep-ph]





$m_l/m_s \to 0^+$: $\Rightarrow$ **larger growth below peak enhanced by chiral symmetry**
$\Rightarrow$ **flattening above peak from more efficient $\chi_S^\kappa \to \chi_P^K$**

$m_l/m_s \to 1$: $\chi_S^\kappa \to \chi_S$:
peak grows and moves towards T_c ($SU(3)$ degeneration $\sigma \leftrightarrow \kappa$)

Thermal interactions crucial again to reproduce expected peak behaviour

$$\chi_S^\kappa(0) = \chi_S^{\kappa,ChPT}(0)$$

LECs Molina, Ruiz de Elvira JHEP2020

$s_p = 678 - i289$ **MeV at** $T=0$

CONCLUSIONS

- ⋆ $U(1)_A$ **breaking** $@T_c$ **stronger for** $N_f = 2+1$ **than** $N_f = 2$.
  Role of strangeness crucial

- ⋆ **WI** $\Rightarrow$ $O(4) \times U(1)_A$ **for exact chiral restoration of S/P nonet.**
  OK with $N_f = 2$ **lattice**

- ⋆ **WI** $\Rightarrow K/\kappa$ **alternative channel for** $O(4) \times U(1)_A$ **restoration.**
  Driven by $\langle \bar{s}s \rangle$ **around** χ_S^κ **peak**

- ⋆ $U(3)$ **ChPT patterns&partners OK with WI and lattice**

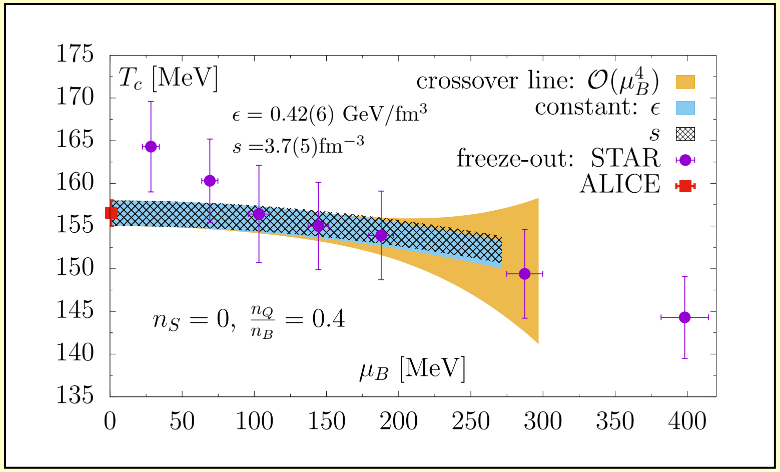
- ⋆ **Scalar thermal resonances crucial for chiral and** $U(1)_A$ **restoration. Generated within finite-**T **unitarized ChPT scattering**

- ⋆ **Saturated** χ_S **with thermal** $f_0(500)$ **reproduces** χ**SR peak at** T_c **in agreement with lattice**

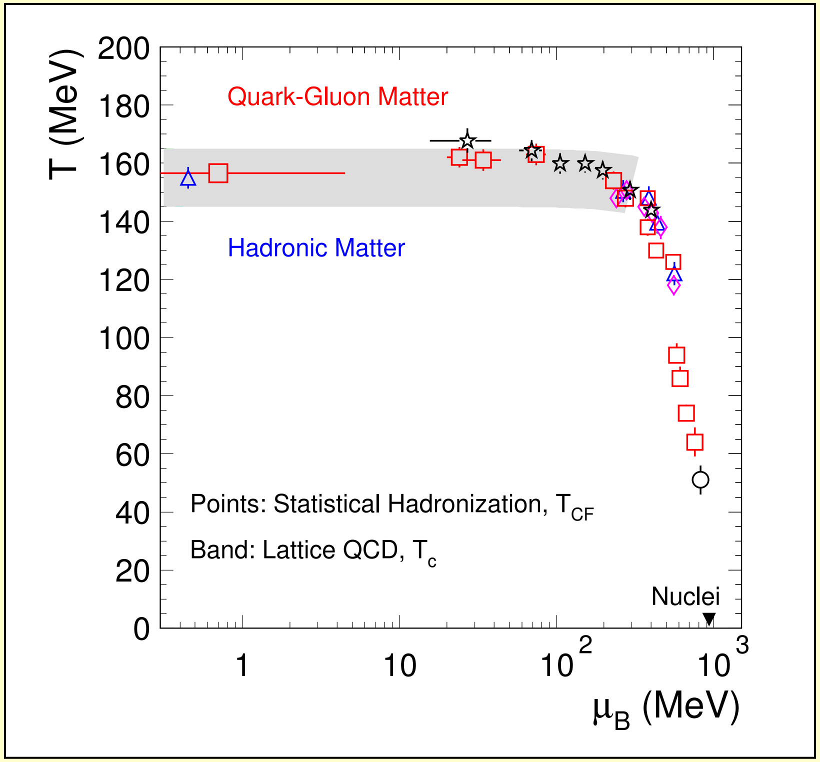
- ⋆ **Saturated** χ_S^κ **with thermal** $K_0^*(700)$ **OK with** $O(4) \times U(1)_A$ **pattern**

BACKUP SLIDES

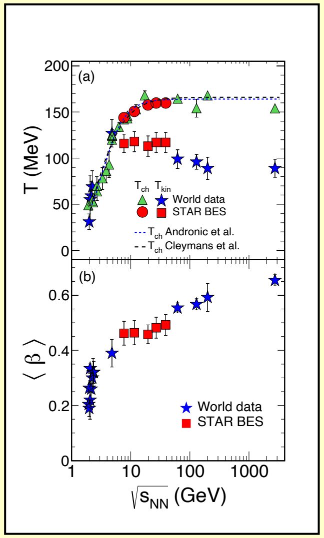
QCD phase diagram explored in HIC $\rightarrow$ chemical freeze-out close to phase boundary

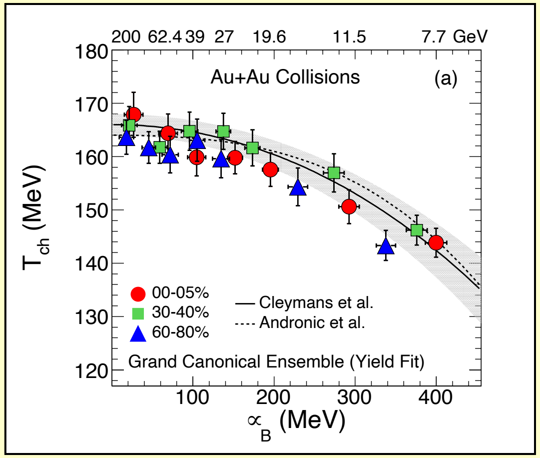


A.Bazavov et al (Hot QCD) 2018
(μ_B through Taylor expansion)



Andronic et al 2018 (ALICE)



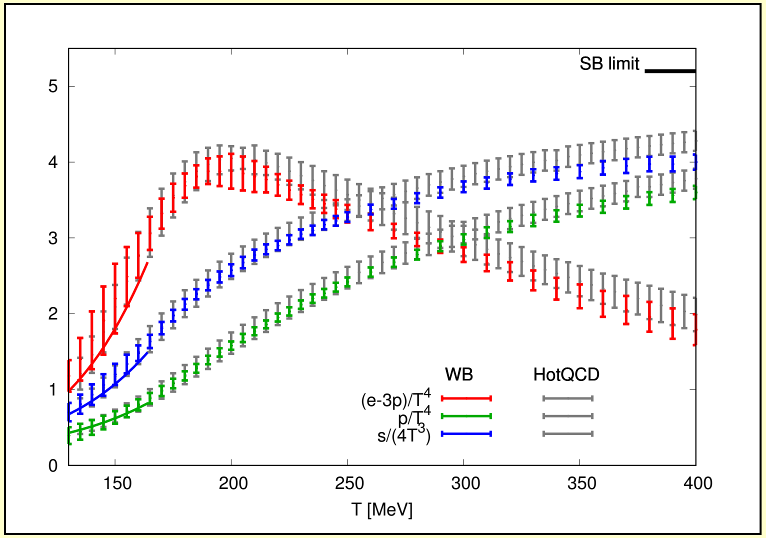


Chemical FO from Hadron Statistical Model fit to hadron yields (central ALICE data)

BES (STAR Adamczyk et al 2017)

OTHER HIGHLIGHTS OF QCD TRANSITION

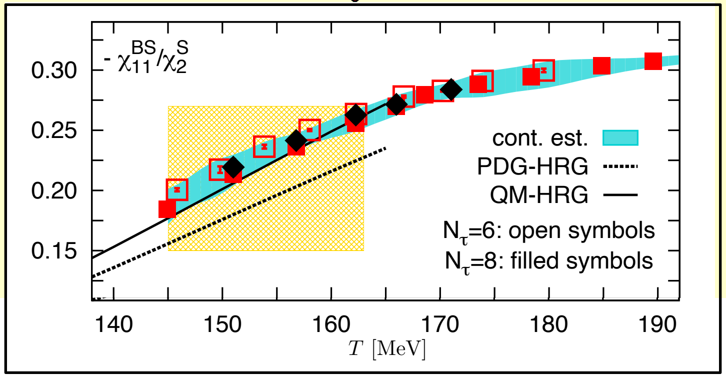
Pressure, entropy, trace anomaly



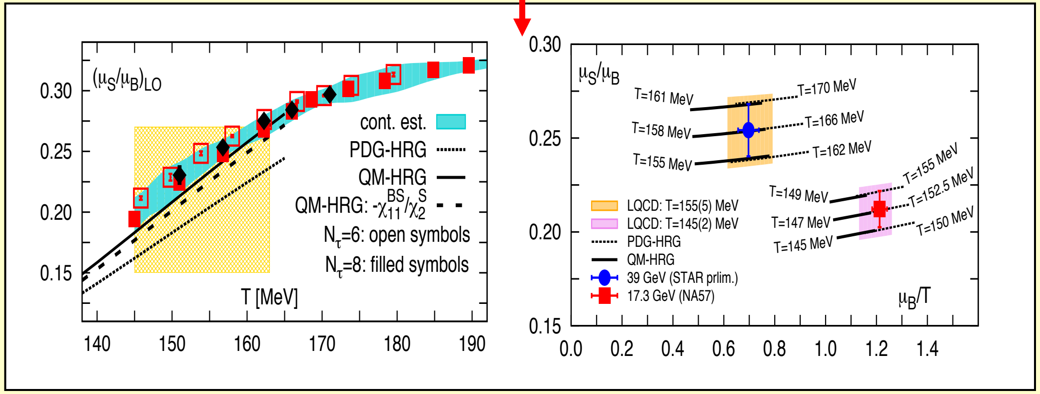
From C.Ratti 2018 (2014 WB, HotQCD data)

Fluctuations of conserved charges (Q,B,S)

Bazavov et al 2014



related to strangeness freeze-out conditions



$$n_S = 0,\ n_Q/n_B = 0.4$$

Chiral Symmetry Restoration vs $U(1)_A$

Additional consequences of $U(1)_A$ restoration around T_c:

- **Reduction of $M_{\eta'}(T)$ (visible also in HIC) and $\chi_{top}(T)$**

  J. I. Kapusta, D. Kharzeev, L. D. McLerran, PRD53, 5028 (1996)

  T. Csorgo et al, PRL105, 182301 (2010)

  G. Grilli di Cortona, et al, JHEP 1601, 034 (2016)

  M. Ishii et al, PRD95, 114022 (2017)

  AGN, J.Ruiz de Elvira, A.Vioque-Rodríguez, JHEP11, 086 (2019)

  M. P. Lombardo, A. Trunin, IJMPA 35 (2020)

  S.Aoki et al (JLQCD coll), PRD103, 074506 (2021)

- **$\eta - \eta'$ mixing ideal ($\eta \sim \eta_l$, $\eta' \sim \sqrt{2}\eta_s$)**

  M. Ishii et al PRD93, 016002 (2016)

  AGN, J.Ruiz de Elvira, JHEP1603, 186 (2016), PRD97, 074016 (2018), PRD98, 014020 (2018)

- **Affects critical point at $\mu_B \neq 0$**

  M. Mitter, B. J. Schaefer, PRD89, 054027 (2014)

AGN, J.Ruiz de Elvira, R.Torres,
PRD88, 076007 (2013)
AGN, J.Ruiz de Elvira,
JHEP1603, 186 (2016)
PRD97, 074016 (2018)
PRD98, 014020 (2018)

$O(4)$ vs $U(1)_A$: Ward Identities

From QCD generating functional:

- π **SECTOR** $\rightarrow \chi_P^\pi(T) = -\dfrac{\langle \bar{q}q \rangle_l (T)}{m_l}$

- K **SECTOR** $\rightarrow \chi_P^K(T) = -\dfrac{\langle \bar{q}q \rangle_l (T) + 2\langle \bar{s}s \rangle (T)}{m_l + m_s}$

- η, A **SECTOR** $\rightarrow \eta/\eta'$ **mixing &** $U_A(1)$ **anomaly:**

$$\begin{aligned}
\chi_P^{\eta_l}(T) &= -\frac{\langle \bar{q}q \rangle_l (T)}{m_l} - \frac{4}{m_l^2}\chi_{top}(T) \\
\chi_P^{\eta_s}(T) &= -\frac{\langle \bar{s}s \rangle (T)}{m_s} - \frac{1}{m_s^2}\chi_{top}(T) \\
\chi_P^{ls}(T) &= -\frac{m_l}{2m_s}\left[\chi_P^\pi(T) - \chi_P^{\eta_l}(T)\right] = -\frac{2}{m_l m_s}\chi_{top}(T)
\end{aligned}$$

- κ **SECTOR** $\rightarrow \chi_S^\kappa(T) = \dfrac{\langle \bar{q}q \rangle_l (T) - 2\langle \bar{s}s \rangle (T)}{m_s - m_l}$

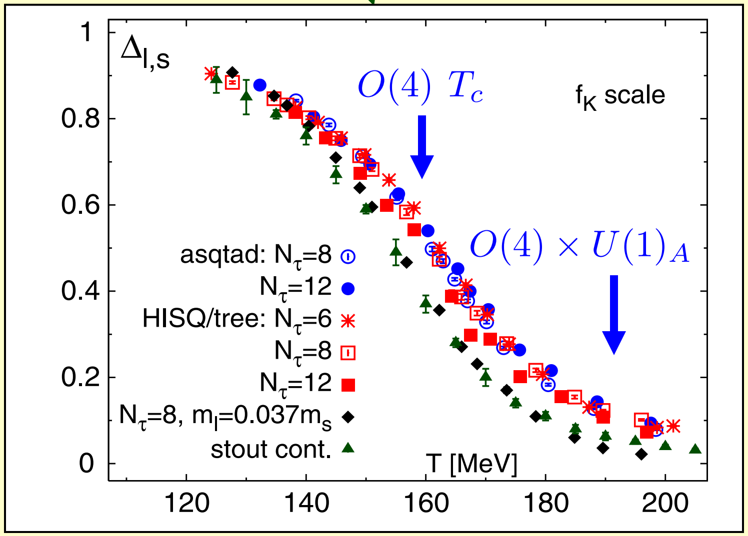
$\chi_P^O \equiv \int_T dx \, \langle O(x)O(0) \rangle$ **susceptibilities**

$\chi_{top} \equiv -\dfrac{1}{36}\int_T dx \langle \mathcal{T} A(x)A(0) \rangle$ **Topological Susceptibility** $\quad A(x) = \dfrac{3\alpha_s}{4\pi} Tr_c G_{\mu\nu}\tilde{G}^{\mu\nu}$

In addition ...

$$\chi_S^\kappa(T) - \chi_P^K(T) = \frac{2m_s}{m_s^2 - m_l^2}\left[\langle \bar{q}q\rangle_l(T) - 2\frac{m_l}{m_s}\langle \bar{s}s\rangle(T)\right]$$

subtracted condensate $\Delta_{l,s}$
(removes lattice finite-size divergences)



$\Rightarrow$ $N_f = 2$, consistent with $O(4) \times U(1)_A$ restoration for $m_l, \langle \bar{q}q\rangle_l \to 0^+$

$\Rightarrow$ In physical limit provides well-determined $O(4) \times U(1)_A$ breaking above T_c via $\Delta_{l,s}$ tail modulated by $\langle \bar{s}s\rangle$

Ward Identities and lattice screening masses

- M_{sc} **well measured in lattice** $\lim_{z\to\infty} K(z) \simeq e^{-M_{sc}z}$

- **With reasonable assumptions of smooth T-behaviour of residues and M_{sc}/M_{pole} in $K_{P,S}$ correlators :**

$$\chi_{P,S} = K_{P,S}(p=0) \sim M_{pole}^{-2} \sim M_{sc}^{-2}$$

$$\frac{M_\pi^{sc}(T)}{M_\pi^{sc}(0)} \sim \left[\frac{\chi_P^\pi(0)}{\chi_P^\pi(T)}\right]^{1/2} = \left[\frac{\langle \bar q q\rangle_l(0)}{\langle \bar q q\rangle_l(T)}\right]^{1/2}$$

$$\frac{M_K^{sc}(T)}{M_K^{sc}(0)} \sim \left[\frac{\chi_P^K(0)}{\chi_P^K(T)}\right]^{1/2} = \left[\frac{\langle \bar q q\rangle_l(0)+2\langle \bar s s\rangle(0)}{\langle \bar q q\rangle_l(T)+2\langle \bar s s\rangle(T)}\right]^{1/2}$$

$$\frac{M_{\bar s s}^{sc}(T)}{M_{\bar s s}^{sc}(0)} \sim \left[\frac{\chi_P^{\bar s s}(0)}{\chi_P^{\bar s s}(T)}\right]^{1/2} \sim \left[\frac{\langle \bar s s\rangle(0)}{\langle \bar s s\rangle(T)}\right]^{1/2}$$

$$\frac{M_\kappa^{sc}(T)}{M_\kappa^{sc}(0)} \sim \left[\frac{\chi_S^\kappa(0)}{\chi_S^\kappa(T)}\right]^{1/2} = \left[\frac{\langle \bar q q\rangle_l(0)-2\langle \bar s s\rangle(0)}{\langle \bar q q\rangle_l(T)-2\langle \bar s s\rangle(T)}\right]^{1/2}$$

Anomalous contrib. $\frac{\hat m}{m_s}$ **suppressed**

WI and Screening Masses

Subtracted Condensates have the right critical behavior in lattice, avoiding $T=0$ finite-size divergences $\langle \bar{q}_i q_i \rangle \sim m_i/a^2 + \ldots$:

$$\Delta_l(T) = \frac{\langle \bar{q}q \rangle_l (T) - \langle \bar{q}q \rangle_l (0) + \langle \bar{q}q \rangle_l^{ref}}{\langle \bar{q}q \rangle_l^{ref}}$$

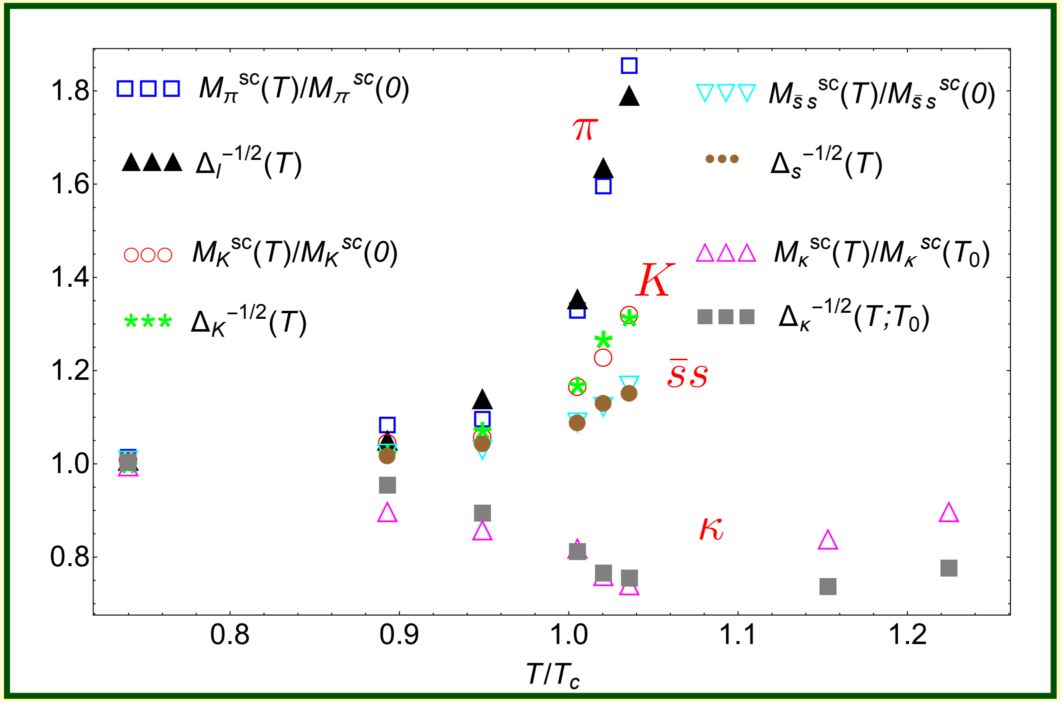
$$\Delta_K(T) = \frac{\langle \bar{q}q \rangle_l (T) - \langle \bar{q}q \rangle_l (0) + 2\left[\langle \bar{s}s \rangle (T) - \langle \bar{s}s \rangle (0)\right] + \langle \bar{q}q \rangle_l^{ref} + \langle \bar{s}s \rangle^{ref}}{\langle \bar{q}q \rangle_l^{ref} + \langle \bar{s}s \rangle^{ref}}$$

$$\Delta_s(T) = \frac{2\left[\langle \bar{s}s \rangle (T) - \langle \bar{s}s \rangle (0)\right] + \langle \bar{s}s \rangle^{ref}}{\langle \bar{s}s \rangle^{ref}}$$

$$\Delta_\kappa(T;T_0) = \frac{\langle \bar{q}q \rangle_l (T) - \langle \bar{q}q \rangle_l (0) - 2\left[\langle \bar{s}s \rangle (T) - \langle \bar{s}s \rangle (0)\right] + \langle \bar{q}q \rangle_l^{ref} - \langle \bar{s}s \rangle^{ref}}{\langle \bar{q}q \rangle_l (T_0) - \langle \bar{q}q \rangle_l (0) - 2\left[\langle \bar{s}s \rangle (T_0) - \langle \bar{s}s \rangle (0)\right] + \langle \bar{q}q \rangle_l^{ref} - \langle \bar{s}s \rangle^{ref}}$$

$$\begin{aligned} r_1^3 \langle \bar{q}q \rangle_l^{ref} &= 0.750 \\ r_1^3 \langle \bar{s}s \rangle^{ref} &= 1.061 \\ r_1 &\simeq 0.31 \text{ fm} \end{aligned}$$

Using lattice masses and condensates from same simulation:
M.Cheng et al (HotQCD) EPJC71, 1564 (2011); PRD77, 014511 (2008)



- **Δ_i subtracted condensates with two fit parameters**
- **Scaling law OK within 5% for $T < T_c$**
- **Rapid increase $M_\pi^{sc} \sim \langle \bar{q}q \rangle_l^{-1/2}$ around T_c**
- **Softer $M_K^{sc} \sim (\langle \bar{q}q \rangle_l + 2\langle \bar{s}s \rangle)^{-1/2}$. and even softer $M_{\bar{s}s}^{sc} \sim \langle \bar{s}s \rangle^{-1/2}$**
- **Minimum of M_κ^{sc} from $\langle \bar{q}q \rangle_l - 2\langle \bar{s}s \rangle$ counterpart of χ_S^κ peak (last two points not fitted)**

Disconnected susceptibility and $O(4) \times U(1)_A$ restoration

$$\boxed{\chi_S^\sigma = 2\chi_S^{con} + 4\chi_S^{dis} = \chi_S^\delta + 4\chi_S^{dis}} \quad (m_u = m_d)$$

On the other hand,

$$\boxed{\chi_{5,disc}(T) = \chi_S^{dis}(T) + \frac{1}{4}\left[\chi_P^\pi(T) - \chi_S^\sigma(T)\right] + \frac{1}{4}\left[\chi_S^\delta(T) - \chi_P^{\eta_l}(T)\right]}$$

$\Rightarrow$ **Is the vanishing of $\chi_{5,disc}$ in conflict with χ_S^{dis} peaking at the chiral transition?**

Disconnected susceptibility and $O(4) \times U(1)_A$ restoration

$$\boxed{\chi_S^\sigma = 2\chi_S^{con} + 4\chi_S^{dis} = \chi_S^\delta + 4\chi_S^{dis}} \quad (m_u = m_d)$$

On the other hand,

$$\boxed{\chi_{5,disc}(T) = \chi_S^{dis}(T) + \frac{1}{4}\left[\chi_P^\pi(T) - \chi_S^\sigma(T)\right] + \frac{1}{4}\left[\chi_S^\delta(T) - \chi_P^{\eta_l}(T)\right]}$$

From ChPT in the chiral limit $M_\pi \to 0^+$(IR),

$$\Rightarrow \chi_{5,disc}(T_c) \sim \mathcal{O}\left(\frac{T_c}{M_\pi}\right) \sim \chi_S^{dis}(T_c)$$

"peak" with same coeff.

$$\chi_{5,disc}(T_c) = \chi_S^{dis}(T_c) + \frac{1}{4}\cancel{\left[\chi_P^\pi(T_c) - \chi_S^\sigma(T_c)\right]} + \underbrace{\frac{1}{4}\left[\chi_S^\delta(T_c) - \chi_P^{\eta_l}(T_c)\right]}_{IR\ regular}$$

Disconnected susceptibility and $O(4)\times U(1)_A$ restoration

$$\boxed{\chi_S^\sigma = 2\chi_S^{con} + 4\chi_S^{dis} = \chi_S^\delta + 4\chi_S^{dis}} \quad (m_u = m_d)$$

On the other hand,

$$\boxed{\chi_{5,disc}(T) = \chi_S^{dis}(T) + \frac{1}{4}\left[\chi_P^\pi(T) - \chi_S^\sigma(T)\right] + \frac{1}{4}\left[\chi_S^\delta(T) - \chi_P^{\eta_l}(T)\right]}$$

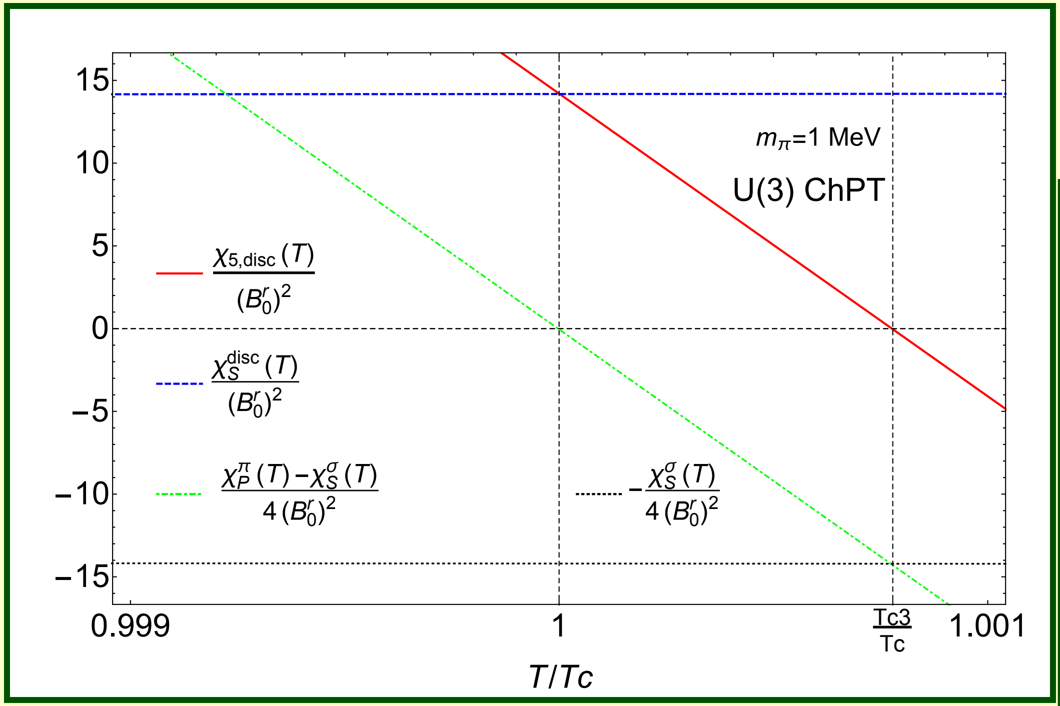
From ChPT in the chiral limit $M_\pi \to 0^+$(IR), $\Rightarrow T_{c3} = T_c + \mathcal{O}\left(M_\pi^2\right)$

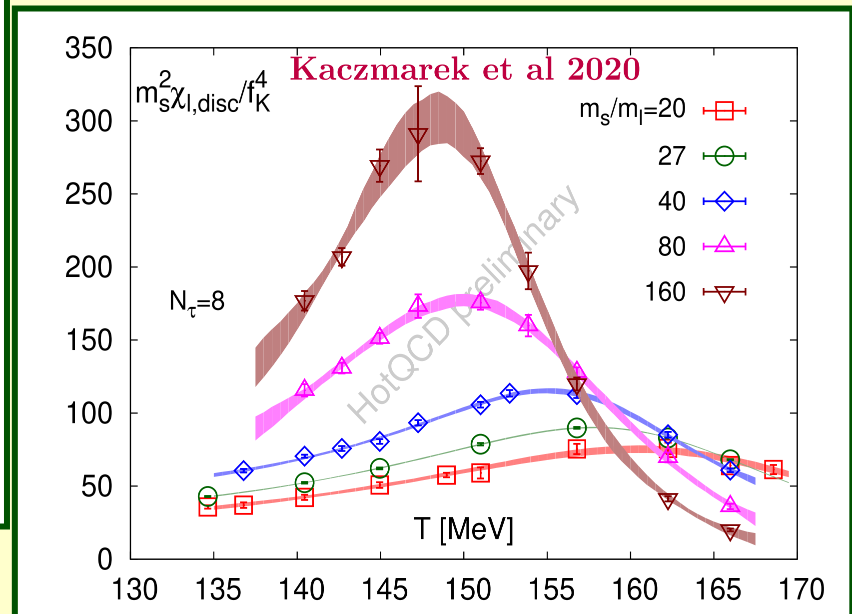
$$\Rightarrow \chi_S^{disc}(T_{c3}) \sim \mathcal{O}\left(\frac{T_{c3}}{M_\pi}\right) \sim \frac{1}{4}\chi_S^\sigma(T_{c3}) \quad \text{"peak" with same coff.}$$

$$\chi_{5,disc}(T_{c3}) = 0 = \chi_S^{dis}(T_{c3}) + \frac{1}{4}\left[\cancel{\chi_P^\pi(T_{c3})} - \chi_S^\sigma(T_{c3})\right] + \frac{1}{4}\left[\underbrace{\chi_S^\delta(T_{c3})}_{IR\ regular} \cancel{-\chi_P^{\eta_l}(T_{c3})}\right]$$

Disconnected susceptibility and $O(4)\times U(1)_A$ restoration

Vanishing of $\chi_{5,dis}$ at $O(4)\times U(1)_A$ rest. near T_c at chiral limit compatible with χ_S^{dis} peak and both peaking at T_c





$\Rightarrow$ χ_S^{dis} **not the best $O(4)\times U(1)_A$ indicator near chiral limit**

Effective Theories

- Needed to describe the hadron gas for $T < T_c$

- SU(2), SU(3) and U(3) ChPT model-independent formalism for light mesons (π, K, η, η'), more abundant at low T

  Chiral counting $1/N_c \sim m_q \sim T^2 \sim p^2$ [(1)]

- Light meson collisions are dominant interactions

  $\rightarrow$ thermal resonances through Unitarization [(2)]

- Spectral modifications of thermal resonances decaying in the hadron gas described within ET[(3)]

- HRG allows to include heavier states for $T < T_c$ [(4)]

- (U)ChPT useful for certain observables regarding chiral and $U(1)_A$ restoration around T_c $\rightarrow\rightarrow\rightarrow$

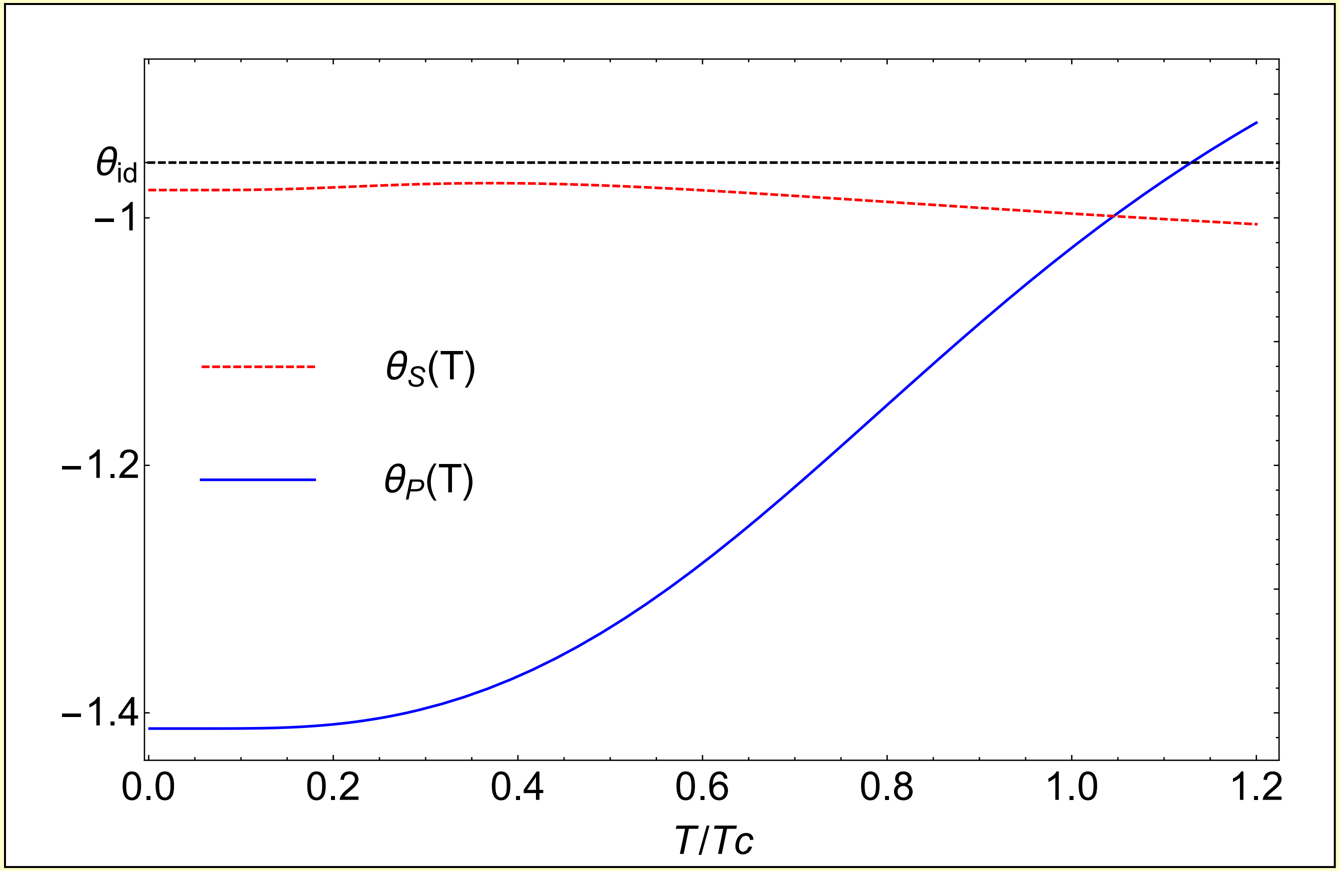
(1) Gasser, Leutwyler, Gerber, Kaiser, Herrera-Siklody et al, AGN, Ruiz de Elvira, ...

(2) Cabrera, Dobado, Fernández-Fraile, AGN, Llanes-Estrada, Peláez, Ruiz de Elvira, Torres-Andrés, Vioque, …

(3) Gale, Rapp, Wambach, Song, Koch, van Hees, Tolos, ...

(4) Karsch, Tawfik, Redlich, Tawfik-Toublan, Huovinen, Petreczcky, Jankowski, Blaschke, Spalinski, ...

Mixing in $U(3)$ ChPT



$$\frac{1}{2}\left[\chi^{88}_{P,S}(T)-\chi^{00}_{P,S}(T)\right]\sin\left[2\theta_{P,S}(T)\right]+\chi^{08}_{P,S}\cos\left[2\theta_{P,S}(T)\right]=0,$$

($\eta\eta'$ correlator vanishing, leading order)

Effective Theories: Topological susceptibility in $U(3)$ ChPT

AGN, J.Ruiz de Elvira, A.Vioque-Rodríguez,
JHEP 11 (2019) 086

$$\epsilon_{vac}(\theta) = \epsilon_{vac}(0) + \frac{1}{2}\chi_{top}\theta^2 + \frac{1}{24}c_4\theta^4 + \dots$$

$\sim$ **Axion mass** $\quad$ $\sim$ **Axion coupling**

$$\chi_{top}^{U(3),LO} = \Sigma \frac{M_0^2 \bar{m}}{M_0^2 + 6B_0\bar{m}} \qquad c_4^{U(3),LO} = -\frac{\Sigma}{\bar{m}^{[3]}}\left(\frac{M_0^2\bar{m}}{M_0^2+6B_0\bar{m}}\right)^4$$

$$\bar{m} = \left[\frac{1}{m_u}+\frac{1}{m_d}+\frac{1}{m_s}\right]^{-1} \qquad \bar{m}^{[3]} = \left[\frac{1}{m_u^3}+\frac{1}{m_d^3}+\frac{1}{m_s^3}\right]^{-1}$$

$(-\Sigma)$ LO quark condensate $\quad$ M_0 anomalous part of $M_{\eta'}$ $(m_{u,d,s}=0)$

- **vanish** $m_q \to 0 \Rightarrow$ **expected to be well described within ChPT**
- $SU(3)$ **for** $M_0 \to \infty$, $SU(2)$ **for** $M_0, m_s \to \infty$
- **Quenched** $m_q \to \infty$**:** $\chi_{top}^{LO} = F^2 M_0^2/6$

  (Witten-Veneziano 1979) $\to$ **meson loops crucial**

Leutwyler,Smilga 1992: $SU(3)$ **LO**
Mao et al 2009; Bernard et al: 2012: $SU(3)$ **NLO**
Grilli et al 2016: $T \neq 0$ $SU(2)$ **NLO**

Effective Theories: Topological susceptibility in $U(3)$ ChPT

→ $T = 0$ **results** ($m_u = m_d$):

$\chi_{top}^{1/4}$ [MeV]	U(3)	SU(2)	SU(3)
LO	74(3)	75(3)	75(3)
NLO	74(3)	78(3)	83(2)
NNLO	81(2)		

$\left[\chi_{top}^{latt}\right]^{1/4} = 73(9)$ **(Bonati et al 2016)**

$b_2 = \frac{c_4}{12\chi_{top}}$	U(3)	SU(2)	SU(3)
LO	-0.01737(4)	-0.02083	-0.01960
NLO	-0.018(2)	-0.029(2)	-0.025(1)
NNLO	-0.023(2)		

$b_2^{latt} = -0.0216(15)$ **(Bonati et al 2016, gluodynamics)**

⇒ $SU(2)$ **dominates,** $U(3)$ η' **loops and** $\eta - \eta'$ **mixing of the same order than** $SU(3)$ K, η **loops**

⇒ **full** $U(3)$ **in agreement with lattice within uncertainties (LEC and lattice, larger lattice uncertainty for** b_2**)**

⇒ **In addition, within** $U(3)$ **ChPT, explicit expressions for the leading and subleading** $1/N_c$ **contributions can be obtained:**

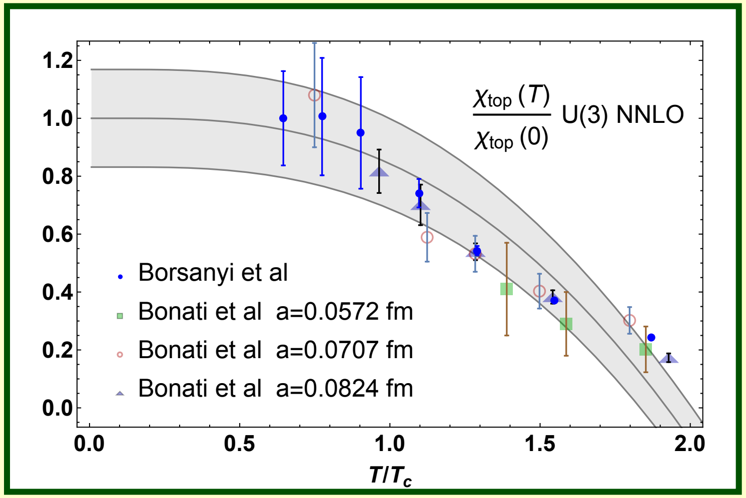
$$\chi_{top} = \underbrace{\mathcal{O}(1)}_{\text{Witten-Veneziano term}} + \mathcal{O}(N_c^{-1}) + \dots$$

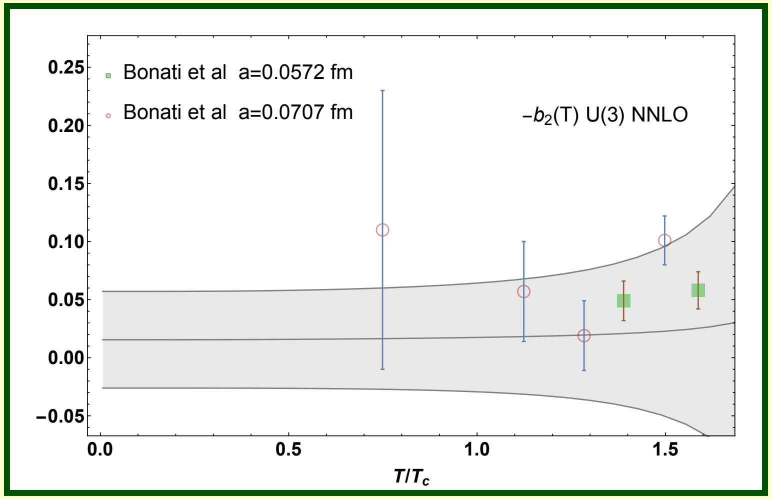
$$c_4 = \underbrace{\mathcal{O}(N_c^{-2})}_{\text{constant } \theta^4 \text{ term in } \mathcal{L}} + \mathcal{O}(N_c^{-3}) + \dots$$

OK with lattice and theo:

⇒ **Vicari, Panagopoulos 2009-2011**
Bonati et al 2016
Vonk, Guo, Meissner 2019

Effective Theories: Topological susceptibility in $U(3)$ ChPT

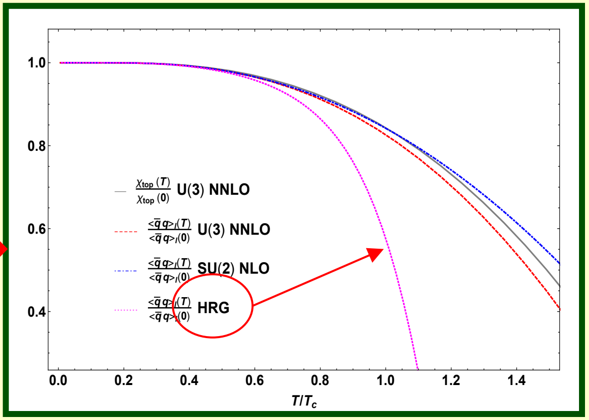




- T-dependence captured by ChPT within uncertainties (larger for c_4) far beyond the low-T applicability regime

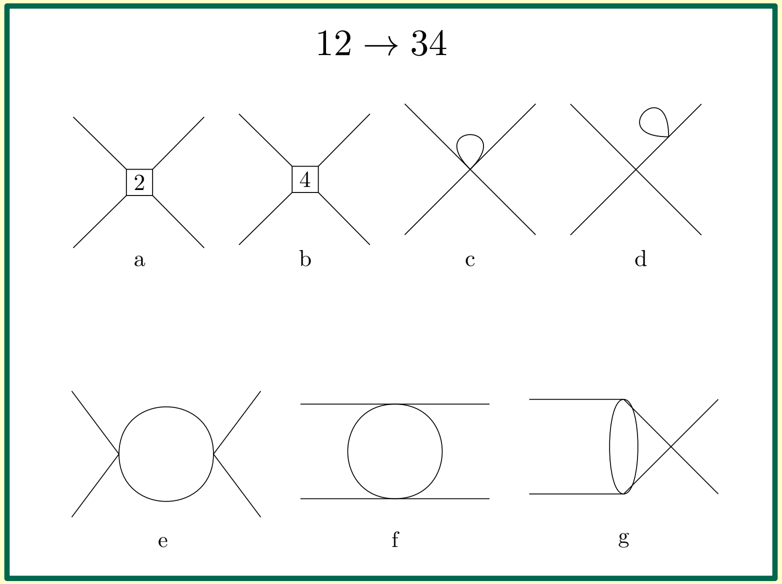
- χ_{top} scales with T dominated by $\langle \bar{q}q \rangle_l$
- However, 2nd term in WI $\chi_{top} = -\frac{1}{4}\left[m_{ud} \langle \bar{q}q \rangle_l + m_{ud}^2 \chi^{\eta_l} \right]$ relevant near T_c

  $\rightarrow$ finite $O(4)$-$U(1)_A$ gap in physical case



Scattering and Resonances within finite-T Unitarized ChPT

A.Dobado, AGN, F.J.Llanes-Estrada, J.R.Peláez,
J.Ruiz de Elvira, A.Vioque-Rodríguez



- **LSZ+finite-T Green function**
- **Thermal bath breaks Lorentz covariance**

  $\rightarrow$ s, t, S^0, T^0, U^0 **on-shell independent**

  $\rightarrow$ **3 independent J_k-loop integrals**

  $\mathbf{S} = p_1 + p_2$, $\mathbf{T} = p_1 - p_3$, $\mathbf{U} = p_1 - p_4$

$$\mathcal{T}(\mathbf{S},\mathbf{T},\mathbf{U};T) = \underset{\textbf{(a)}}{\mathcal{T}_2(s,t,u)} + \underset{\textbf{(b)}}{\mathcal{T}_4^{tree}(s,t,u)} + \underset{\textbf{(c-g)}}{\mathcal{T}_4^F(\mathbf{S},\mathbf{T},\mathbf{U};T)} + \underset{\textbf{(e-g)}}{\mathcal{T}_4^J(\mathbf{S},\mathbf{T},\mathbf{U};T)}$$

$$F_{\beta a}(T) = T \sum_{n=-\infty}^{\infty} \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \frac{1}{q^2 - M_a^2} \quad \textbf{tadpole-like}$$

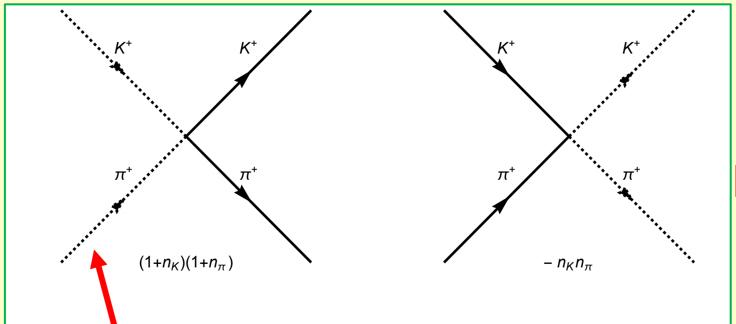
$$J_k^{ab}(Q_0, |\vec{Q}|; T) = T \sum_{n=-\infty}^{\infty} \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \frac{q_0^k}{[q^2 - M_a^2]\,[(q-Q)^2 - M_b^2]} \quad (k = 0, 1, 2)$$

$q_0 = \omega_n \equiv 2\pi i n T$, $Q_0 = \omega_m$ $n, m \in \mathbb{Z}$, $\omega_m \rightarrow -i(Q_0 + i\epsilon)$ **AC to** $Q_0 \in \mathbb{R}$

Scattering and Resonances within finite-T Unitarized ChPT

A.Dobado, AGN, F.J.Llanes-Estrada, J.R.Peláez, J.Ruiz de Elvira, A.Vioque-Rodríguez

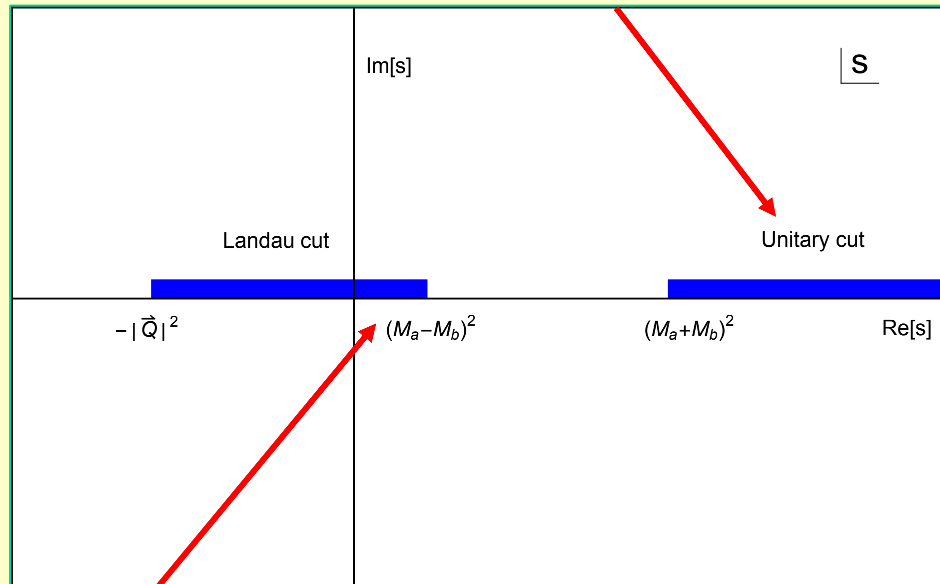
- **Unitarity modified from physical thermal bath processes:**

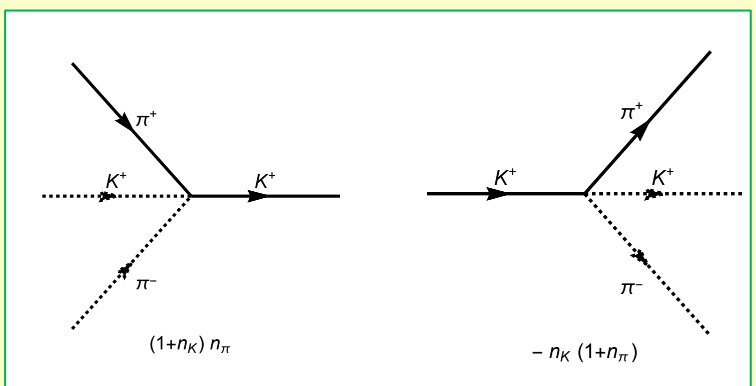


Correction to standard unitary cut

thermal bath particles

$n_i \equiv n_B(E_i) = \frac{1}{e^{E_i/T}-1}$ **BE distrib.**



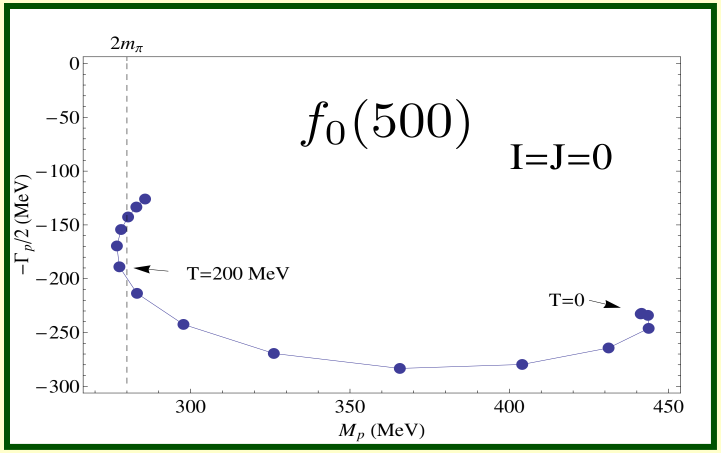


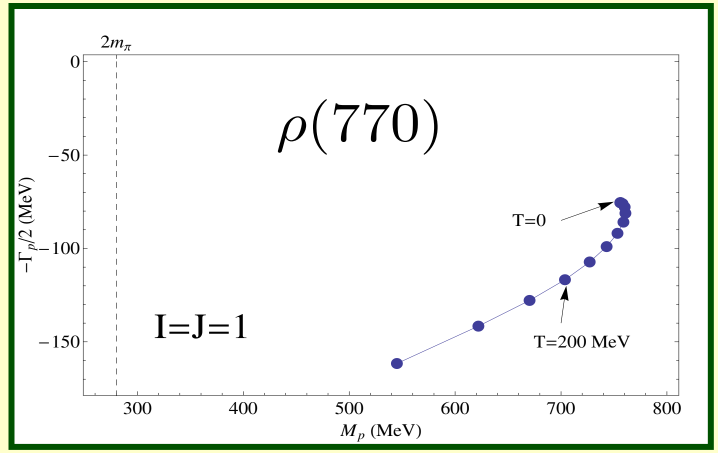
Landau cut
(purely thermal, vanishes for $T=0$ or $M_a=M_b$)

Scattering and Resonances within finite-T Unitarized ChPT

$\pi\pi$ scattering

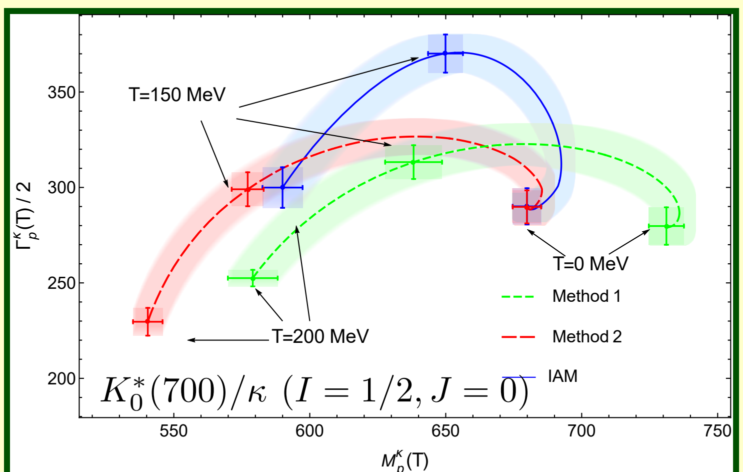
A.Dobado, AGN, F.J.Llanes-Estrada, J.R.Peláez, PRC66, 055201 (2002)

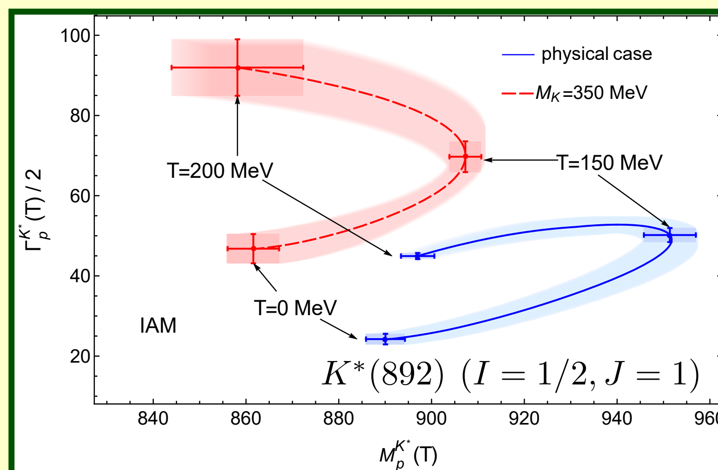




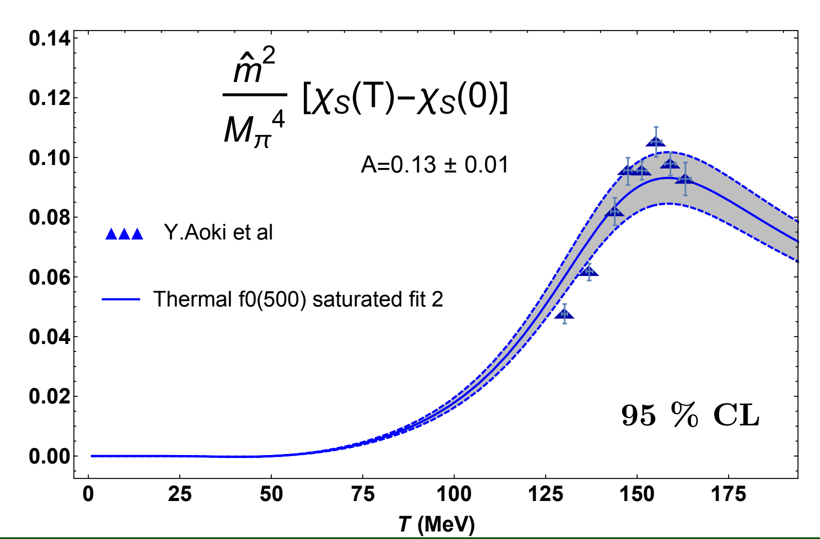
Kπ scattering

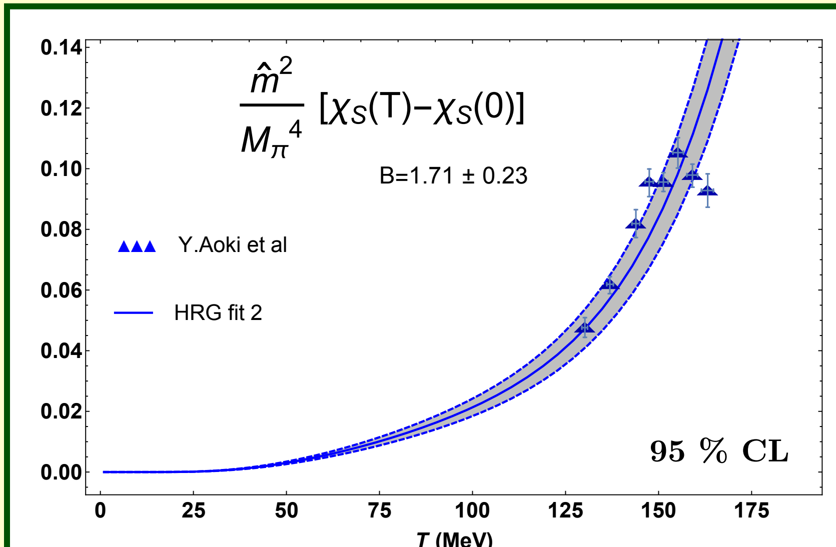
AGN, J.Ruiz de Elvira, A.Vioque-Rodríguez, 2304.08786 [hep-ph]





Saturating scalar susceptibilities with light thermal resonances

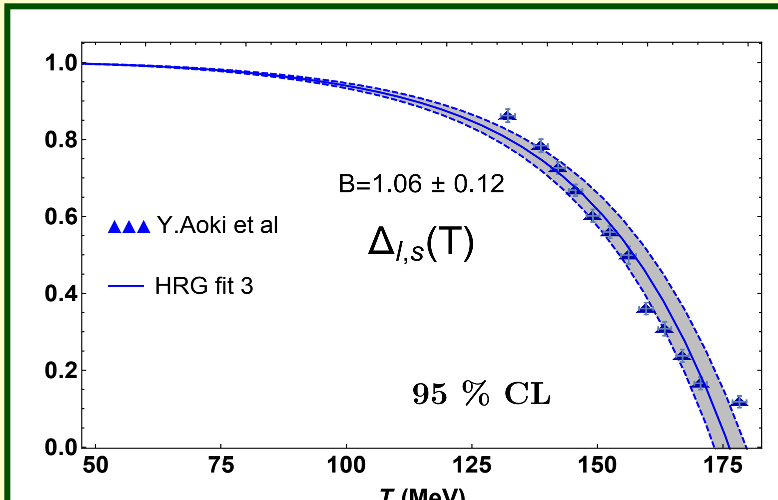




$$\chi_S(T) = A\frac{m_\pi^4}{4m_q^2}\frac{M_S^2(0)}{M_S^2(T)} \qquad (A_{ChPT} \simeq 0.14)$$

HRG Jankowski et al PRD87, 105018 (2013)
Fitted with free-energy normalization *B*

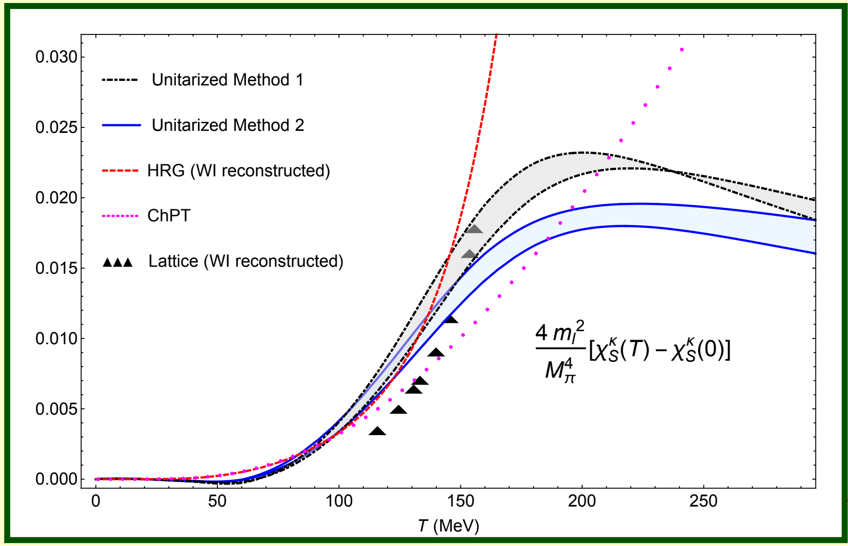
Fit	A	B	χ^2/dof	T_{max}(MeV)
Thermal f_0 fit 1	0.13± 0.02	—	6.25	155
Thermal f_0 fit 2	0.13± 0.01	—	4.93	165
HRG fit 1	—	1.90± 0.02	1.33	155
HRG fit 2	—	1.71± 0.23	10.30	165
HRG fit 3	—	1.06± 0.12	3.77	155



- **Thermal $f_0(500)$ fits better around T_c (peak behaviour)**
- **HRG increasing χ_S, in conflict with condensate fit**

Saturating scalar susceptibilities with light thermal resonances

$\Rightarrow$ χ_S^κ **saturated by thermal** $I = 1/2$ $K_0^*(700)$ **scalar pole from unitarized** πK **scattering** $\rightarrow$ $\chi_S^{\kappa,U}(T) = \chi_S^\kappa(0)\dfrac{M_\kappa^2(0)}{M_\kappa^2(T)}$



Thermal interactions crucial again to reproduce expected peak

$\chi_S^\kappa(0) = \chi_S^{\kappa,ChPT}(0)$

AGN, J.Ruiz de Elvira, A.Vioque-Rodríguez, D. Álvarez-Herrero, EPJC81 (2021) 637

with simplified unitarized methods:

subtracted CU-like approach (R.Gao et al PRD 2019)

$$t_U(s;T) = \frac{t_2^2(s)}{t_2(s) - \tilde{t}_4(s,T)} \Rightarrow \begin{array}{llcl} \textbf{M1:} & \tilde{t}_4(s;T) & = & 16\pi\, t_2(s)^2 \tilde{J}_0^{K\pi}(s;T) \\ \textbf{M2:} & \tilde{t}_4(s;T) & = & t_4(s;0) + 16\pi\, t_2(s)^2 \left[J_0^{K\pi}(s;T) - J_0^{K\pi}(s;0)\right] \end{array}$$

LECs Molina, Ruiz de Elvira JHEP2020

$T = 0$ **poles:**

$\sqrt{s_p}^{(1)} = (731 \pm 7) - i(280 \pm 9)$ **MeV**

$\sqrt{s_p}^{(2)} = (679 \pm 6) - i(289 \pm 8)$ **MeV**

Saturated Scalar susceptibility in the LSM

S.Ferreres, AGN, A.Vioque, 2018

$$\mathcal{L}_{LSM} = \frac{1}{2}\partial_\mu \Phi^T \partial^\mu \Phi - \frac{\lambda}{4}\left[\Phi^T\Phi - v_0^2\right]^2 + h\sigma,$$

$$\chi_S(T) = \left(\frac{d^2h}{dm_q^2}\right) v(T) + \left(\frac{dh}{dm_q}\right)^2 \Delta_\sigma(k=0;T),$$

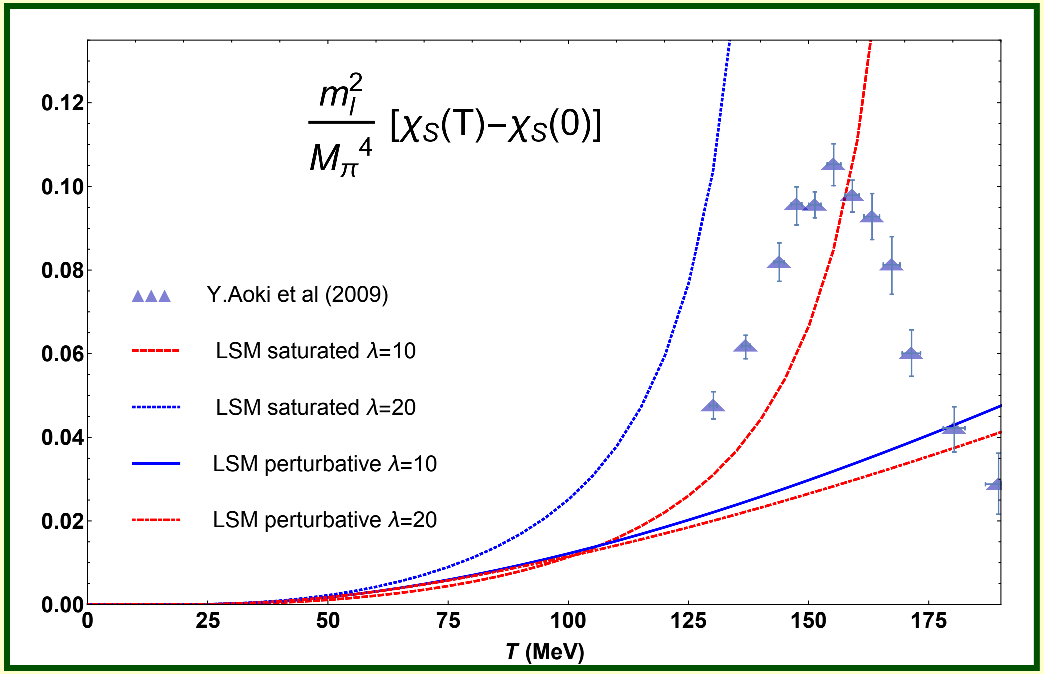
suppressed near T_c

$$M_{0\pi}^2 = \frac{h}{v} = \lambda(v^2 - v_0^2) \quad , \quad M_{0\sigma}^2 = M_{0\pi}^2 + 2\lambda v^2, \quad v = \langle\sigma\rangle$$

calculating the self-energy Σ to one loop:

M_π (MeV)	M_p (MeV)	Γ_p (MeV)	λ
0	450.0	172.5	8.4
0	775.1	550.0	20.0
140	450.0	159.2	9.6
140	750.1	550.0	21.2

$$\frac{\chi_S(T)}{\chi_S(0)} \simeq \frac{M_{0\sigma}^2 + \Sigma\,(k=0;T=0)}{M_{0\sigma}^2 + \Sigma\,(k=0;T)}$$



Chiral Imbalance in ChPT

D.Espriu, AGN, A.Vioque, 2020

- μ_5 **chem. pot. for approx. (local) conservation of chiral charge** Q_5
  (characteristic time $t_{L-R} \sim m_q^{-1} \gg t_{fireball}$ **for light quarks)**

$\Rightarrow \mathcal{L}_{QCD} \to \mathcal{L}_{QCD} + \mu_5 \bar{\psi}\gamma^0\gamma_5\psi$ **in that region**

$$Q_5 = \int_{vol} d^3\vec{x}\ \bar{\psi}\gamma^0\gamma_5\psi$$

- **GOAL: Construct the most general meson eff. lagr. for** $\mu_5 \neq 0$

Chiral Imbalance in ChPT

D.Espriu, AGN, A.Vioque, 2020

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- **GOAL: Construct the most general meson eff. lagr. for** $\mu_5 \neq 0$

- **"local" chiral invariant ChPT with axial** $U(1)$ **external source**

D_μ **& explicit source-dep new terms**

$N_f = 2$

$Z \sim 0.8$ **(EM pion mass dif)**

$$\mathcal{L}_2 \to \mathcal{L}_2 + 2\mu_5^2 F^2 (1 - Z + \kappa_0)$$

κ_i **new LEC to be fixed**

$$\mathcal{L}_4 \to \mathcal{L}_4 + \kappa_1 \mu_5^2 \mathbf{tr}\left(\partial^\mu U^\dagger \partial_\mu U\right) + \kappa_2 \mu_5^2 \mathbf{tr}\left(\partial_0 U^\dagger \partial^0 U\right) + \kappa_3 M_\pi^2 \mu_5^2 \mathbf{tr}\left(U + U^\dagger\right) + \kappa_4 \mu_5^4$$

- **NLO dispersion relation** $\Longrightarrow$

No lattice data

$$v_\pi(\mu_5) = \left.\frac{|\vec{p}|}{p_0}\right|_{M=0} = 1 + 2\kappa_2 \frac{\mu_5^2}{F_\pi^2} \qquad (\kappa_2 < 0)$$

$$\left[M_\pi^2\right]^{pole}(\mu_5) = M_\pi^2 \left[1 - 4(\kappa_1 + \kappa_2 - \kappa_3)\frac{\mu_5^2}{F^2}\right]$$

$$\left[M_\pi^2\right]^{scr}(\mu_5) = M_\pi^2 \left[1 - 4(\kappa_1 - \kappa_3)\frac{\mu_5^2}{F^2}\right]$$

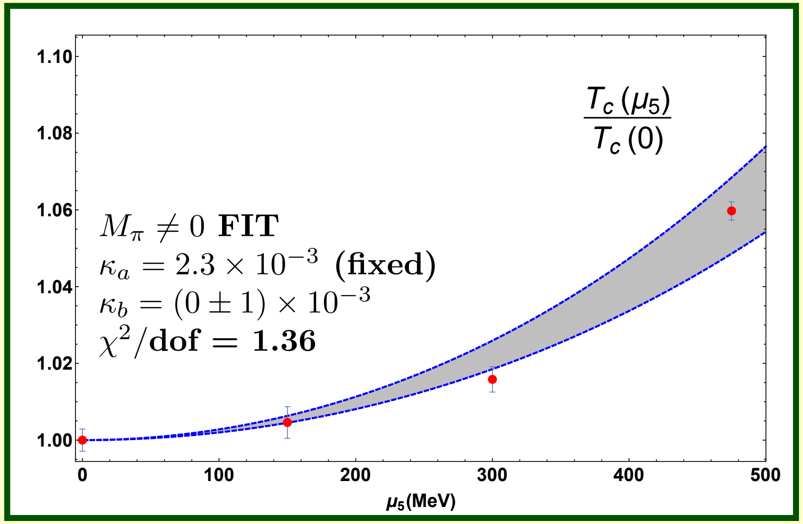
- **NNLO quark condensate** $\Longrightarrow$

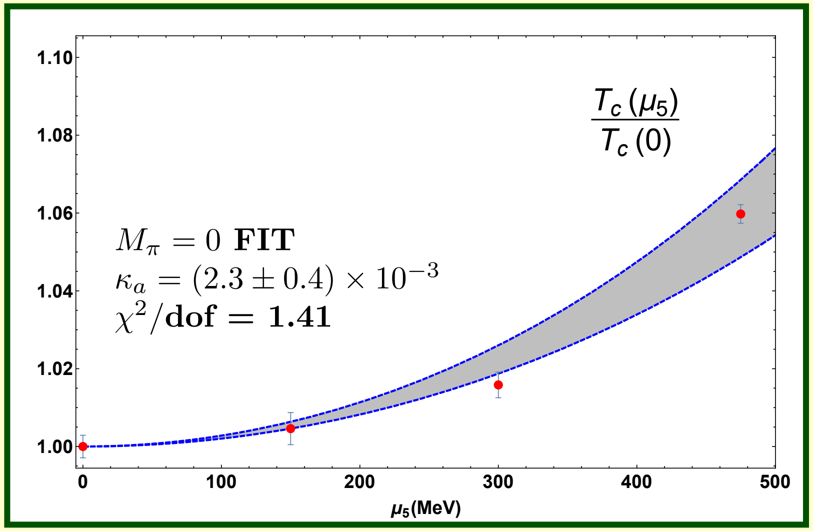
$$\left.\frac{\langle \bar{q}q\rangle_l (T,\mu_5)}{\langle \bar{q}q\rangle_l (0,\mu_5)}\right|_{M=0} = 1 - \frac{T^2}{8F^2}\left[1 - 2\kappa_a \frac{\mu_5^2}{F^2}\right] - \frac{T^4}{384F^4}$$

$\kappa_a = 2\kappa_1 - \kappa_2, \quad \kappa_b = \kappa_1 + \kappa_2 - \kappa_3$

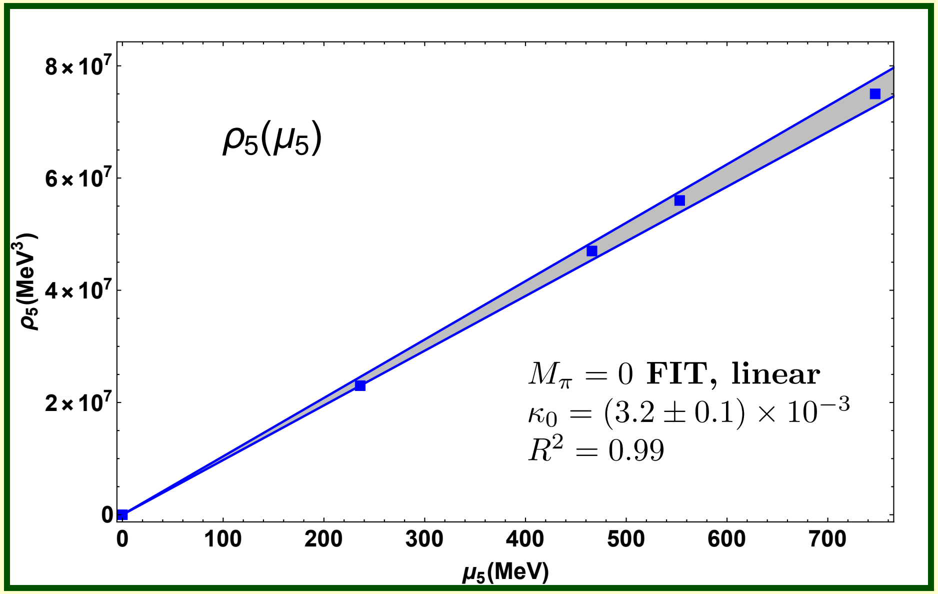
Well acommodated by chiral limit ChPT:

Lattice data Braguta et al 2015 ($N_c = 2$)





- NLO chiral charge density $\Longrightarrow$



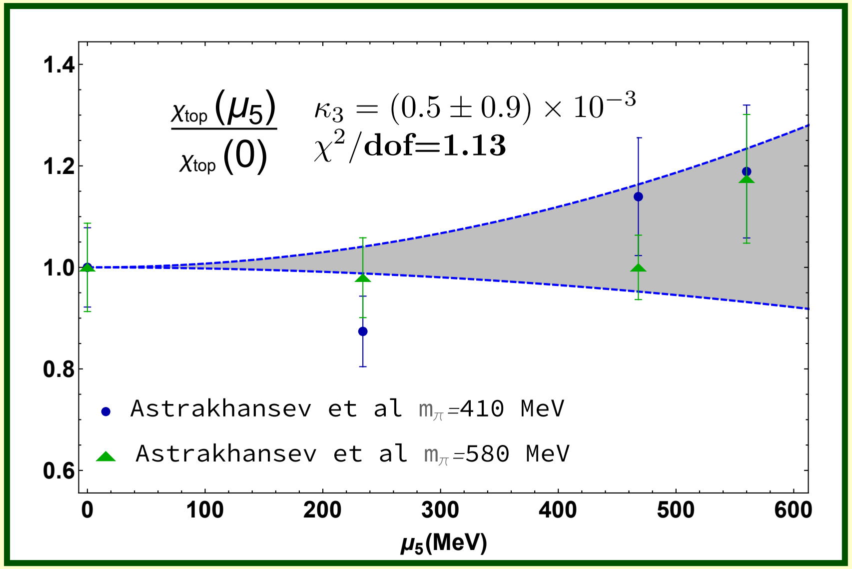
Linear $\mathcal{O}(\mu_5)$ dominant for moderate μ_5

Very insensitive to M_π

Lattice data Astrakhantsev et al 2019

- NLO topological susceptibility $\Longrightarrow$

$$\frac{\chi_{top}(\mu_5)}{\chi_{top}(0)} = 1 + 4\frac{\kappa_3 \mu_5^2}{F^2}$$



More lattice data needed to better pin down κ_i

Lattice data Astrakhantsev et al 2019