Moments of the proton charge density

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- **New approach** to extract the **moments of a probability density function** through integral forms of its Fourier transform **Hoballah *et al.* Phys . Let. B 808 135669 (2020)**

- Application to **proton electric form factor data** **M. A. *et al.*, ArXiv:2304.1352 [nucl-ex]**

M. Atoui, M.B. Barbaro, M. Hoballah, C. Keyrouz, M. Lassaut, D. Marchand, G. Quéméner, E.Voutier, R. Kunne, J. Van De Wiele

What is the value of the radius of the proton?

Electron proton elastic scattering

$r_p = 0.879(8)$ fm

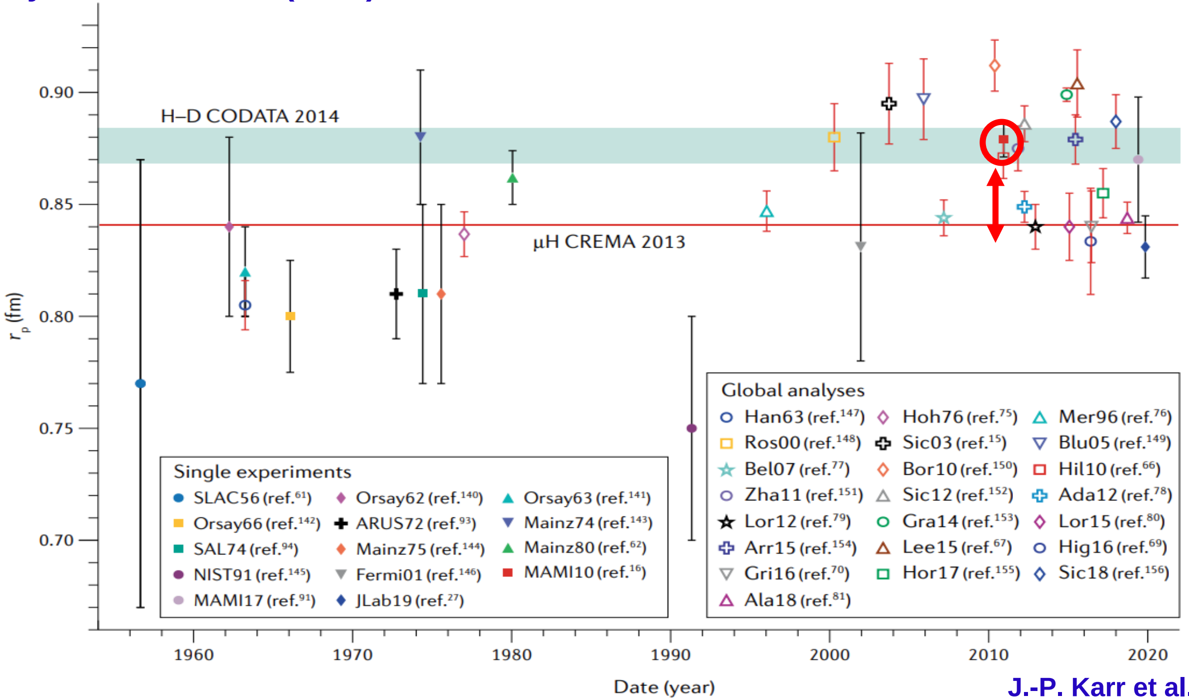
J. Bernauer et al.., Phys. Rev. Lett 105 (2010)

5σ

Muonic hydrogen spectroscopy

$r_p = 0.84184(67)$ fm

R. Pohl et al., Nature 466, 213 (2010)



J.-P. Karr et al., Nature Rev. Phys. 2 , 601 (2020)

Subject no longer under pressure

(CODATA 2018 average)

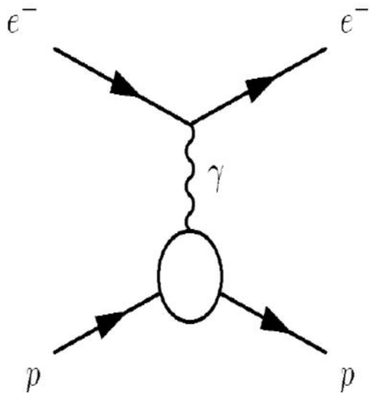
but

discrepancies not fully understood

=> motivation for this work

Proton radius from elastic ep Scattering

Measurement of the **e-p scattering cross section:** **Sachs Form factor**



$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}\Big|_{\text{Mott}} \times \left[\frac{G_E^2(k^2) + \tau\, G_M^2(k^2)}{1+\tau} + 2\,\tau \tan^2\left(\frac{\theta}{2}\right) G_M^2(k^2)\right]$$

The **radius** is defined as

$$r_p = \sqrt{-6\,\frac{\partial G_E^2(k^2)}{\partial k^2}\Big|_{k^2=0}}$$

Indirect measurement of the proton charge radius through **extrapolation of the form factor to zero squared four-momentum transfer** $(k^2 = 0)$

Issues faced when evaluating the radius:

- **What is the best k^2 range to fit and extrapolate** the Form Factor?
- What function to use? **Model assumption of the functional behavior** of the form factor
- **Sensitivity** to variations of the Form Factor **at low k^2**

Goal: Another method to evaluate the moments of the charge density from experimental data

Moments of the charge density: **Refer to how charge is distributed inside the nucleon**

$$< r^\lambda >= (r^\lambda, \rho_E) = \int \mathrm{d}^3\mathbf{r} \quad r^\lambda \quad \rho_E(\boldsymbol{r})$$

Proton charge density defined, in a non relativistic approach, and in Breit Frame, as the inverse Fourier transform of the **Electric Form Factor** G_E

- **Spatial density** $\rho_E(\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int_{R^3} d^3\boldsymbol{k}\, e^{i\boldsymbol{kr}}\, \boldsymbol{G_E}(\boldsymbol{k})$
- **Moments** $(\boldsymbol{r}^\lambda, \boldsymbol{\rho}_E) = \int \mathrm{d}^3\mathbf{r}\, r^\lambda\, \boldsymbol{\rho}_E(\boldsymbol{r})$

$$(\boldsymbol{r}^\lambda, \boldsymbol{\rho}_E) = \frac{1}{2\pi^3} \int_{R^3} d^3\boldsymbol{k}\, \boldsymbol{G_E}(\boldsymbol{k}) \int_{\boldsymbol{R^3}} d^3\boldsymbol{r} \boldsymbol{e}^{\boldsymbol{ikr}} r^\lambda$$

Finite term

Divergent term needs to be **regularized** as by definition the moment r^λ is finite

Exponential regularization

The integral:

$$g_\lambda(k) = \int_{R^3} d^3r\ e^{ikr}\ r^\lambda$$

can be taken as the limit of the convergent integral

$$\boldsymbol{g_\lambda(k)} = \lim_{\epsilon\to 0^+} \int_{R^3} \boldsymbol{d^3r\ e^{-\epsilon r} e^{ikr}\ r^\lambda} = \lim_{\epsilon\to 0^+} \boldsymbol{I_\lambda(k,\epsilon)}$$

- Moments r^λ can be written as:

$$(\boldsymbol{r^\lambda}, \boldsymbol{\rho_E}) = \frac{2}{\pi}\,\Gamma(\lambda+2) \lim_{\epsilon\to 0^+} \int_0^\infty dk\ \boldsymbol{G_E}(k) \frac{k\ sin\left[(\lambda+2)\text{Arctan}\left(\frac{k}{\epsilon}\right)\right]}{(k^2+\epsilon^2)^{\frac{\lambda}{2}+1}}$$

Condition: $\lambda > -3$

Access moments with real orders $\lambda > -3$

- For integer values of λ:

$$(r^m, \rho_E) = \frac{2}{\pi}(m+1)! \lim_{\epsilon\to 0^+} \epsilon^{m+2} \int_0^\infty dk\ \boldsymbol{G_E(k)} \frac{k}{(k^2+\epsilon^2)^{m+2}} \Phi_m\left(\frac{k}{\epsilon}\right)$$

with

$$\Phi_m\left(\frac{k}{\epsilon}\right) = \sum_{j=0}^{m+2} \sin\left(\frac{j\pi}{2}\right) \frac{(m+2)!}{j!(m+2-j)!} \left(\frac{k}{\epsilon}\right)^j$$

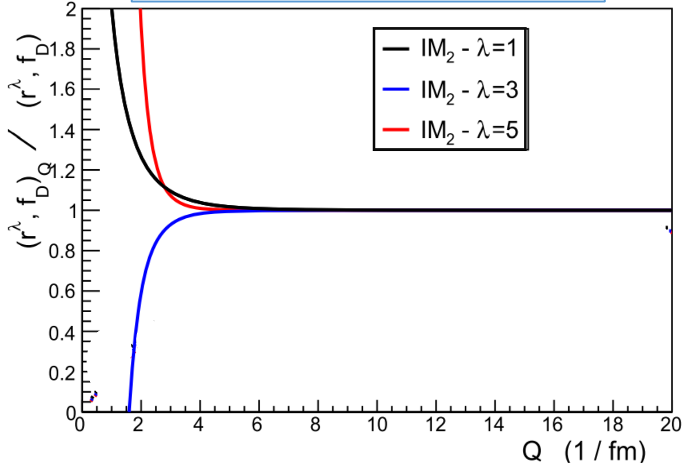
For even order moments : IM recovers formally the same quantities as the derivative

Experimental measurements of the Form Factor do not extend to infinite k^2:

- **But**: Integrals are most likely to saturate at a squared four-momentum transfer value well below infinity.
- **Hence**: **Cut-off Q** replaces the infinite integral boundary : **truncated moments**.

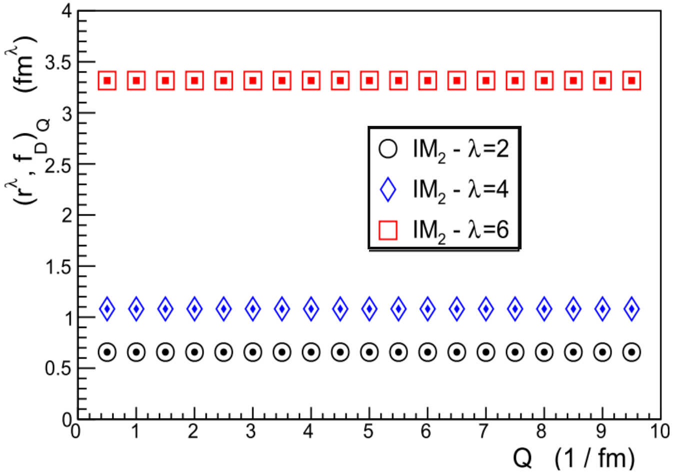
Use the polynomial ratio parametrization $\mathbf{G_E}(\boldsymbol{k}) = \frac{1+a_1k^2}{1+b_1k^2+b_2k^4+b_3k^6}$ J.J. Kelly, Phys. Rev. C 70 (2004) 068202.

Odd truncated moments



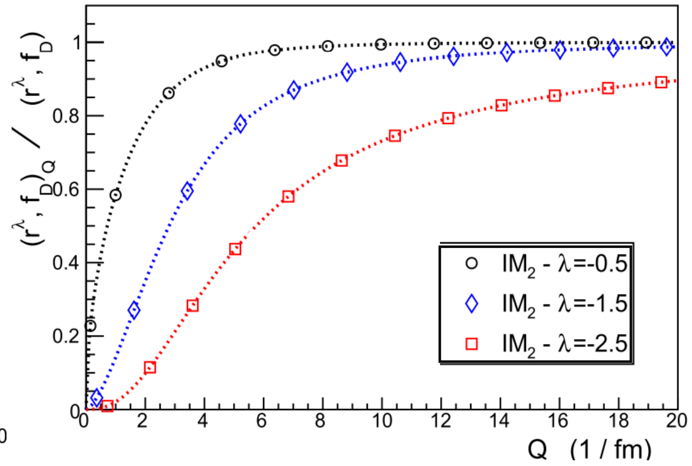
The method rapidly saturates about 6 fm^{-1}, in a momentum region well covered by proton electromagnetic FF data.

Positive even truncated moments



Q-independence is reproduced

Negative truncated moments



Convergence is not guaranteed within the domain covered by experimental data

Application to experimental data

- **Select G_E from elastic electron scattering experiments**
  - **Rosenbluth Separation** : Measure σ_R at a fixed k^2

    for different values of beam energy and scattering angle
  - **G_M contribution is strongly suppressed:** at very low k^2

    → **21 data sets**:

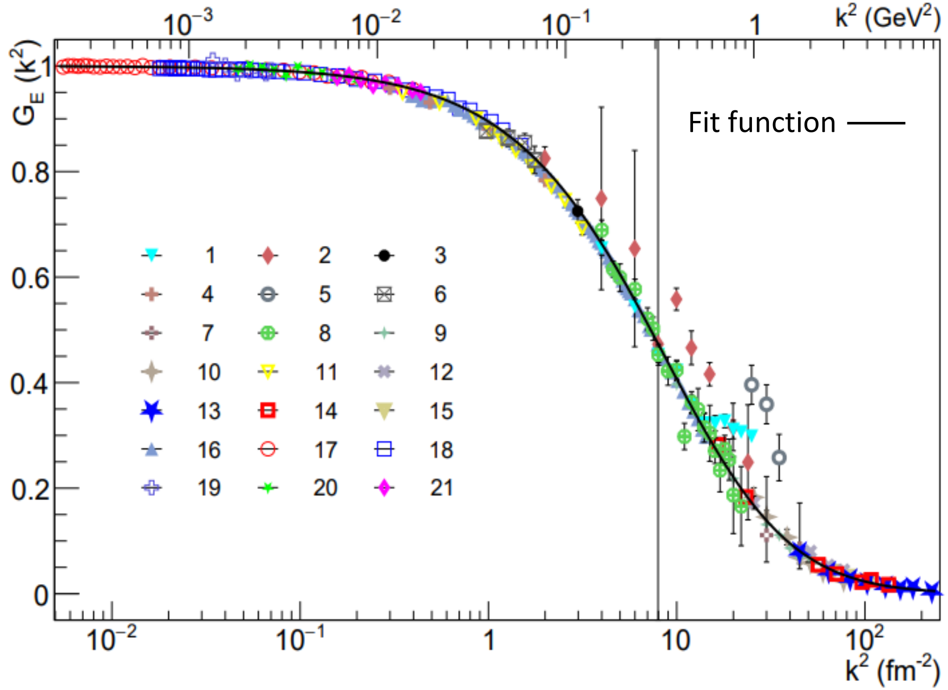
    $$[2.15 \times 10^{-4}\mathrm{GeV}^2]\ \ 5.51 \times 10^{-3} \leq k^2(\mathrm{fm}^{-2}) \leq 226\ [8.8\ \ \mathrm{GeV}^2]$$

- **Fit simultaneously the different datasets using the functional form**

$$G_E(k) = \eta_i \frac{1 + a_1 k^2}{1 + b_1 k^2 + b_2 k^4 + b_3 k^6}$$

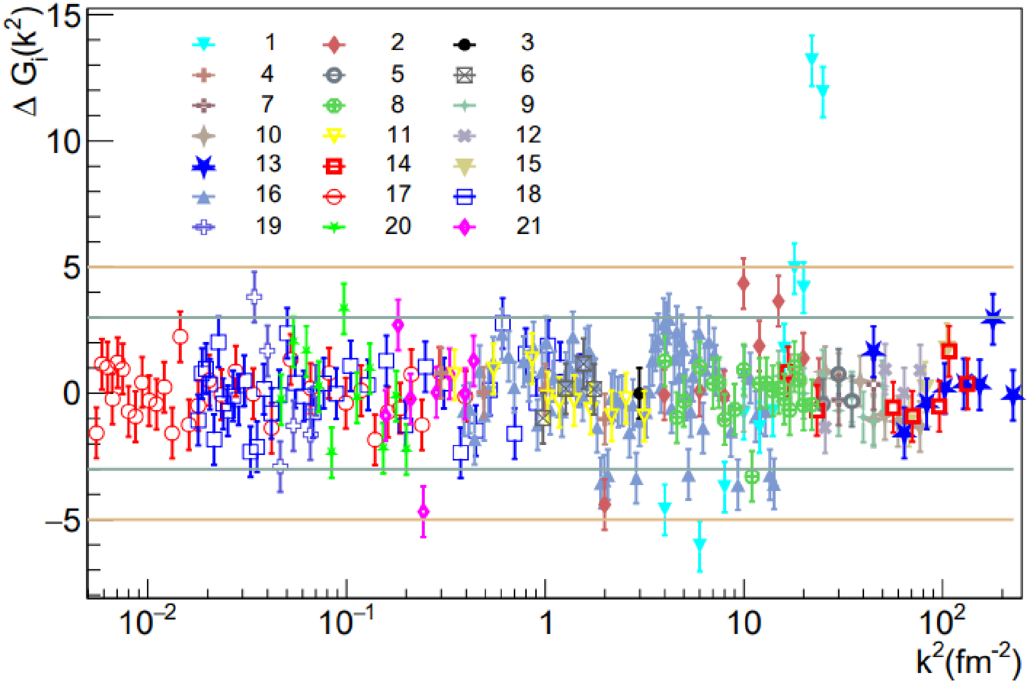
  - The **same functional behavior** is assumed for each dataset
  - A **separate normalization parameter η_i** is considered for each dataset number i

Data Set Number	Year	Author	Number of data	k^2-range k^2_{min} (fm^{-2})	k^2_{max} (fm^{-2})
1	1961	Bumiller *et al.*	11	4.00	25.0
2	1961	Littauer *et al.*	9	2.00	24.0
3	1962	Lehmann *et al.*	1	2.98	2.98
4	1963	Dudelzak *et al.*	4	0.30	2.00
5	1963	Berkelman *et al.*	3	25.0	35.0
6	1966	Frèrejacque *et al.*	4	0.98	1.76
7	1966	Chen *et al.*	2	30.0	45.0
8	1966	Janssens *et al.*	20	4.00	22.0
9	1971	Berger *et al.*	9	1.00	50.0
10	1973	Bartel *et al.*	8	17.2	77.0
11	1975	Borkowski *et al.*	10	0.35	3.15
12	1994	Walker *et al.*	4	25.7	77.0
13	1994	Andivahis *et al.*	8	44.9	226.
14	2004	Christy *et al.*	7	16.7	133.
15	2005	Qattan *et al.*	3	67.8	105.
16	2014	Bernauer *et al.*	77	0.39	14.2
17	2019	Xiong *et al.* - 1.1 GeV	33	5.51×10^{-3}	3.96×10^{-1}
18	2019	Xiong *et al.* - 2.1 GeV	38	1.79×10^{-2}	1.49
19	2021	Mihovilovič *et al.* - 195 MeV	6	3.43×10^{-2}	6.99×10^{-2}
20	2021	Mihovilovič *et al.* - 330 MeV	11	4.69×10^{-2}	2.00×10^{-1}
21	2021	Mihovilovič *et al.* - 495 MeV	8	1.57×10^{-1}	4.37×10^{-1}



- $\chi^2 = 2.9$

$$\Delta G_i(k) = \frac{G^i_{E,data}(k)/\eta_i - G_{E,Fit}(k)}{\delta G^i_E(k)}$$



- **Residuals within $\pm 3\ \sigma$ with some outlyers**

Fit parameters

Fit parameters of the Functional form:

$$G_E(k) = \eta_i \frac{1 + a_1 k^2}{1 + b_1 k^2 + b_2 k^4 + b_3 k^6}$$

	a_1 $[10^{-1}\ \mathrm{fm}^2]$	b_1 $[10^{-1}\ \mathrm{fm}^2]$	b_2 $[10^{-1}\mathrm{fm}^4]$	b_3 $[10^{-3}\mathrm{fm}^6]$
	8.8030	**9.9402**	**1.0454**	**2.7020**
Statistical errors	**0.0012**	**0.0025**	**0.0013**	**0.0153**
Systematic errors	**0.0096**	**0.0019**	**0.0031**	**0.0317**

Normalization parameters η_i

- **Recent experiments (2010-2021)**

Deviation from unity is smaller than 1%

- **Old Experiments**

Deviations up to 15%

How systematic errors on parameters are evaluated?

1. The **data are shifted with respect to their systematic errors** (upwards or downwards): 2^{21} configurations
2. A **fit is performed** and **parameters are extracted**
3. Systematics are evaluated from **the difference of the parameter value w.r.t the reference fit**

Evaluation of moments

Evaluation of moments for **different values of order λ**: Negative, positive (even and odd)

Truncated moments evaluated for the cutoff $Q^2 = 52\text{fm}^{-2}$	**Moments** evaluated in the limit $k^2 \to \infty$	**Even moments** from the derivative of G_E at $k^2 = 0$

λ	$\langle r^\lambda \rangle_Q$ $[\text{fm}^\lambda]$	$\langle r^\lambda \rangle$ $[\text{fm}^\lambda]$	$\langle r^{2p} \rangle_d$ $[\text{fm}^\lambda]$
-2	6.5826	8.9093	–
-1	1.9752	2.1043	–
1	0.7186	0.7158	–
2	0.6824	0.6824	0.6824
3	0.7966	0.7970	–
4	1.0208	1.0208	1.0208
5	0.9219	0.9217	–
6	-3.6823	-3.6823	-3.6823
7	-49.6804	-49.6802	–

Advantage of the approach with respect to the derivative method

Positive even order moments: As can be obtained from derivative forms of the Form Factor

- **Evaluations are compatible for positive valued order moments**
- **Negative order moments show discrepancy when a cutoff is taken into account (as expected)**

Propagation of statistical errors to the evaluated moments using MC methods

Take into account correlations between parameters to all orders

Procedure:

- Make **replicas of parameters (50000)** following the assumption of each error source
- The **moments are estimated from each replica**
- A dedicated study of the **variance of the replicas** is performed from which the error sources are obtained

Statistical error coming from experimental data

λ	$\langle r^\lambda \rangle_Q$ $[\mathrm{fm}^\lambda]$	$\delta\left[\langle r^\lambda \rangle_Q\right]$ $[\mathrm{fm}^\lambda]$
-2	6.5826	0.0039
-1	1.9752	0.0005
1	0.7186	0.0004
2	0.6824	0.0020
3	0.7966	0.0096
4	1.0208	0.0515
5	0.9219	0.3098
6	-3.6823	2.0835
7	-49.6804	15.8544

Larger statistical errors for high order positive moments (probing the large distance behavior of the charge density): lack of measurements at ultra low k^2

Evaluation of systematic errors of the moments

Sources of **systematic errors**:

1. Originating from the systematic error that is reported by each considered **experiment on G_E**
2. Discrepancy between **truncated and exact moments**
3. Bias that could be generated on the fit parameters from **the fitting model itself**
4. Error coming from the **choice of the fitting model** **(ex:** Polynomial ratios, polynomials, Inverse polynomials, Constant Fraction (CF) expansion)

λ	$\langle r^\lambda \rangle_Q$ [fm$^\lambda$]	Systematic Error: Dat. [fm$^\lambda$]	Int. [fm$^\lambda$]	Fun. [fm$^\lambda$]	Mod. [fm$^\lambda$]
-2	6.5826	0.0141	2.3267	0.0008	0.0183
-1	1.9752	0.0024	0.1291	0.0002	0.0022
1	0.7186	0.0025	0.0030	0.0001	0.0008
2	0.6824	0.0113	0	0.0001	0.0053
3	0.7966	0.0500	0.0004	0.0005	0.0300
4	1.0208	0.2498	0	0.0042	0.1752
5	0.9219	1.4388	0.0002	0.0273	1.0995
6	-3.6823	9.5186	0	0.2372	7.5914
7	-49.6804	71.544	0.0002	1.7403	58.198

Systematic errors related to **experimental data of FF** (Dat.)

Determination method (Int.)

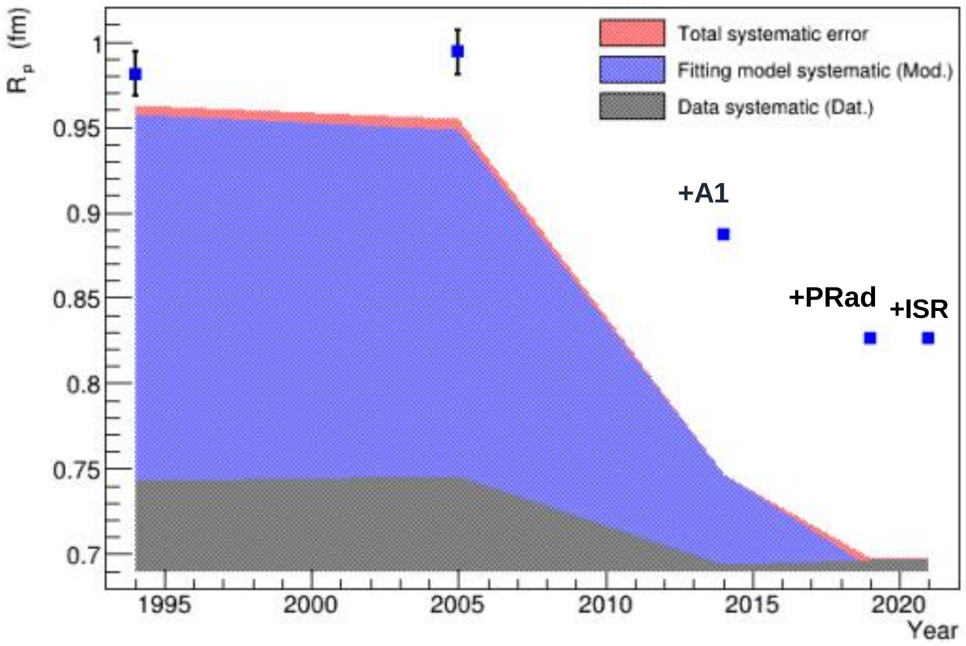
Bias that could be generated from the **fit function** (Fun.)

Errors attached to **the choice of the fitting model** (Mod.)

The Proton charge radius

Evaluation of $R_p = \sqrt{< r^2 >}$ within different time periods:

- all **evaluations are consistent** once systematic errors are taken into account
- Up to 2014 the **major source of systematic uncertainty: choice of the fitting model**
- With **data at low k^2** (Mainz A1, PRad and ISR**):** **constraints on the fit model are reinforced, and this systematic is reduced**



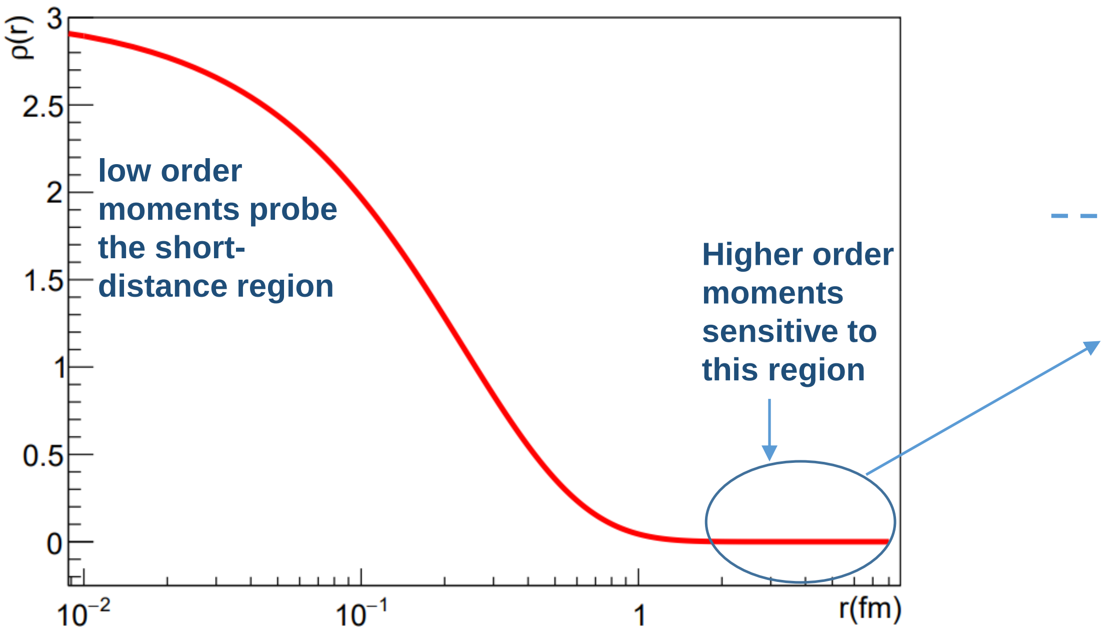
$$R_p = 0.8261 \pm 0.0012 \pm 0.0076 \;\; \text{fm}$$

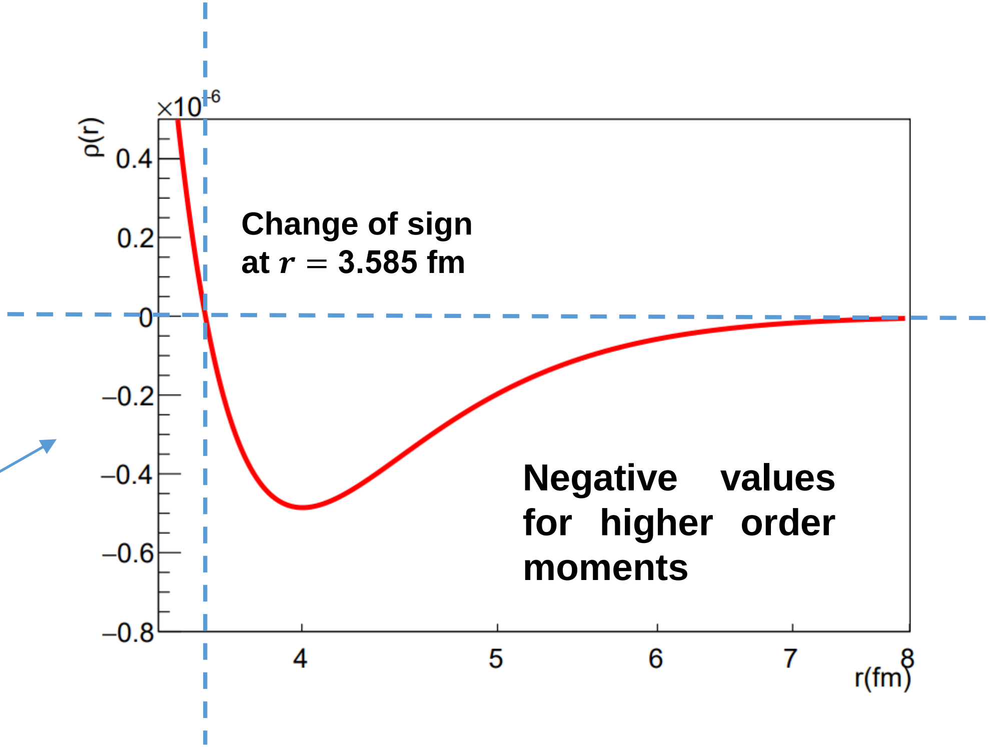
In agreement with CODATA recommended value ($R_p = 0.8414 \pm 0.0019 \;\; \text{fm}$)

This study suggests that **the disagreement between measurements of the Proton Radius** from elastic electron proton scattering **originates essentially from systematic uncertainties**

Spatial density $\rho_E(r) = \frac{1}{2\pi^3} \int_{R^3} d^3k\, e^{ikr}\, G_E(k)$

Fourier transform of our functional form $G_E(k)$





high positive order moments describe the tail of the charge distribution

Conclusion and perspectives

In summary:

- **Novel method** for the **determination of the moments** of the charge density via **integral forms of the electric form factor**.
- **Reanalysis** of some **GE experimental data (Rosenbluth + low k^2)**
  - Extraction of **several moments of the charge density taking all error sources** into consideration
  - Discussion of the **value for the proton radius over years**

Conclusions:

- **The disagreement between the proton radius values** extracted from elastic ep scattering data **originates from systematic uncertainties**
- **Necessity to have experimental data at low k^2 for a better determination of high order positive moments (large-distance effects)**
- **Importance to have data at high k^2 necessary in the evaluation of negative order moments (short-distance effects)**

Conclusion and perspectives

In summary:

- **Novel method** for the **determination of the moments** of the charge density via **integral forms of the electric form factor**.
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Conclusions:

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Perspectives:

Extend this study to include measurements of $G_M(k^2) + \frac{\mu G_E(k^2)}{G_M(k^2)}$ and access **magnetic moments** as well as **Zemach moments**

Thank you for your attention

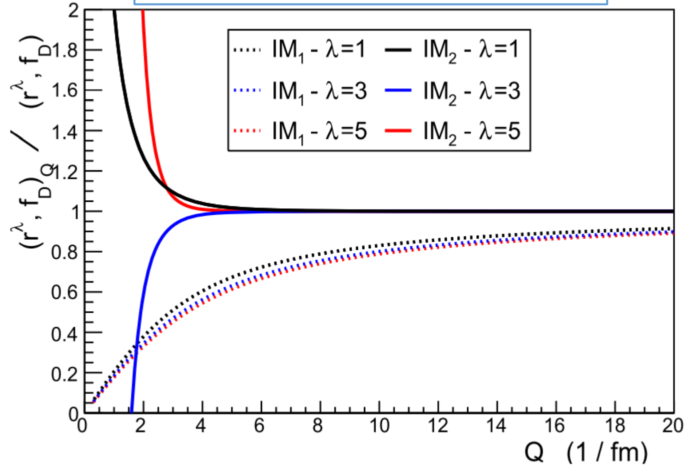
This project has received funding from the European Union’s Horizon 2020 research and innovation program under grant agreement Nº 824093

Backup slides

Application of the method

Use the polynomial ratio parametrization $\mathbf{G_E(k)} = \frac{1+a_1k^2}{1+b_1k^2+b_2k^4+b_3k^6}$ J.J. Kelly, Phys. Rev. C 70 (2004) 068202.

Odd truncated moments



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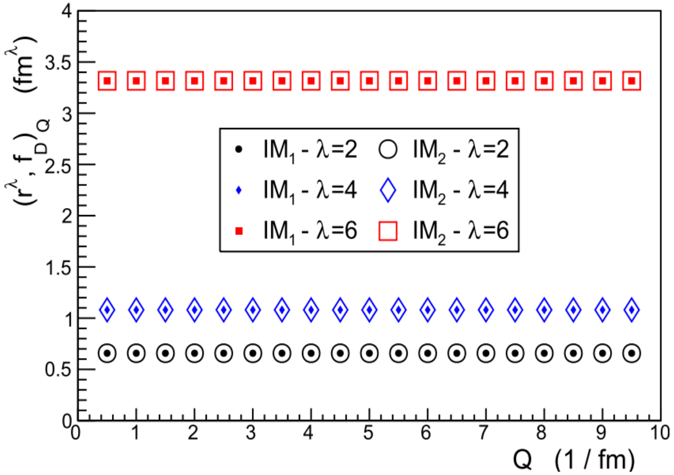
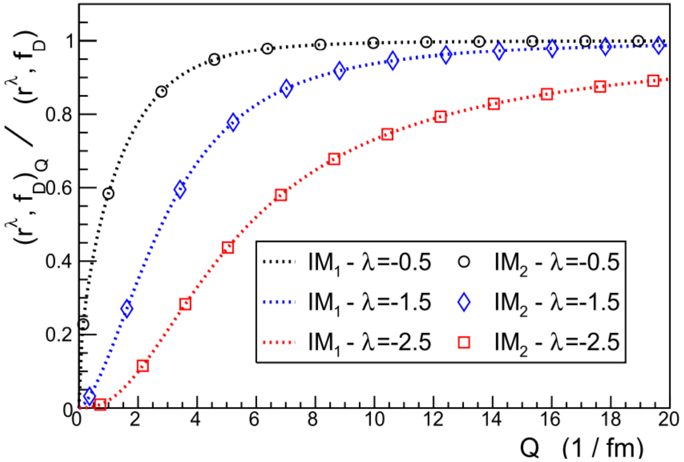
Positive even truncated moments



Q-independence is reproduced by each prescription.

Negative truncated moments



Convergence is not guaranteed within the domain covered by experimental data

The principle value regularization IM1

- **The integral** $g_\lambda(k) = \int_{R^3} d^3r\ e^{ikr}\ r^\lambda$ satisfies the relation $g_\lambda(t\mathbf{k}) = \frac{1}{t^{\lambda+3}}\ g_\lambda(\mathbf{k})$

- Moments can be written as:

$$(r^\lambda, f) = N_\lambda \int_0^\infty \left\{\frac{\tilde{f}(k)}{k^{\lambda+1}}\right\} dk$$ with N_λ is the normalization coefficient: $N_\lambda = \frac{2^{\lambda+2}}{\sqrt{\pi}}\ \frac{\Gamma(\frac{\lambda+3}{2})}{\Gamma(-\frac{\lambda}{2})}$

With $\lambda > -3$: The integral is to be considered as a distribution and **counter terms** $(\tilde{f}_{2j})$ **need to be subtracted to insure convergence**

$$\left\{\frac{\tilde{f}(k)}{k^{\lambda+1}}\right\} \equiv \frac{1}{k^{\lambda+1}}\left(\tilde{f}(k) - \sum_{j=0}^{n} \tilde{f}_{2j}\ k^{2j}\right)$$ with: $\tilde{f}_{2j} = \frac{1}{j!}\ \frac{d^j \tilde{f}(k)}{d(k^2)^j}\Big|_{k=0}$

- The regularized moment: $(r^\lambda, f) = N_\lambda \int_0^\infty dk\ \frac{\left(\tilde{f}(k) - \sum_{j=0}^{n} \tilde{f}_{2j}\ k^{2j}\right)}{k^{\lambda+1}}$
- **The divergence appearing in the normalization term is canceled by the divergence in the integral**

$$(r^m, f) = \lim_{\eta \to 0^+} (r^{m-\eta}, f) \qquad m \text{ even}$$

$$(r^m, f) = (r^{m-\eta}, f)|_{\eta=0} \qquad m \text{ odd}.$$

Some results

Normalization parameters η_i

Experiments at large k2: deviations up to 15%

Recent experiments: Deviation from unity is smaller than 1%

Data Set Number	η_i	$(\delta\eta_i)_{Sta.}$ $(\times 10^{-2})$	$(\delta\eta_i)_{Sys.}$ $(\times 10^{-2})$
1	1 .078	0.248	4.930
2	1.144	0.516	1.144
3	0.993	0.612	9.929
4	0.983	0.167	0.752
5	2.389	5.495	11.946
6	0.991	0.641	0.208
7	0.892	1.659	4.493
8	1.003	1.120	0.802
9	0.990	0.398	1.980
10	1.000	1.277	1.051
11	0.981	0.132	1.766
12	1.134	1.878	0.609
13	0.931	0.813	6.660
14	1.021	1.652	0.544
15	1.023	1.209	0.763
16	0.991	0.018	0.991
17	1.000	0.005	0.215
18	0.998	0.004	0.119
19	1.001	0.030	0.370
20	1.000	0.026	0.365
21	0.998	0.018	0.435

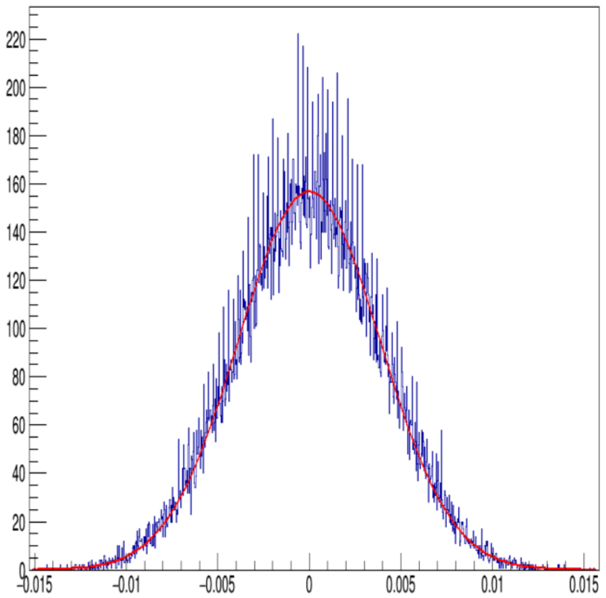
Proton radius through the years

Time period	Data set range	R_p [fm]	$(\delta R_p)_{Sta.}$ [fm]	$(\delta R_p)_{Sys.}$ [fm]
1961-1994	1 - 13	0.9812	0.0130	0.2726
1961-2005	1 - 15	0.9938	0.0126	0.2646
1961-2014	1 - 16	0.8870	0.0029	0.0572
1961-2019	1 - 18	0.8261	0.0014	0.0075
1961-2021	1 - 21	0.8261	0.0012	0.0076

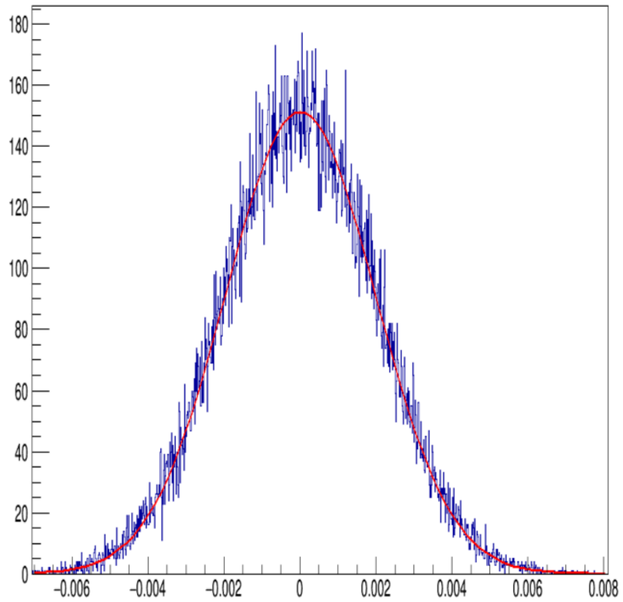
Distributions of moments

50000 replicas
Plot: (fitted-expected) value for each moment

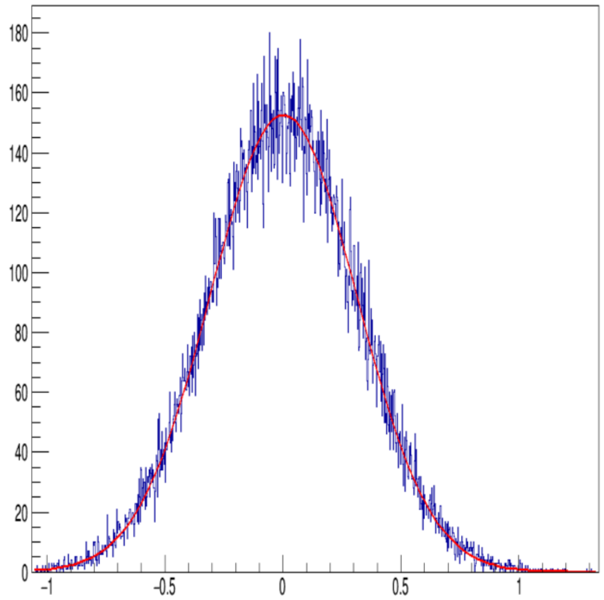
$< r^{-2} >$



$< r^{2} >$



$< r^{5} >$



Systematic error from experimental data:

- Systematics of the fit parameter are propagated to the moments **by shifting upwards or downwards each parameter value with its systematic error**: 2^4 combinations
- For each combination : **Moment and difference with respect to the reference value are evaluated**
- Error on moments: arithmetic average of the evaluations

Systematic error from the fit function:

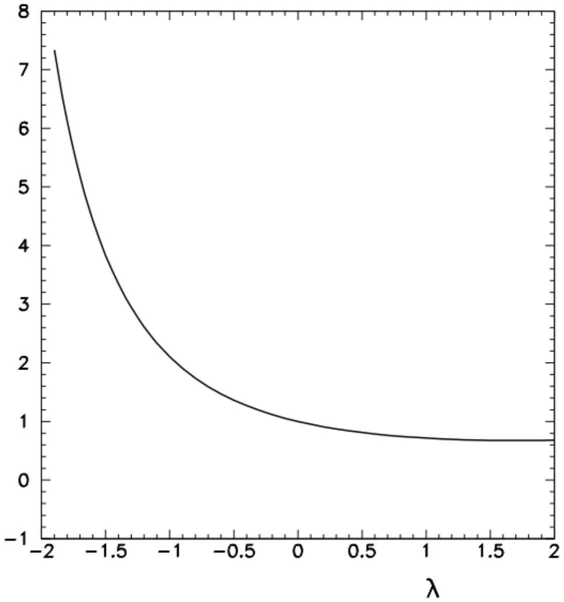
- Generate pseudo-data according to a Gaussian($\boldsymbol{G_{E,f}(k^2)}, \boldsymbol{\sigma = \text{statistical error of real data}}$): 50000 replicas
- **Fit Pseudo-data** with the chosen fit function, extract parameters and evaluate moments
- The **mean values of the distributions of moments** correspond to the fit function systematics

Systematic error from the choice of model function:

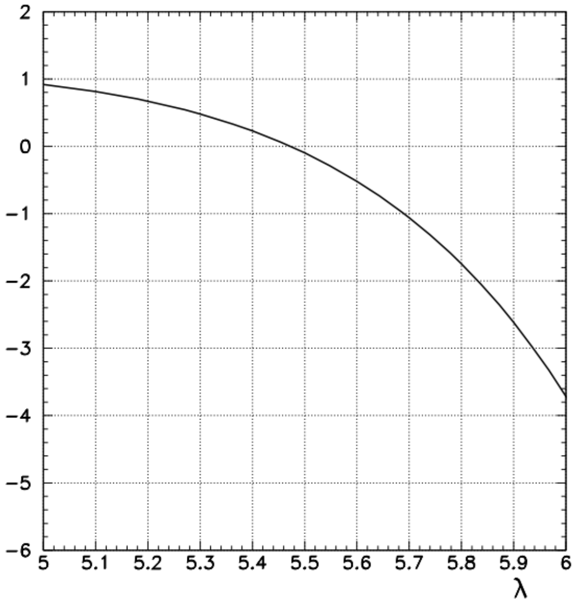
- **Fit the data with several functional forms** (Polynomial ratios, polynomials, Inverse polynomials, Constant Fraction (CF) expansion)
- Choose **the one having a comparable χ^2 to the standard fit** (χ^2 < 3.5, that is 20% larger than the χ_r^2 of the reference fit) : Inverse polynomial of order 2 and a CF (n=3)
- **Evaluate the corresponding moments and errors**

Plot of Moments (fractional and integer orders)

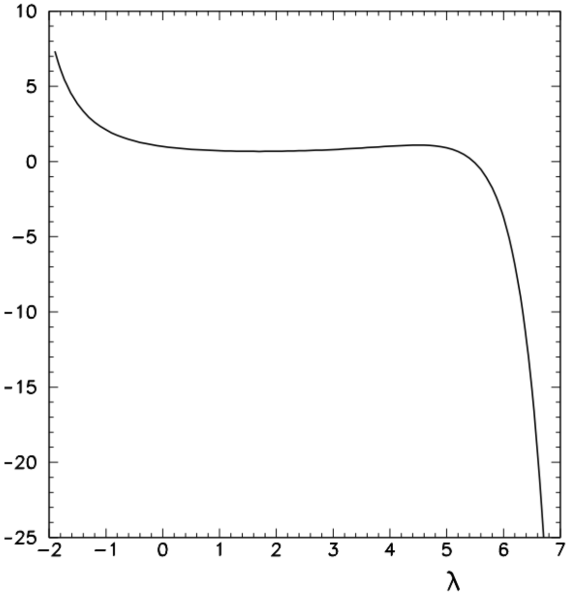
r^{λ} **for** $-2 < \lambda < 2 \quad \lambda_{i+1} = \lambda_i + 0.1$



r^{λ} **for** $5 < \lambda < 6$



r^{λ} **for** $-2 < \lambda < 7$



Radial density from ratio polynomial parametrization

- The inverse Fourier Transfom for a polynomial ratio function (Form Factor):

$$f_K(\mathbf{r}) \equiv f_K(r) = \frac{1}{2\pi^2}\frac{1}{r}\int_0^\infty dk\, k\tilde{f}_K(k)\, \sin(kr)\,.$$

Can be expanded in partial fractions

$$k\tilde{f}_K(k) = \sum_{i=1}^{3}\left[\frac{A_i}{k-k_i} + \frac{\overline{A_i}}{k-\overline{k_i}}\right]$$

- k_i are the poles of $\tilde{f}(k)$
- A_i are the residues of $k\,\tilde{f}(k)|_{k=k_i}$

- After integration:

$$f(r) = \frac{1}{2\pi}\frac{1}{r}\sum_{j=1}^{n} e^{-k_{jI} r}\left[A_{jR}\cos(k_{jR}\, r) - A_{jI}\sin(k_{jR}\, r)\right]$$

With the values from our Functional form parametrization:

i	$k_{i,R}(\mathrm{fm}^{-1})$	$k_{i,I}(\mathrm{fm}^{-1})$	$A_{i,R}(\mathrm{fm}^{-2})$	$A_{i,I}(\mathrm{fm}^{-1})$
1	0	0.1067e+01	−0.1e−02	0
2	0	0.4899e+01	−0.155e+02	0
3	0	0.367996e+01	0.156e+02	0

Distributions of the charge density function

$$\rho_E(r) = \frac{1}{2\pi^3}\int_{R^3} d^3\boldsymbol{k}\, e^{ikr}\, G_E(k)$$

$$\rho_D(r) = \frac{1}{2\pi^3}\int_{R^3} d^3\boldsymbol{k}\, e^{ikr}\, G_{E,D}(k) = \frac{\Lambda^3}{8\pi}\, e^{-\Lambda r}$$

Fourier transform of our functional form $G_E(k^2)$

Fourier transform of the dipole parametrization: $G_{E,D} = \frac{1}{\left(\frac{k^2}{\Lambda^2}+1\right)^2}$

$\Lambda^2 = 18.2\ \mathrm{fm}^{-2}$