Dispersive determination of the σ resonance from lattice QCD

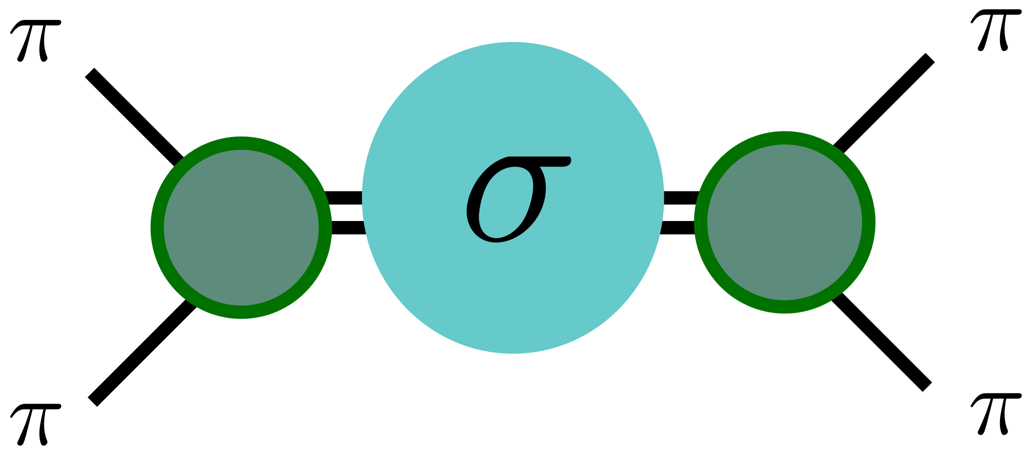
HADRON 2023

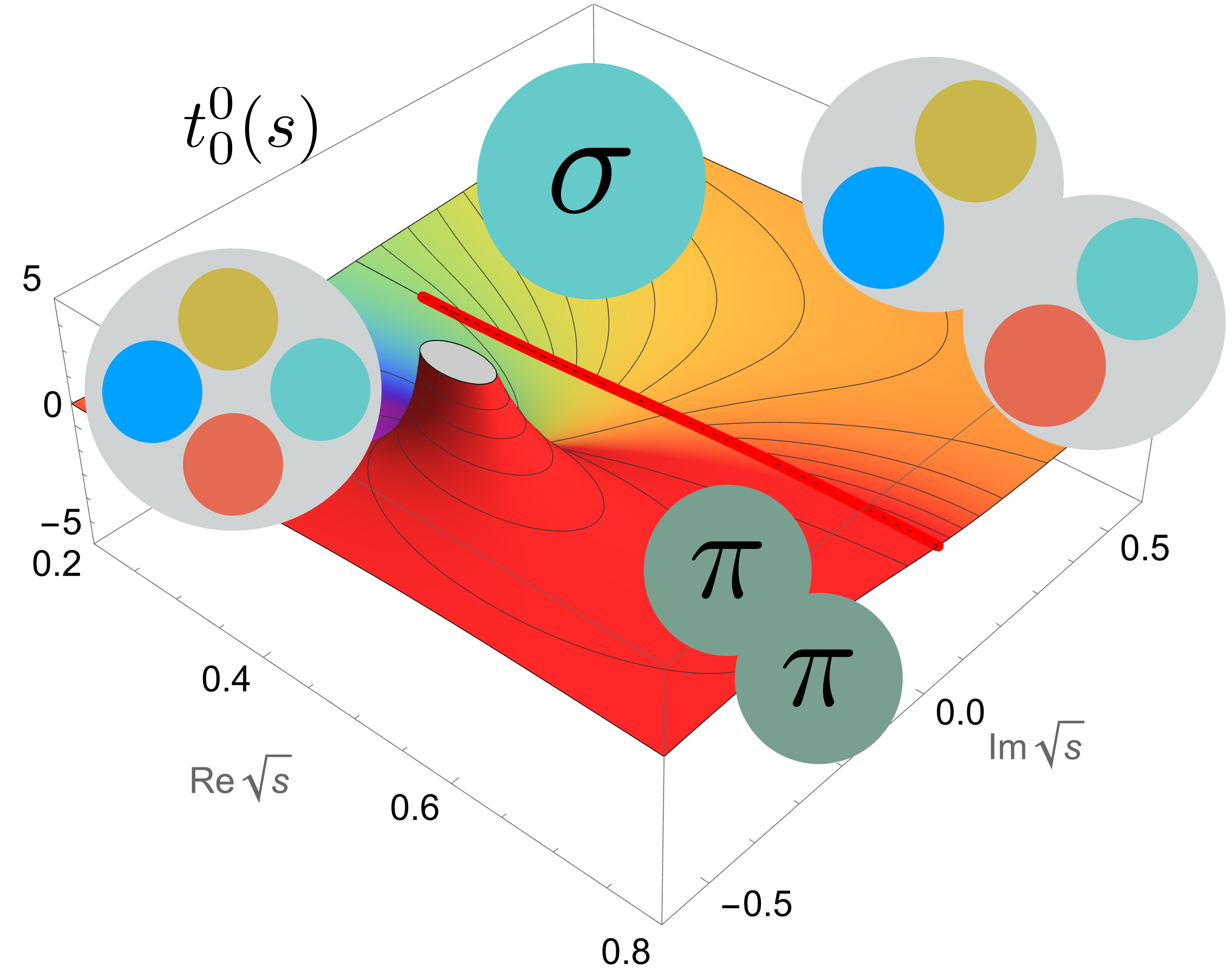
Arkaitz Rodas

Jefferson Lab
Thomas Jefferson National Accelerator Facility

hadspec

Light Scalars: the σ





Lightest resonance in QCD

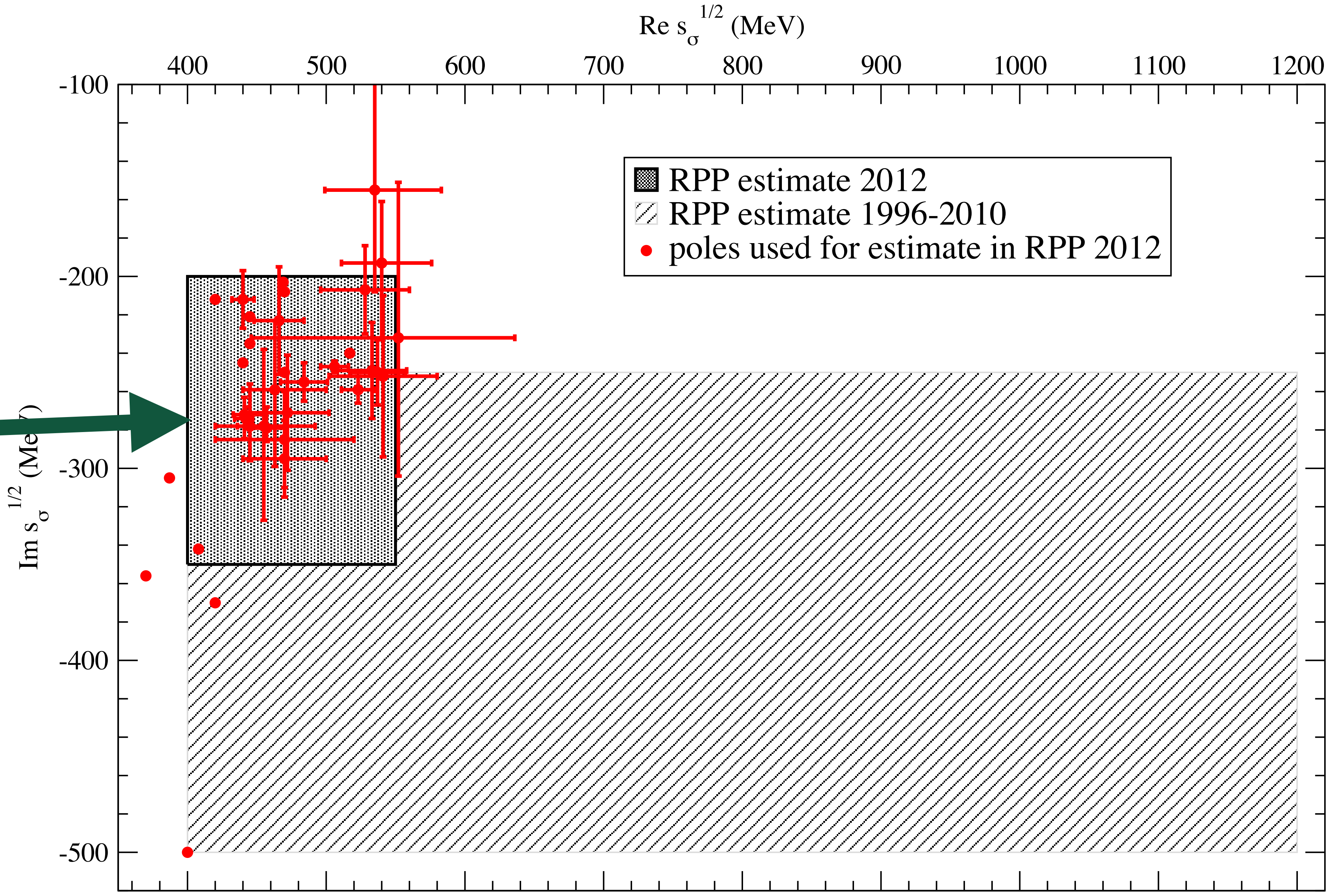
Extremely broad → extremely short-lived

Correlated with chiral symmetry-breaking phenomena

Not well-understood → new observables ??

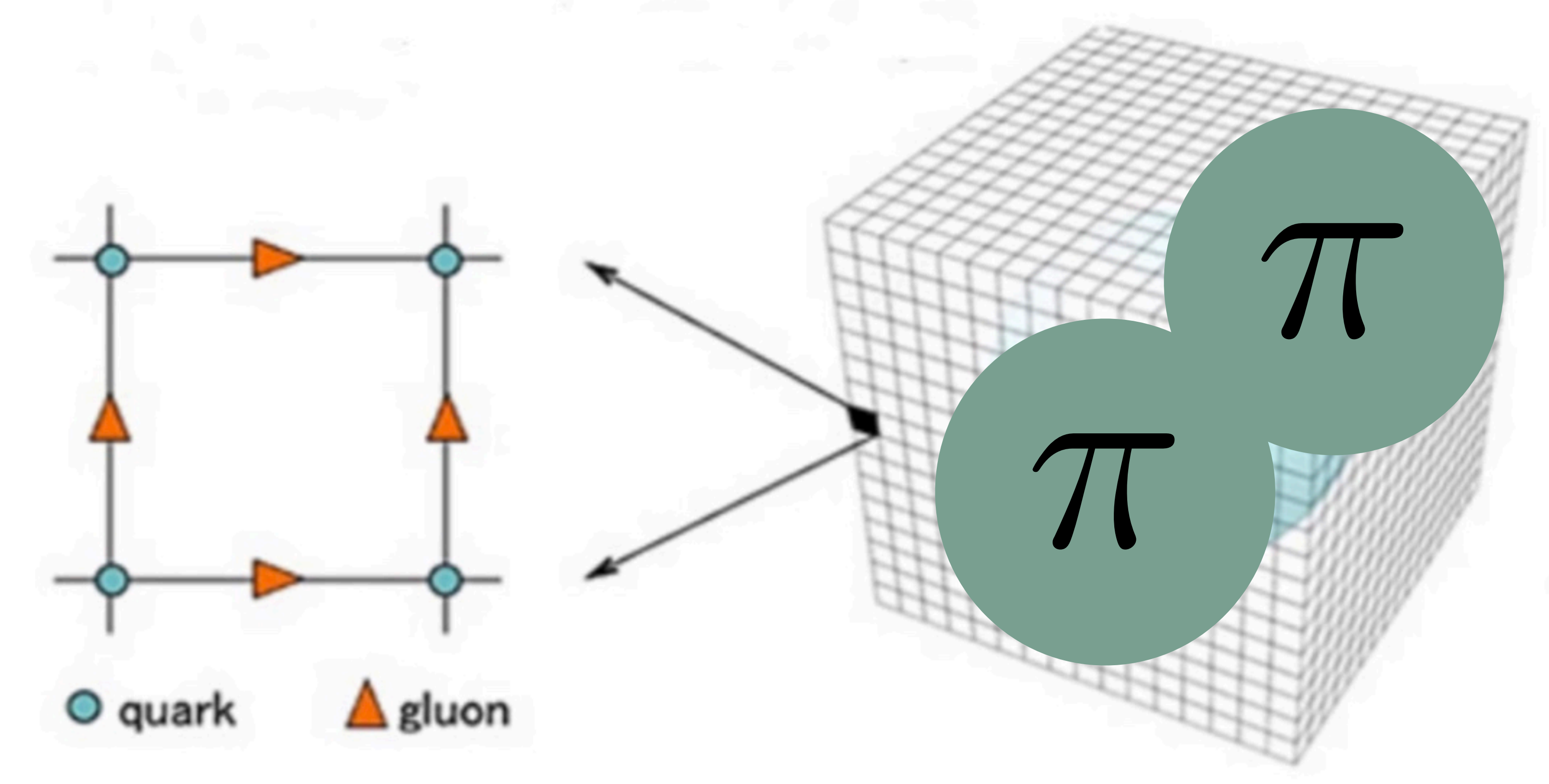
Input to hadron physics observables

Very challenging experimental extraction



What happens for Lattice QCD ??

Lattice QCD



Discretized, euclidean spacetime

$e^{-S[U]}$

Statistics

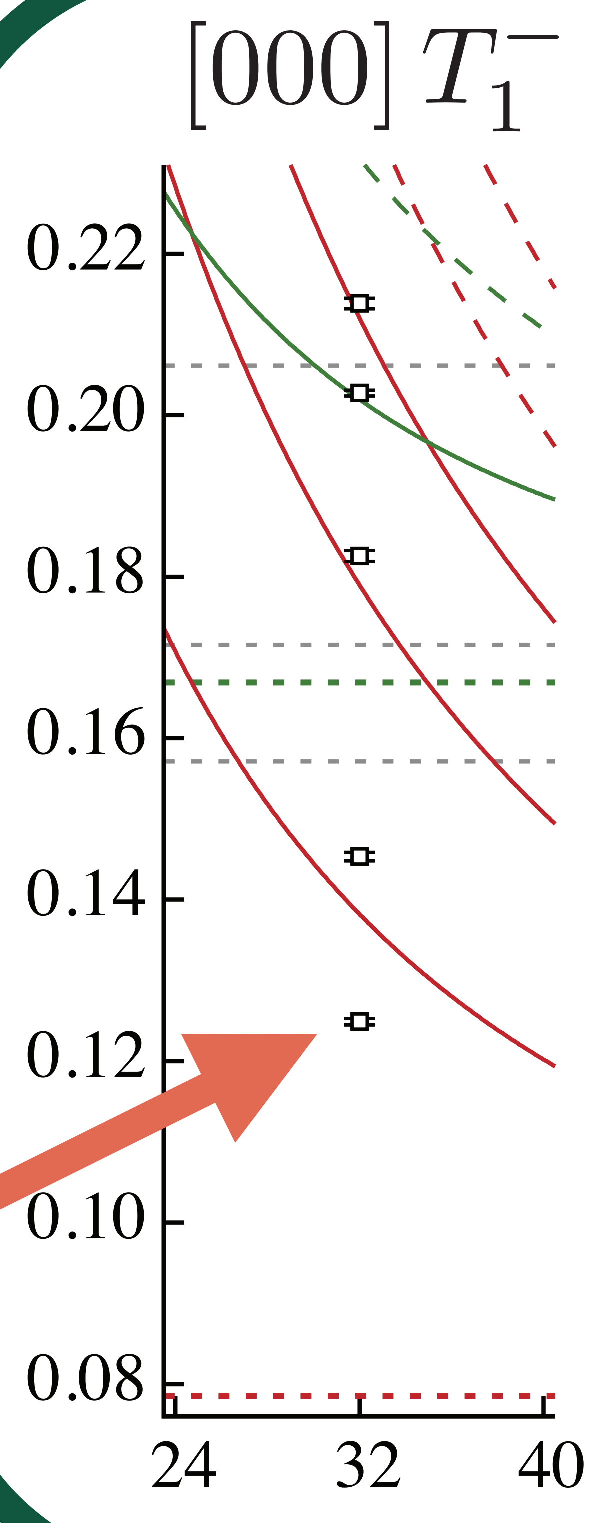
$$\langle O\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} O\left[U_n\right]$$

Time evolution

$$\sum_{\mathrm{n}} e^{-E_{\mathrm{n}}t}\left\langle 0\left|O_{\mathrm{f}}(0)\right|\mathrm{n}\right\rangle\left\langle \mathrm{n}\right|O_{\mathrm{i}}^{\dagger}(0)\left|0\right\rangle$$

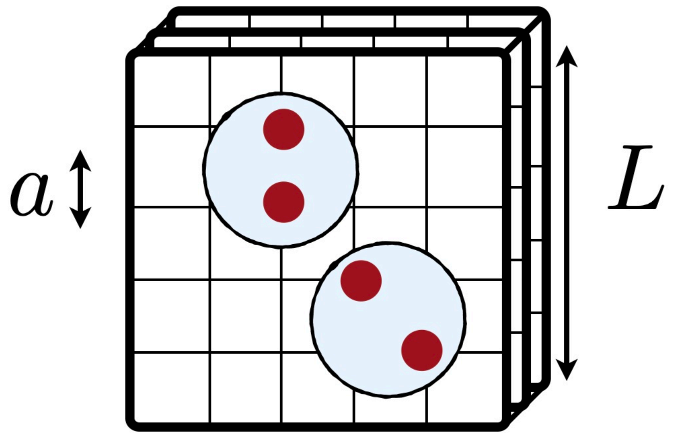
Desired energies

ππ



Lattice QCD

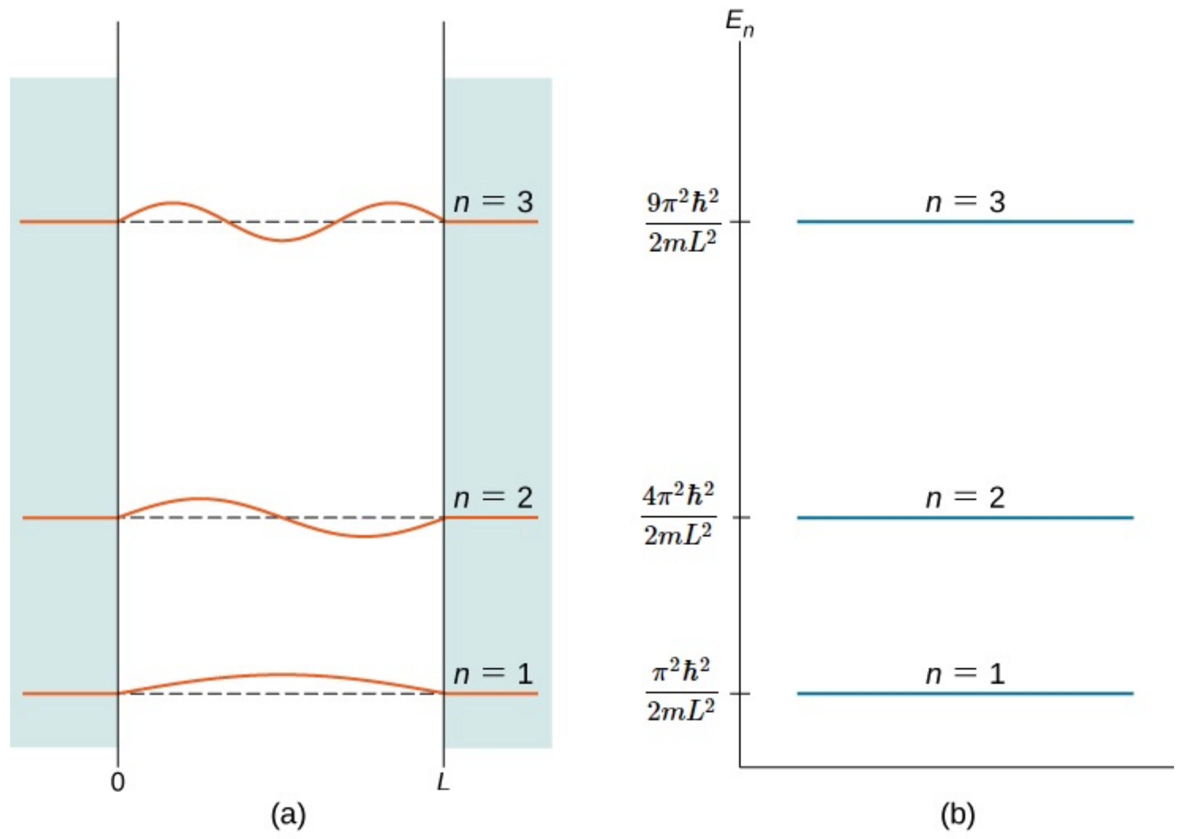
Non-perturbative



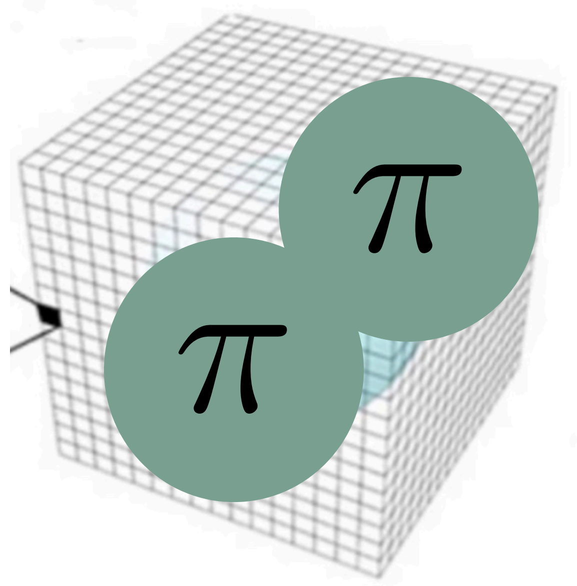
$$D_\mu = \left(\quad \right) (L/a)^3 \times (T/a)$$

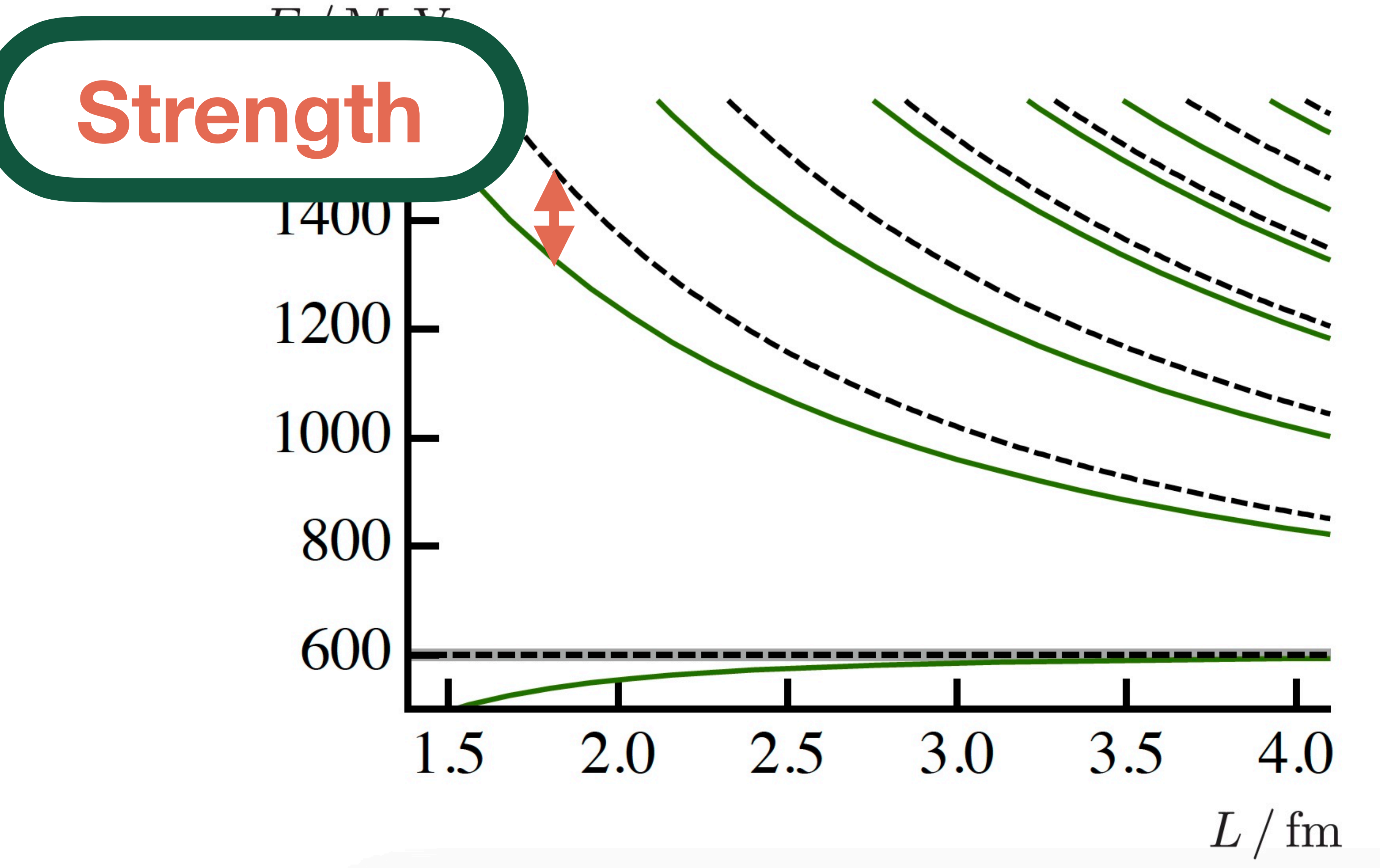
$$\Downarrow$$

$$\langle (\pi^-(t)\pi^+(t))(\pi^-(0)\pi^+(0)) \rangle \sim e^{-E_n t}$$



$\Leftarrow$





Lattice QCD

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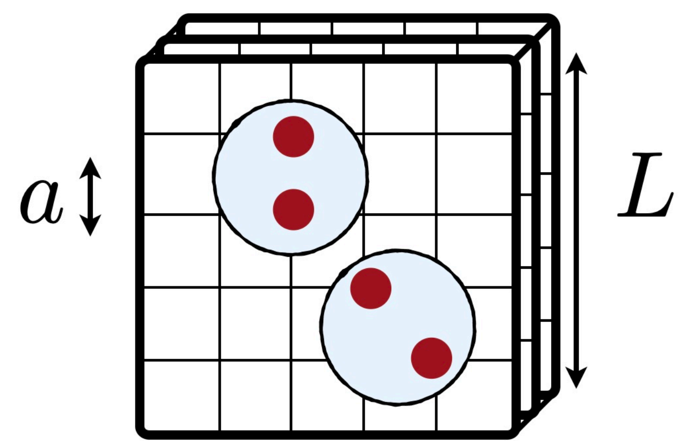
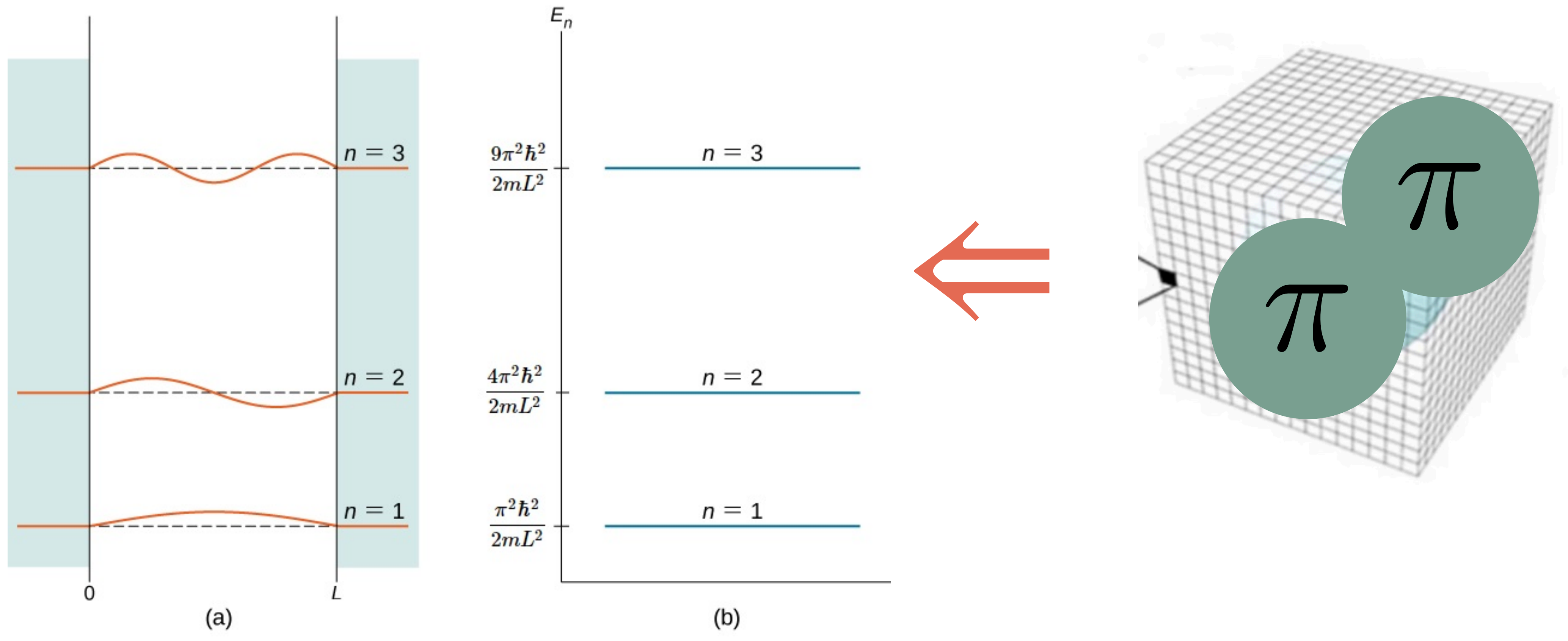
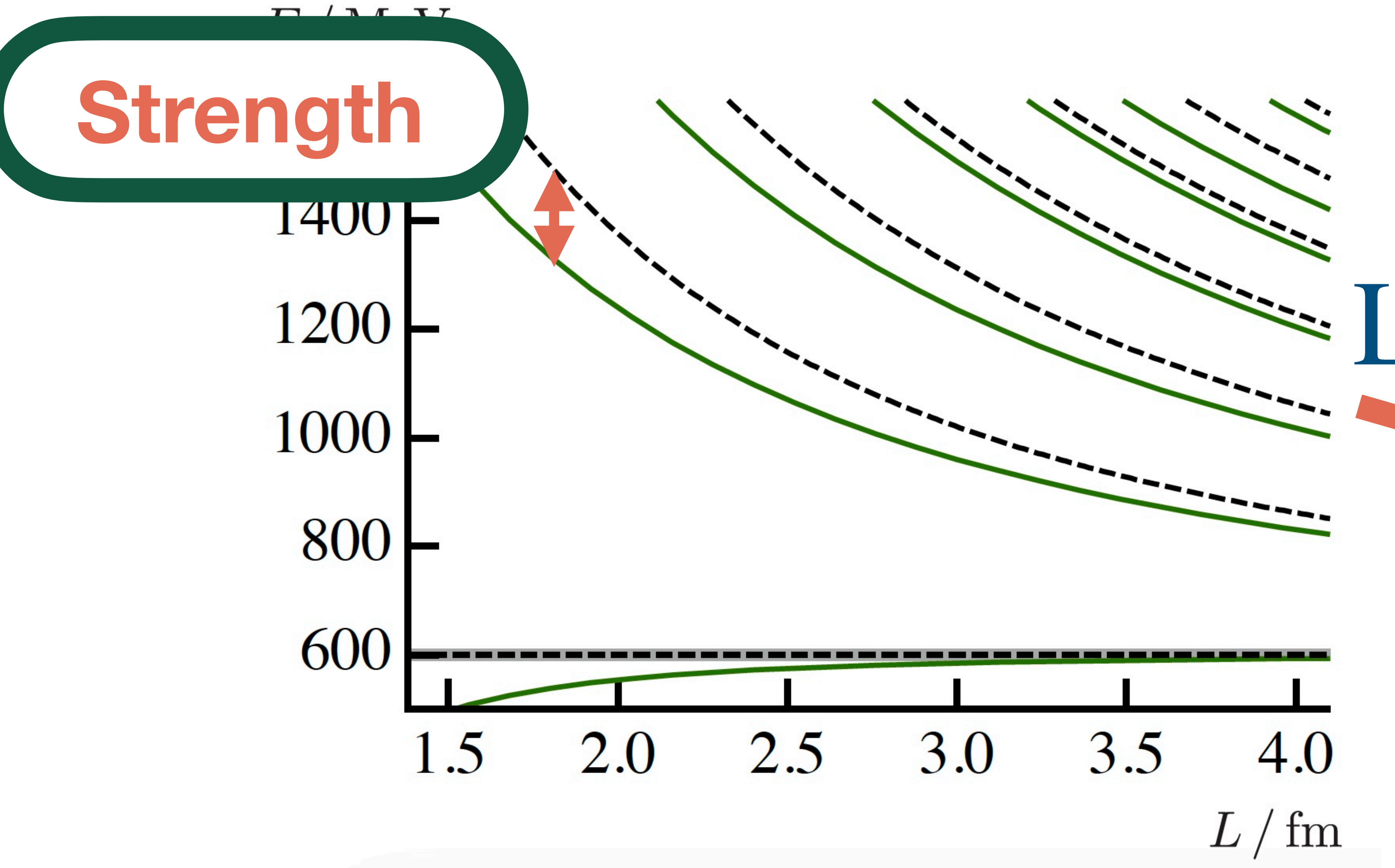


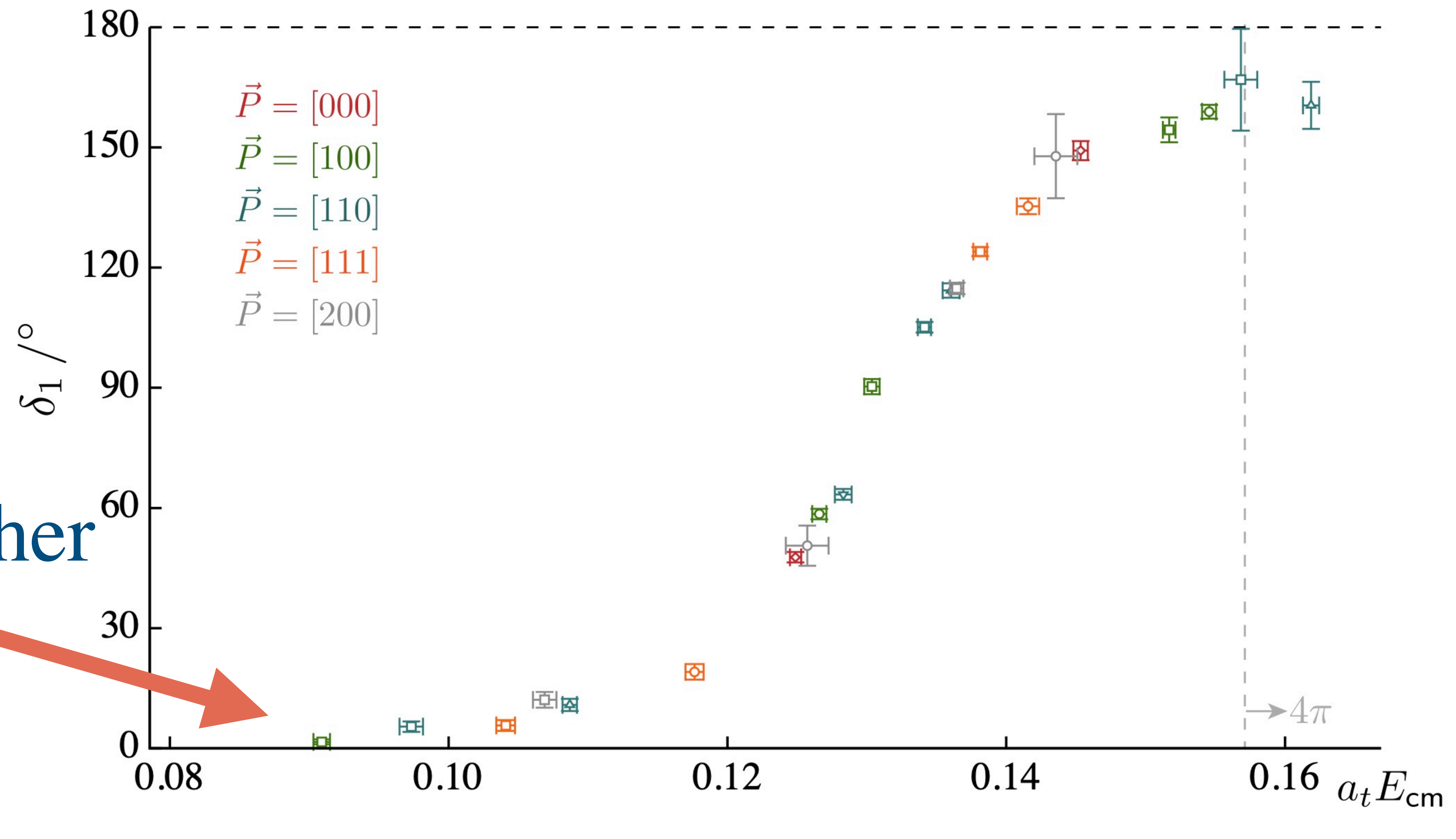
$$D_\mu = \Big(\quad \Big) \updownarrow (L/a)^3 \times (T/a)$$

$$\big\langle (\pi^-(t)\pi^+(t))(\pi^-(0)\pi^+(0)) \big\rangle \sim e^{-E_n t}$$



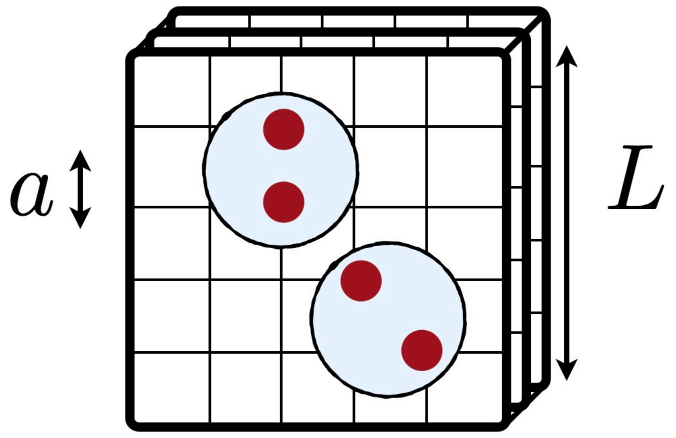


Lüscher

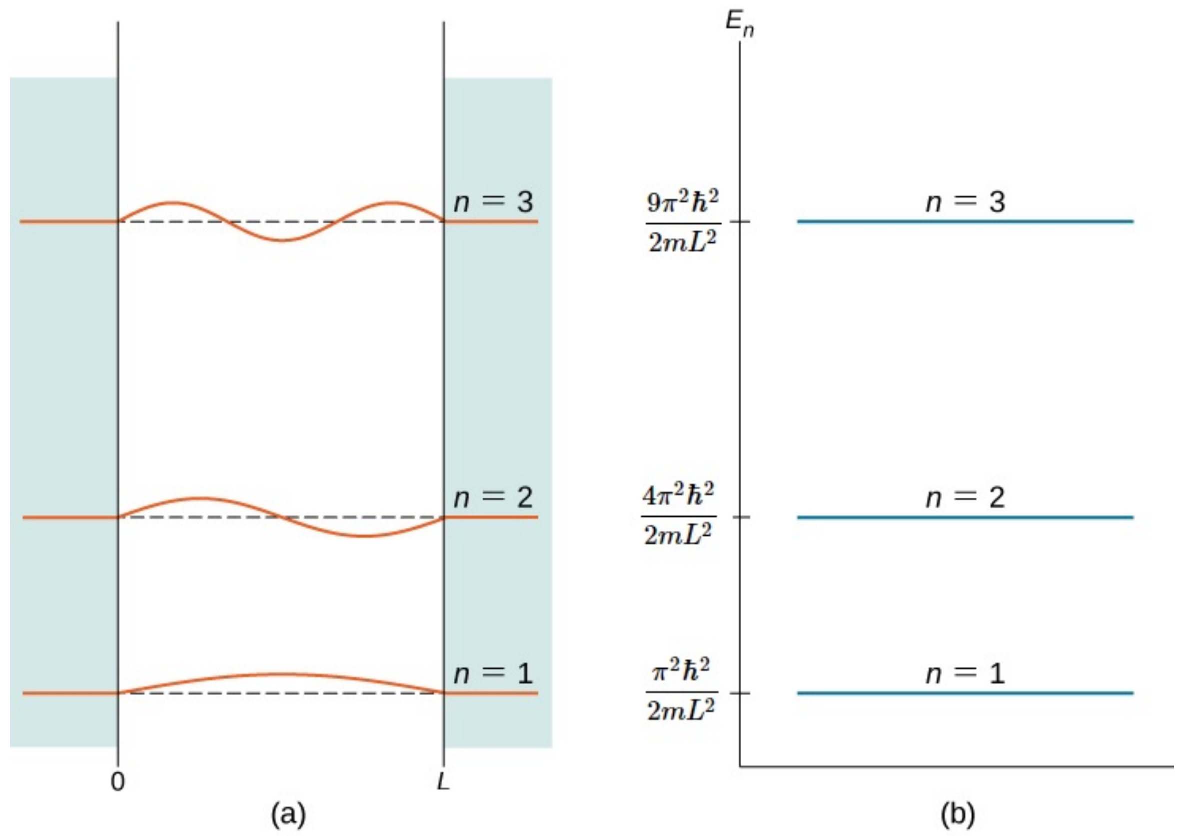


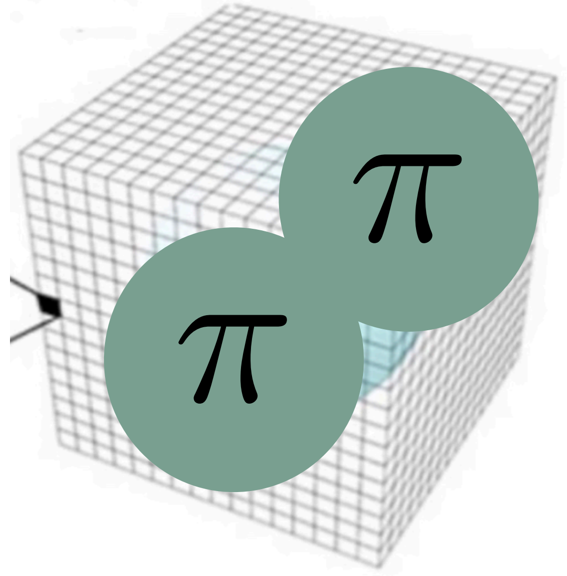
Lattice QCD

Non-perturbative



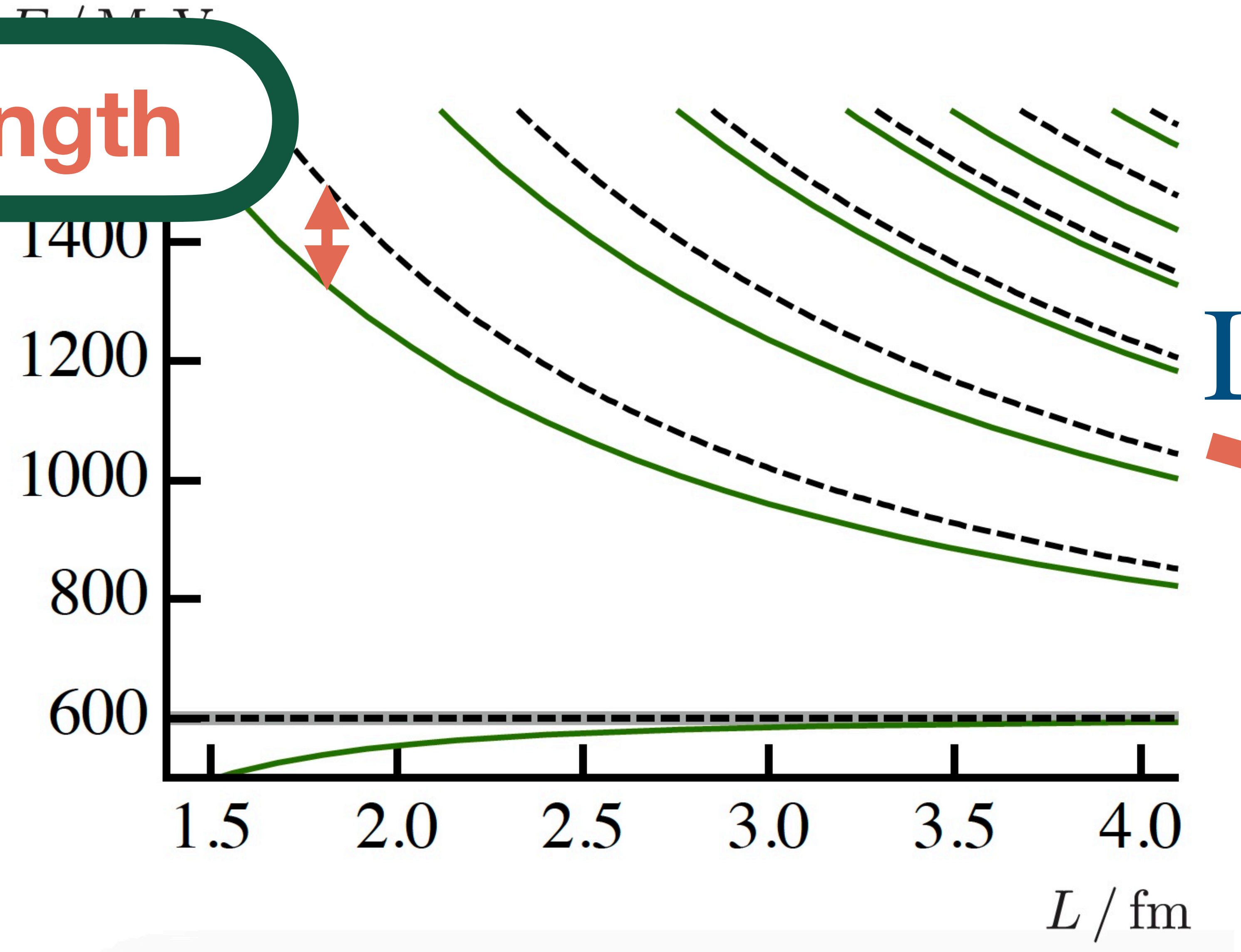
$$D_\mu = \left(\quad \right) (L/a)^3 \times (T/a)$$



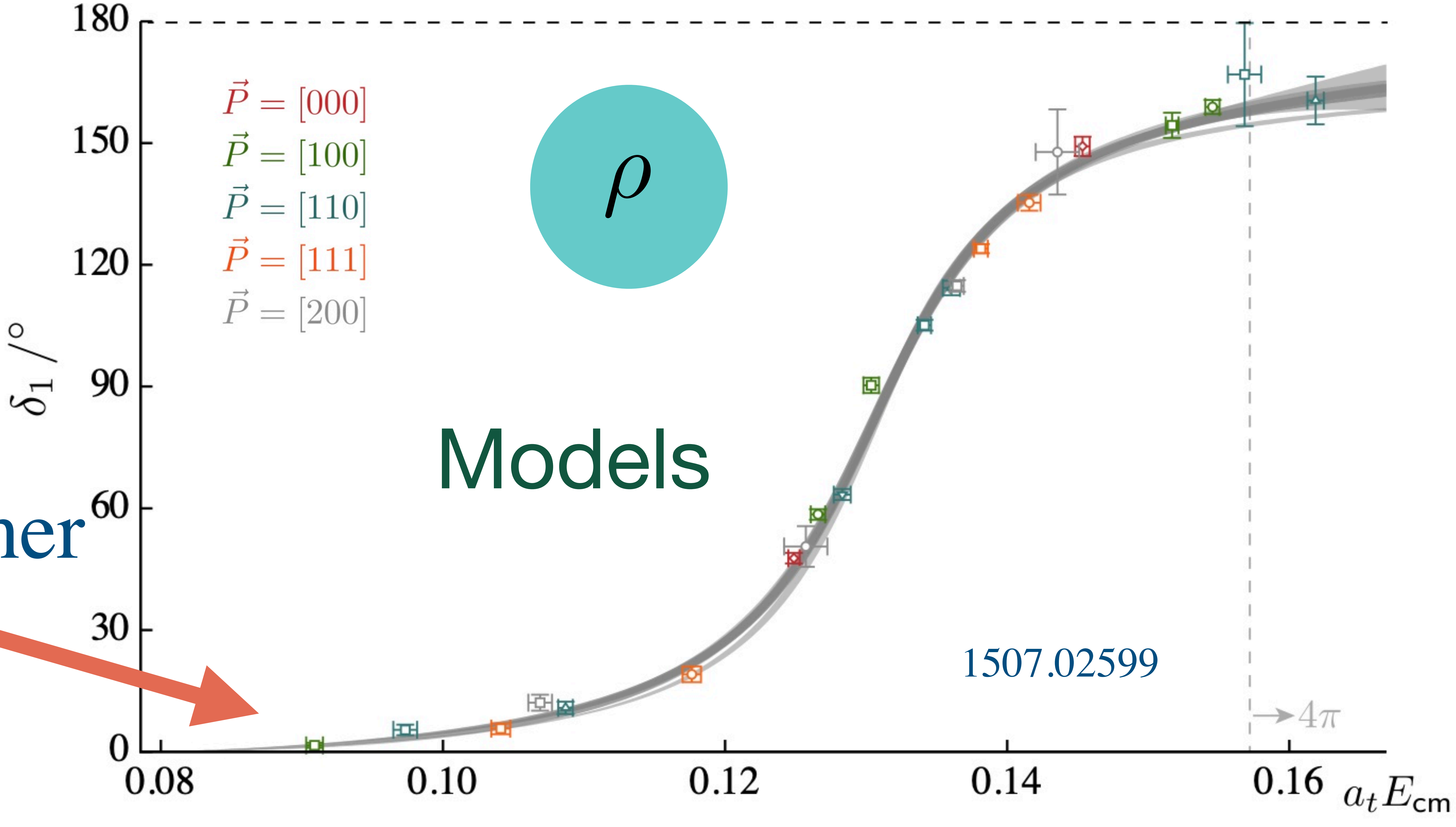


$$\langle(\pi^-(t)\pi^+(t))(\pi^-(0)\pi^+(0))\rangle \sim e^{-E_n t}$$

Strength

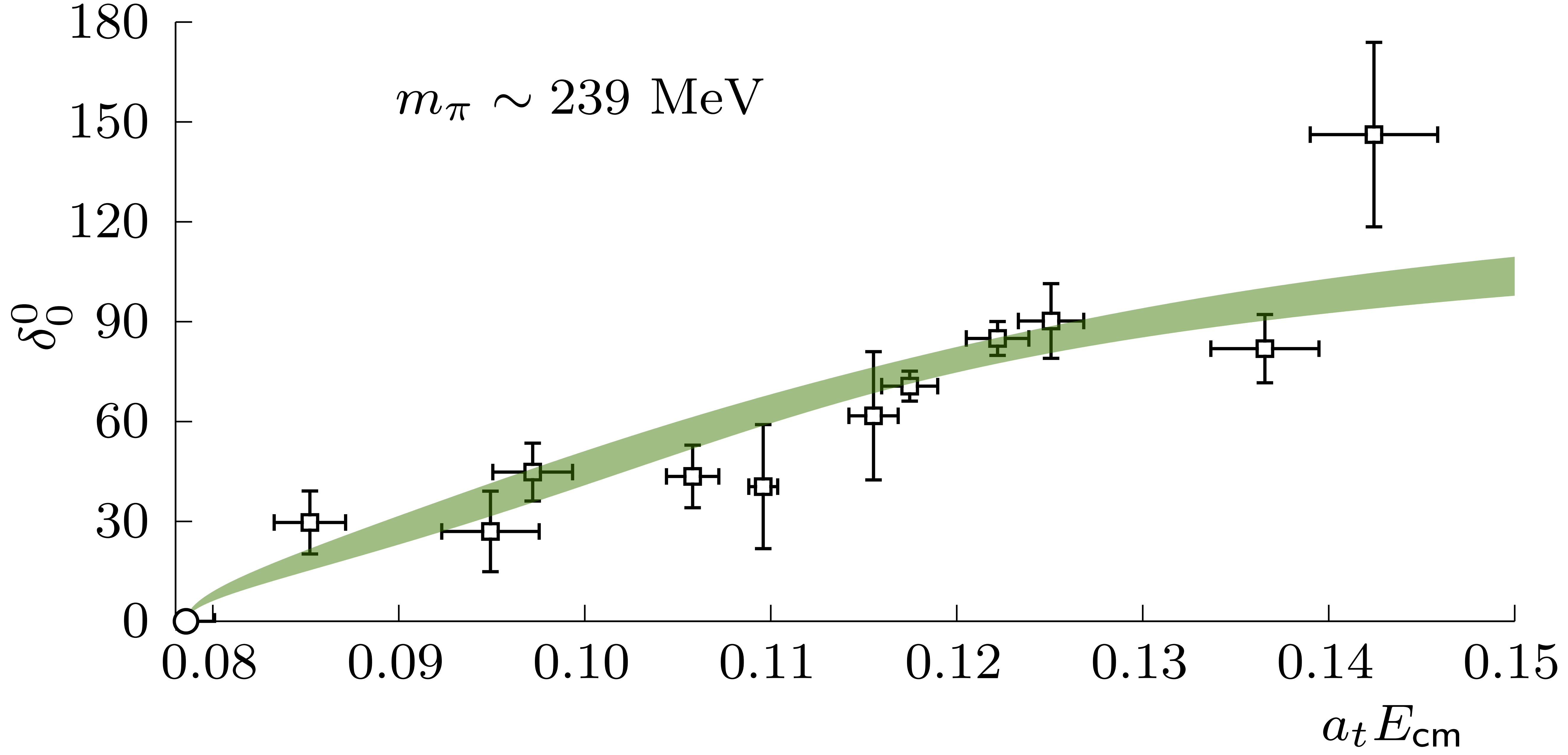


Lüscher



Light Scalars: the σ

Data is precise !!



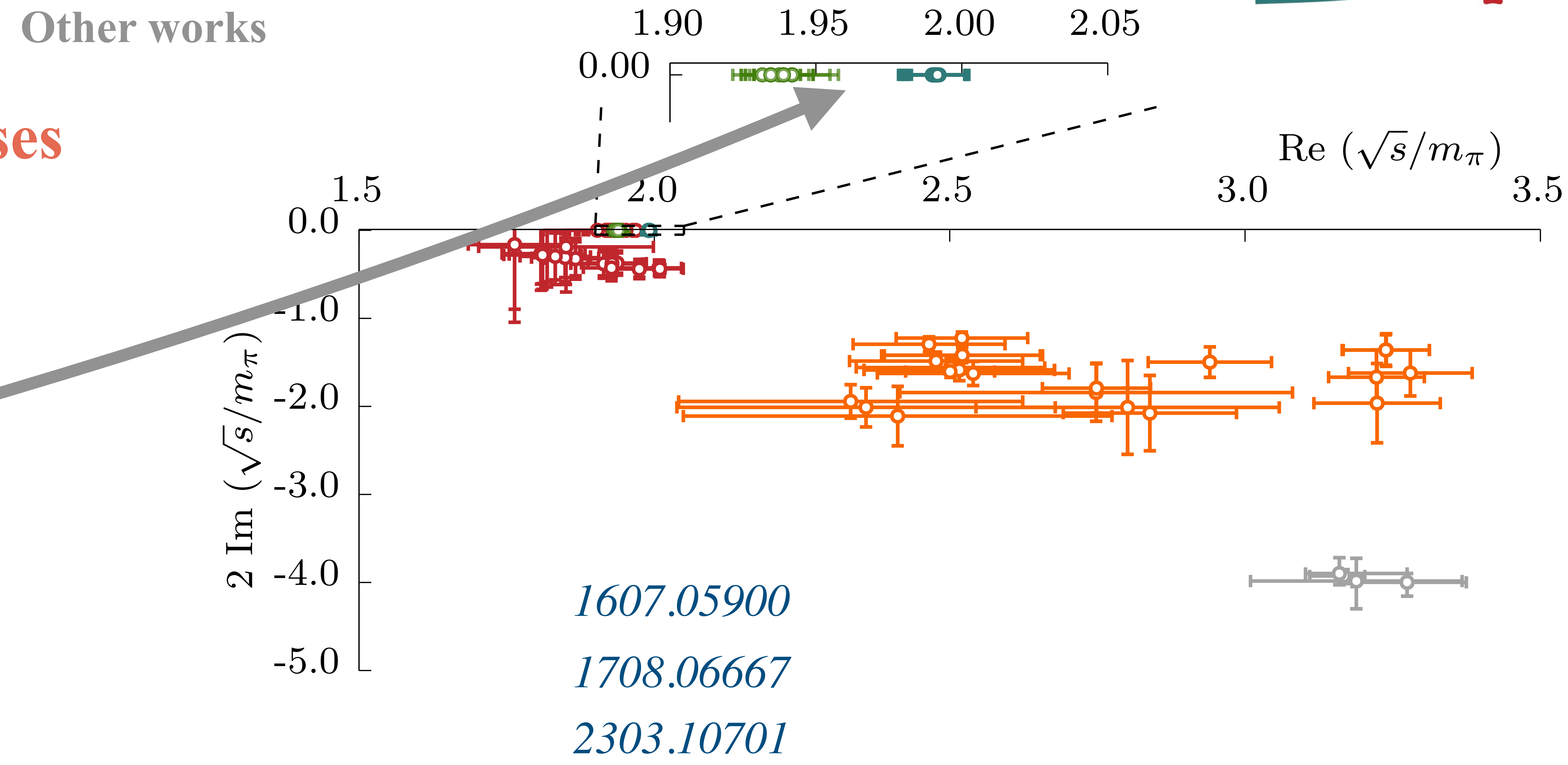
Light Scalars: the σ

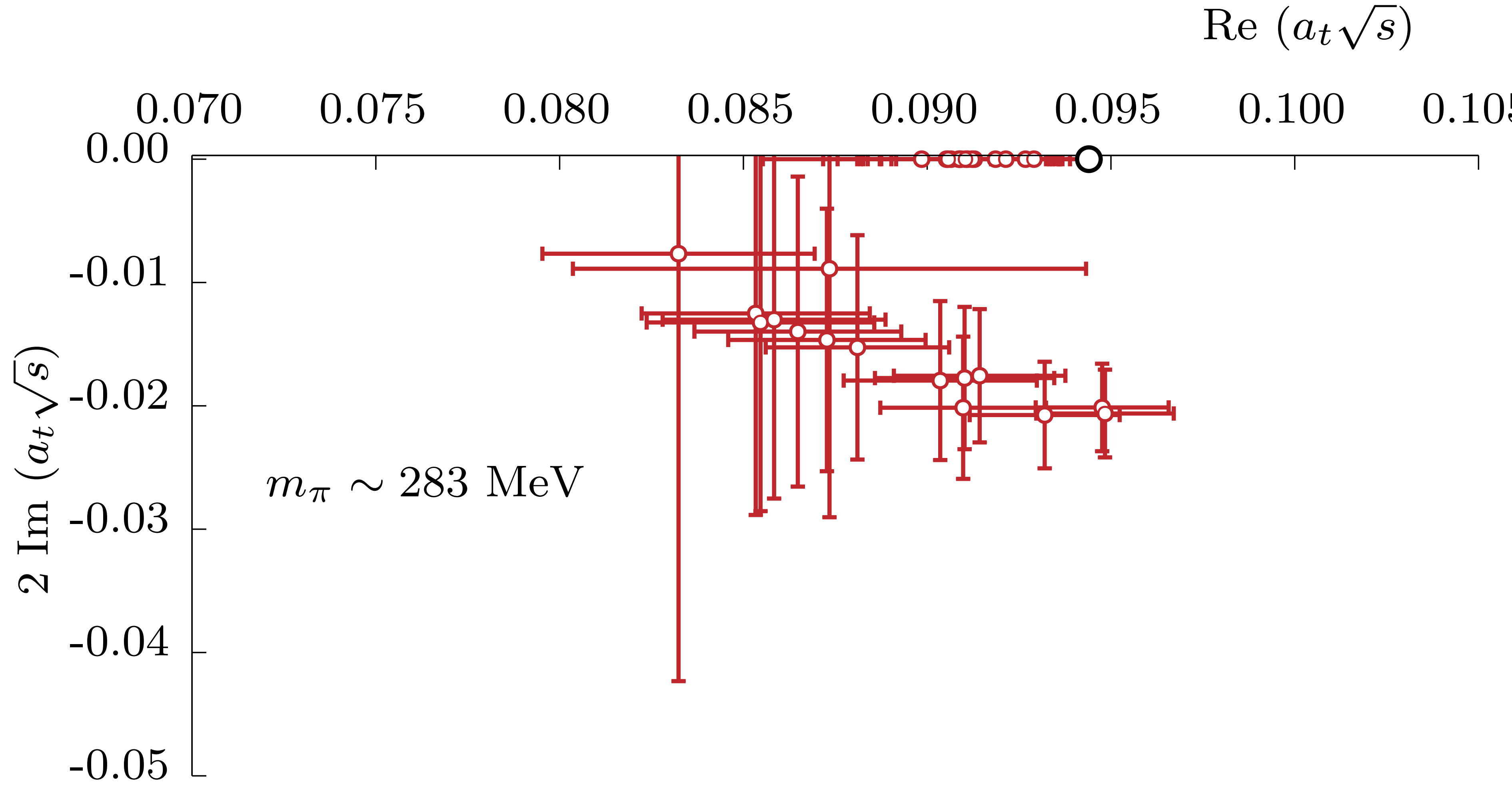
16010.10070 1803.02897 1908.01847

hadspec

Stable and "easy" to extract at higher masses

- $m_\pi \sim 391\, MeV \rightarrow$ **Stable**
- $m_\pi \sim 330\, MeV \rightarrow$ **Stable**
- $m_\pi \sim 239\, MeV \rightarrow$ **Broad resonance**
- $m_\pi \sim 283\, MeV \rightarrow$ **??**

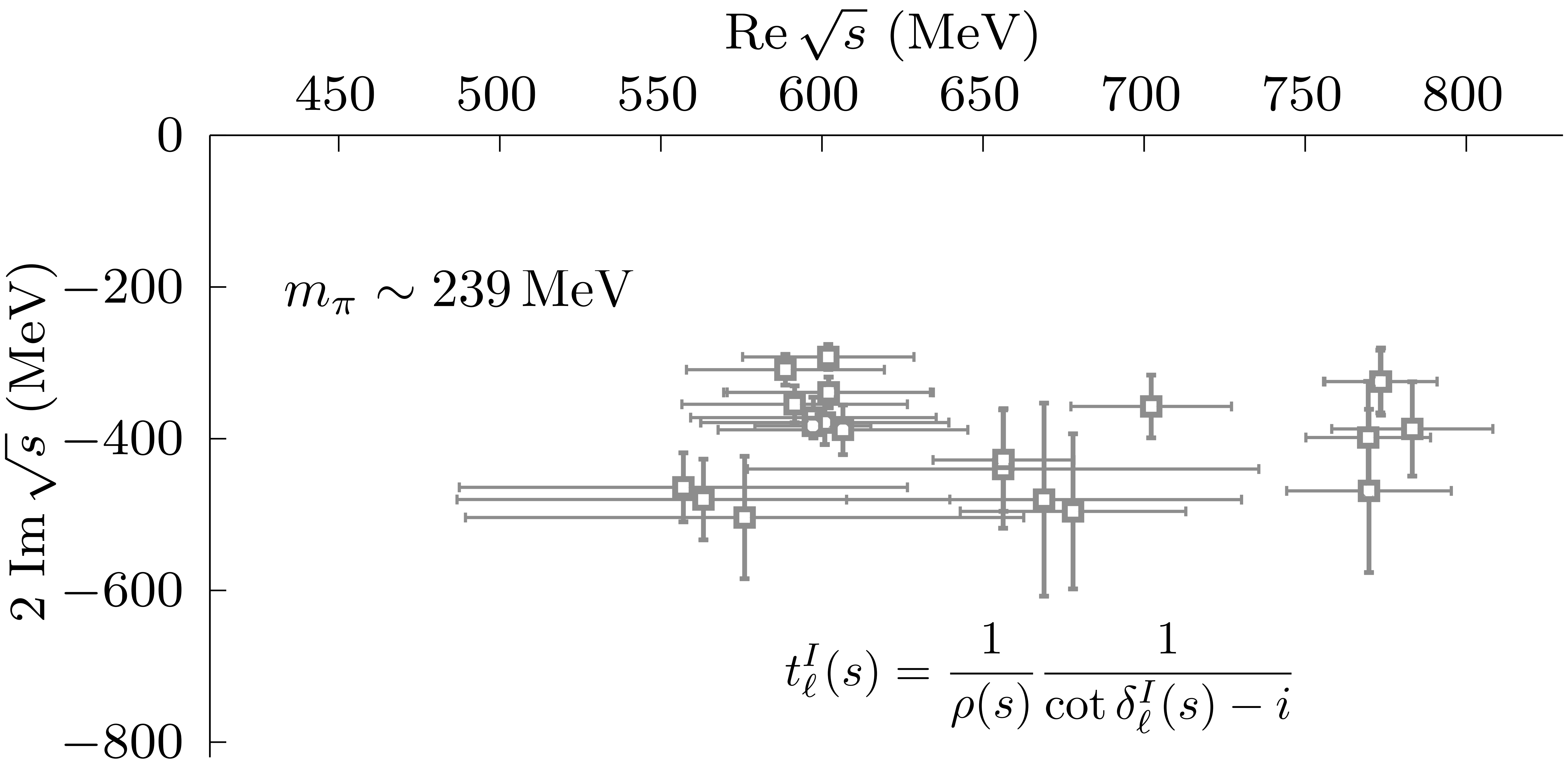




Light Scalars: the σ

hadspec

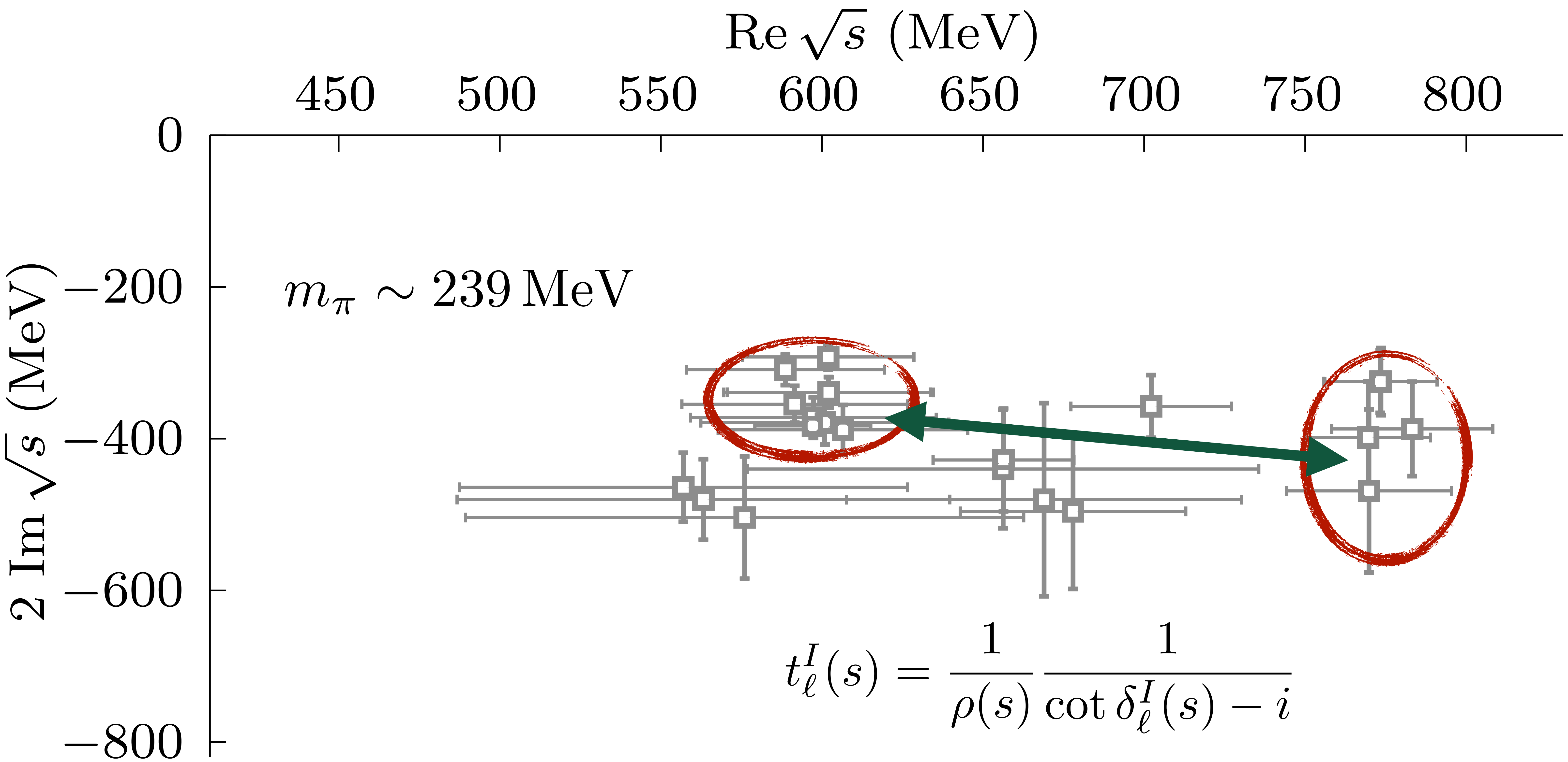
Total error becomes really large when decreasing the pion mass



$$t^I_\ell(s) = \frac{1}{\rho(s)}\frac{1}{\cot\delta^I_\ell(s) - i}$$

Light Scalars: the σ

Total error becomes really large when decreasing the pion mass

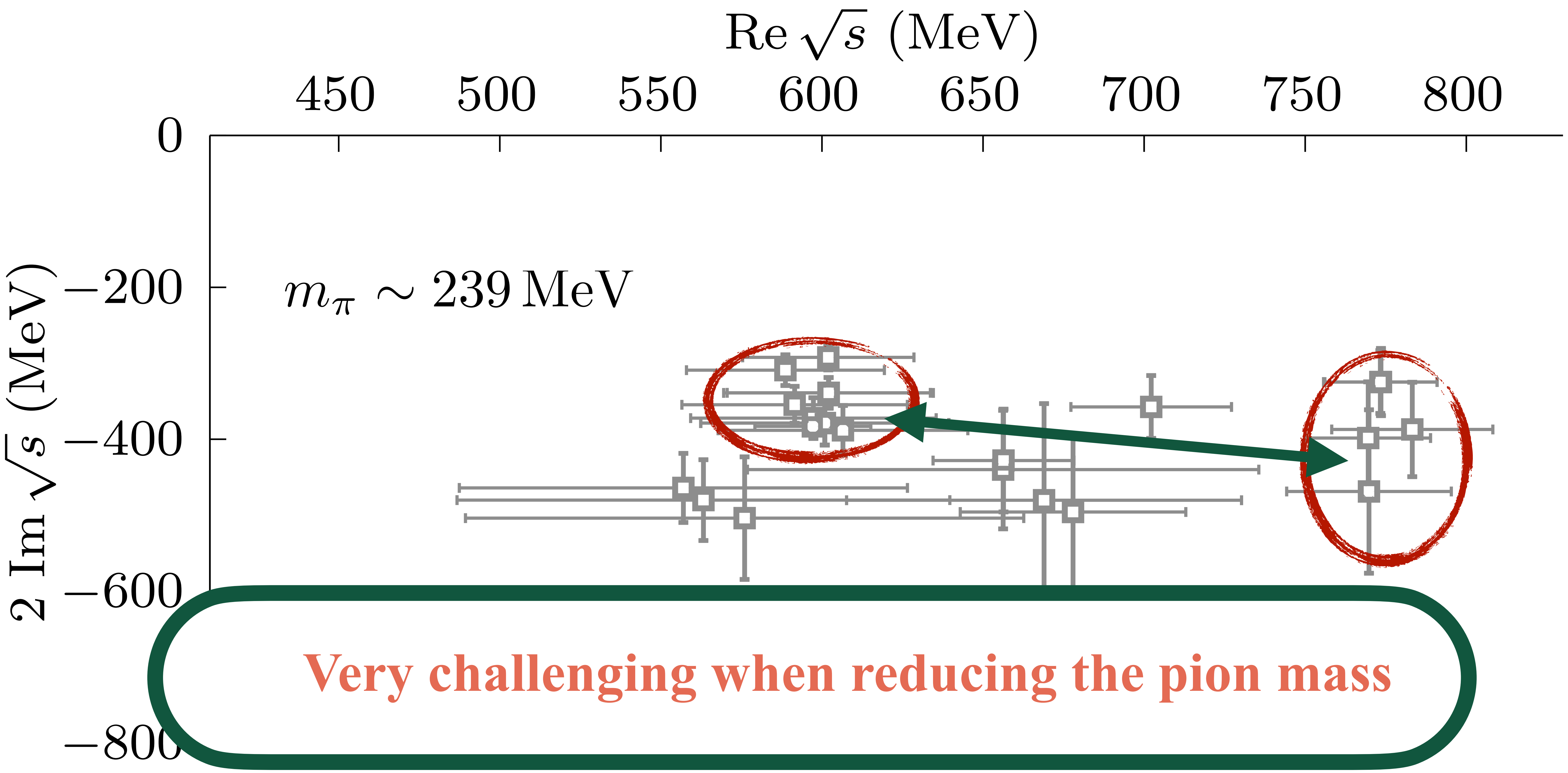


$$t_\ell^I(s) = \frac{1}{\rho(s)}\frac{1}{\cot\delta_\ell^I(s) - i}$$

Models are incompatible with one another

Light Scalars: the σ

Total error becomes really large when decreasing the pion mass



Models are incompatible with one another

Light Scalars: the σ

hadspec

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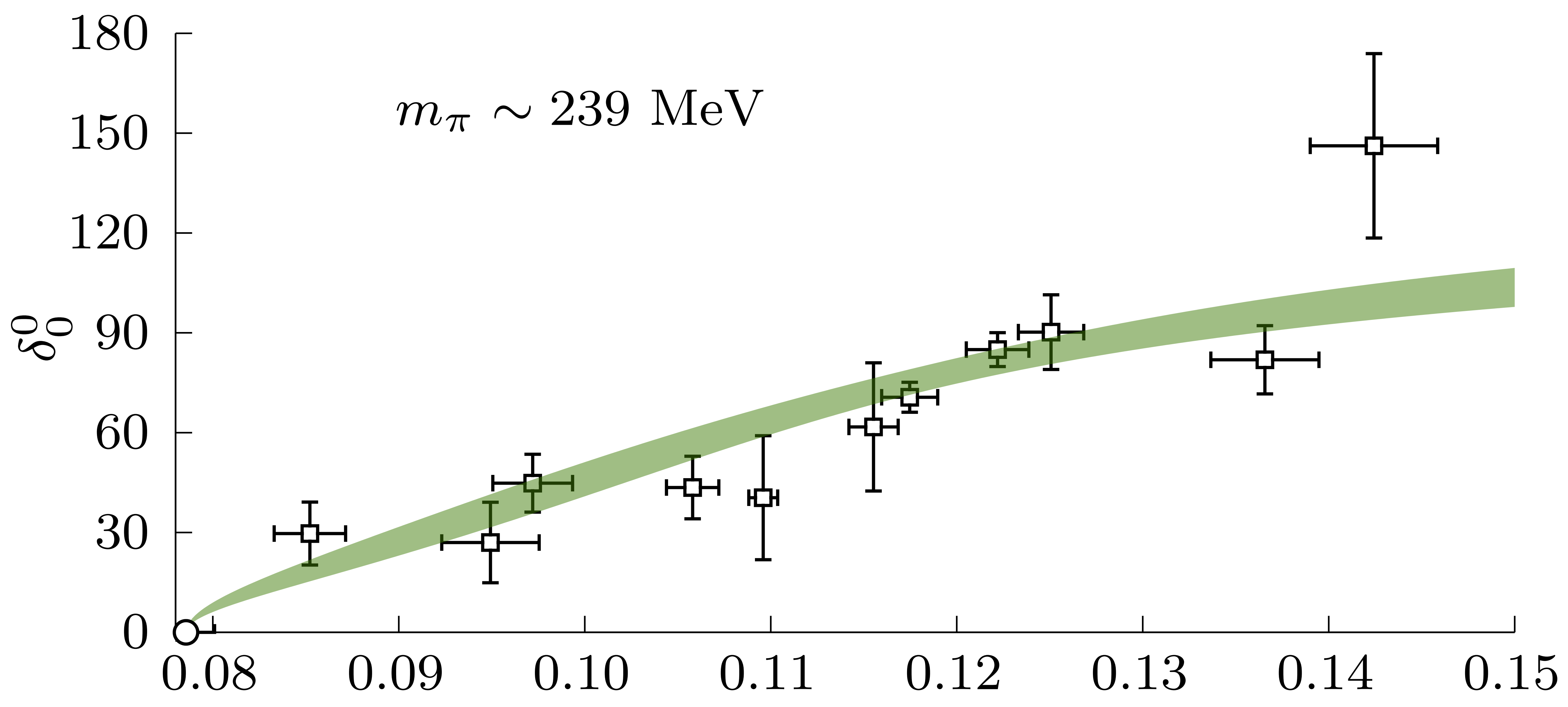
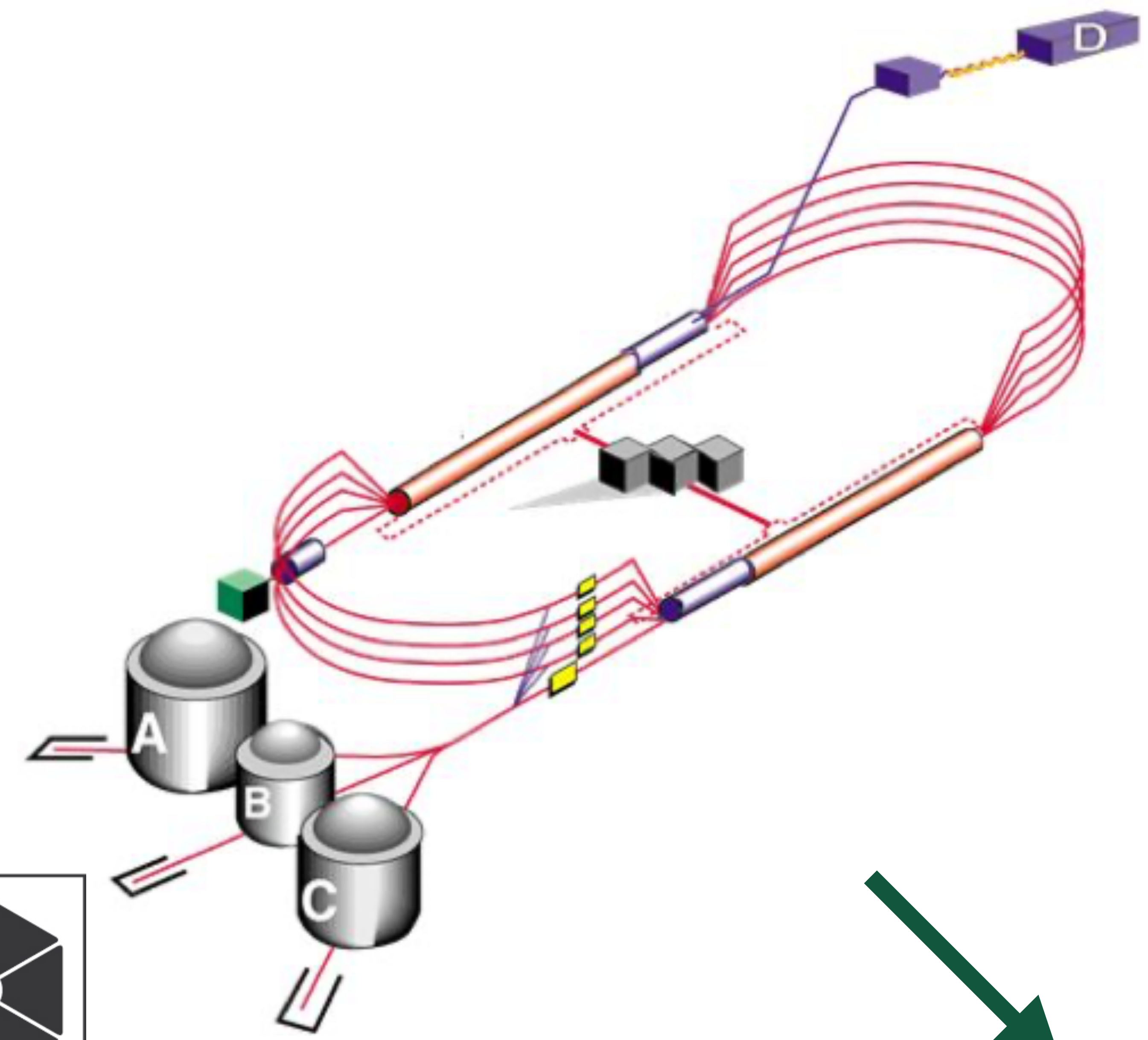
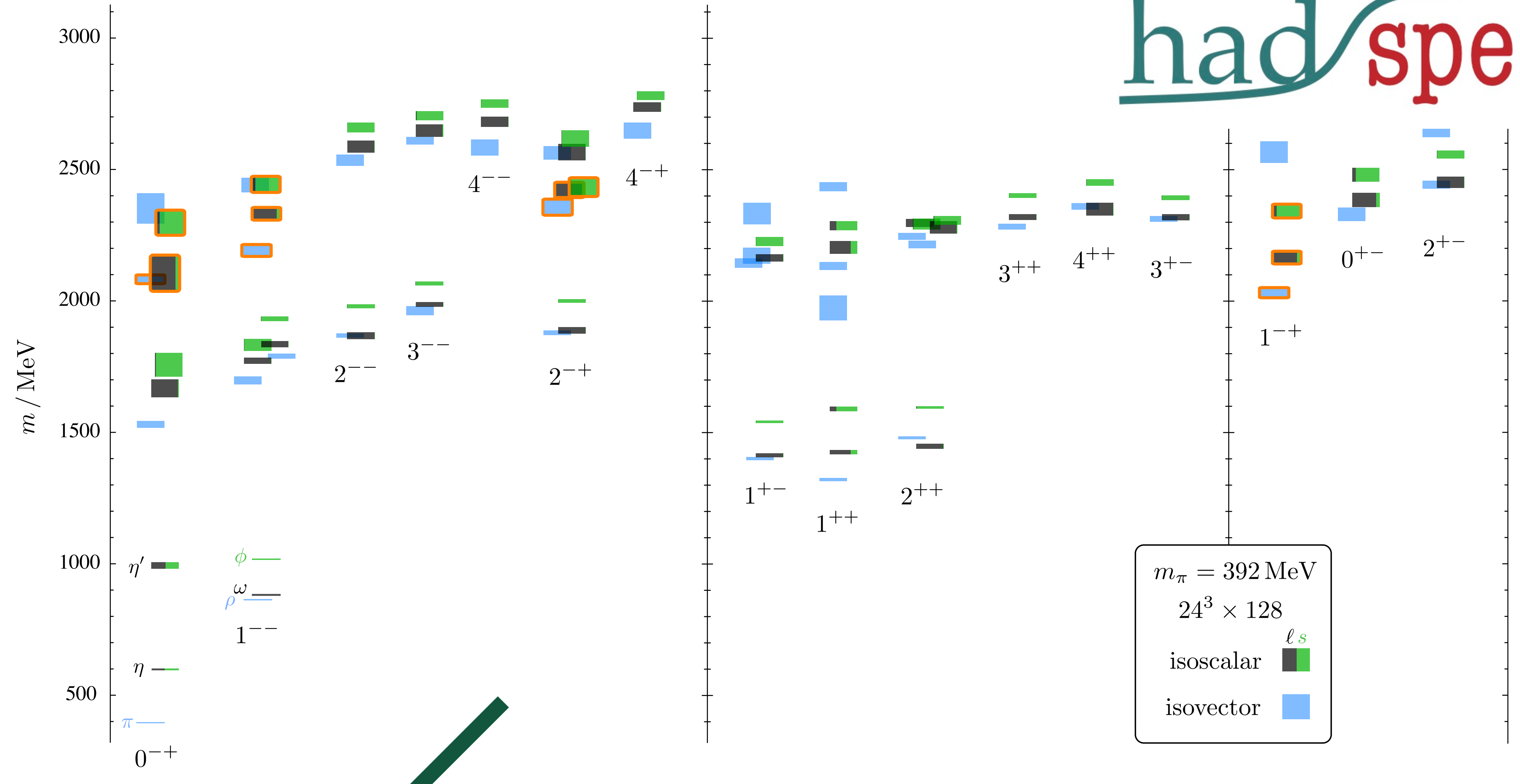


σ very challenging when reducing the pion mass

But data is precise !!



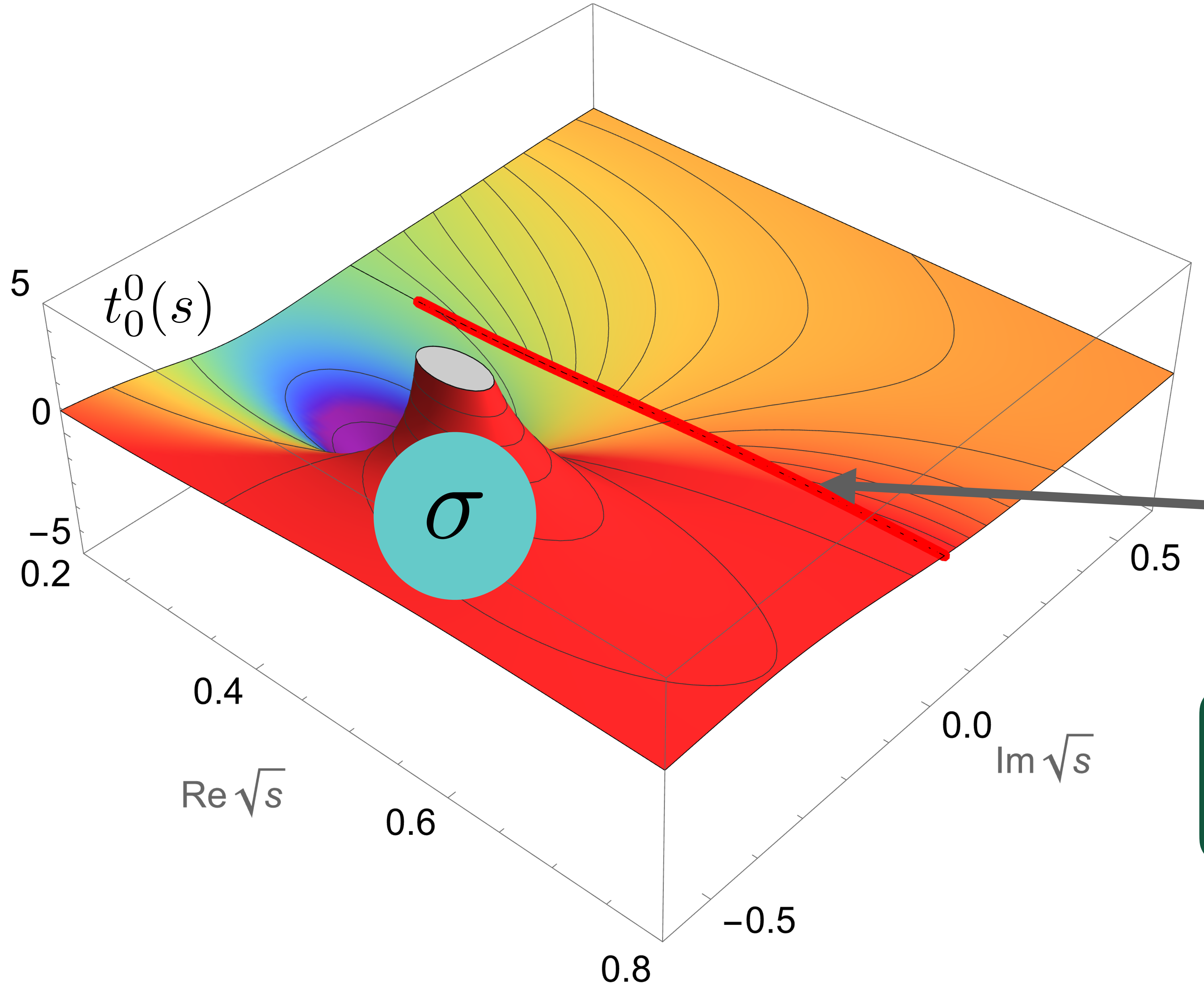


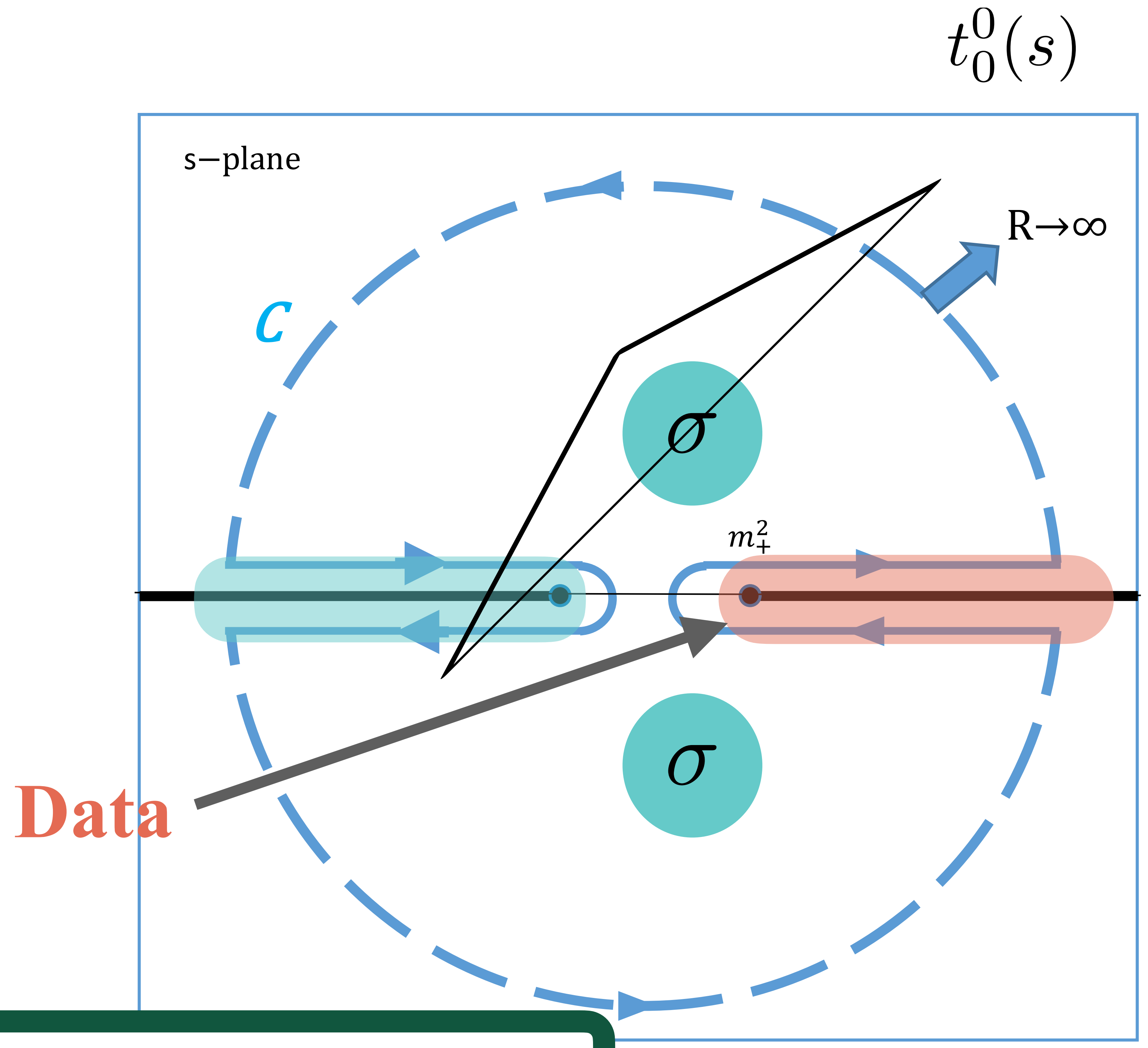
Amplitude analyses

- Unitarity
- Analiticity
- Crossing

Observables

Crossing





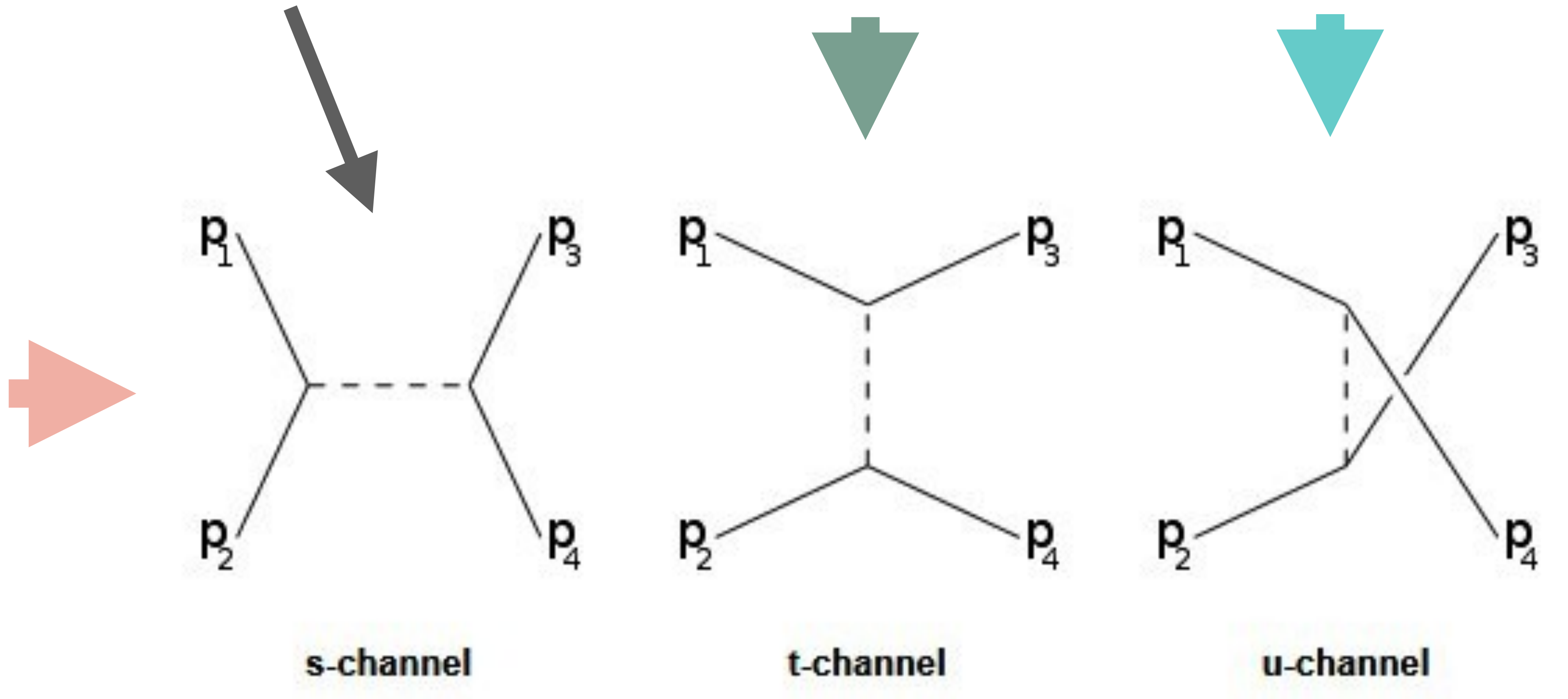
$$\mathrm{Im}\, T_{i\,i}(s, t=0) \propto \sum_n T_{i\,n} T^\dagger_{n\,i} \propto \sigma^i_{\rm tot}$$

Particles and anti-particles are related

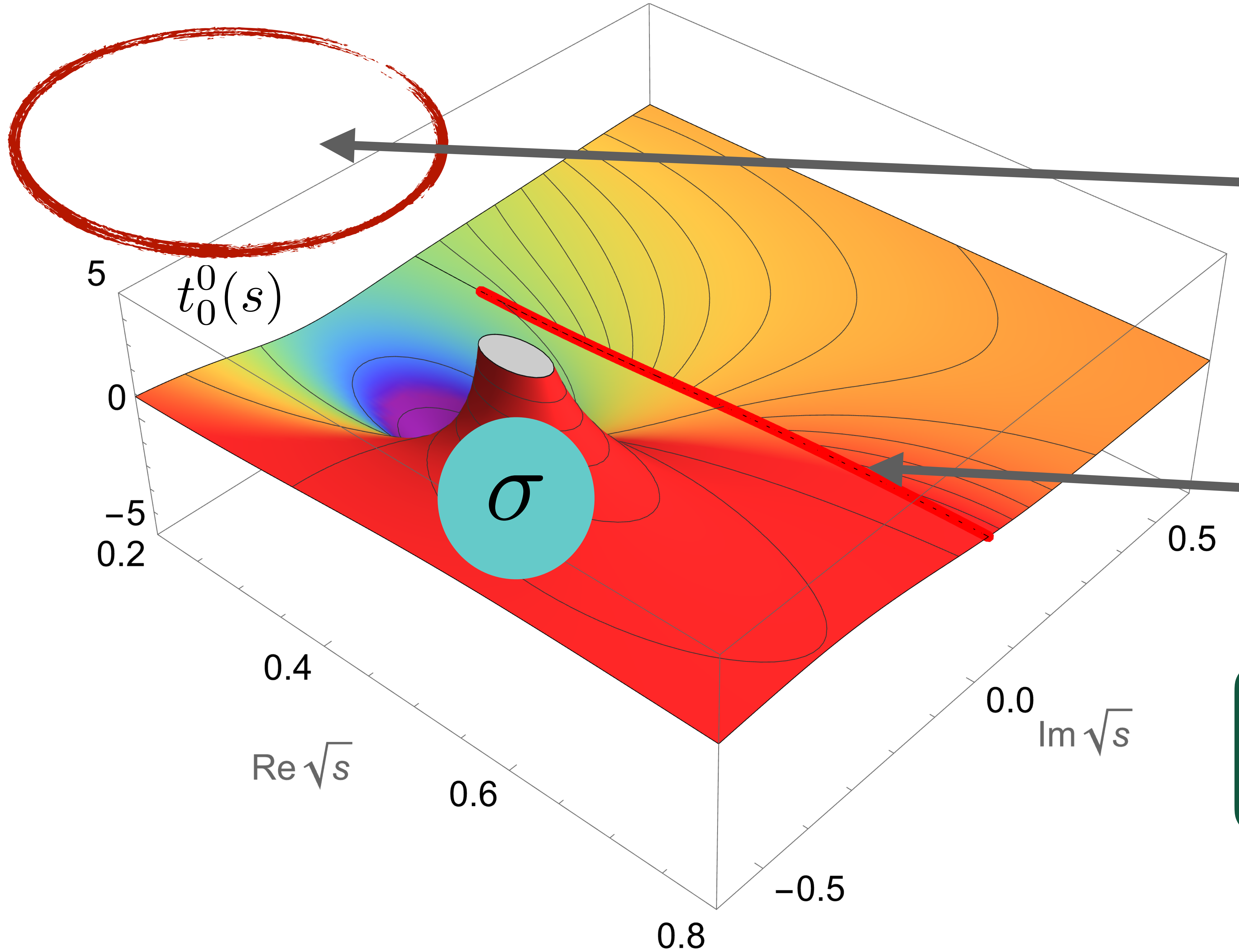
s-channel

t-channel

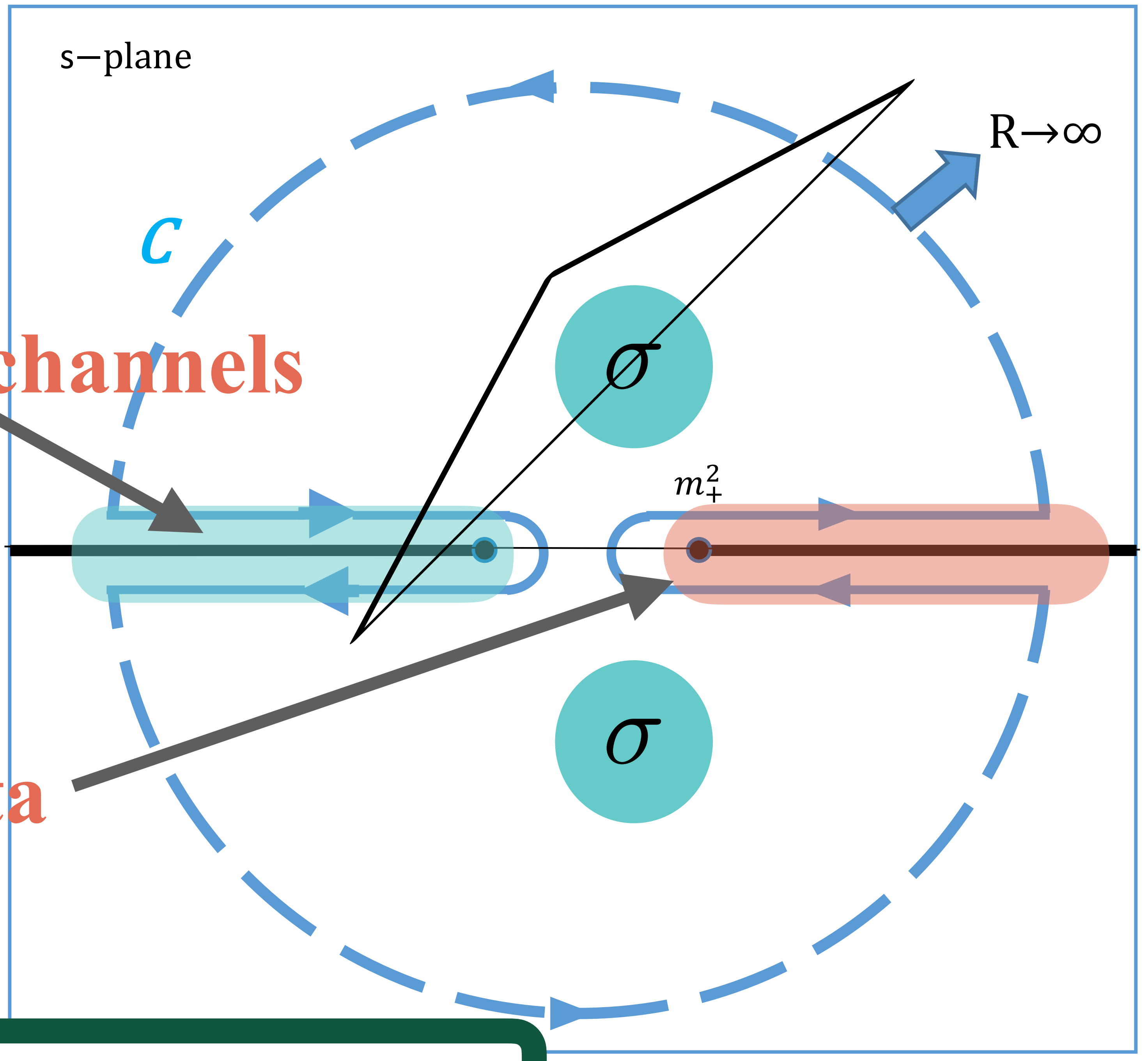
u-channel



Crossing



$t_0^0(s)$



Cross-channels

Data

$$\mathrm{Im}\, T_{i\,i}(s, t=0) \propto \sum_n T_{i\,n} T^\dagger_{n\,i} \propto \sigma^i_{\mathrm{tot}}$$

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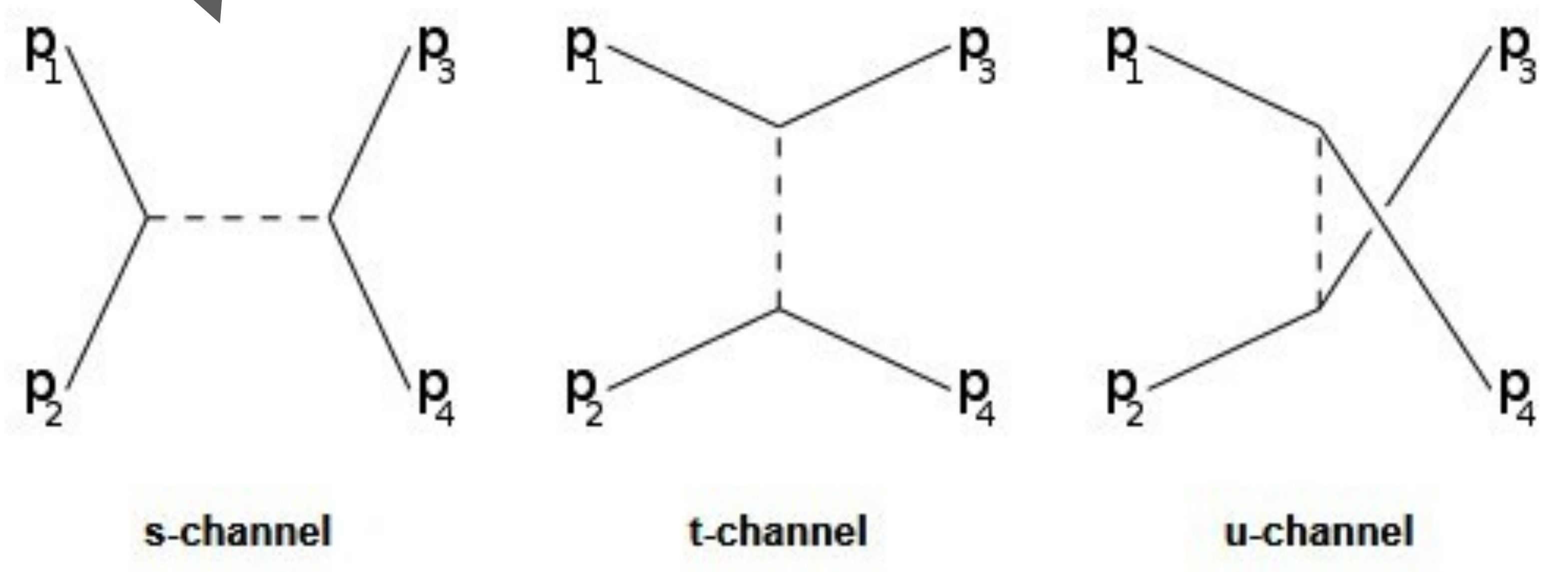
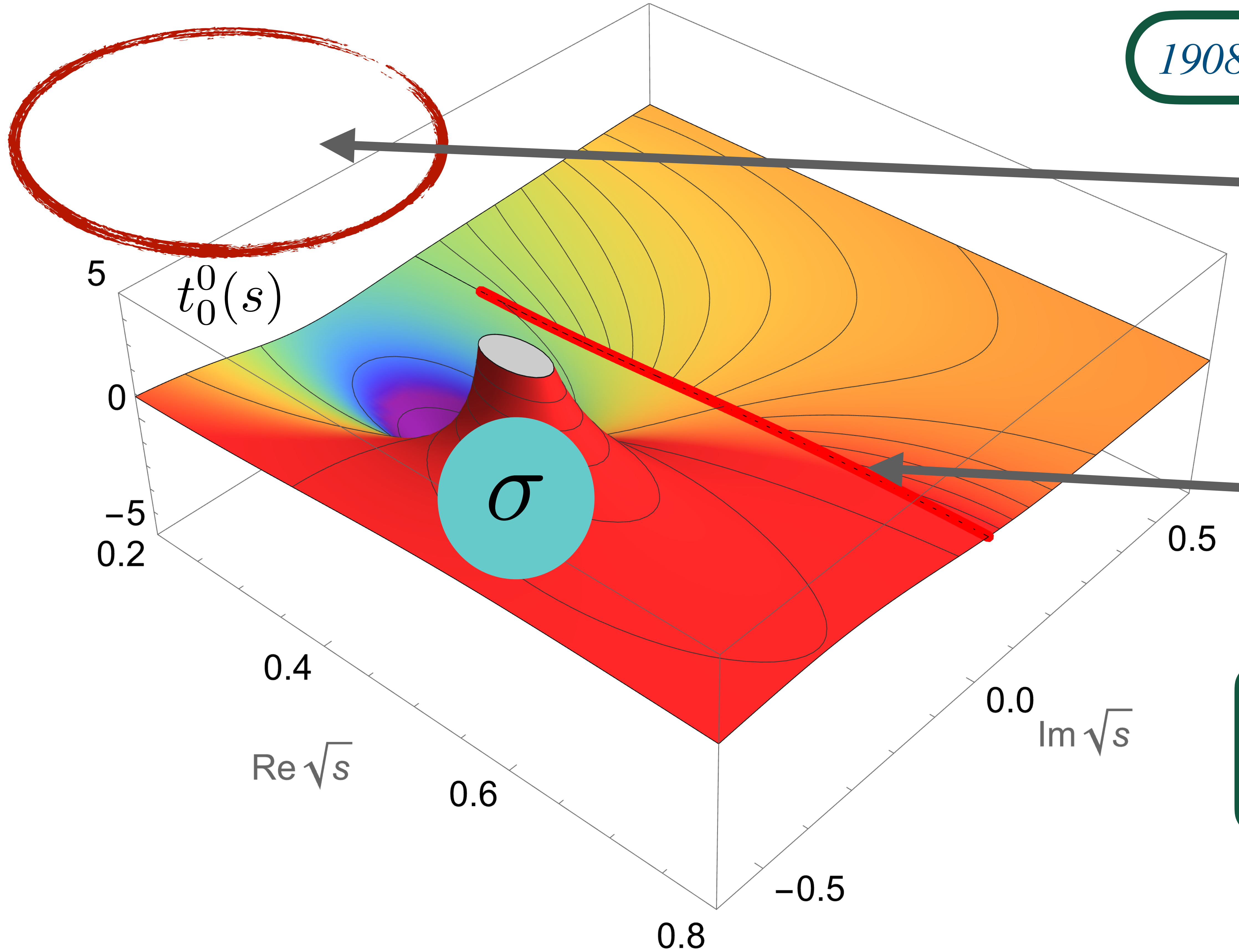


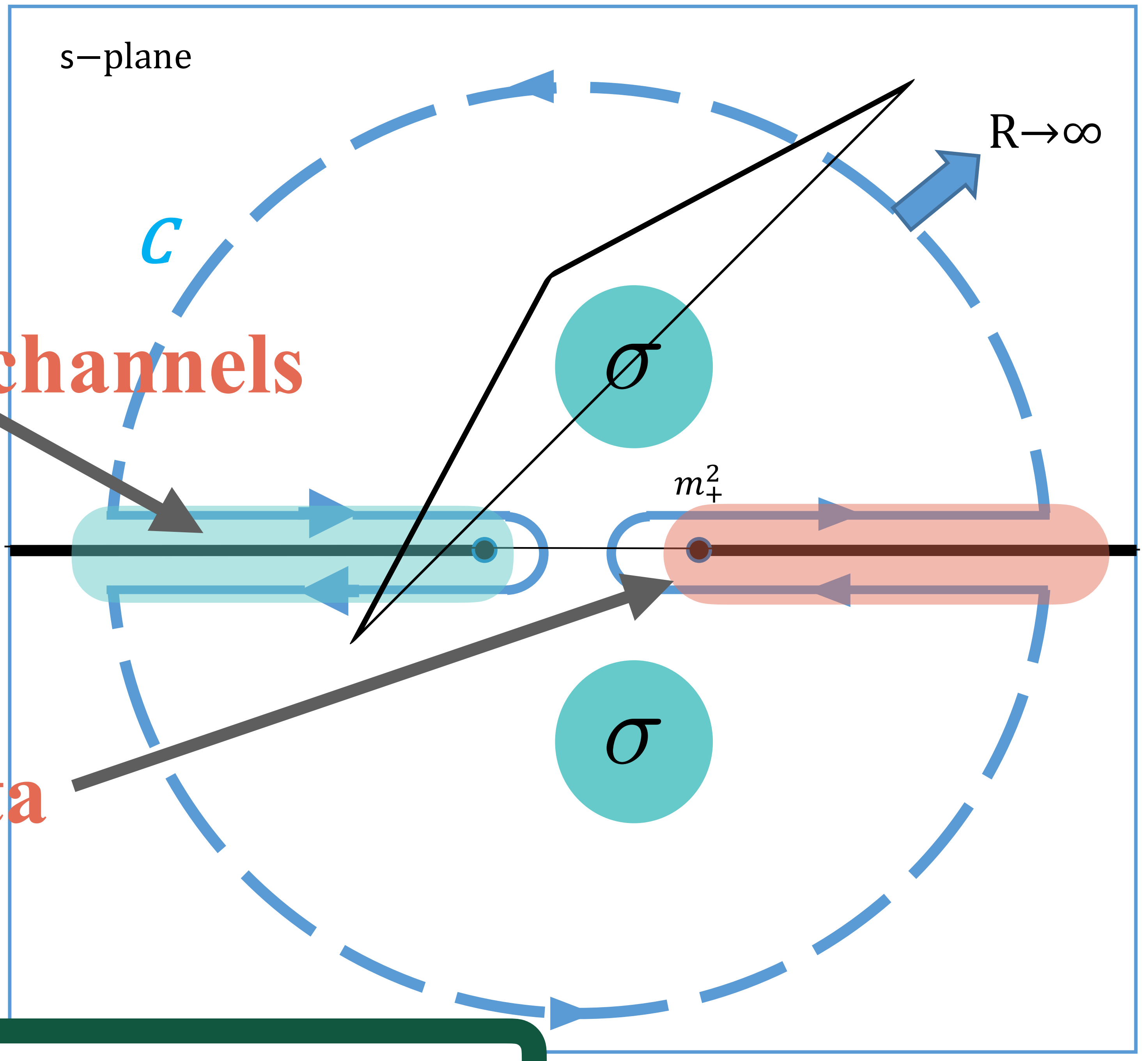
Crossing

UChPT

1908.01847



$t_0^0(s)$



Cross-channels

Data

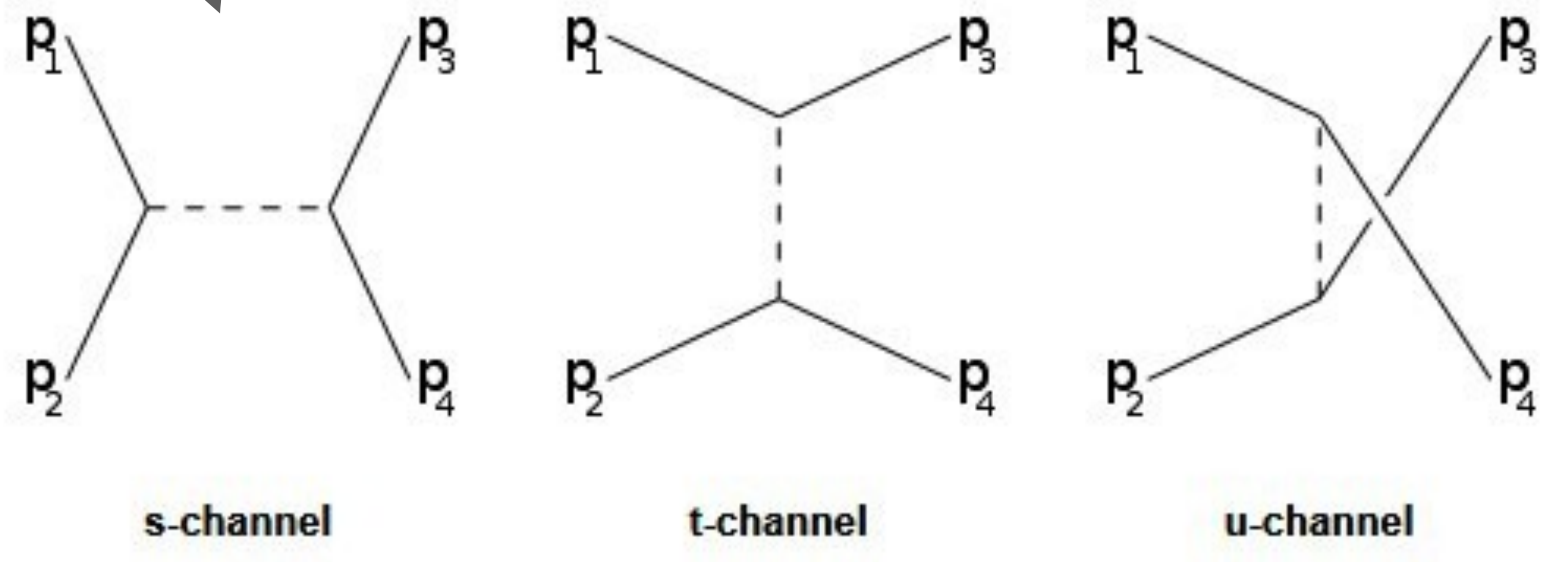
$$\mathrm{Im}\, T_{i\,i}(s,t=0) \propto \sum_n T_{i\,n} T^{\dagger}_{n\,i} \propto \sigma^i_{\mathrm{tot}}$$

Particles and anti-particles are related

s-channel

t-channel

u-channel

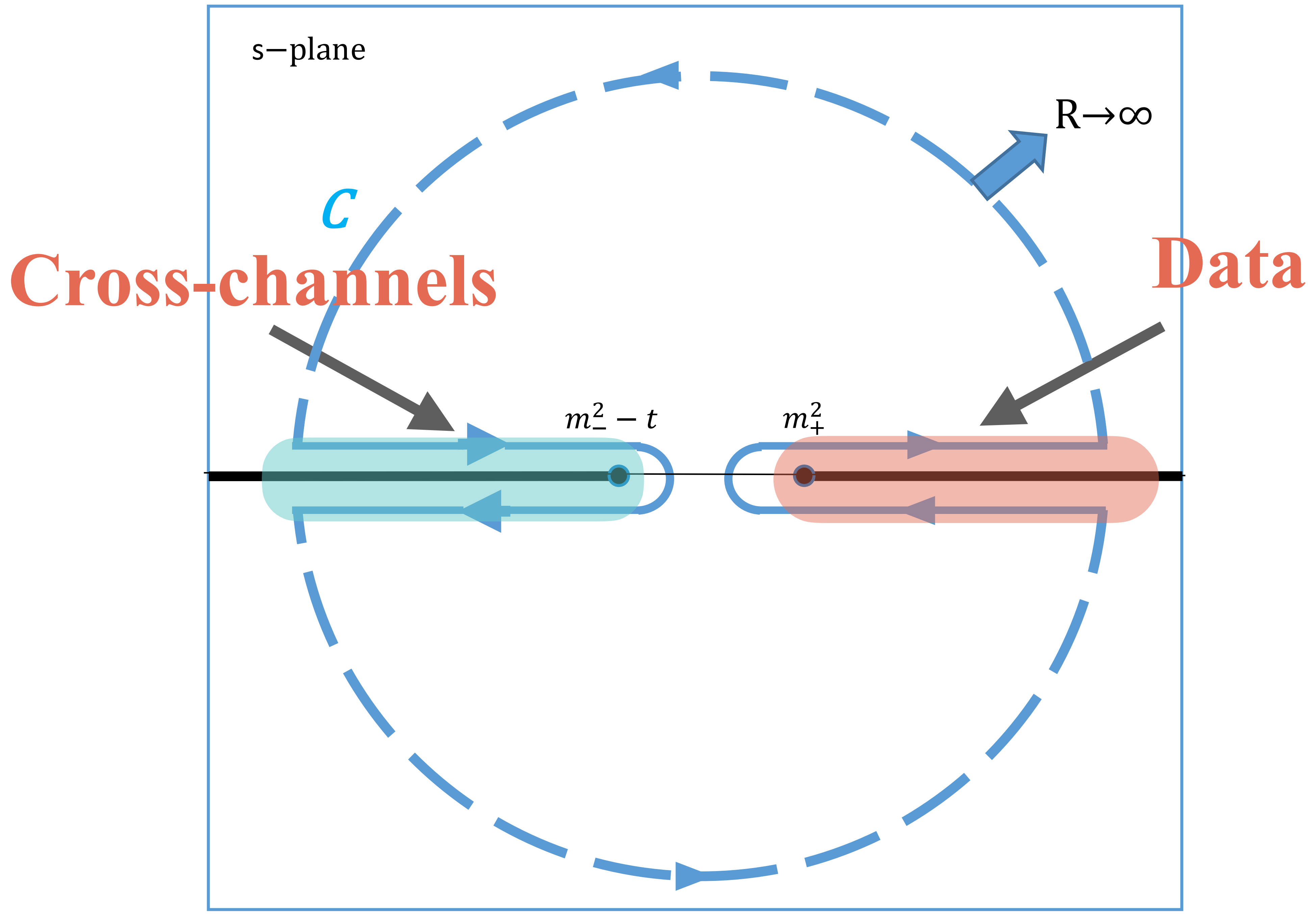


Dispersion relations

Causality ↔ Analiticity

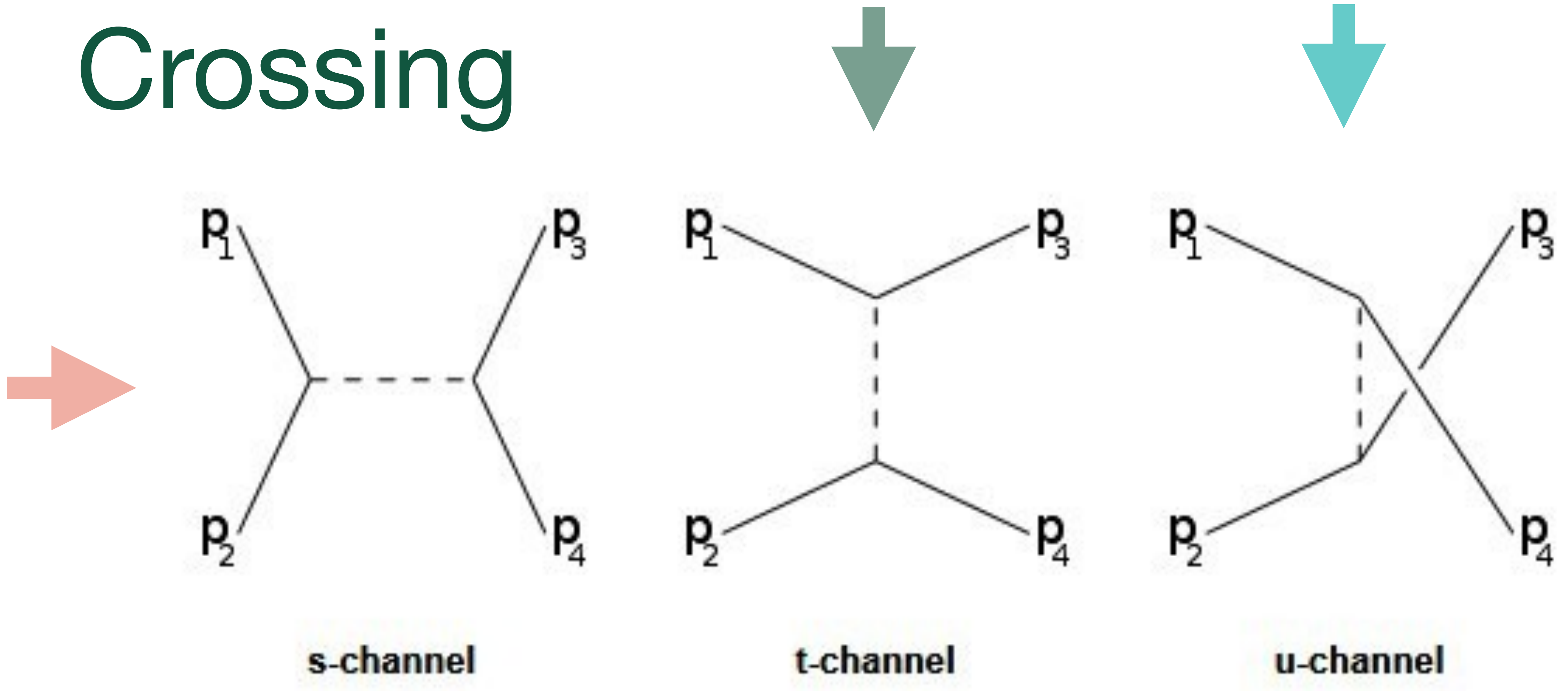
Built using Cauchy's theorem

$$t(z) = \oint_C \frac{t(z')}{z'-z} dz'$$



They can implement both analyticity AND crossing

Crossing

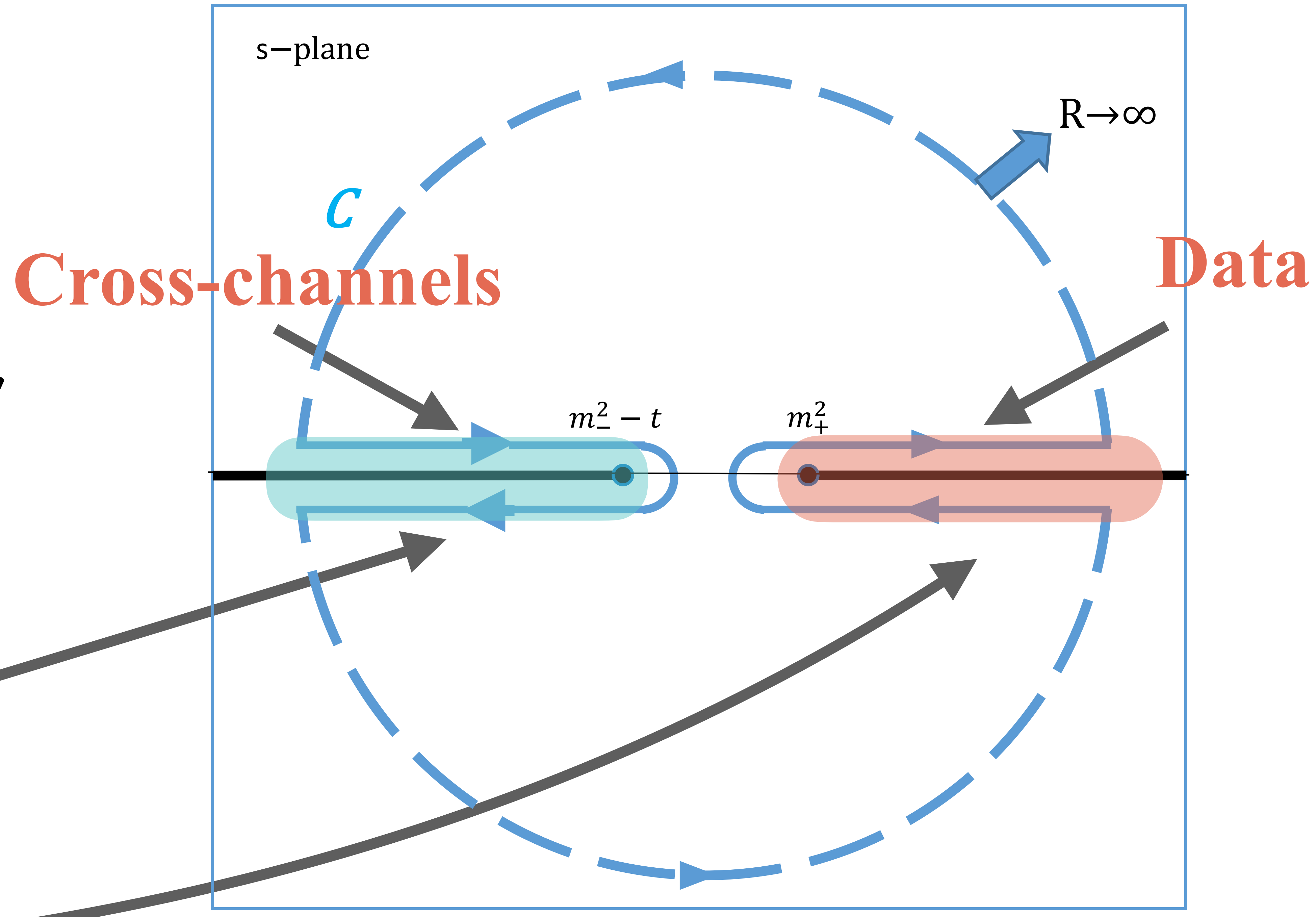


Dispersion relations

Cauchy

$$t(z)=\oint_C \frac{t(z')}{z'-z}dz'$$

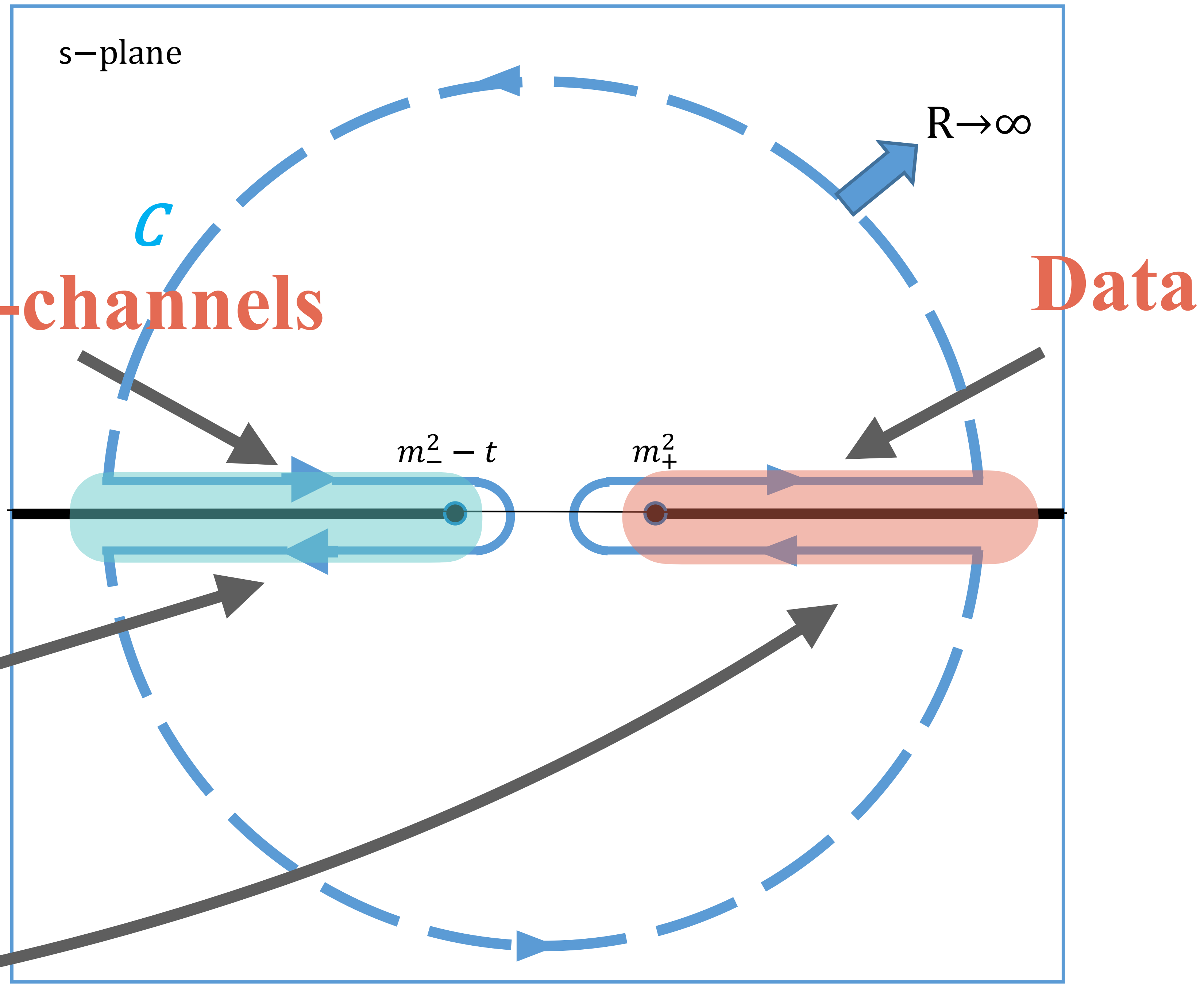
$$T^I(s,t)=\frac{1}{\pi}\int_{s_{th}}^{\infty} ds' \frac{\operatorname{Im} T^I(s',t)}{s'-s} + LHC$$



Dispersion relations

Cauchy

$$t(z) = \oint_C \frac{t(z')}{z' - z} dz'$$



$$T^I(s,t) = \frac{1}{\pi} \int_{s_{th}}^{\infty} ds' \frac{\text{Im}\, T^I(s',t)}{s' - s} + LHC$$

Crossing

$$T^I(s,t) = \frac{1}{\pi} \int_{s_{th}}^{\infty} ds' \left(\frac{\text{Im}\, T^I(s',t)}{(s' - s)} + \frac{\sum C_{su}^{I\,I'}\, \text{Im}\, T^{I'}(s',t)}{(s' - u)} \right)$$

Fit → In

DR → Out

$$\tilde{t}_0^0(s) = a_0^0 + \frac{1}{12m_\pi^2}\left(2a_0^0 - 5a_0^2\right)\left(s - 4m_\pi^2\right) + \sum_{I',\ell'} \int_{4m_\pi^2}^{\infty} ds' K_{0\ell'}^{0I'}(s',s)\, \mathrm{Im}\, t_{\ell'}^{I'}(s')$$

"Model-independent"

- [x] **Obtain your DRs** ⟷ **Crossing+analyticity**
- [ ] **Use all PWs available** ⟷ **Necessary Input**
- [ ] **Make** *Fit → In* *DR → Out* **compatible** ⟷ **Unitarity**

Use all PWs available

Scalar $\ell = 0$ waves dominate the DRs

But we extracted/fitted several more waves

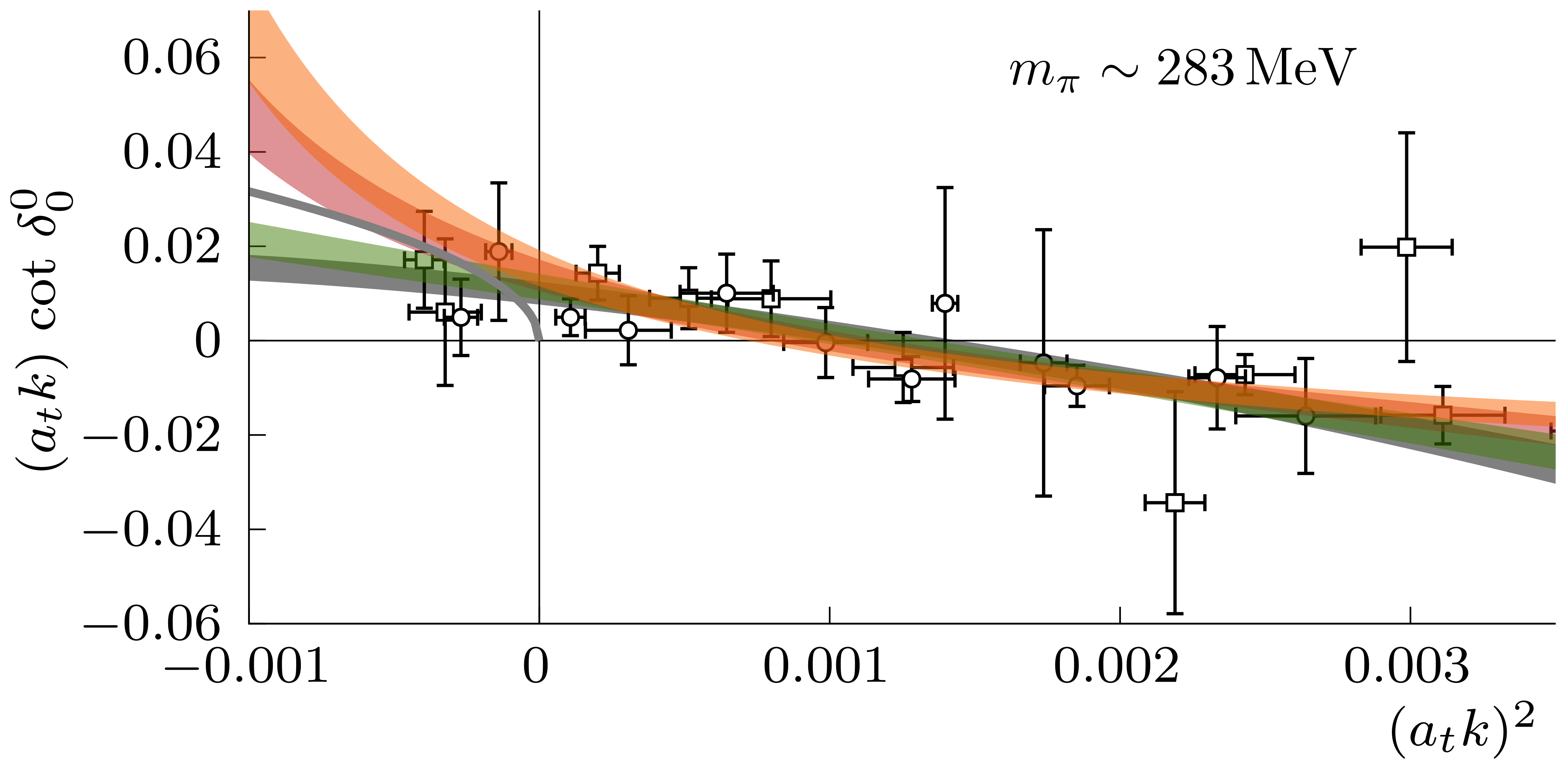
Every band is a different model fit

$$t^I_\ell(s) = \frac{1}{\rho(s)} \frac{1}{\cot \delta^I_\ell(s) - i}$$

Large SL spreads at threshold

$$k \cot \delta^I_0(s) \sim 1/a^I_0$$

$\ell = 0,\ I = 0\ \pi\pi$



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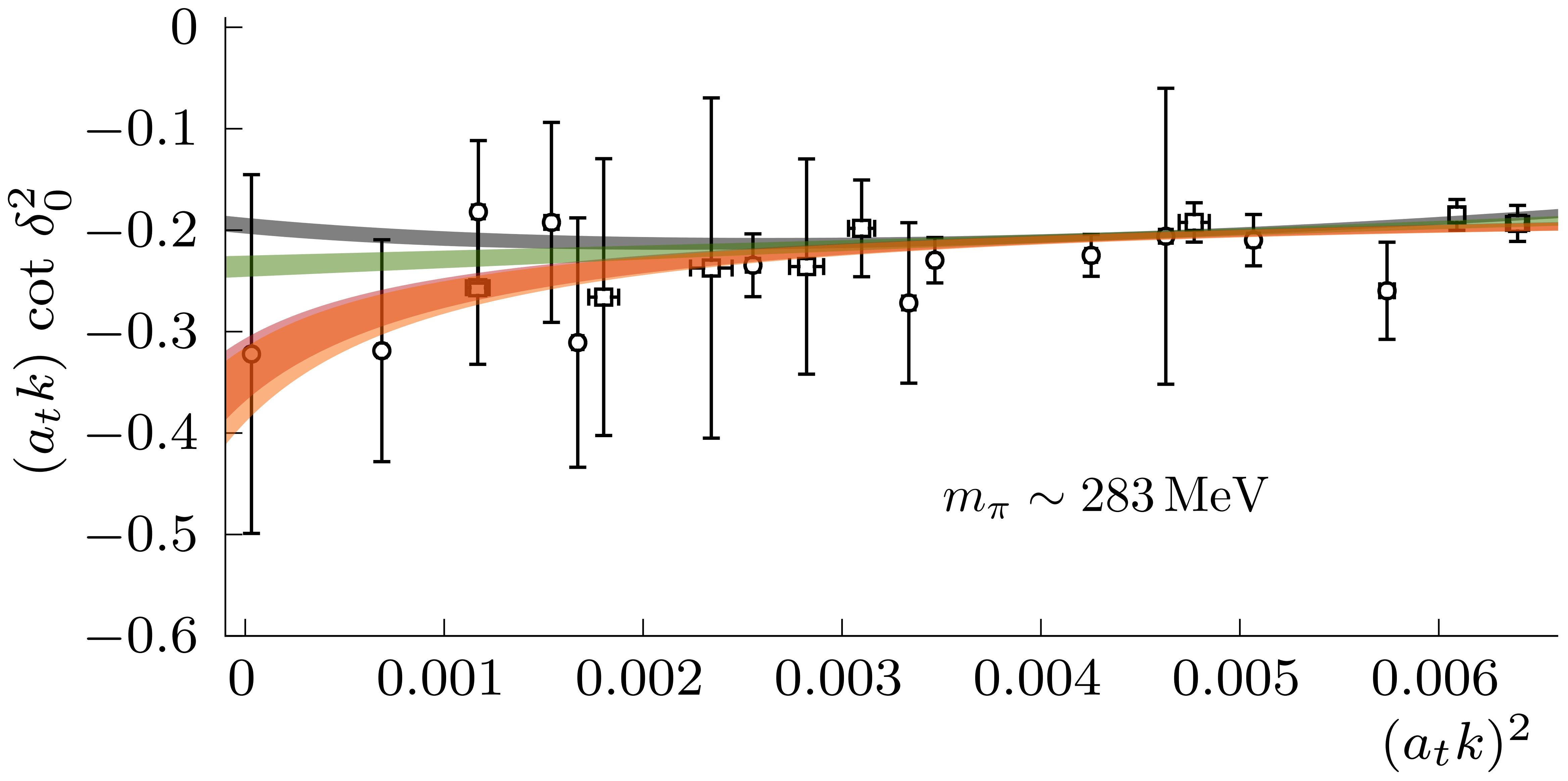
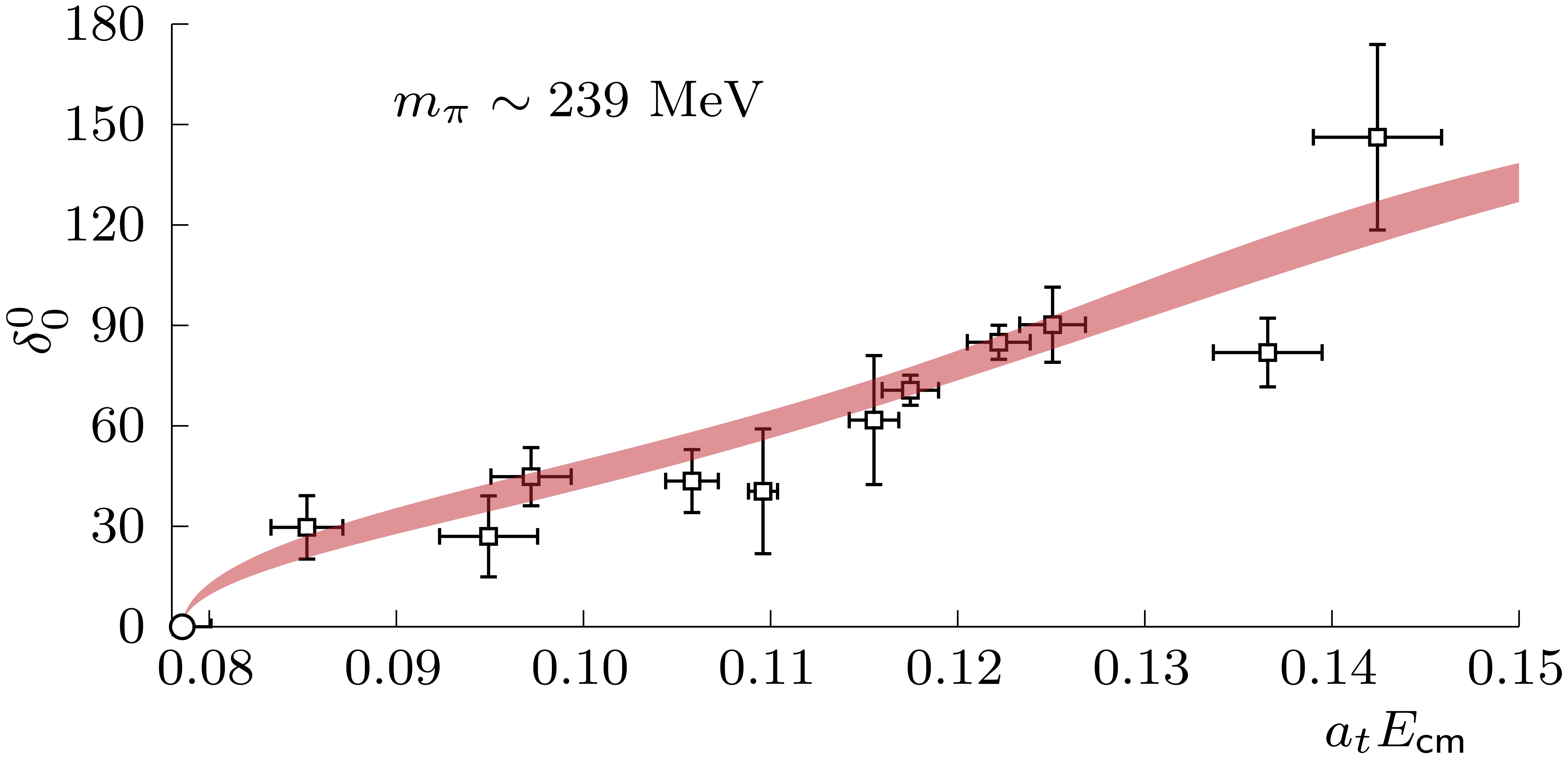
$\ell = 0,\ I = 2\ \pi\pi$



Model 1

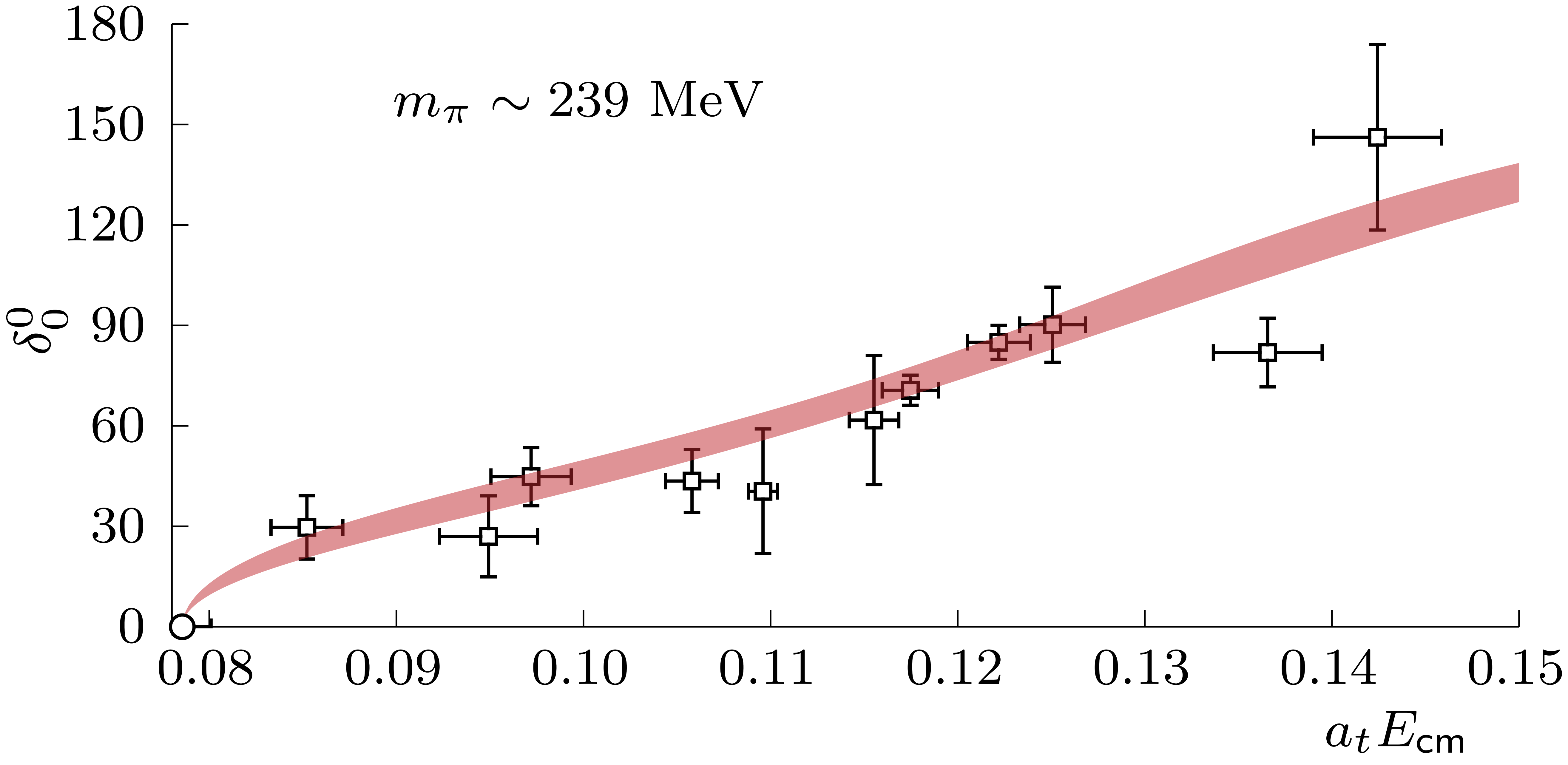


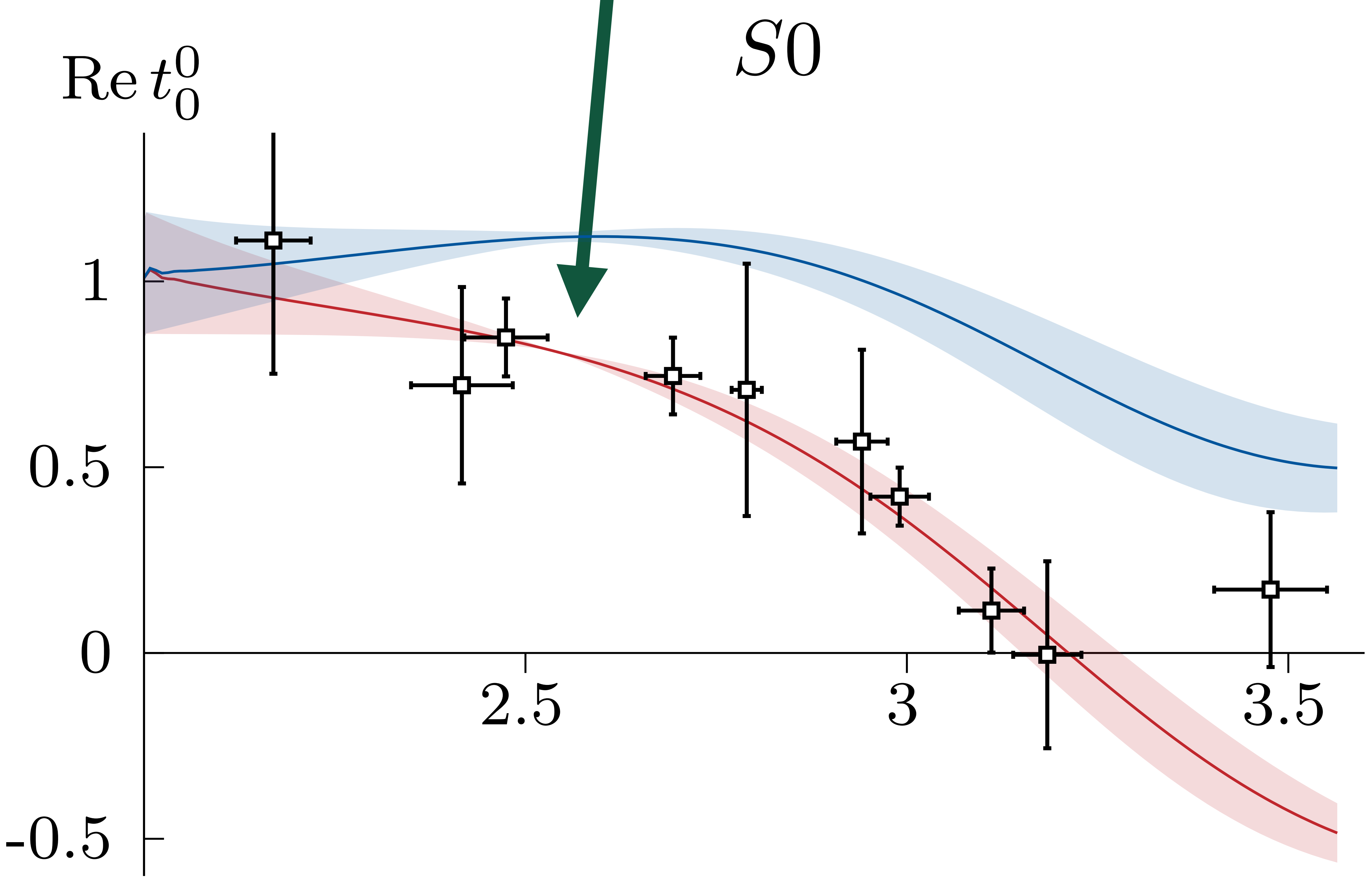
$m_\pi \sim 239$ MeV
δ_0^0
180
150
120
90
60
30
0
0.08
0.09
0.10
0.11
0.12
0.13
0.14
0.15
$a_t E_{\mathrm{cm}}$

Make *Fit → In* *DR → Out* compatible

Model 1

$$a_0^0 + \frac{1}{12m_\pi^2}\left(2a_0^0 - 5a_0^2\right)(s - 4m_\pi^2) + \sum_{I',\ell'} \int_{4m_\pi^2}^{\infty} ds' K_{0\ell'}^{0I'}(s',s)\,\mathrm{Im}\, t_{\ell'}^{I'}(s') = \tilde{t}_0^0(s)$$

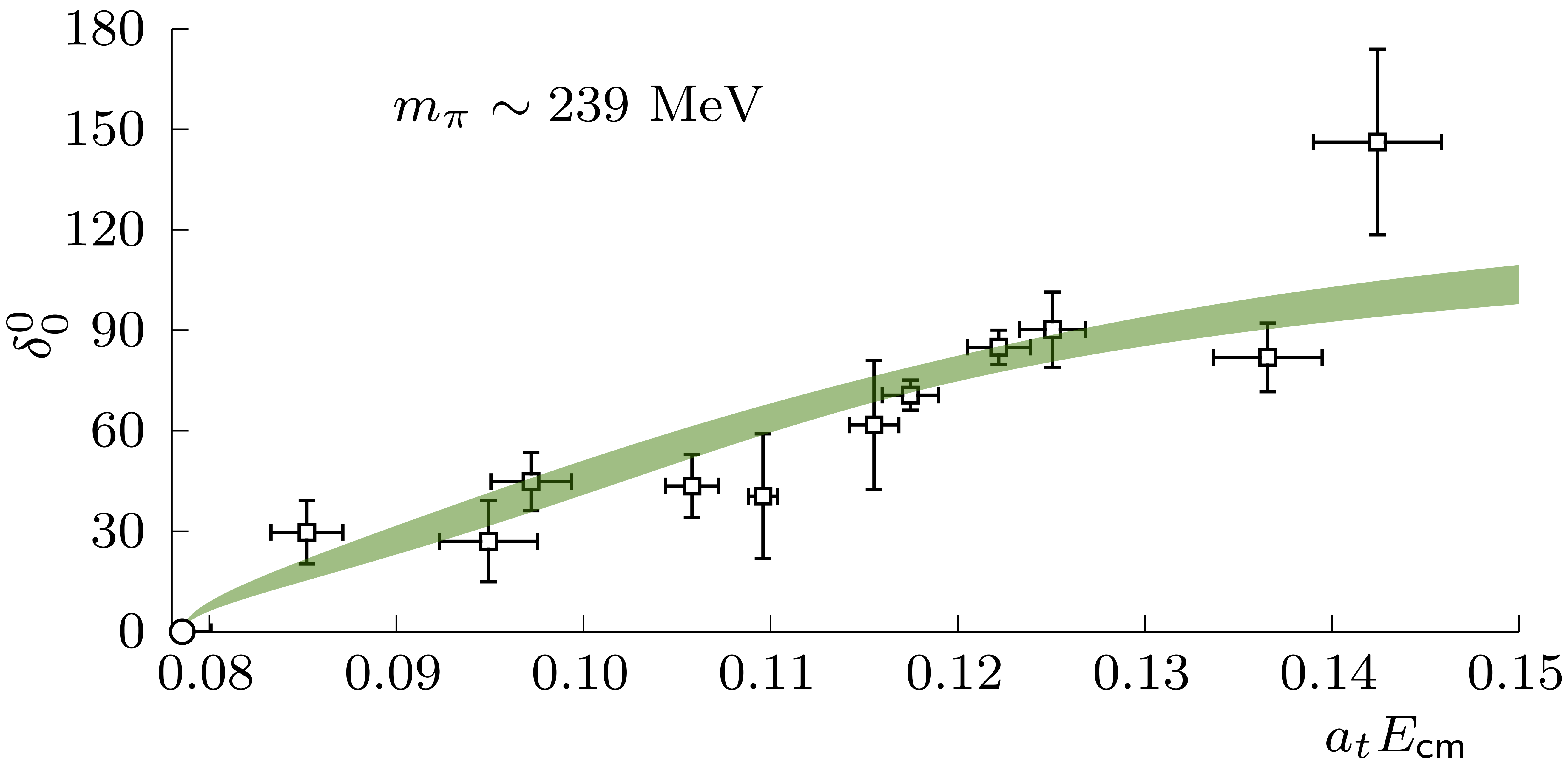


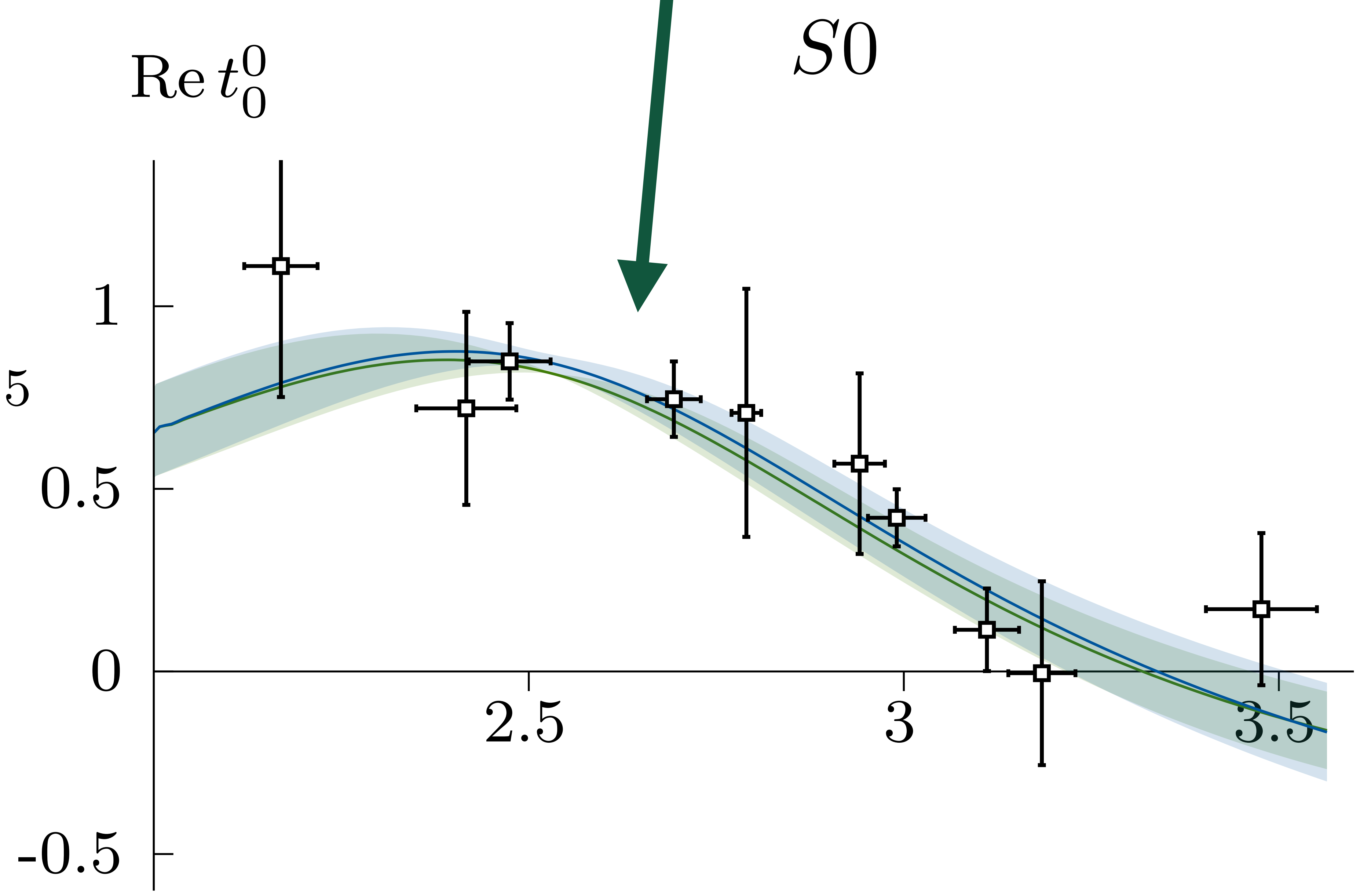


Make *Fit → In* *DR → Out* compatible

Model 2

$$a_0^0 + \frac{1}{12m_\pi^2}(2a_0^0 - 5a_0^2)(s-4m_\pi^2) + \sum_{I',\ell'}\int_{4m_\pi^2}^\infty ds' K_{0\ell'}^{0I'}(s',s)\,\mathrm{Im}\,t_{\ell'}^{I'}(s') = \tilde{t}_0^0(s)$$





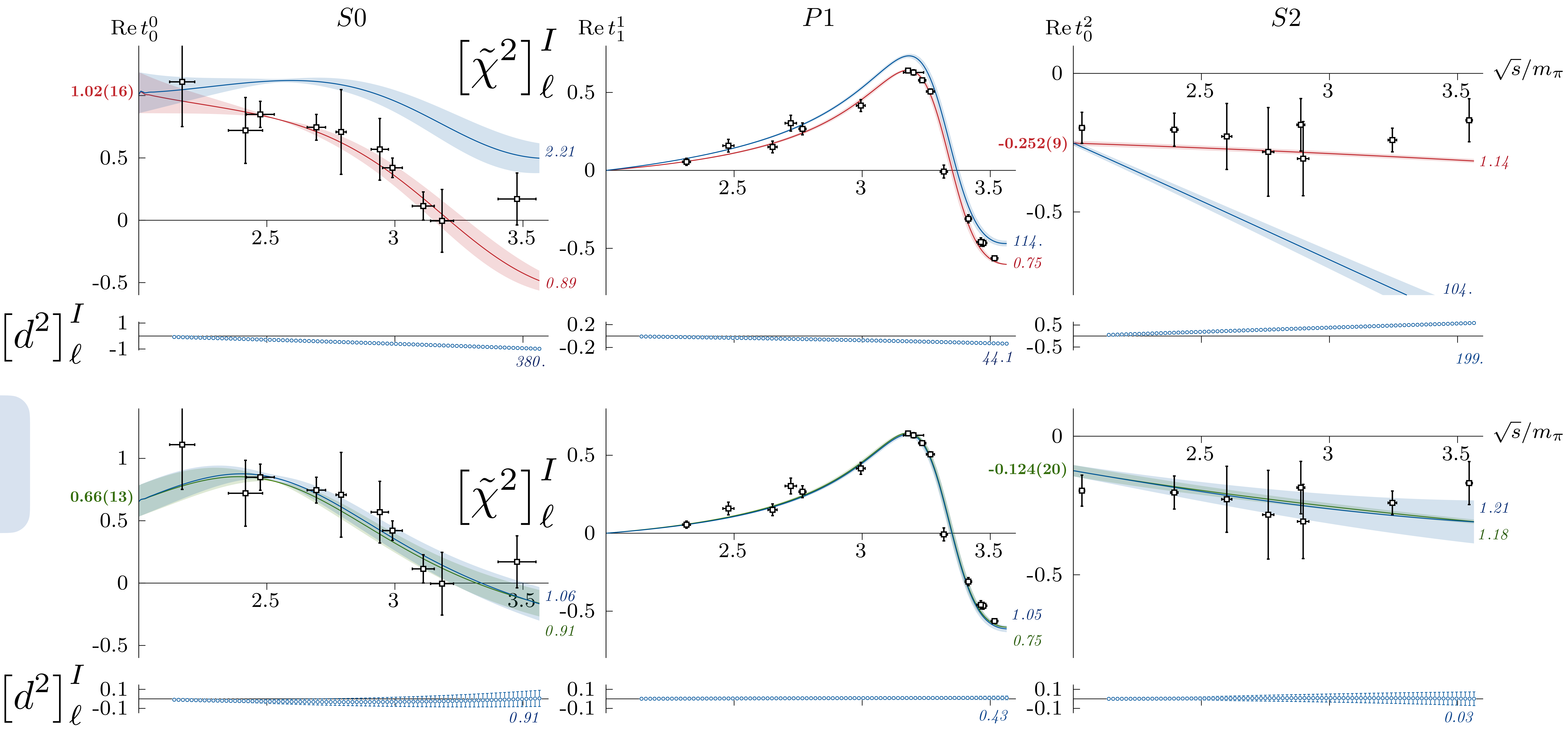
Tests: good vs bad

$m_\pi \sim 239\,\mathrm{MeV}$

Fit combination 1

Dispersive output

Fit combination 2



We select those models that respect the DRs

Scattering plane

$\langle d^2/N_{\rm smpl}\rangle_{\rm pw}$ $\langle \tilde{\chi}^2/N_{\rm lat}\rangle_{\rm pw}$

Black **ROY**

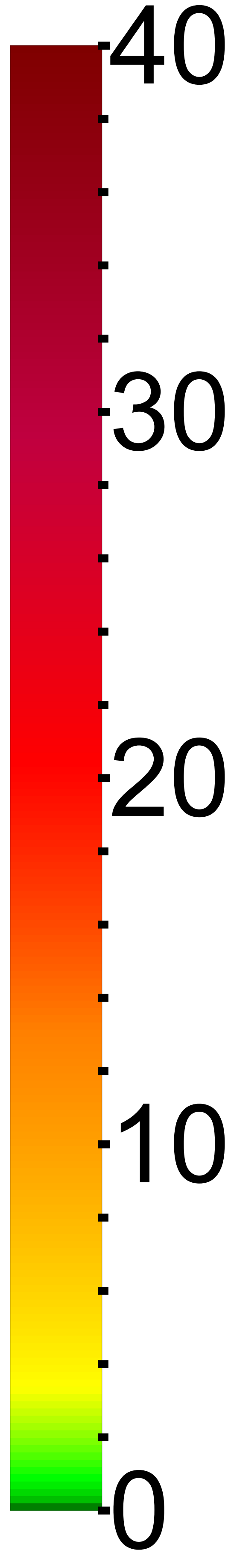
$d^2/N_{\rm smpl} < 1, \quad \tilde{\chi}^2/N_{\rm lat} < 2$

$N_I \ell_{\rm max} N_{\rm params} \sim 300 - 400$

$m_\pi \sim 239\,{\rm MeV}$

S0: 0.49(9), 0.54(13), 0.58(10), 0.58(11), 0.61(12), 0.63(12), 0.64(12), 0.64(12), 0.64(12), 0.66(13), 0.70(10), 0.75(16), 0.76(16), 0.77(17), 0.78(16), 0.78(20), 0.80(16), 0.82(13), 0.85(12), 0.96(15), 1.01(15), 1.02(16), 1.09(17), 1.09(17)

S2: −0.081(6), −0.090(7), −0.094(7), −0.121(22), −0.122(16), −0.124(20), −0.126(20), −0.130(16), −0.134(16), −0.163(14), −0.179(8), −0.194(4), −0.252(9)



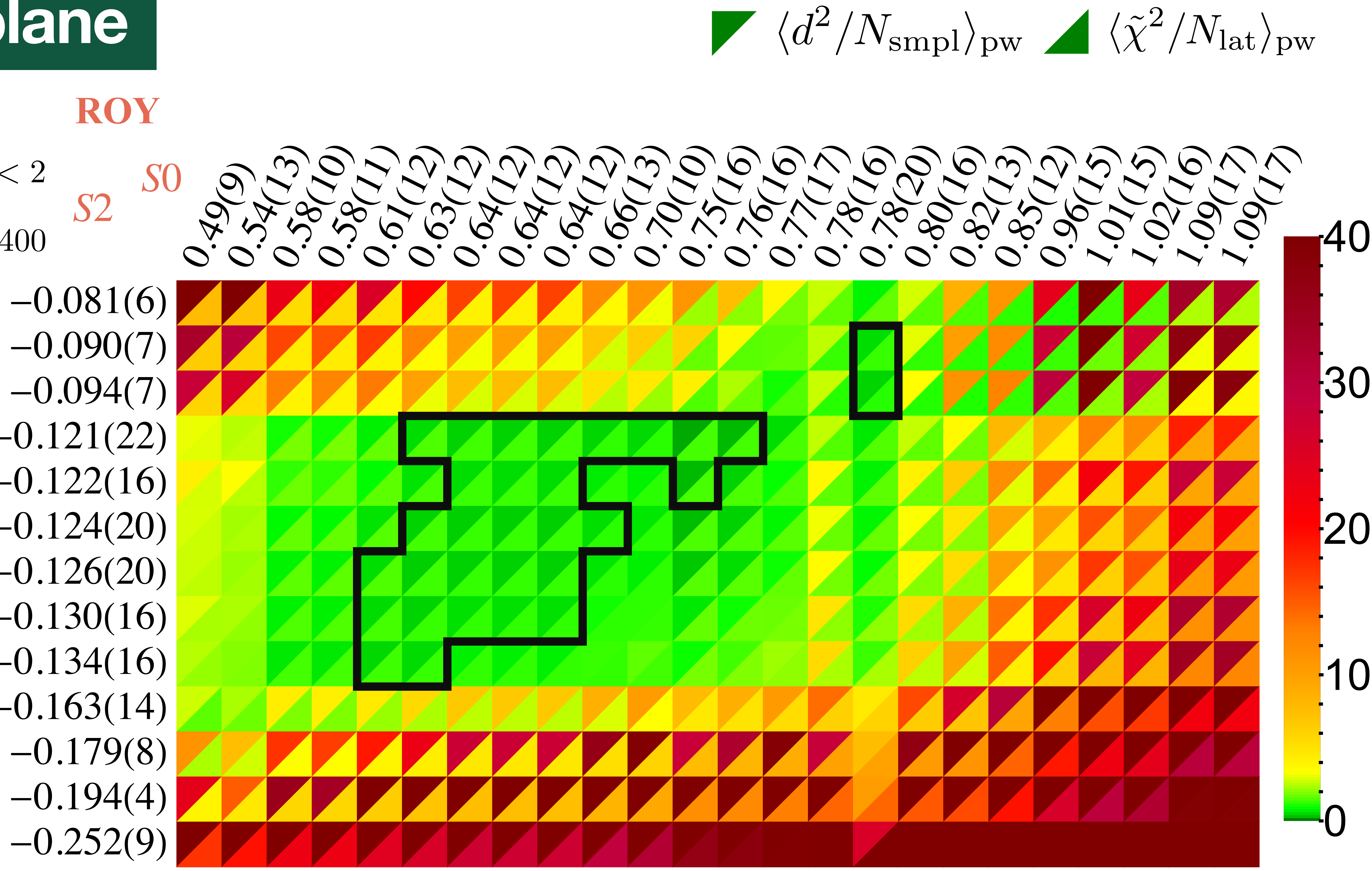
Scattering plane

Black ROY

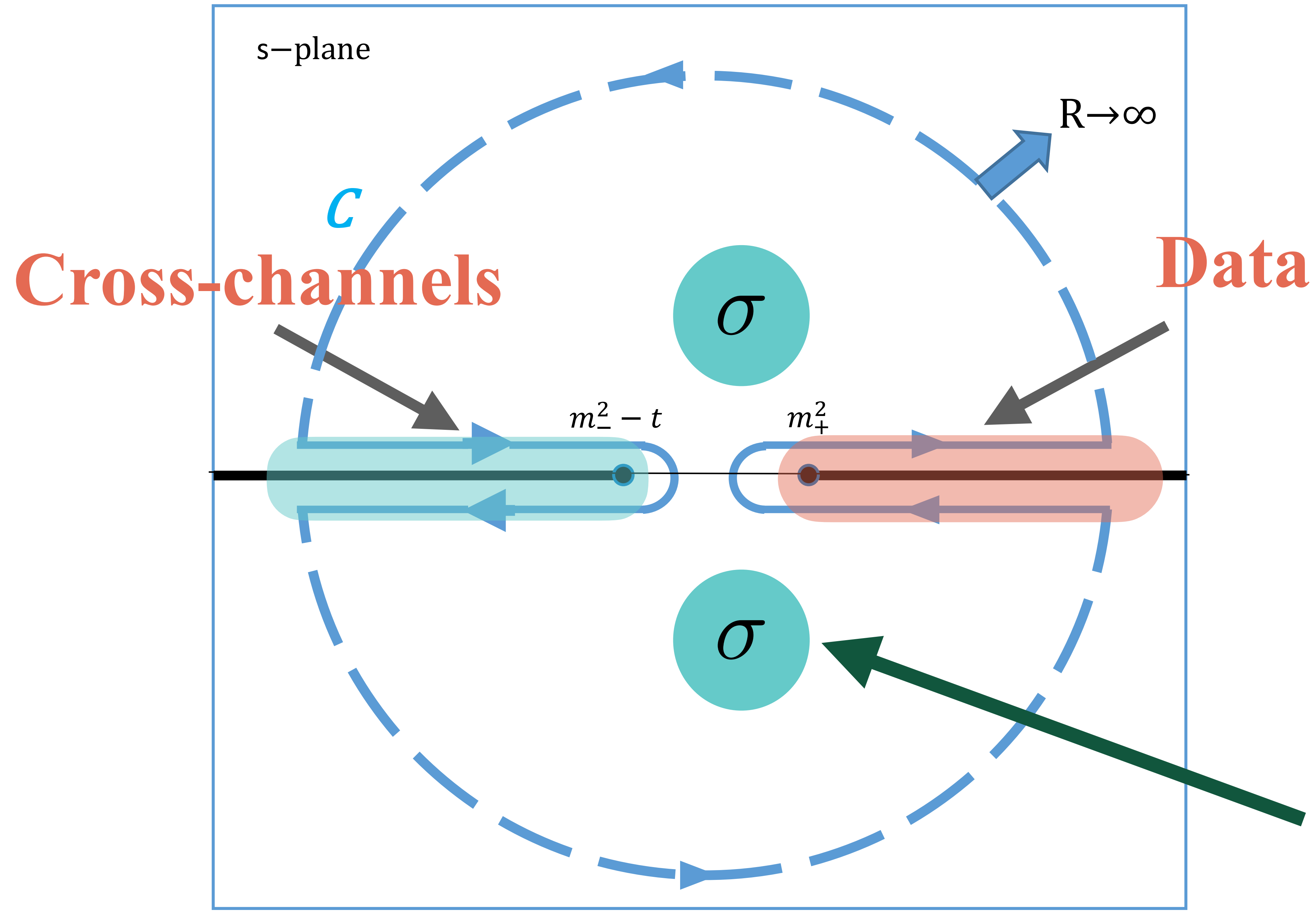
$d^2/N_{\rm smpl} < 1, \quad \tilde{\chi}^2/N_{\rm lat} < 2$

$N_I \ell_{\max} N_{\rm params} \sim 300 - 400$

$m_\pi \sim 239\,\mathrm{MeV}$



Outside the physical region

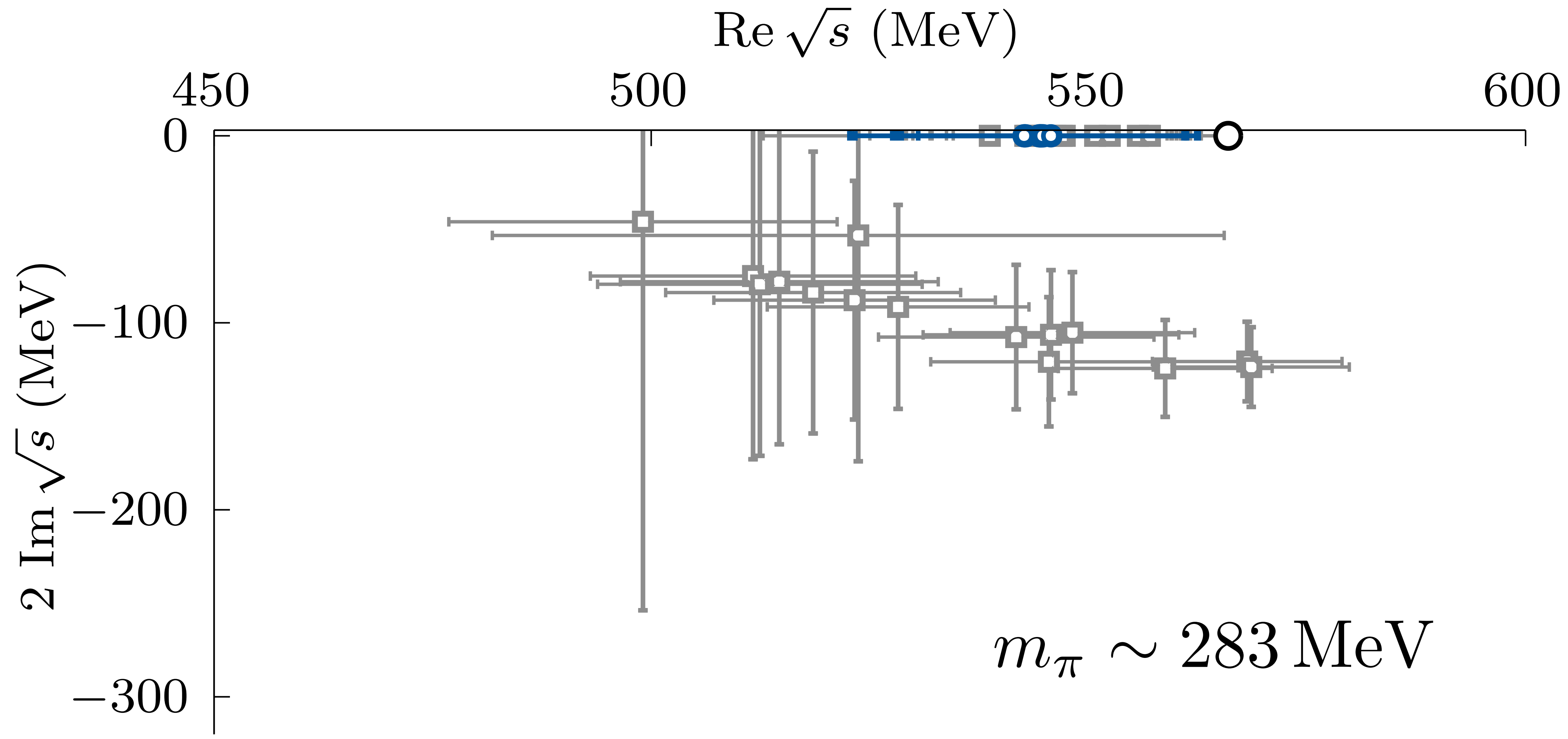


Both sides are good now

What happens everywhere else??

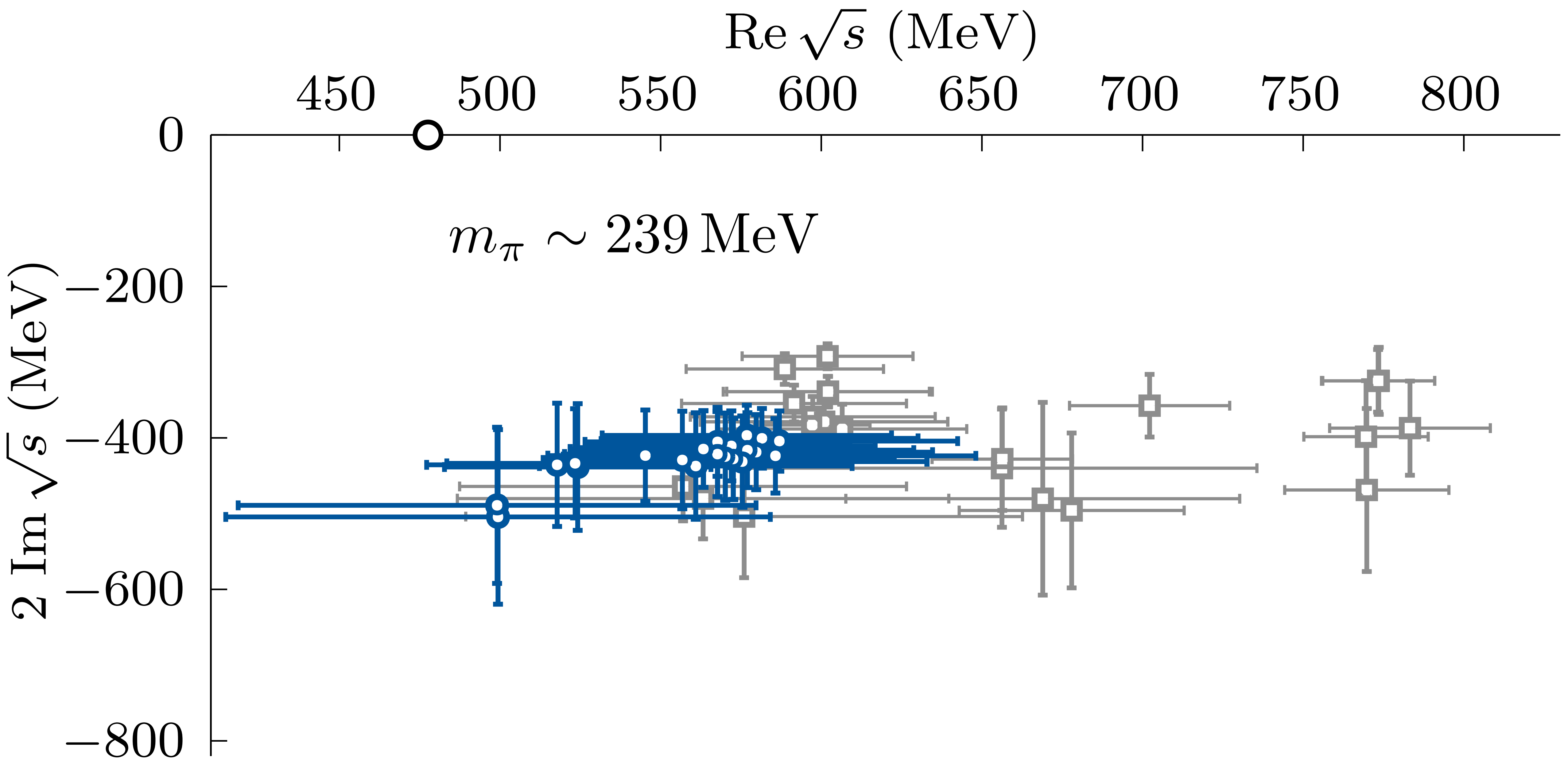
What happens here??

Dispersive σ



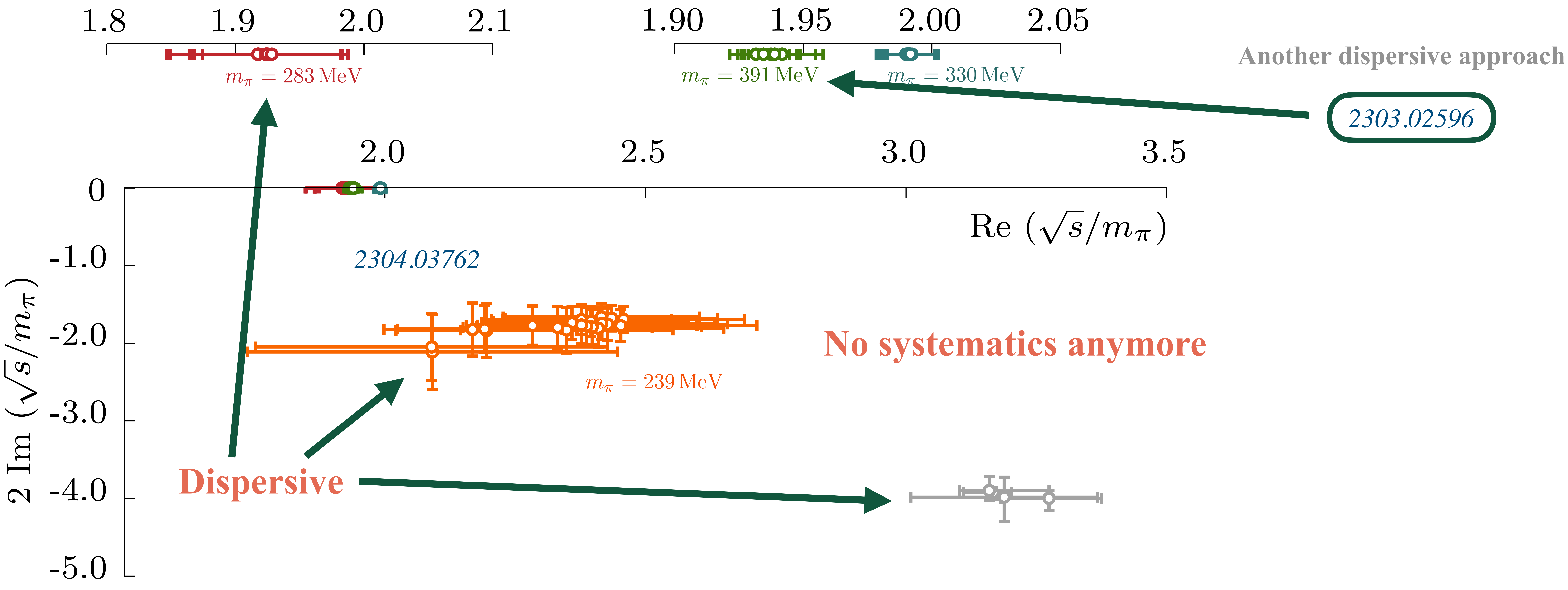
No tension anymore

2304.03762



We traded statistical uncertainty increase by large systematic reduction

Dispersive σ



We traded statistical uncertainty increase by large systematic reduction

Summary and outlook

First-principles extraction of a broad resonance directly from QCD

The lighter the π, the more relevant this approach is

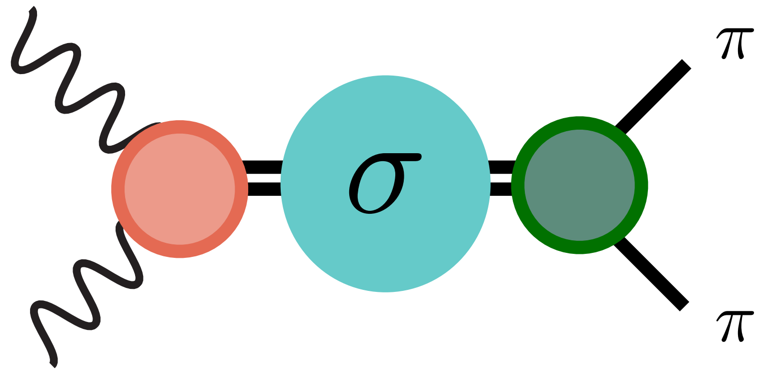
Better constraints over scattering lengths

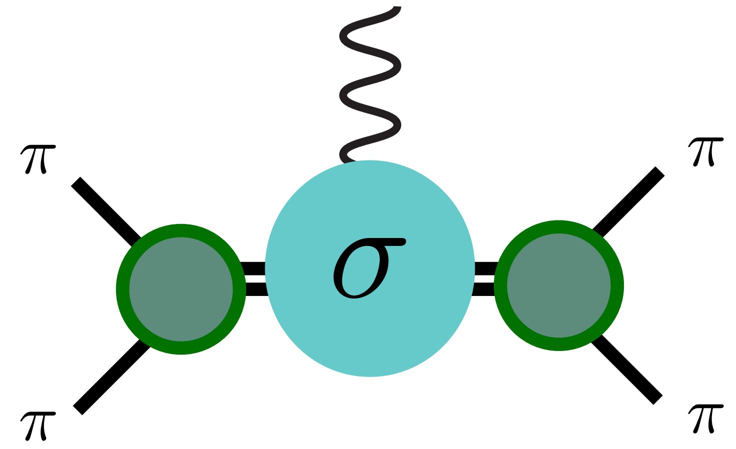
Future

Include second, larger volume for the lighter pion mass

Extract the $f_0(980)$??

Study new observables ??

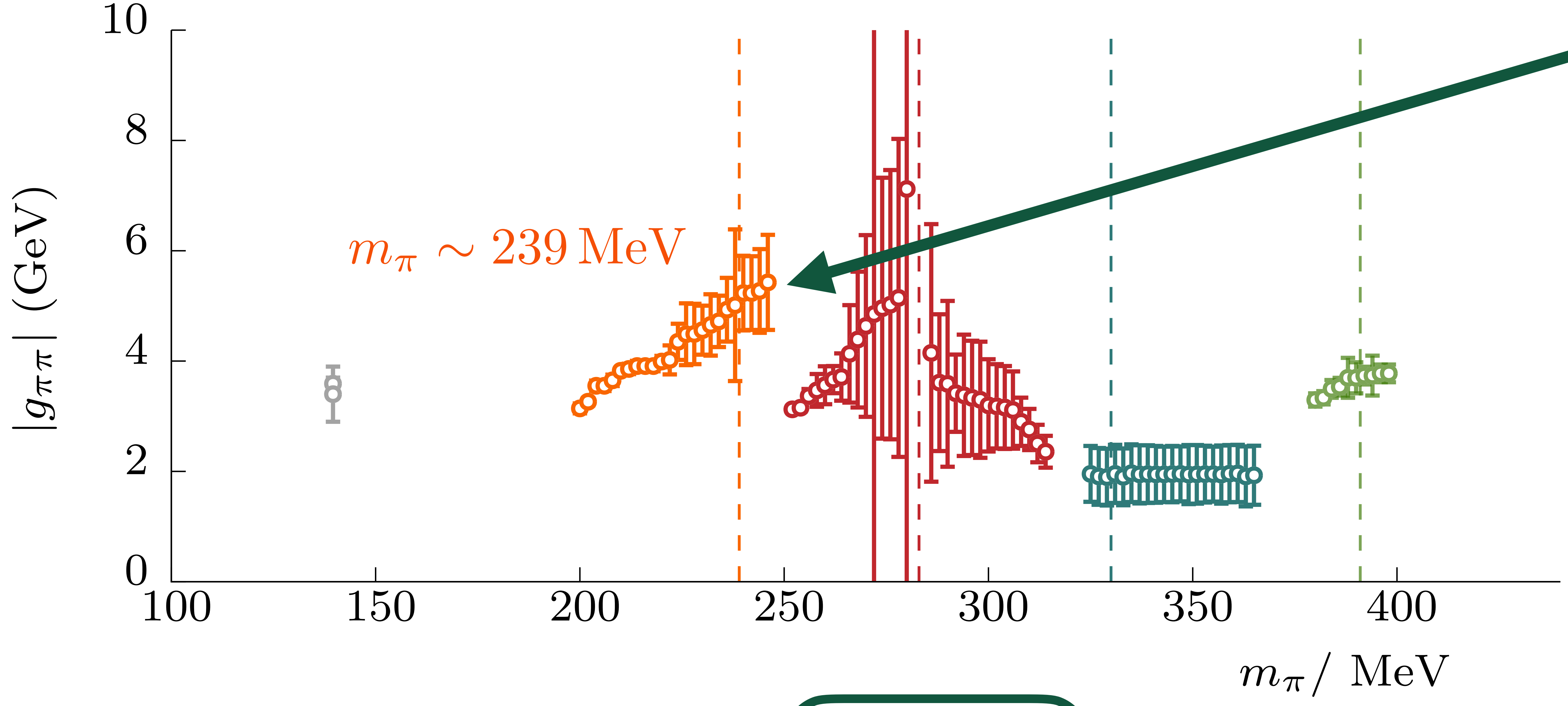




Thank you!!

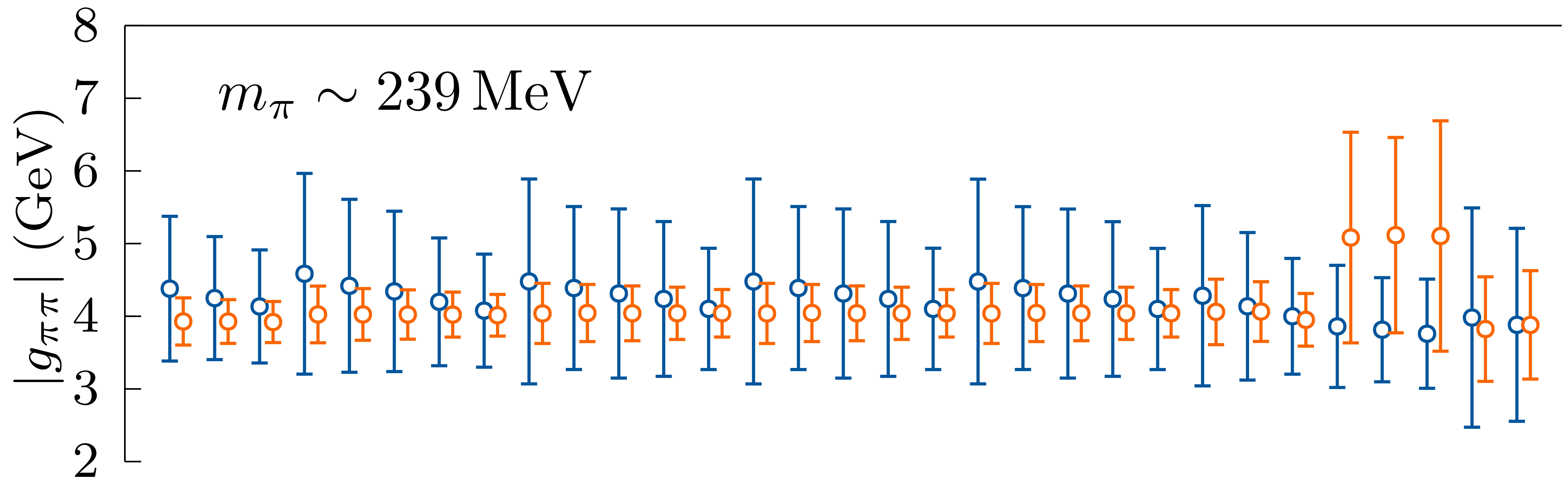
Couplings

Models: Large systematic dependence



DRs: No systematics

2304.03762

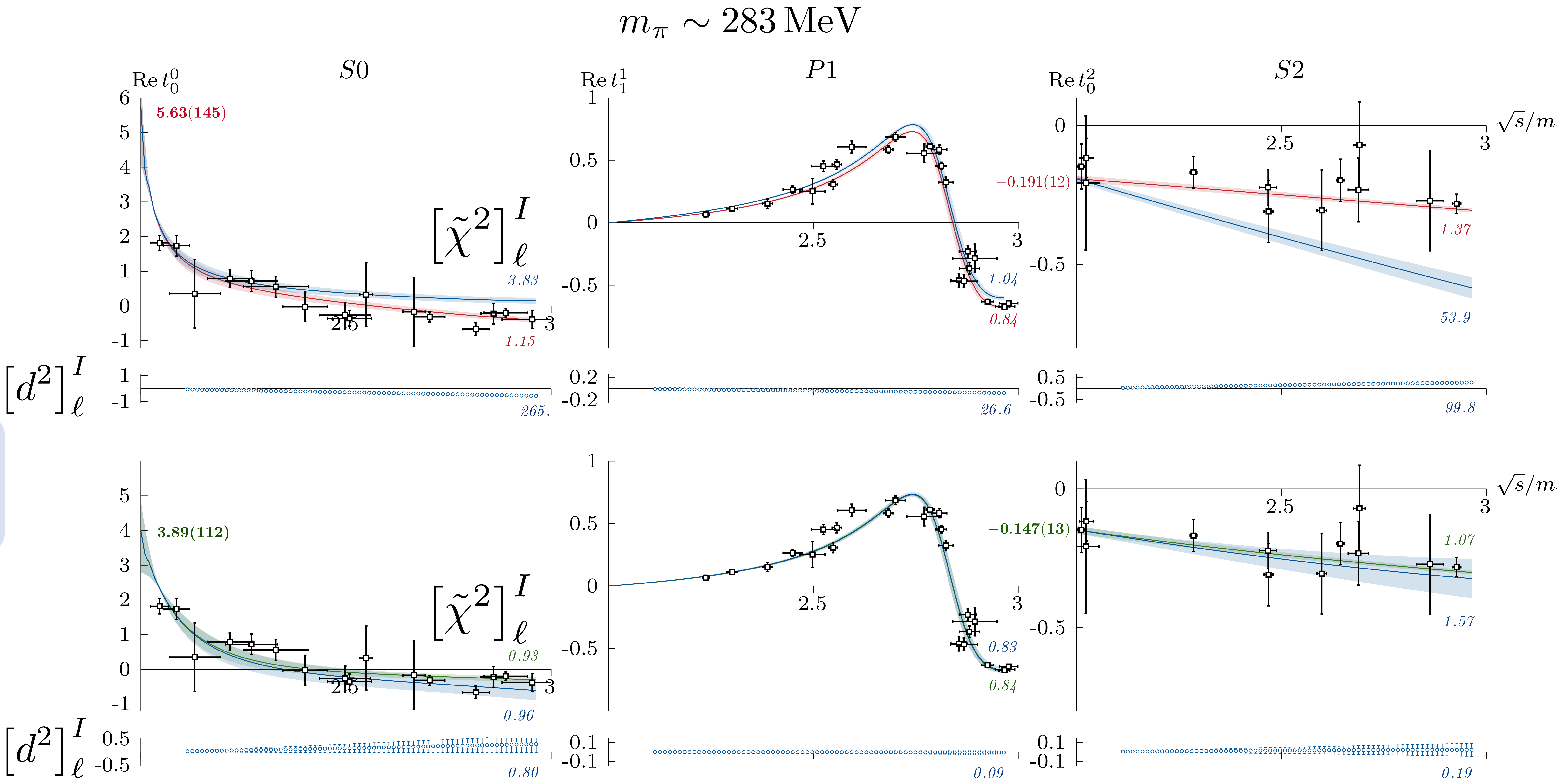


Tests: good vs bad

Fit combination 1

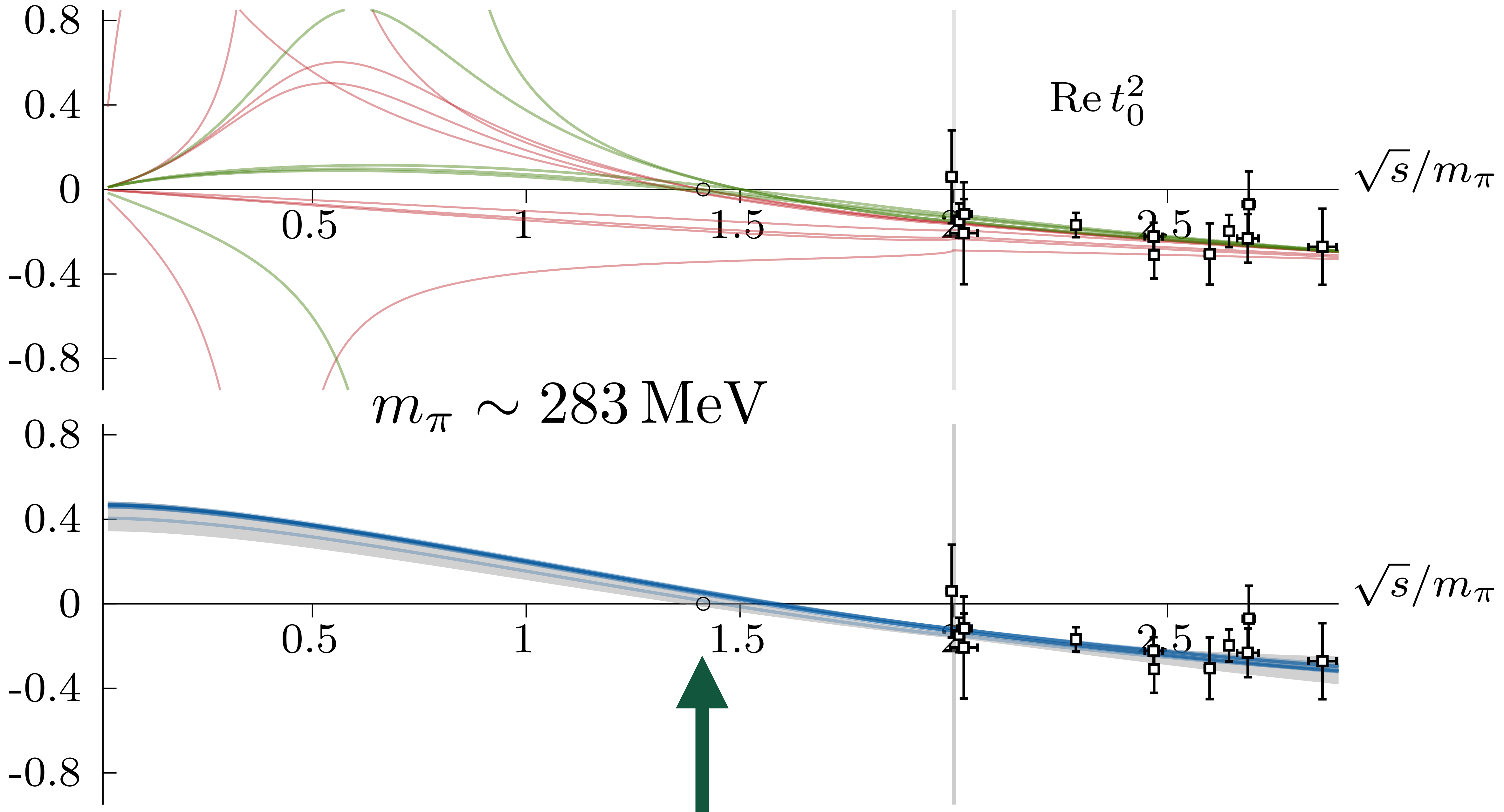
Dispersive output

Fit combination 2

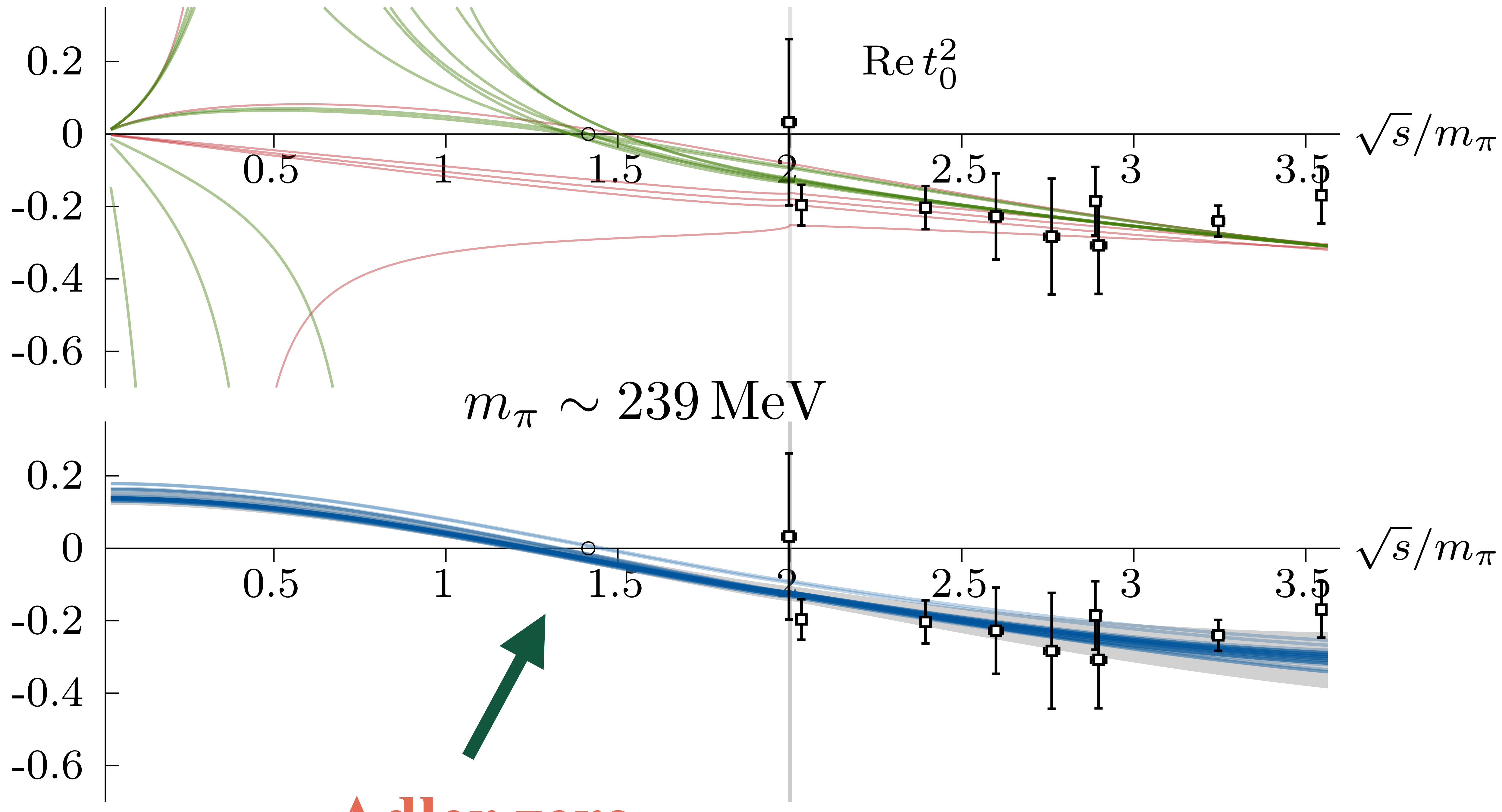


Sub-threshold

Very “stable” for $I = 2\ \pi\pi$



Adler zero

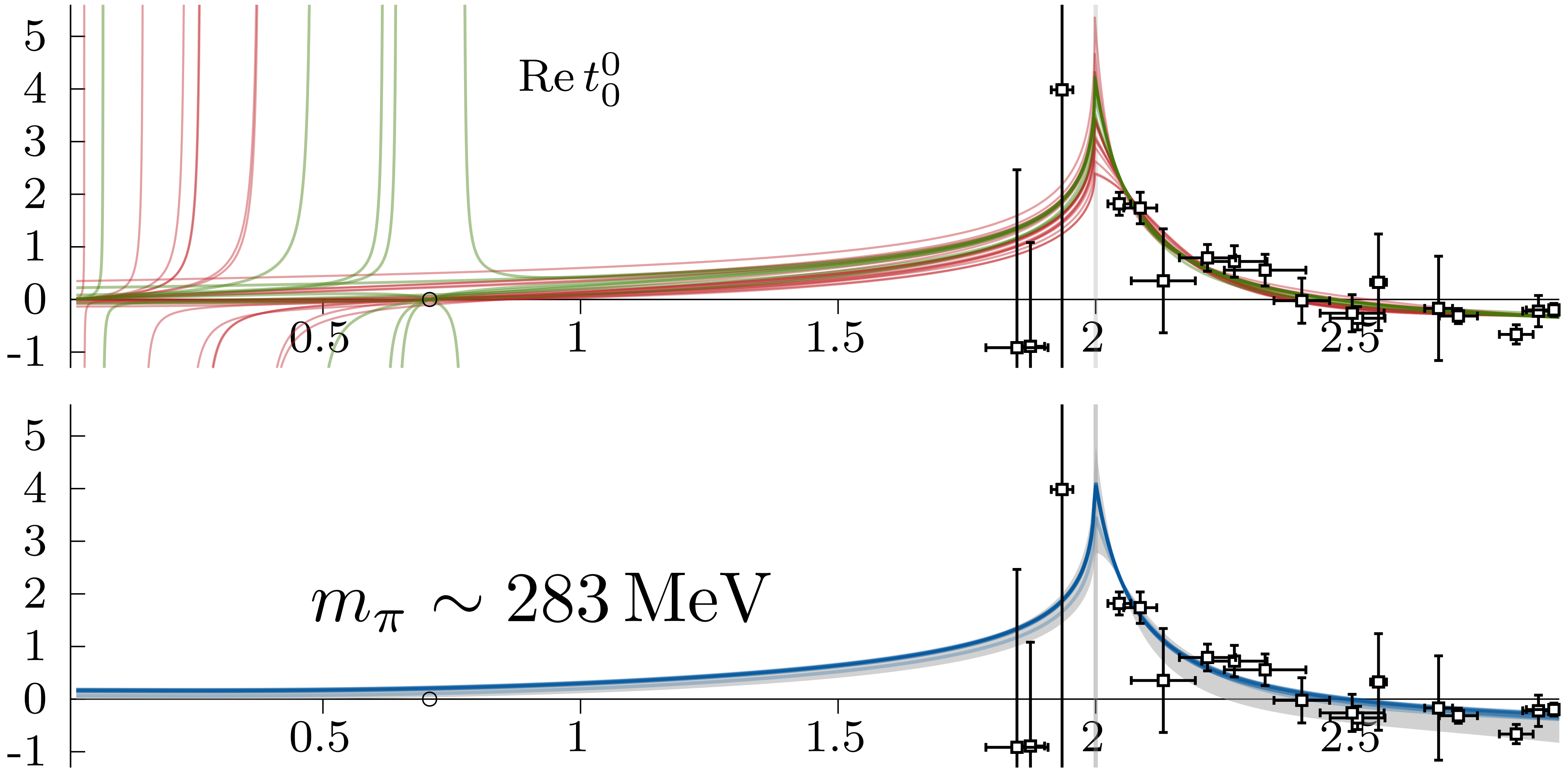


Adler zero

Even “bad” DRs produce Adler zeroes for I=2

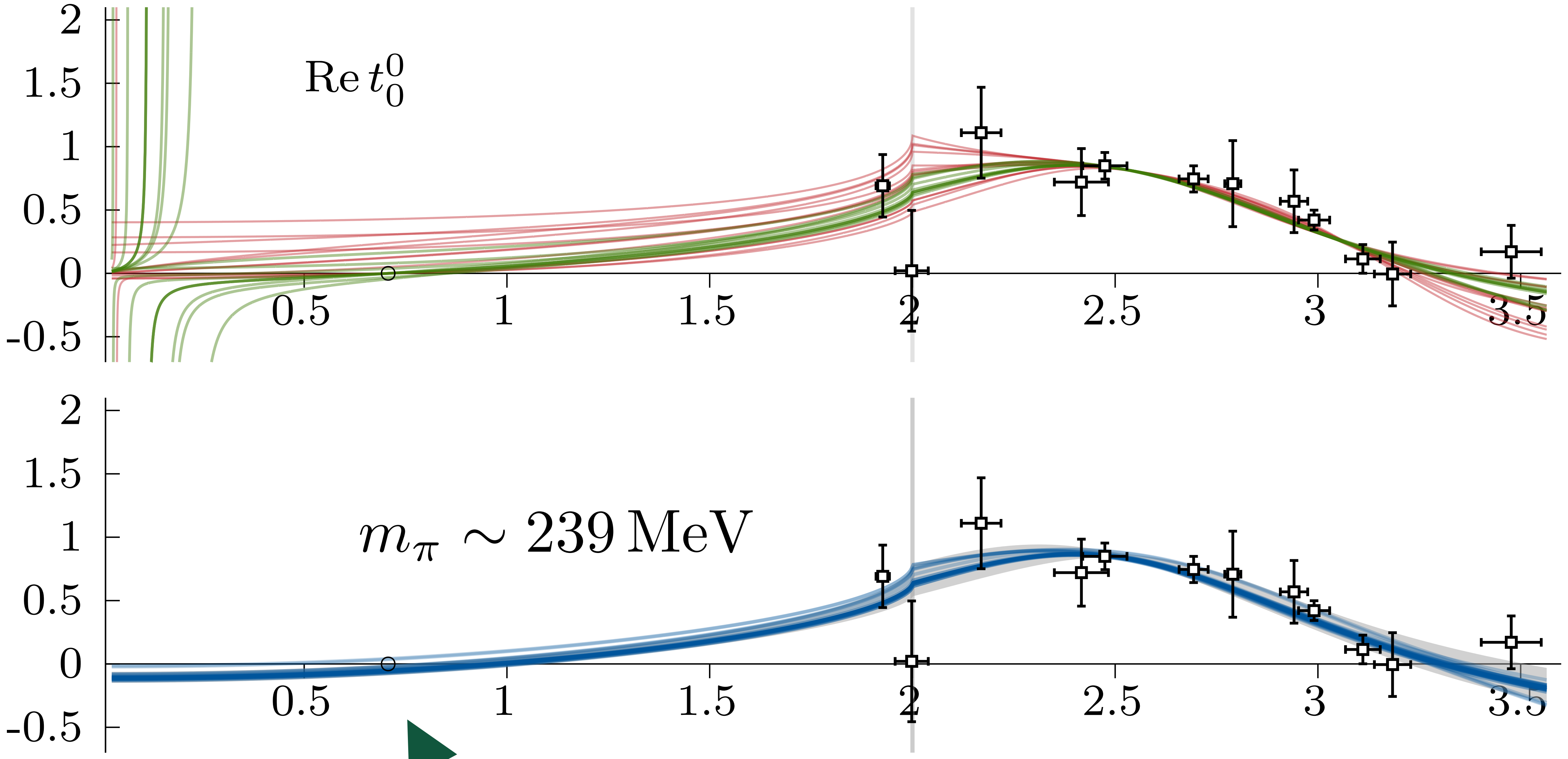
Sub-threshold

No good DR produces an $I = 0$ $\pi\pi$ Adler zero
for the heavier mass



NO Adler zero

All good DRs produce an $I = 0$ $\pi\pi$ Adler zero
for the lighter mass



Adler zero

Make *Fit → In* *DR → Out* compatible ⟷ Unitarity

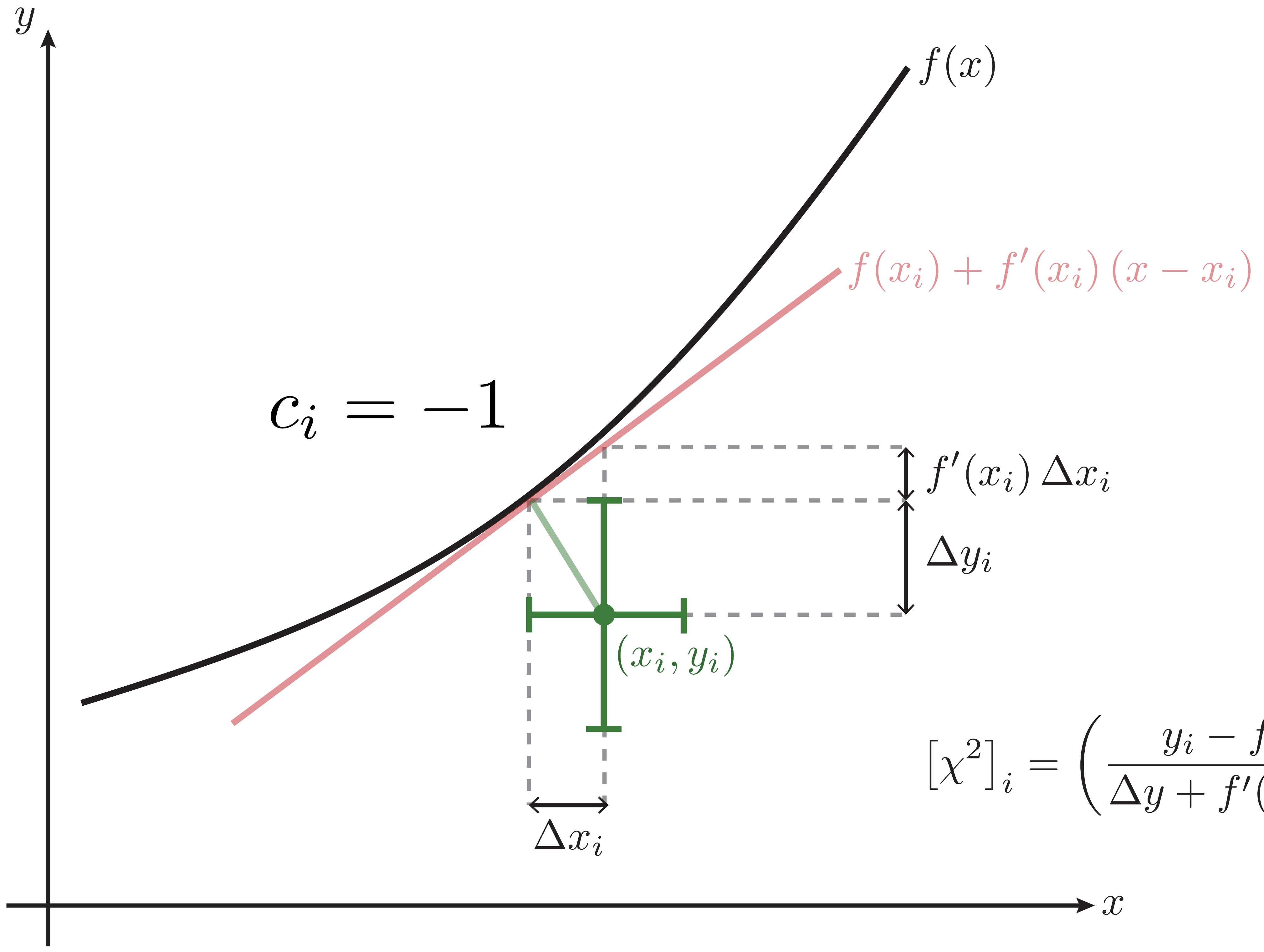
$$\left[d^2\right]_\ell^I \equiv \sum_{i=1}^{N_{\rm smpl}} \left(\frac{\mathrm{Re}\,\tilde{t}_\ell^I(s_i) - \mathrm{Re}\,t_\ell^I(s_i)}{\Delta\left[\mathrm{Re}\,\tilde{t}_\ell^I(s_i) - \mathrm{Re}\,t_\ell^I(s_i)\right]} \right)^2$$

Make *DR → Out* and data compatible ⟷ Lattice QCD data description

$$\left[\tilde{\chi}^2\right]_\ell^I \equiv \sum_{i,j=1}^{N_{\rm lat}} \left(\frac{\mathfrak{f}_i - \mathrm{Re}\,\tilde{t}_\ell^I(s_i)}{\Delta_i} \right) \mathrm{corr}\left(\mathfrak{f}_i, \mathfrak{f}_j\right) \left(\frac{\mathfrak{f}_j - \mathrm{Re}\,\tilde{t}_\ell^I(s_j)}{\Delta_j} \right)$$

Make $DR \to Out$ and data compatible ⟷ Lattice QCD data description

$$[\tilde{\chi}^2]^I_\ell \equiv \sum_{i,j=1}^{N_{\rm lat}} \left(\frac{\mathfrak{f}_i - {\rm Re}\,\tilde{t}^I_\ell(s_i)}{\Delta_i}\right) {\rm corr}(\mathfrak{f}_i,\mathfrak{f}_j) \left(\frac{\mathfrak{f}_j - {\rm Re}\,\tilde{t}^I_\ell(s_j)}{\Delta_j}\right)$$



$$[\chi^2]_i = \left(\frac{y_i - f(x_i)}{\Delta y + f'(x_i)\Delta x_i}\right)^2$$

$$\Delta_i^2 = \begin{pmatrix} \Delta\mathfrak{f}_i & \frac{d\tilde{f}^I_\ell(s_i)}{dE_i}\Delta E_i \end{pmatrix} \begin{pmatrix} 1 & -c_i \\ -c_i & 1 \end{pmatrix} \begin{pmatrix} \Delta\tilde{f}_i \\ \frac{d\tilde{f}^I_\ell(s_i)}{dE_i}\Delta E_i \end{pmatrix}$$

Partial wave dispersion relations

Amplitudes are decomposed in partial waves

$$T^I(s,t) = 32\pi \sum_\ell (2\ell+1) P_\ell(\cos\theta) t_\ell^I(s)$$

Fit to the data

Fit → In

DR → Out

Dispersive's result

$$\tilde{t}_\ell^I(s) = \tau_\ell^I(s) + \sum_{I',\ell'} \int_{4m_\pi^2}^{\infty} ds' \, K_{\ell\ell'}^{II'}(s',s) \, \mathrm{Im}\, t_{\ell'}^{I'}(s')$$

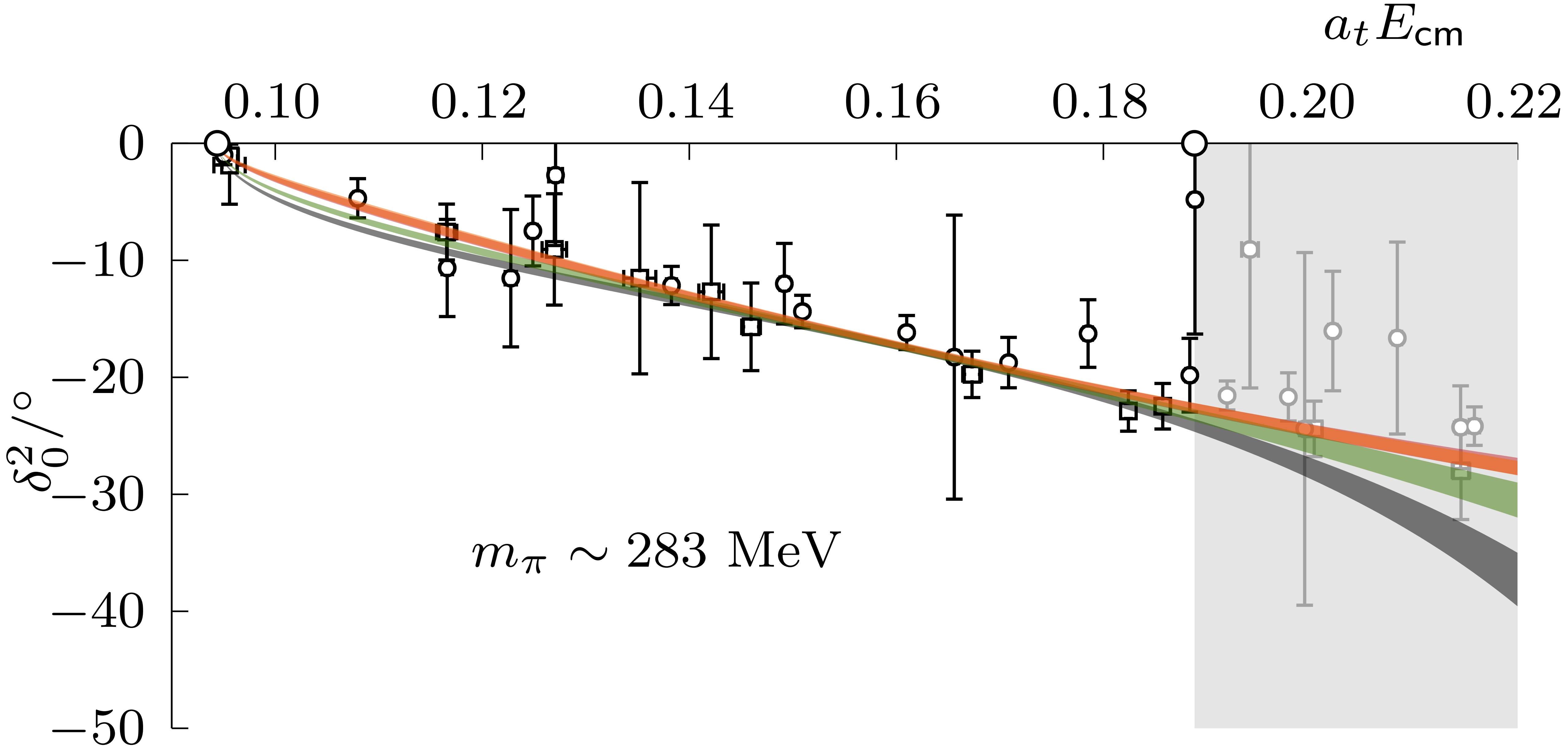
The most well-known are the ROY eqs., for example, for $I = \ell = 0$ they look

Roy PLB (1971)

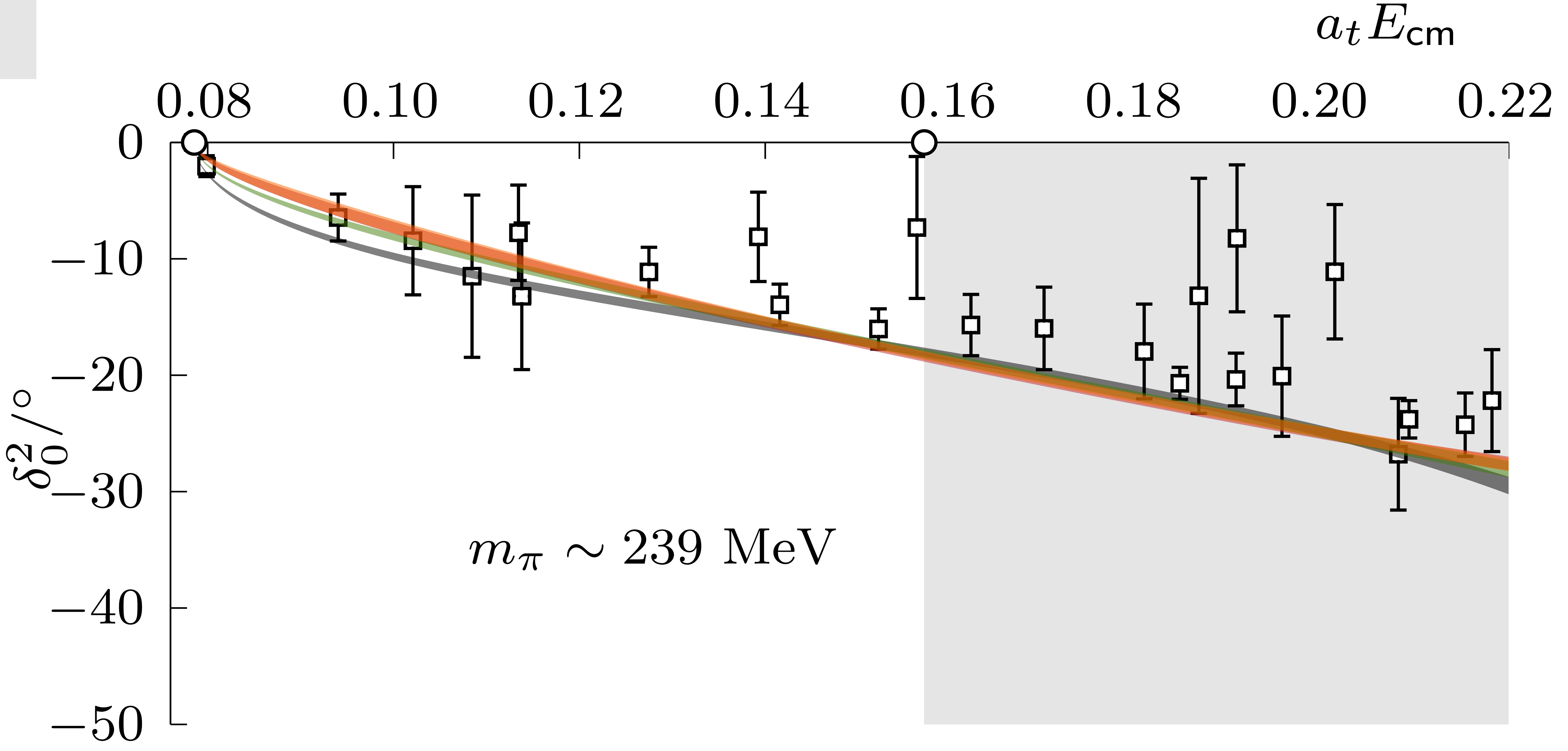
$$\tilde{t}_0^0(s) = a_0^0 + \frac{1}{12m_\pi^2}\left(2a_0^0 - 5a_0^2\right)\left(s - 4m_\pi^2\right) + \sum_{I',\ell'} \int_{4m_\pi^2}^{\infty} ds' \, K_{0\ell'}^{0I'}(s',s) \, \mathrm{Im}\, t_{\ell'}^{I'}(s')$$

$I = 2\ \pi\pi$

2303.10701



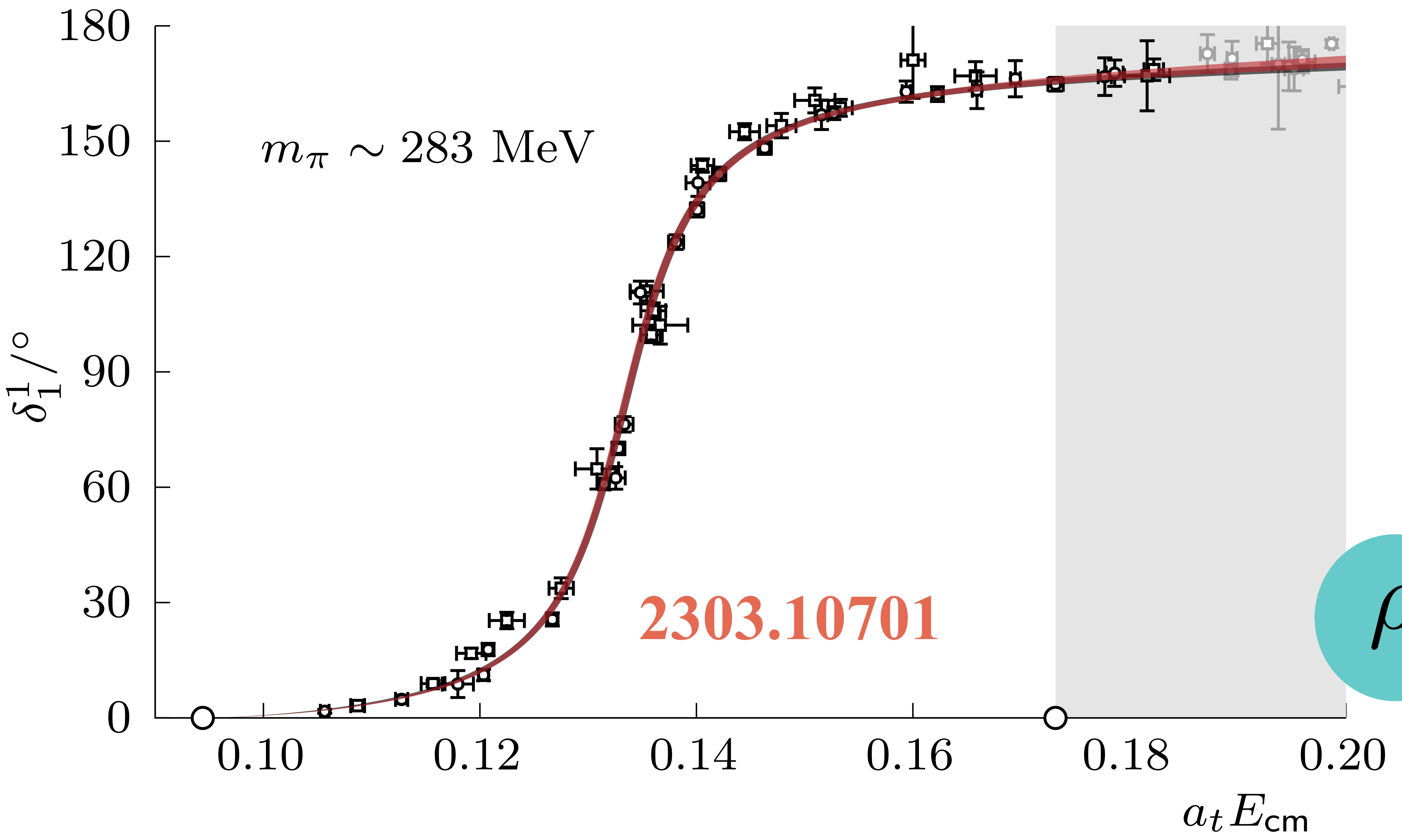
Percent error for $\delta(s)$



10+ parameterizations

Systematic spread at threshold

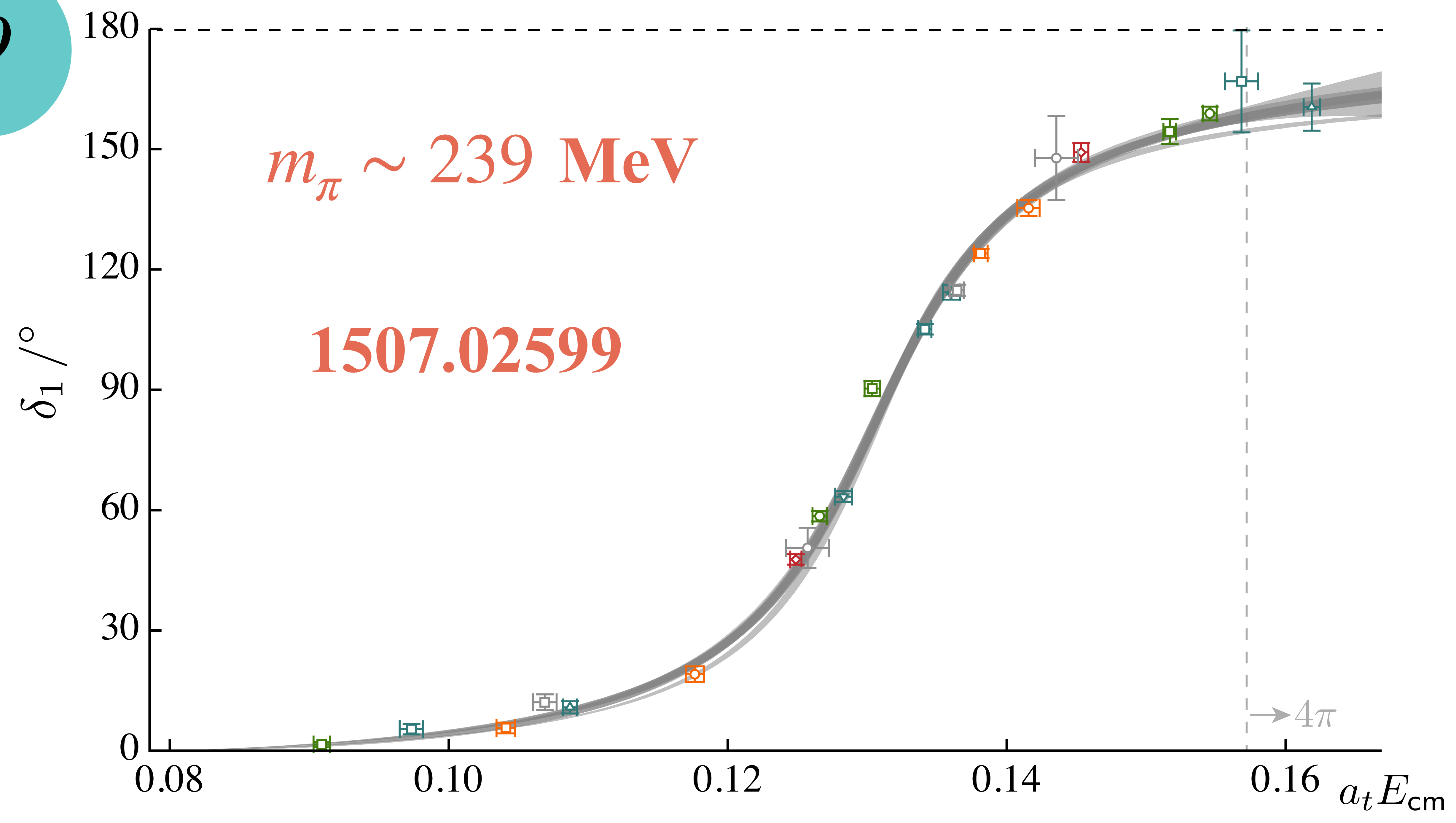
$I = 1\ \pi\pi$



2303.10701

Percent error for $\delta(s)$

ρ



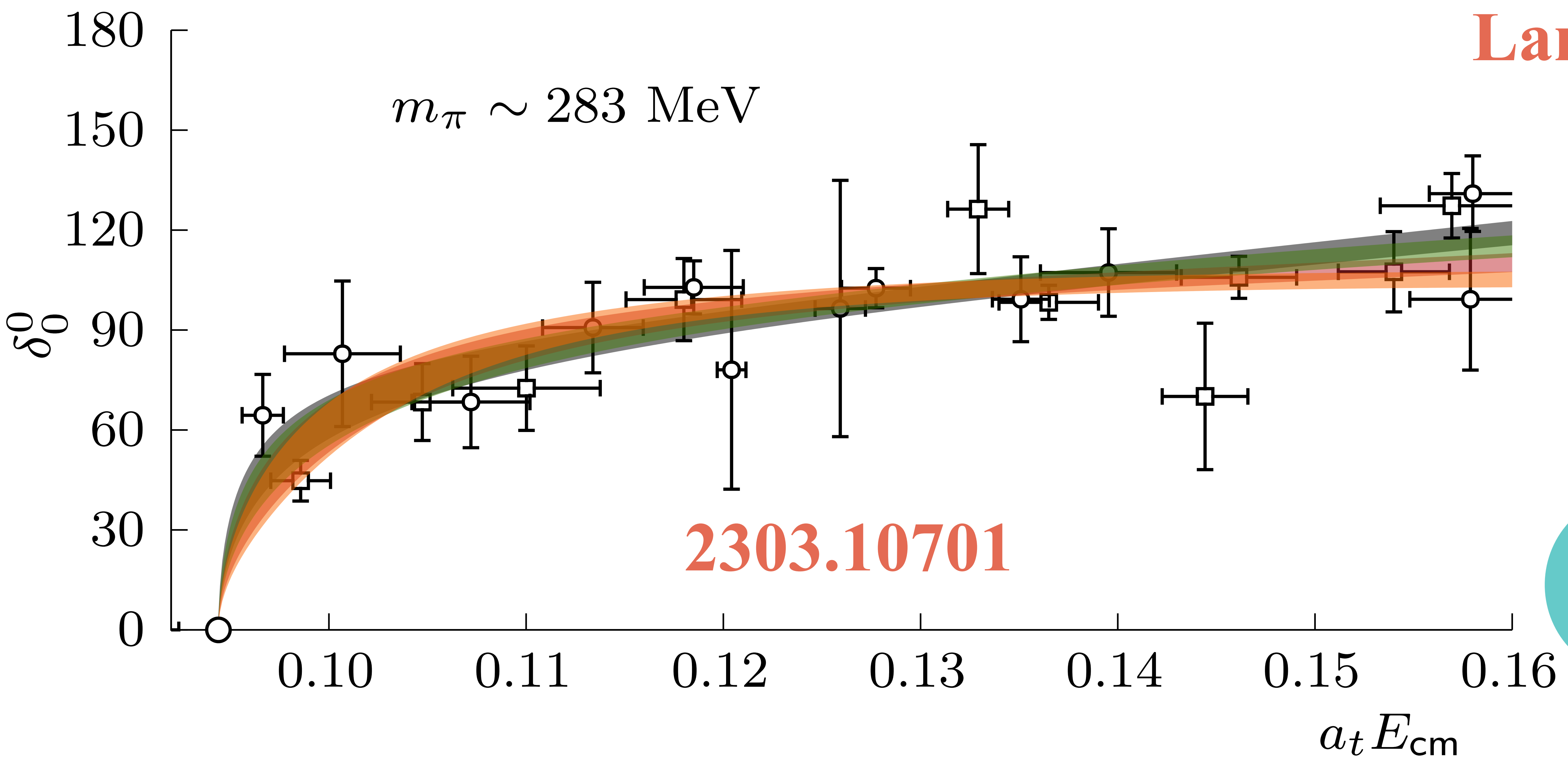
$m_\pi \sim 239$ **MeV**

1507.02599

Around 10 parameterizations

Very consistent amplitude fits

$I = 0\ \pi\pi$

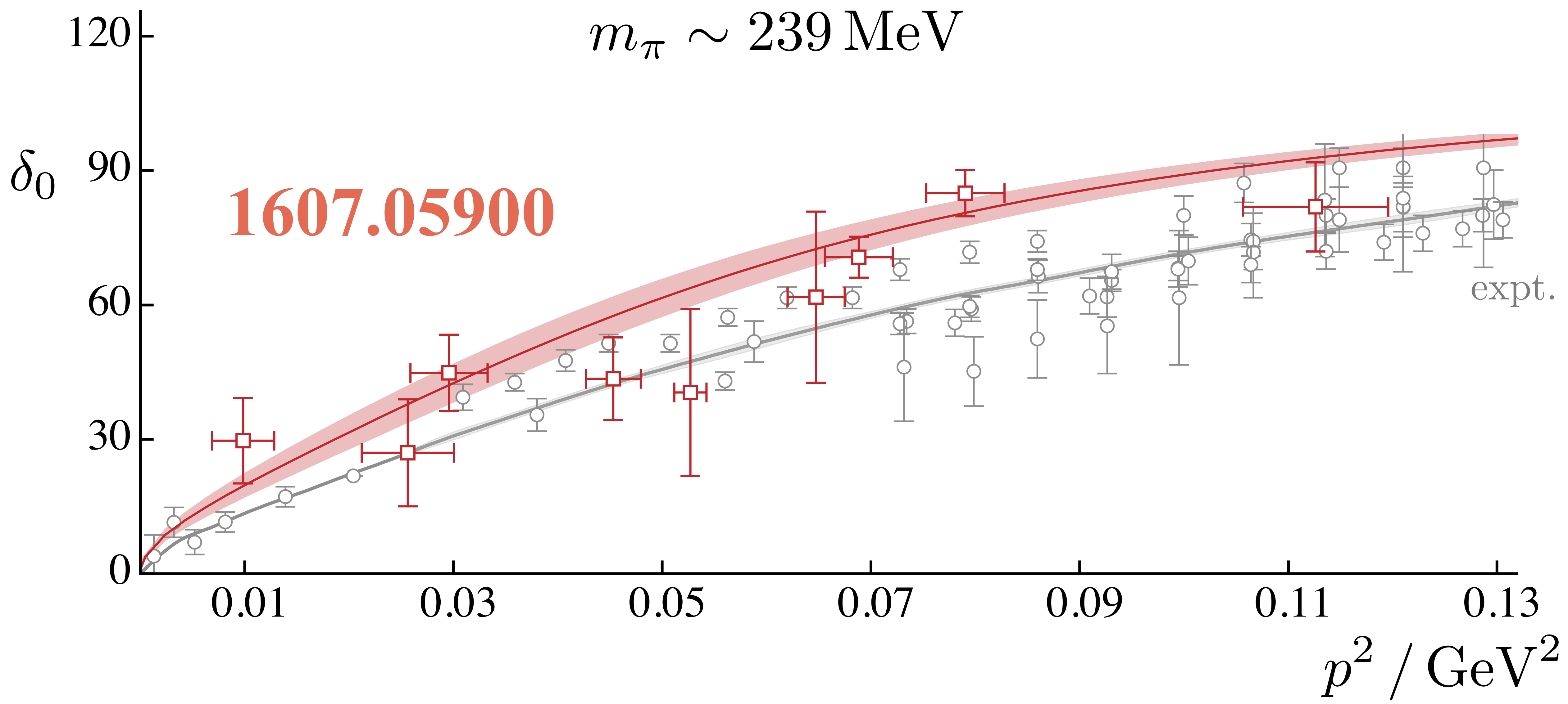


2303.10701

Large derivative at threshold

$\delta(s) = \pi/2$ is far from threshold

σ



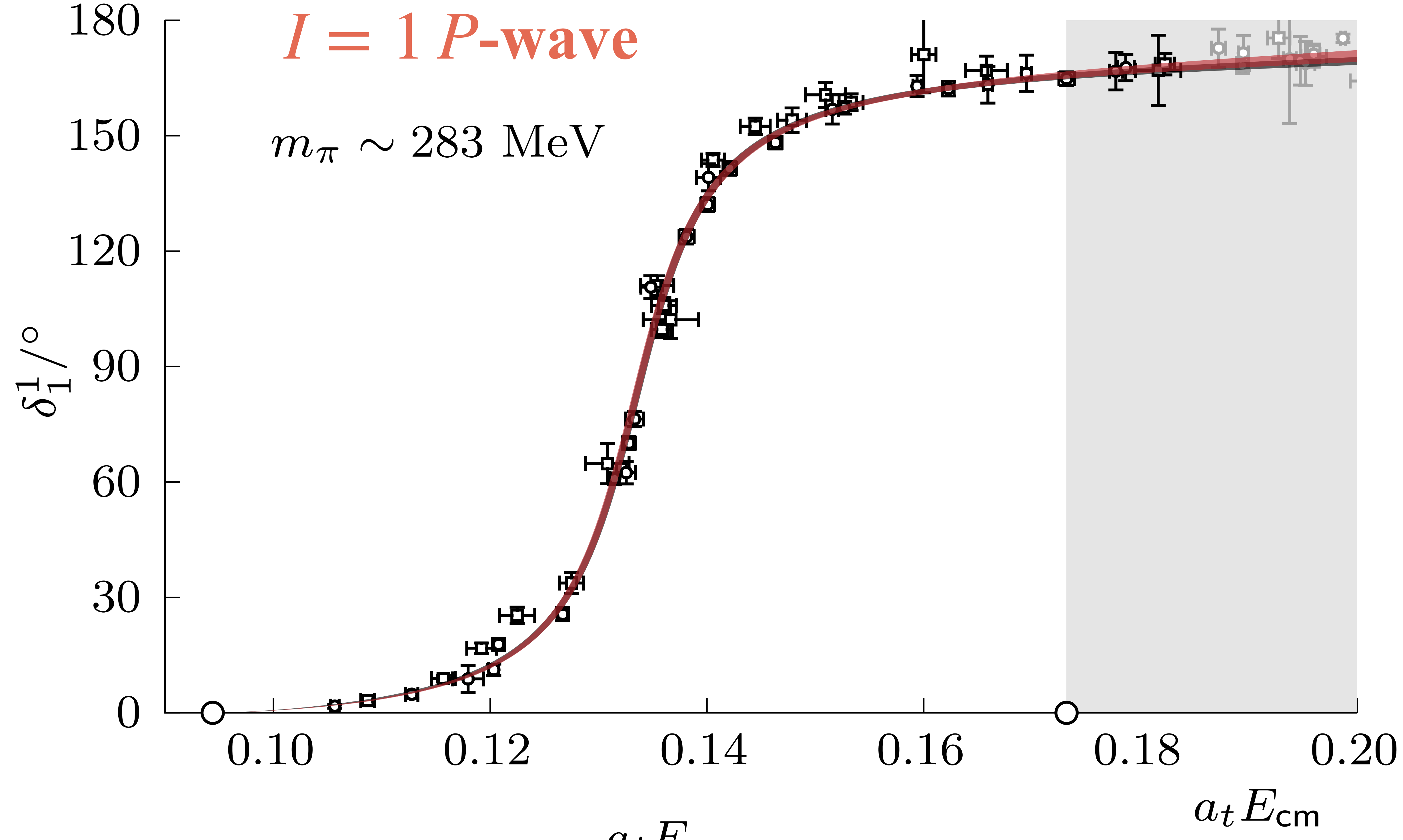
1607.05900

Over 20 parameterizations

Smaller derivative at threshold

Permutations

$$\sum_{I',\ell'} \int_{4m_\pi^2}^{\infty} ds'\, K_{\ell\ell'}^{II'}(s',s)\, \mathrm{Im}\, t_{\ell'}^{I'}(s')$$

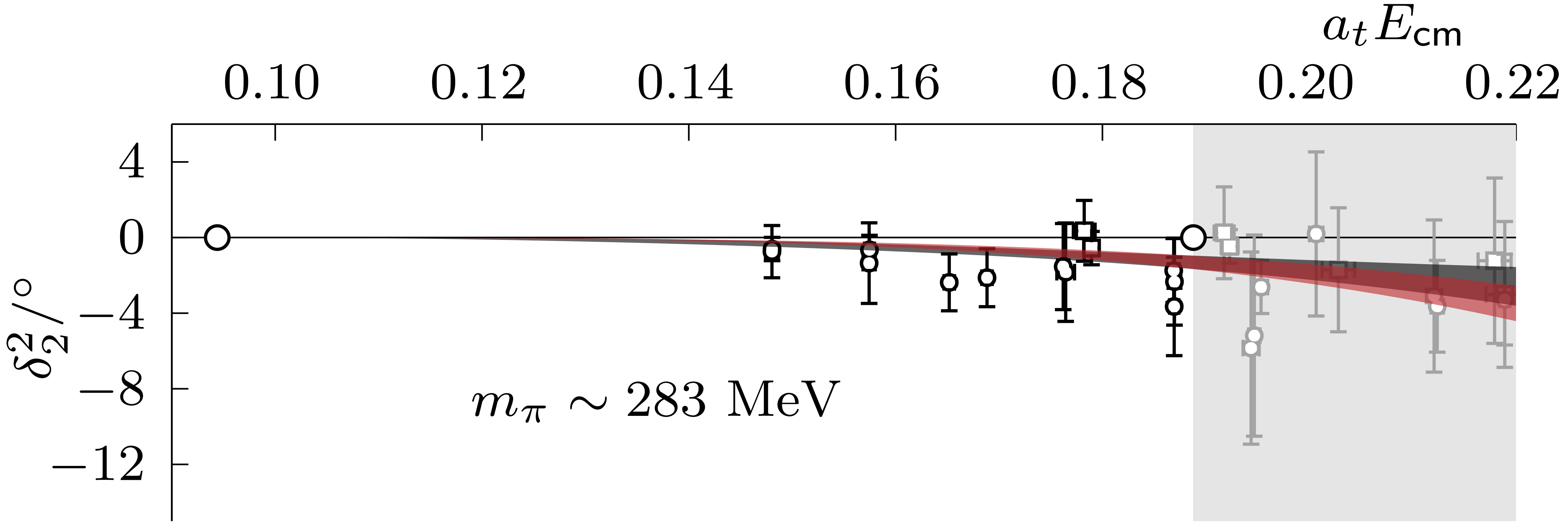


For ℓ_{max} partial waves

$$N_I \ell_{max} N_{params} \sim 10^5$$

We can fix most

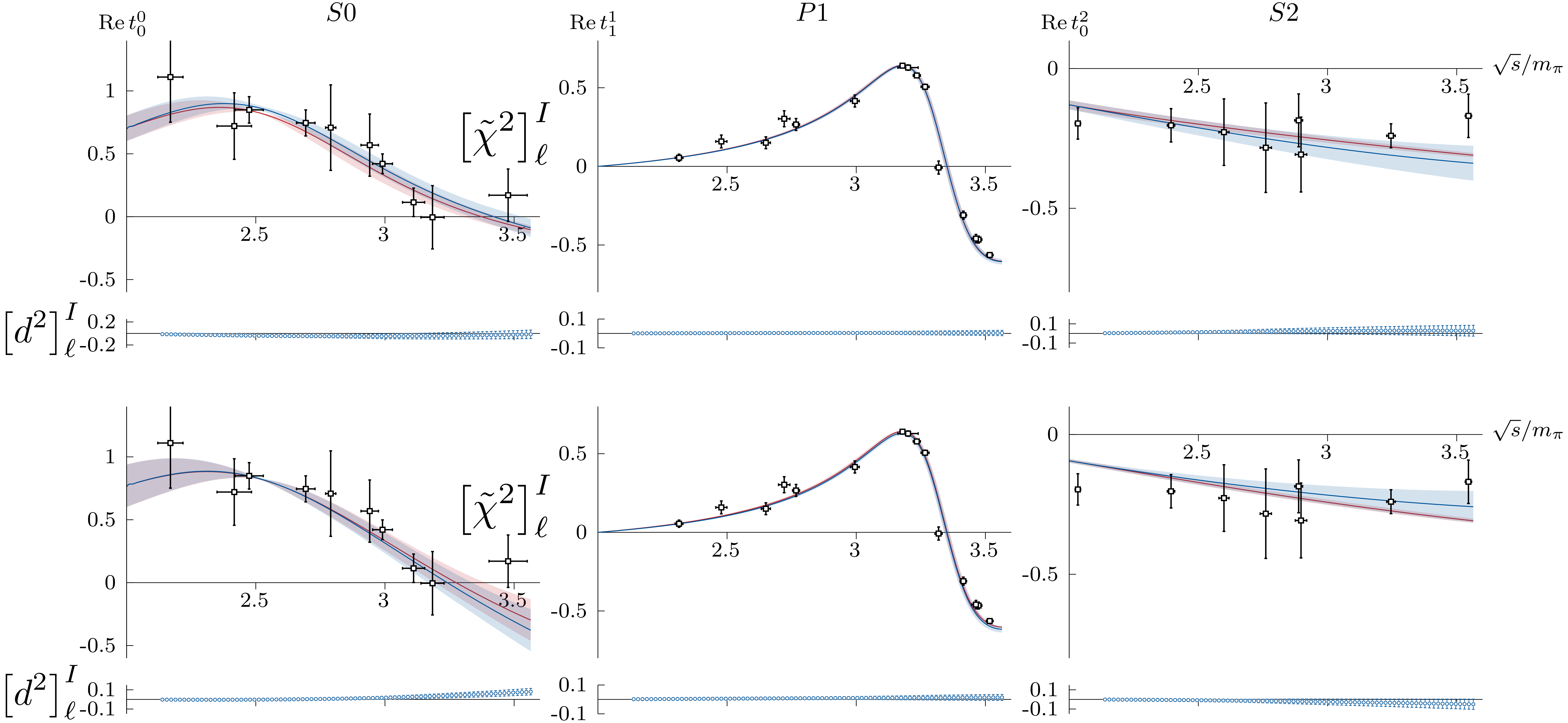
$I = 2$ D-wave



Ok but not great

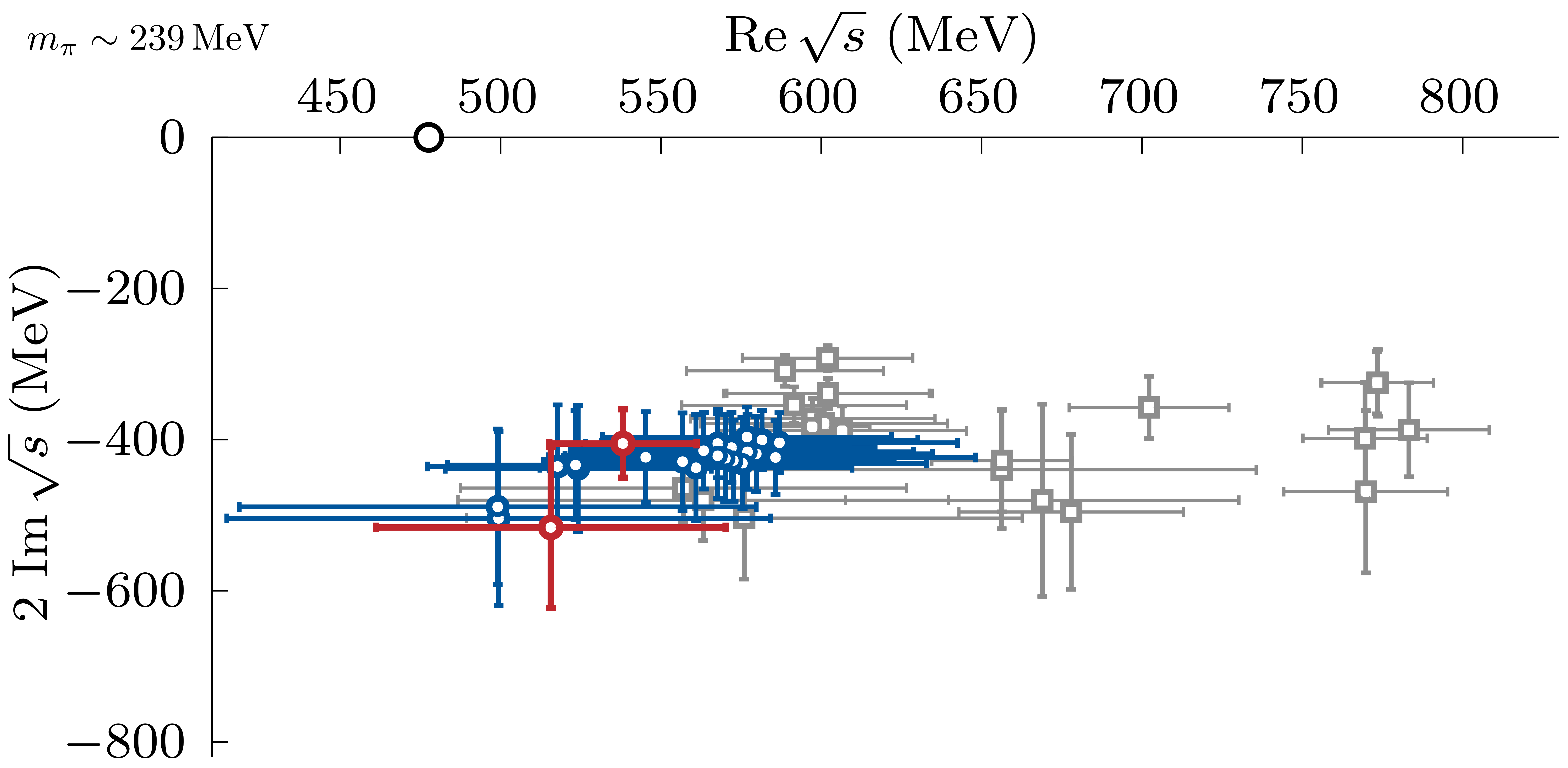
Visually, they describe the data and fit, but they are not perfect

$m_\pi \sim 239\,\text{MeV}$



Ok but not great

Visually, they describe the data and fit, but they are not perfect

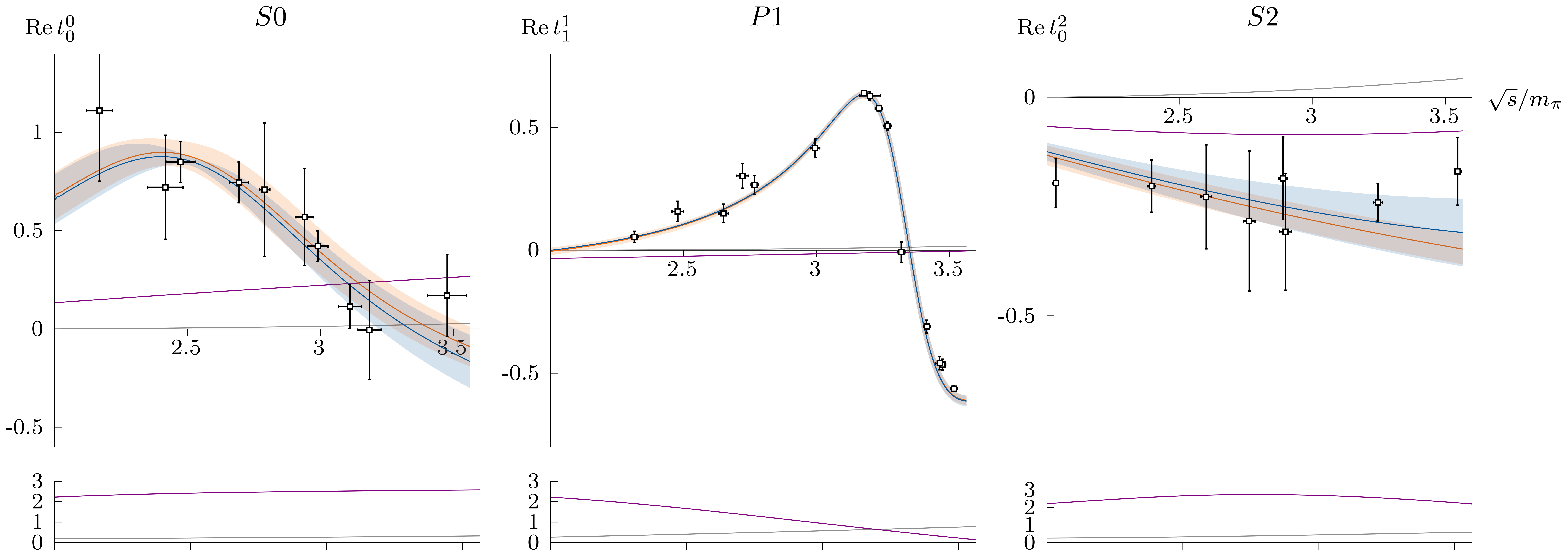


GKPY vs ROY

GKPY: Minimally subtracted → one less subtraction than ROY

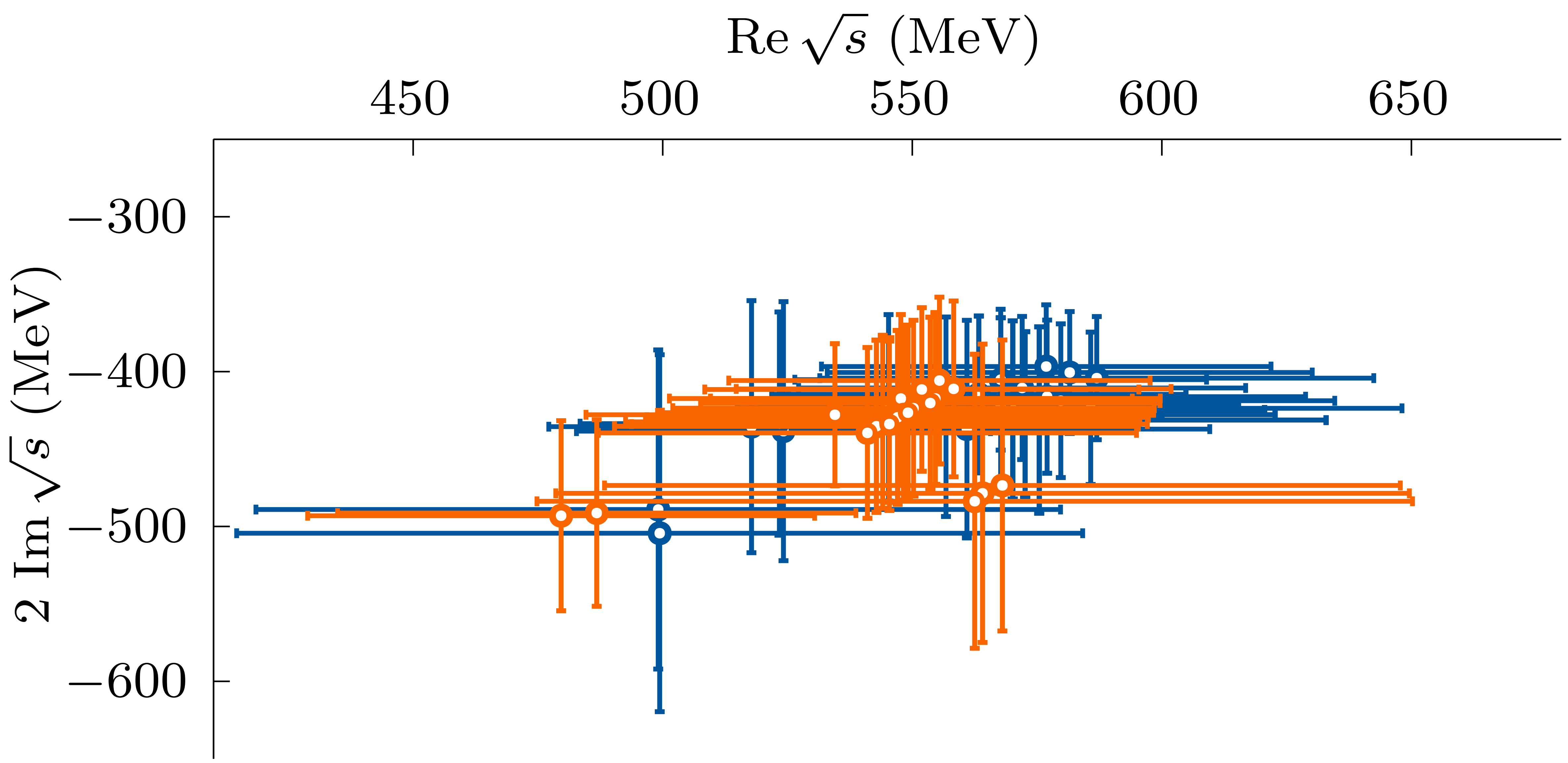
For our analysis, Regge contribution too large for d^2

$m_\pi \sim 239\,\mathrm{MeV}$

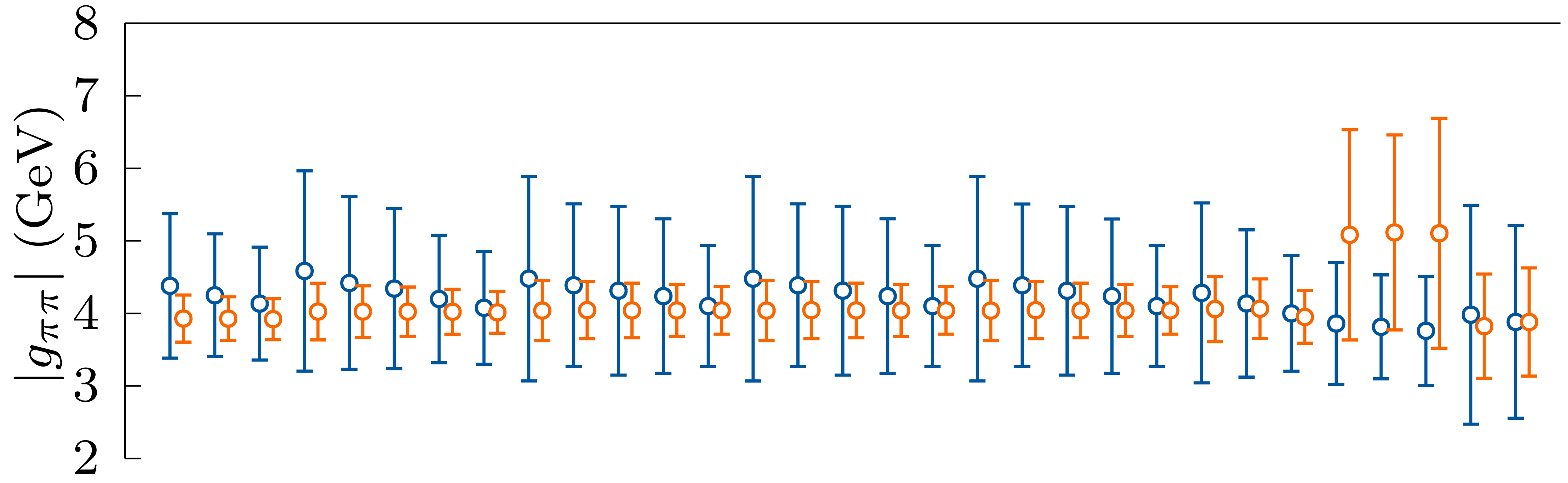


GKPY vs ROY

However, pole extraction is more accurate in most cases, particularly for the coupling



GKPY produces less than half the uncertainty in most cases



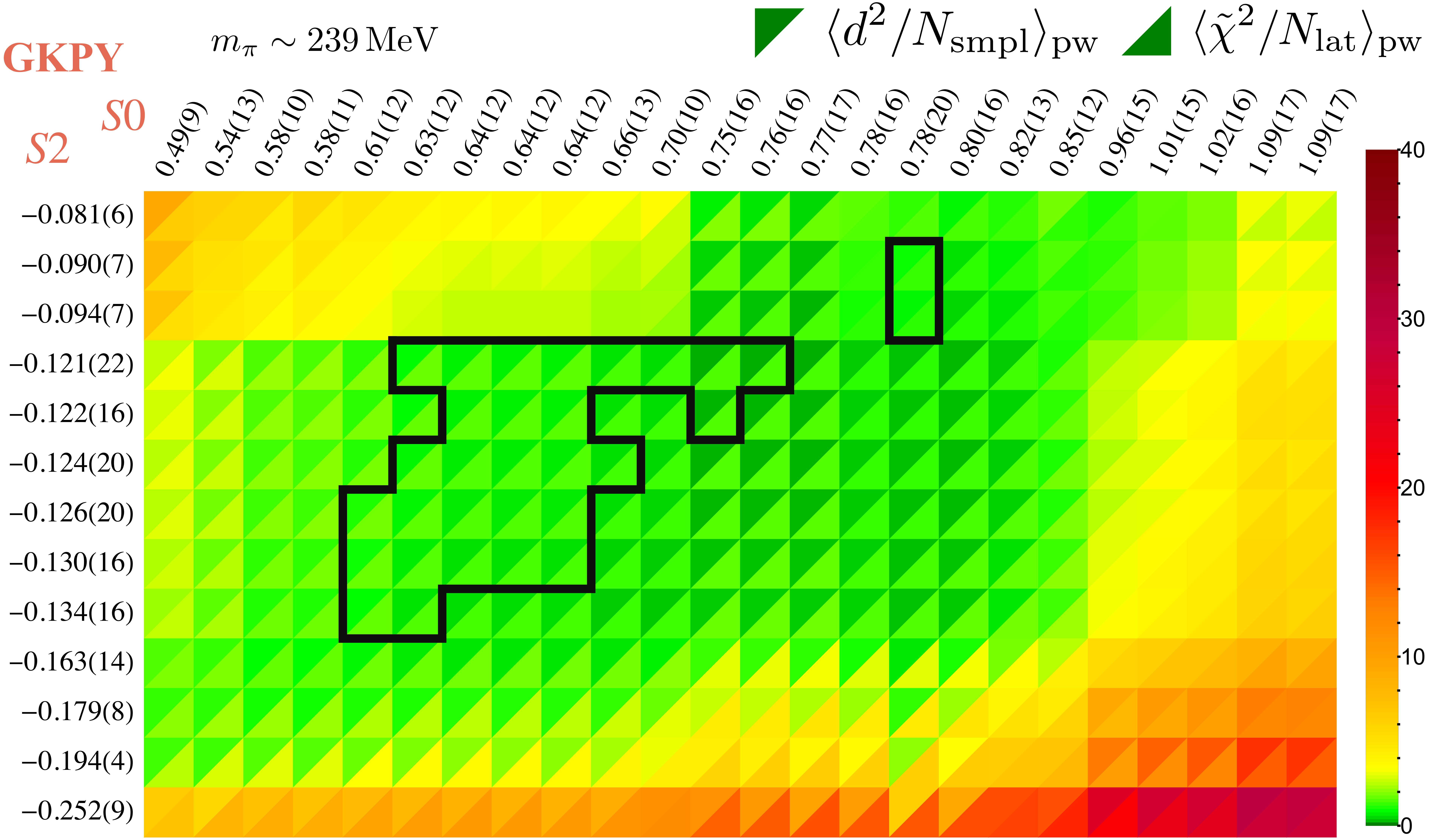
Scattering plane

$N_I \ell_{\max} N_{\text{params}} \sim 300 - 400$

$m_\pi \sim 239\,\text{MeV}$

Black **GKPY**

$d^2/N_{\text{smpl}} < 1, \quad \tilde{\chi}^2/N_{\text{lat}} < 2$



Scattering plane

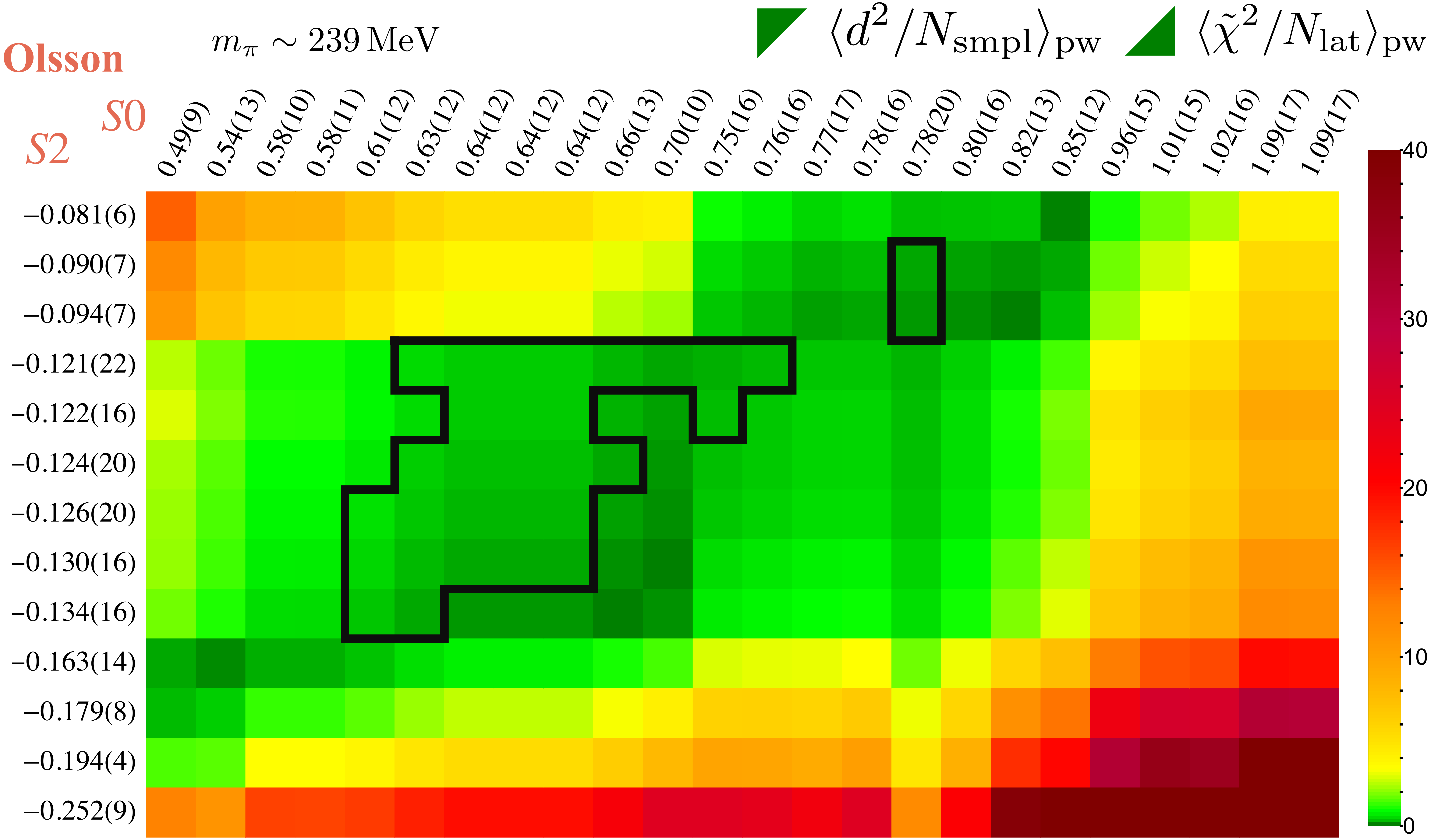
$N_I \ell_{\rm max} N_{\rm params} \sim 300 - 400$

$m_\pi \sim 239\,{\rm MeV}$

Black

Olsson

$d^2/N_{\rm smpl} < 1, \quad \tilde{\chi}^2/N_{\rm lat} < 2$



Scattering plane

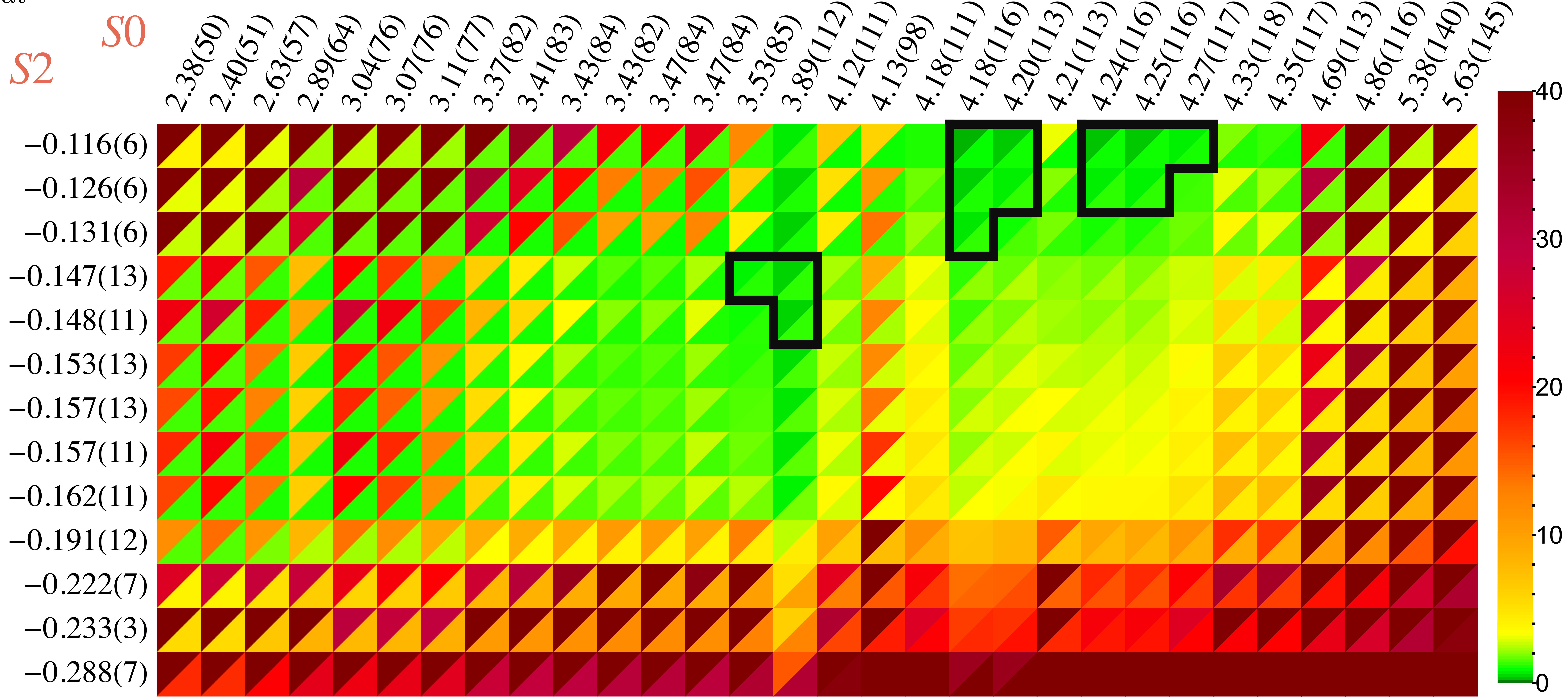
$N_I \ell_{\max} N_{\text{params}} \sim 300 - 400$

Black **ROY**

$m_\pi \sim 283\,\text{MeV}$

$\langle d^2/N_{\text{smpl}} \rangle_{\text{pw}}$ $\langle \tilde{\chi}^2/N_{\text{lat}} \rangle_{\text{pw}}$

$d^2/N_{\text{smpl}} < 1, \quad \tilde{\chi}^2/N_{\text{lat}} < 2$



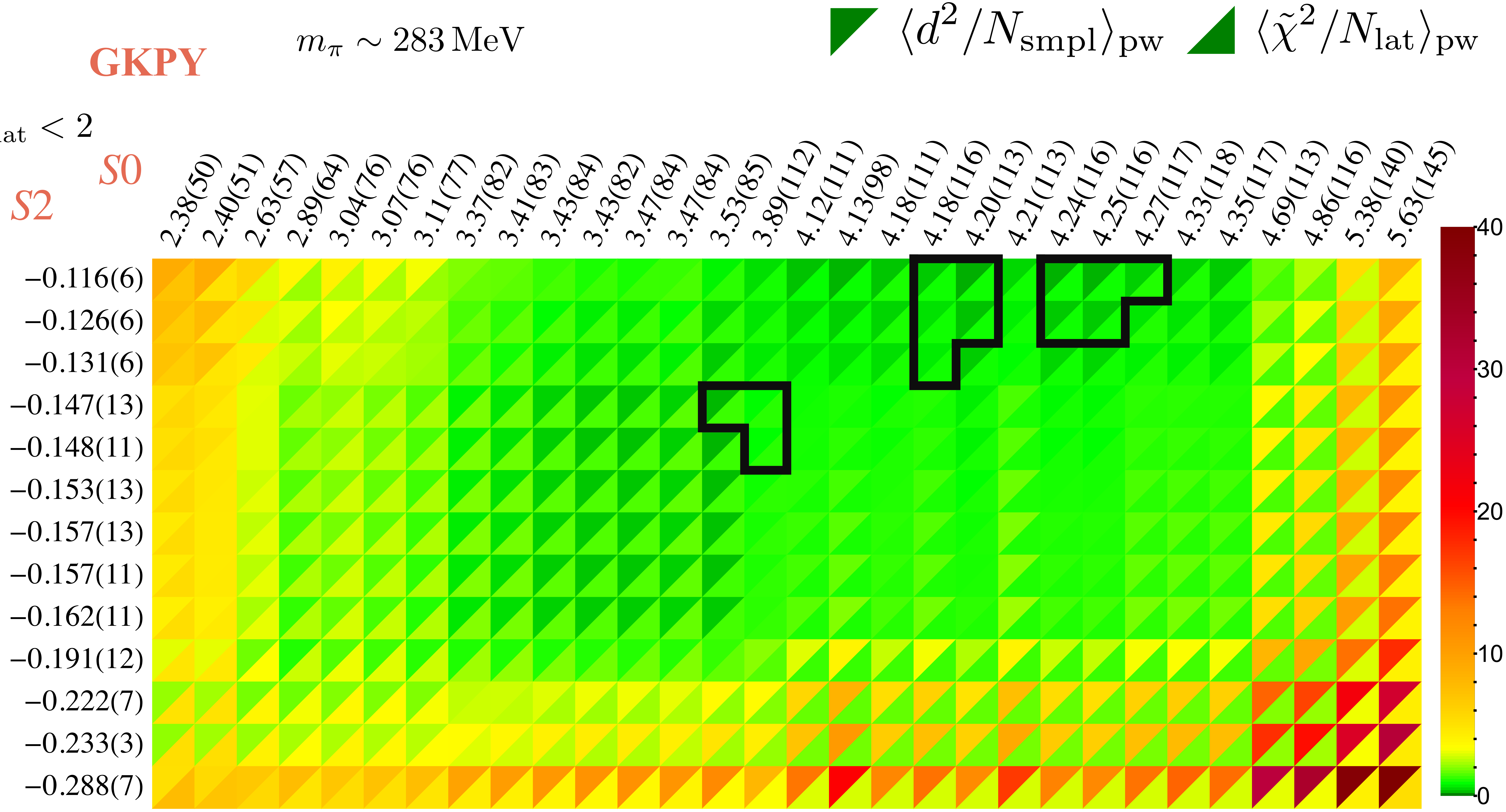
Scattering plane

$N_I \ell_{\max} N_{\text{params}} \sim 300-400$

$m_\pi \sim 283\,\text{MeV}$

Black   GKPY

$d^2/N_{\text{smpl}} < 1, \quad \tilde{\chi}^2/N_{\text{lat}} < 2$

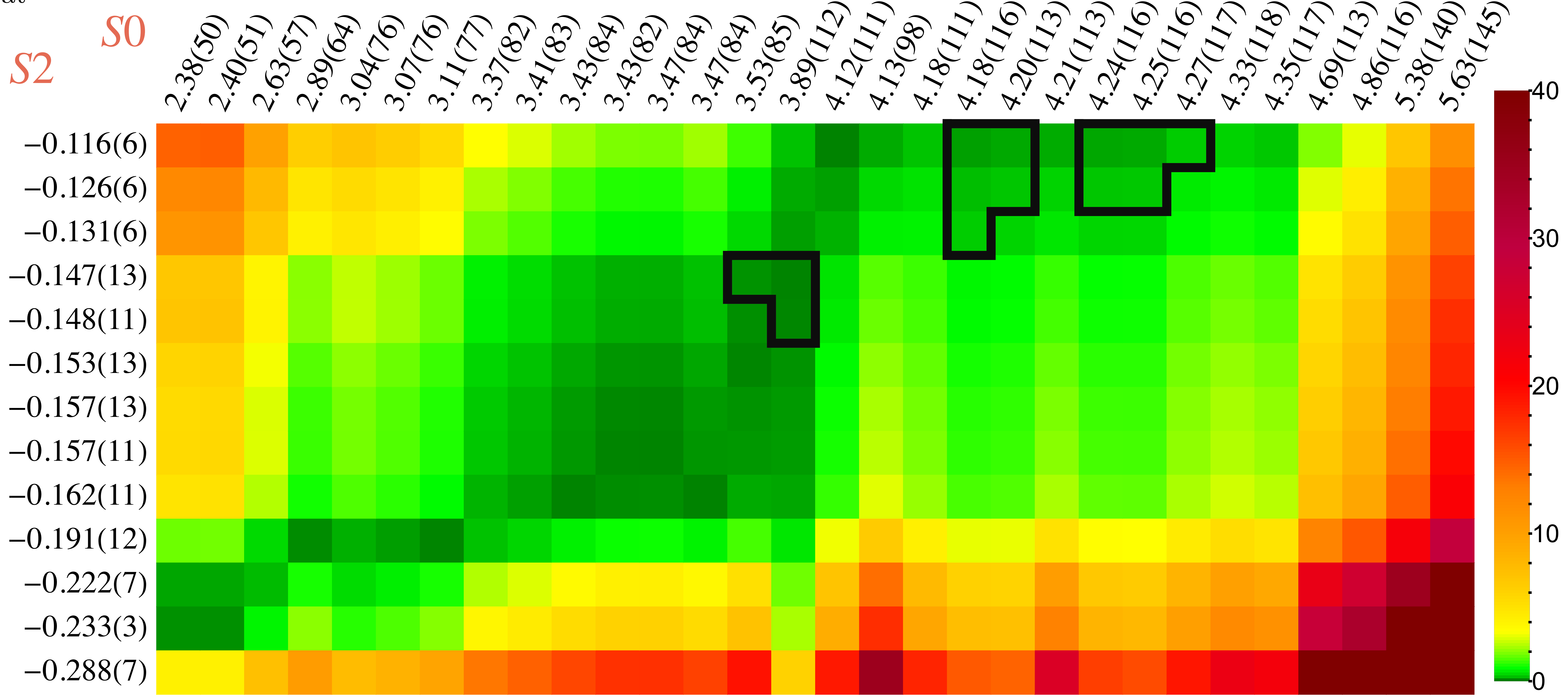


Scattering plane

$N_I \ell_{\max} N_{\text{params}} \sim 300 - 400$

$m_\pi \sim 283\,\text{MeV}$

$d^2/N_{\text{smpl}} < 1, \quad \tilde{\chi}^2/N_{\text{lat}} < 2$



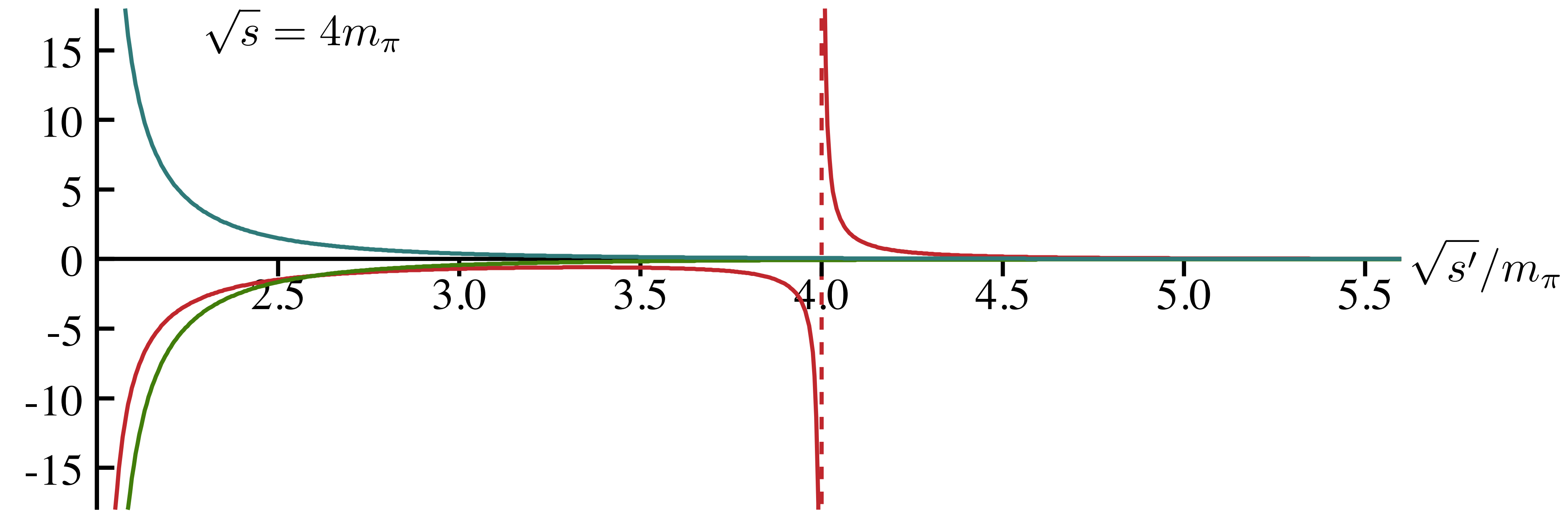
The good

Fit → In

DR → Out

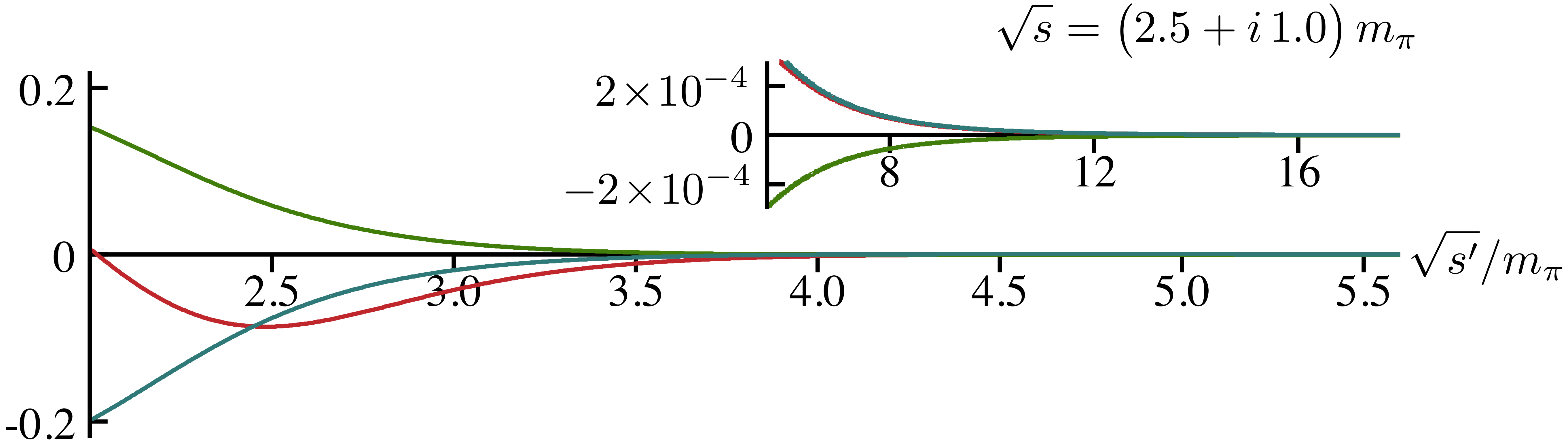
$$\tilde{t}^I_\ell(s) = \tau^I_\ell(s) + \sum_{I',\ell'} \int_{4m_\pi^2}^{\infty} ds' \, K^{II'}_{\ell\ell'}(s',s) \, \mathrm{Im}\, t^{I'}_{\ell'}(s')$$

Real s



Smeared over a large energy region

Complex s



An ϵ on the real axis $\rightarrow \epsilon'$ in the complex plane

The bad

Not happening

$$\sum_{I',\ell'} \int_{4m_\pi^2}^{\infty} ds' \, K_{\ell\ell'}^{II'}(s',s) \, \mathrm{Im}\, t_{\ell'}^{I'}(s')$$

Partial waves

Extrapolated

$$\int_{4m_\pi^2}^{\infty} = \int_{4m_\pi^2}^{s_{max}} + \int_{s_{max}}^{\infty} = \int_{4m_\pi^2}^{s_{fit}} + \int_{s_{fit}}^{s_{max}} + \int_{s_{max}}^{\infty}$$

Regge

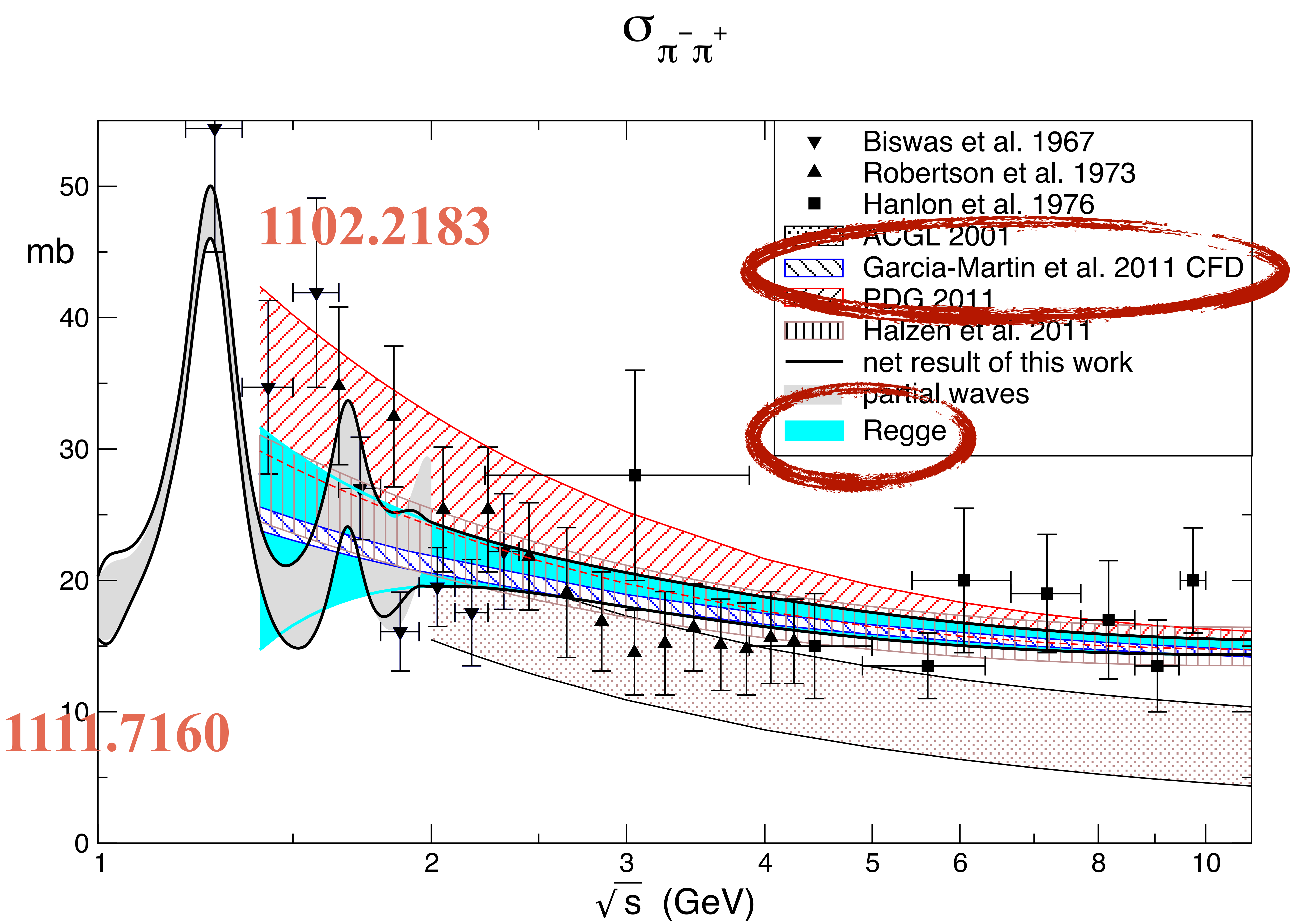
- **Regge must be extrapolated from phys. m_π**
- **Regge is wrong below $a_t m_\pi \sim 0.22 - 0.25$**

The Regge

Regge must be extrapolated from phys. m_π

$\mathbb{P}$ → gluon exchanges → constant over m_q

ρ, f_2 → resonances, not constant → $\lambda \sim \Gamma/M$

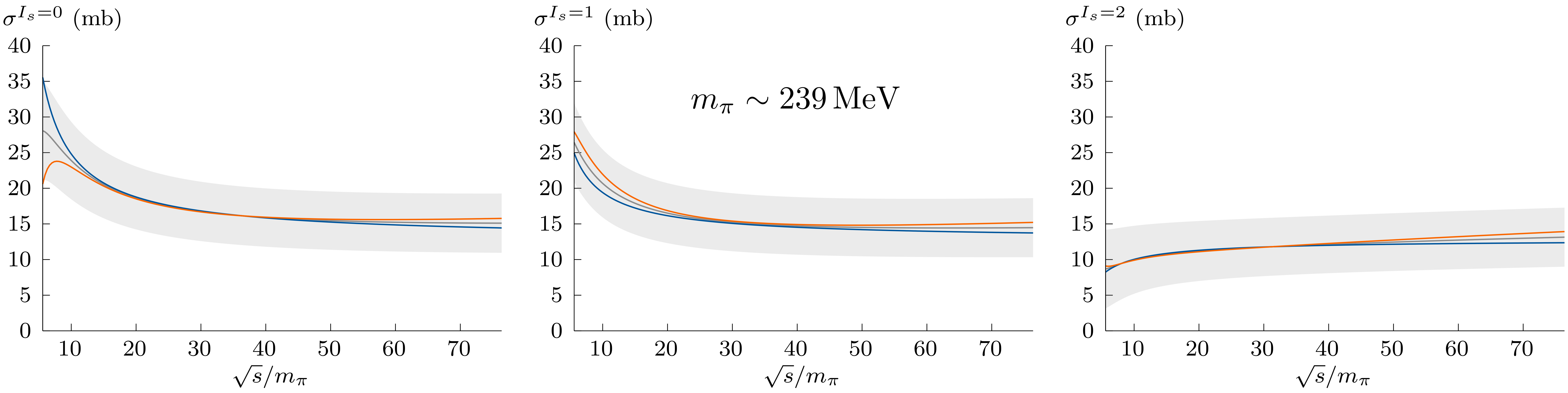


Our $F_{Regge} = \dfrac{F_{Regge1} + F_{Regge2}}{2}$

Big uncertainty $\Delta F_{Regge} = 0.3\, F_{Regge}$

Regge

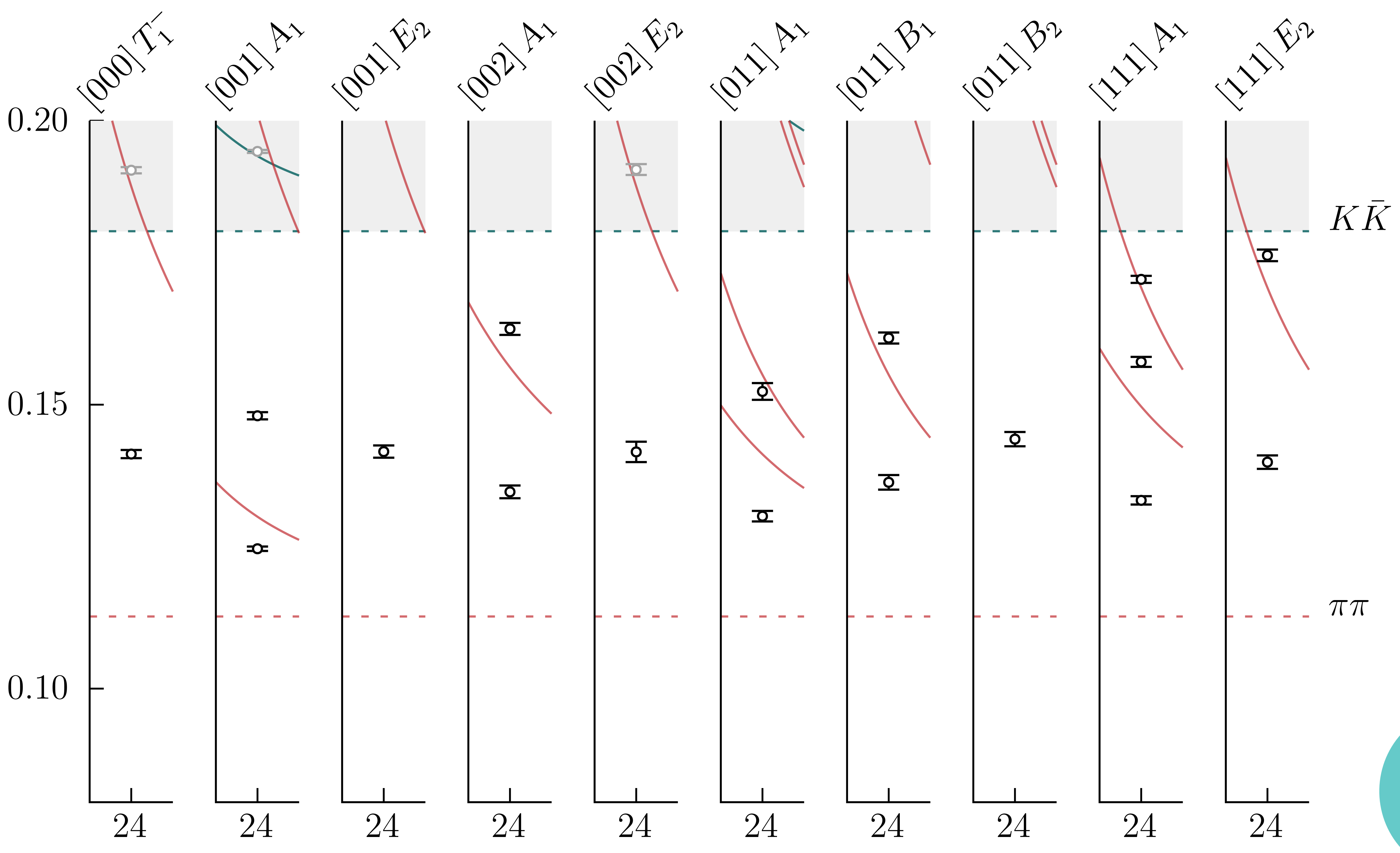
Regge must be extrapolated from phys. m_π



$$\textbf{Our } F_{Regge} = \frac{F_{Regge1} + F_{Regge2}}{2}$$

Big uncertainty $\Delta F_{Regge} = 0.3\, F_{Regge}$

$I = 1\ \pi\pi$

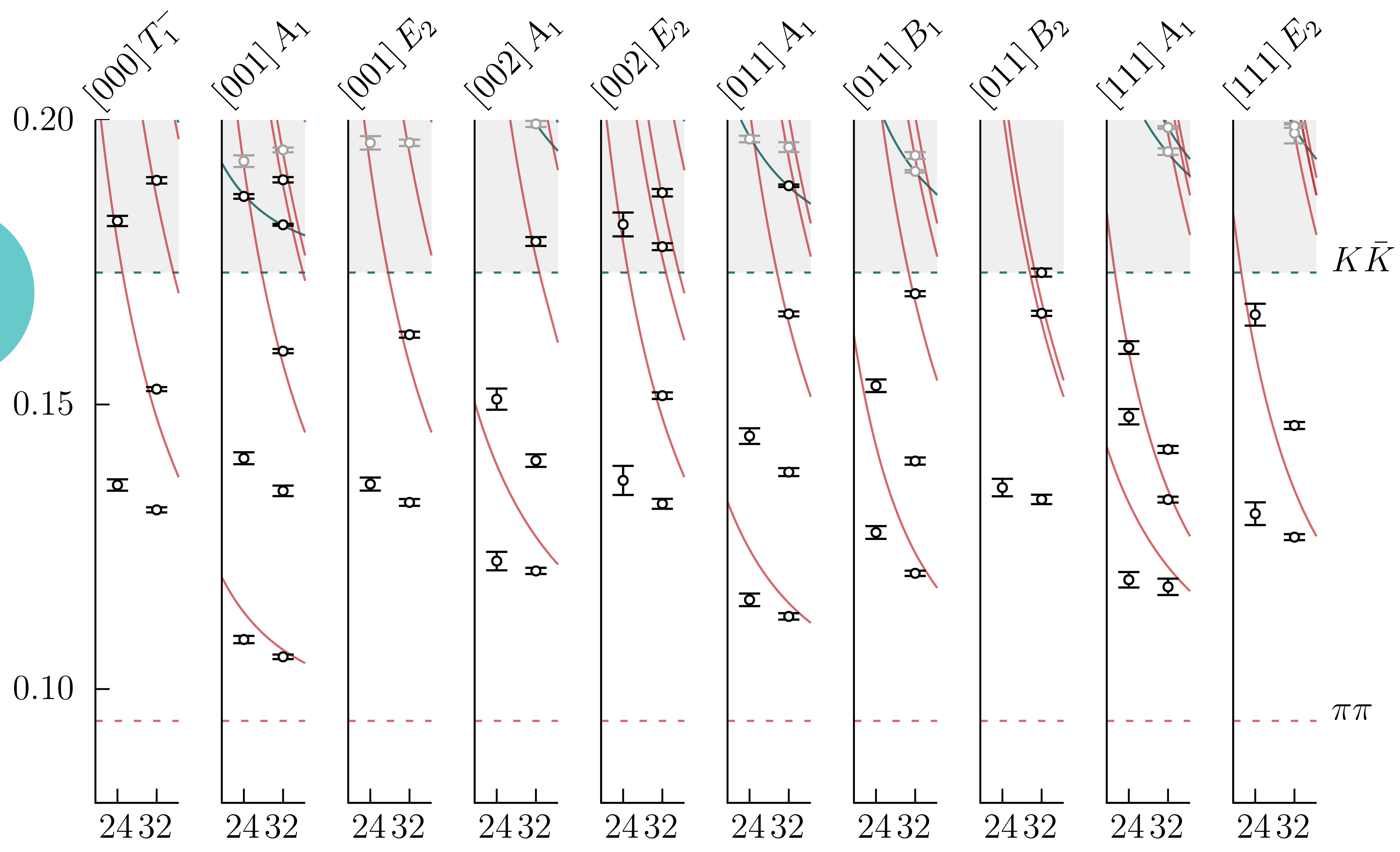


$m_\pi \sim 330\,\text{MeV}$

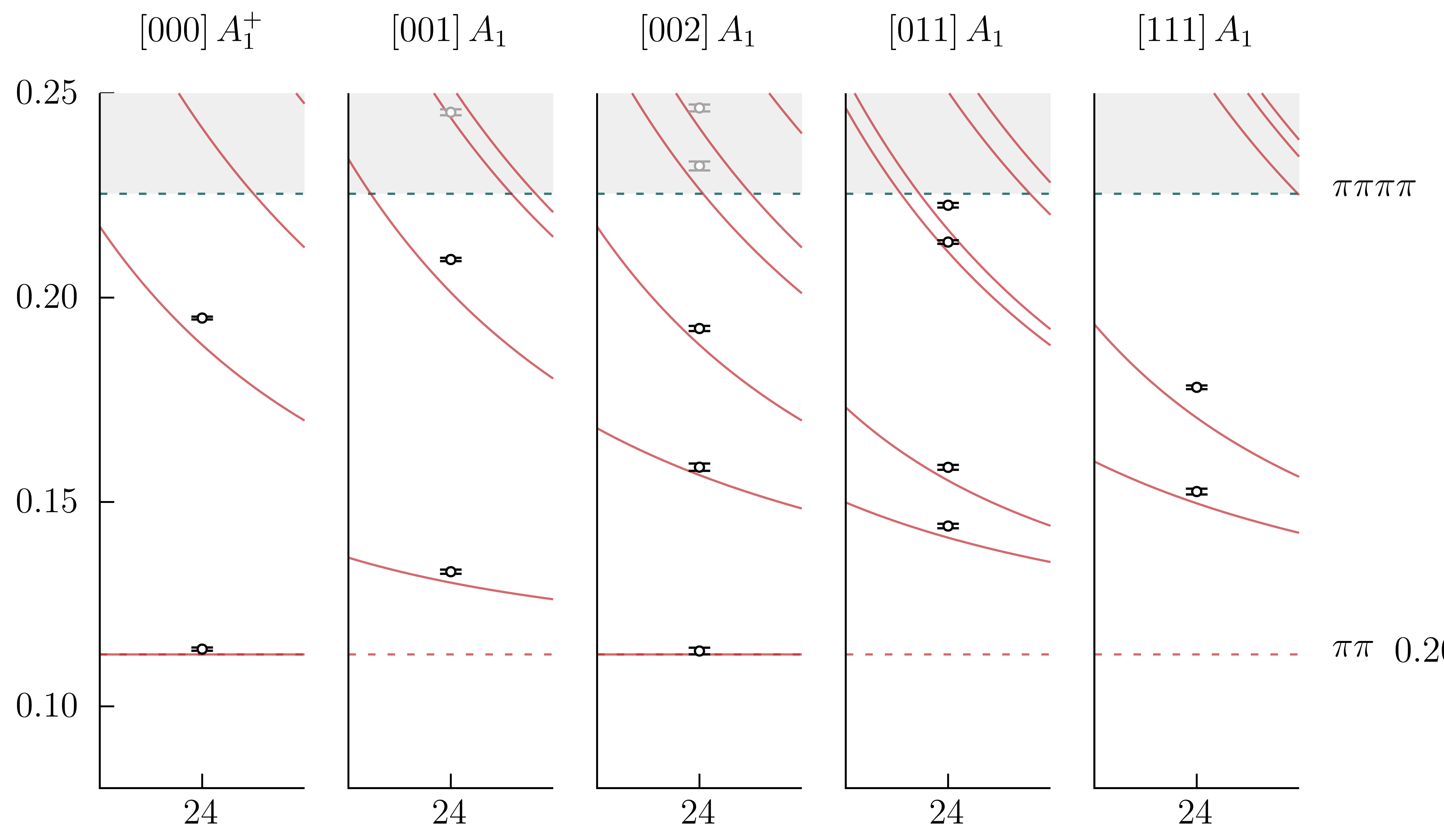
Similar spectrum to previous masses

ρ

$m_\pi \sim 283\,\text{MeV}$



Allows us to study the ρ resonance m_q dependence



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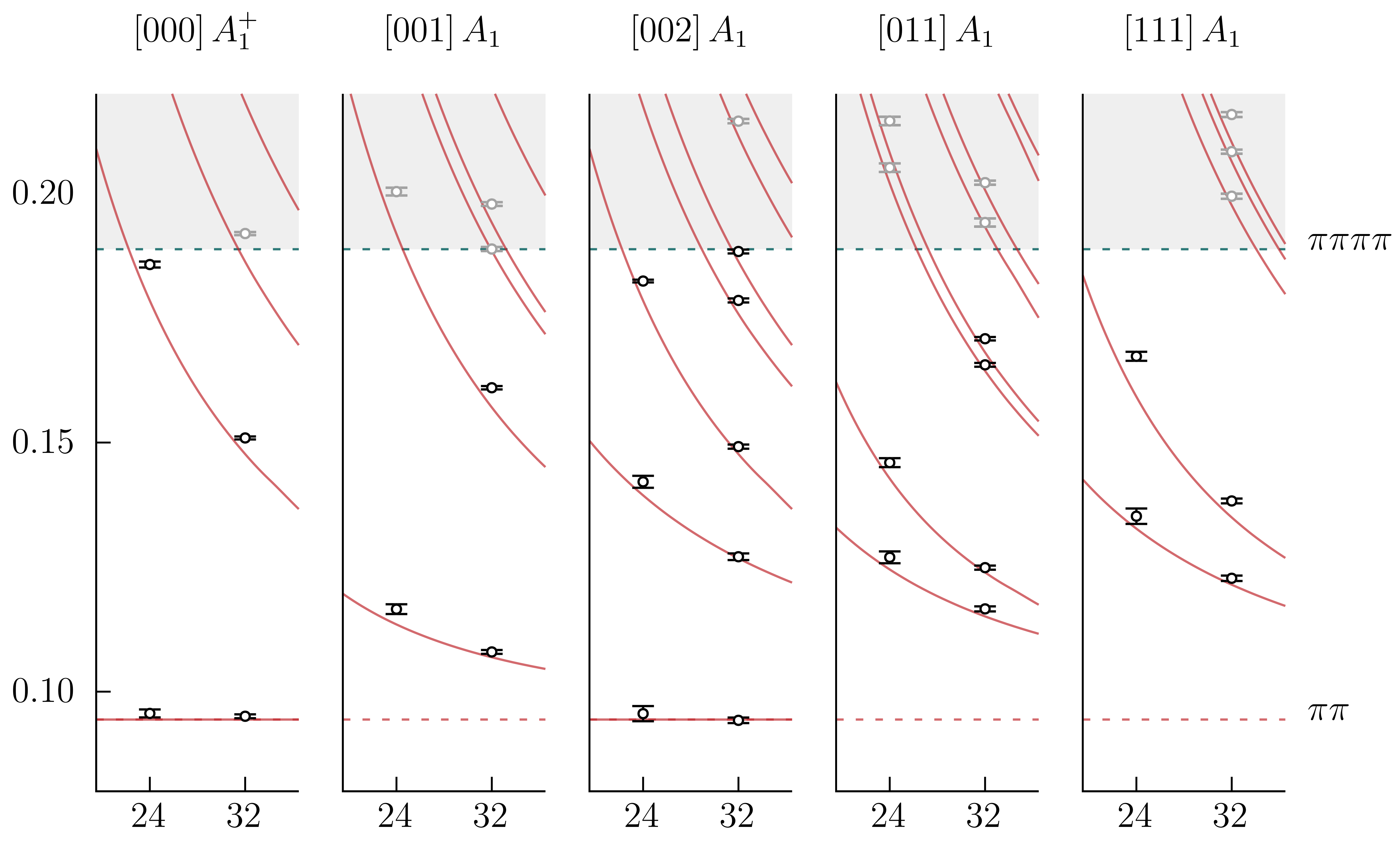
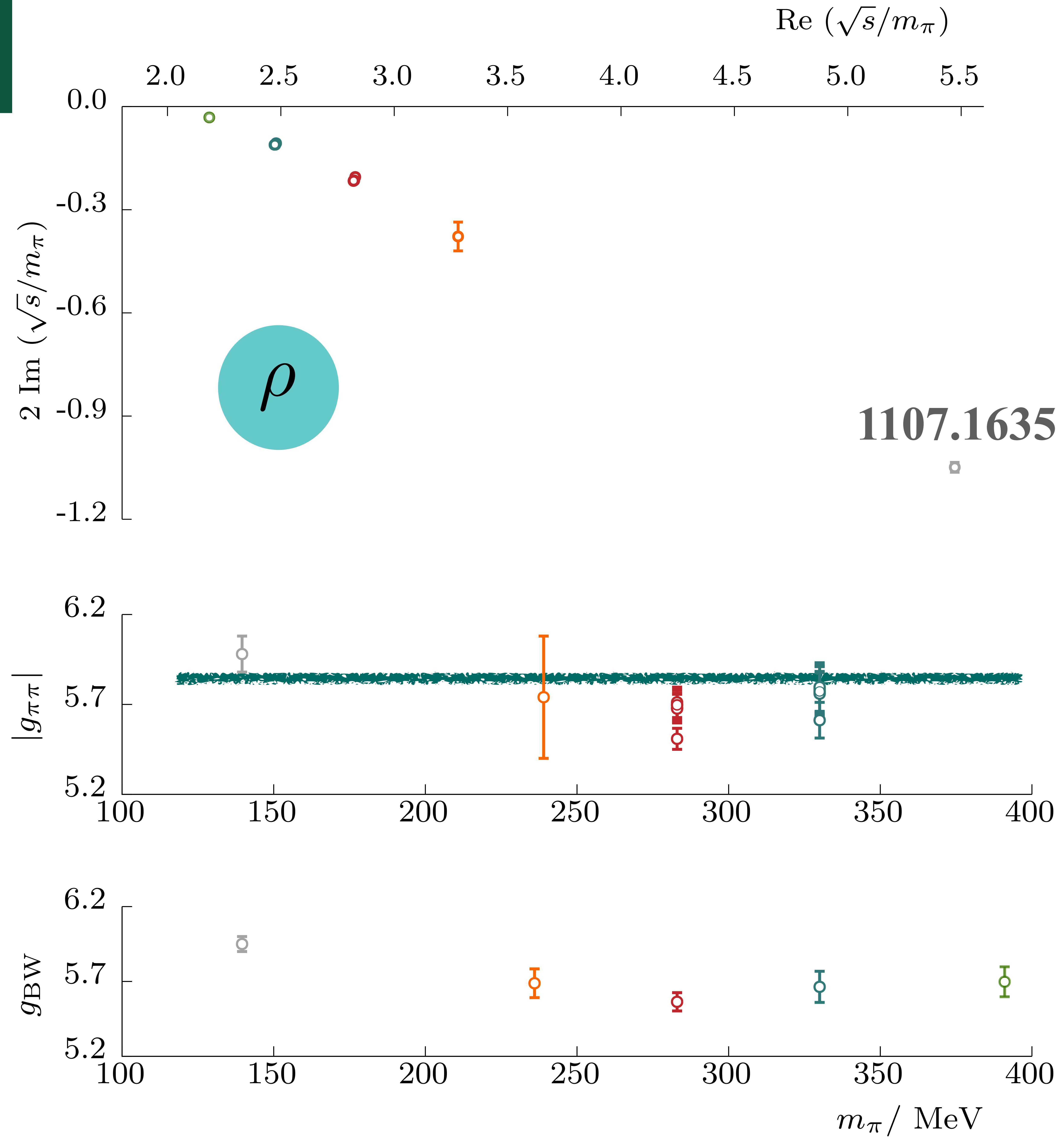


$I = 1\ \pi\pi$

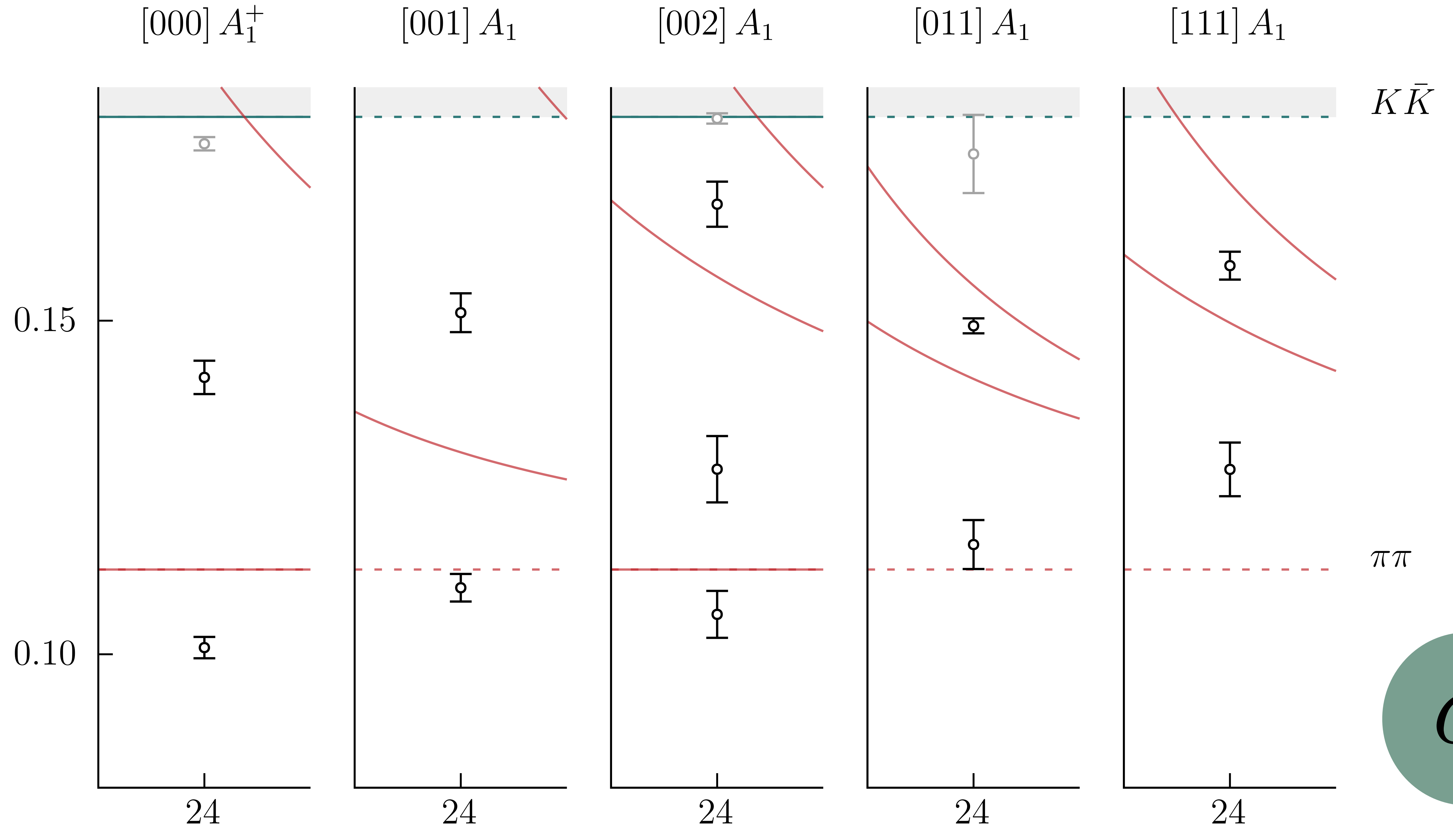
hadspec

Ordinary m_q dependence

g **constant**



$I = 0\ \pi\pi$



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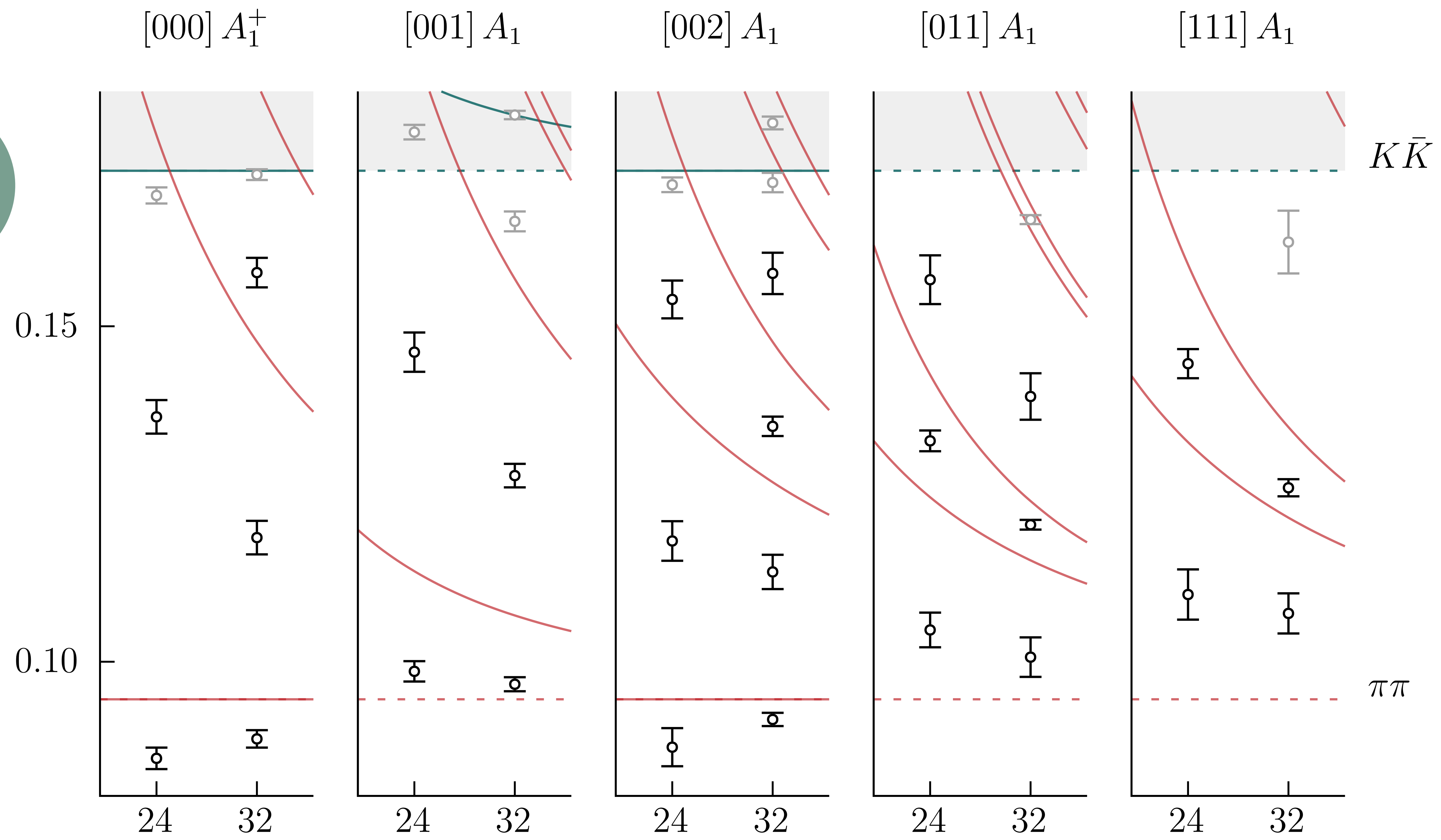


σ

Over 60 "elastic" levels for I=0

$I = 0\ \pi\pi$

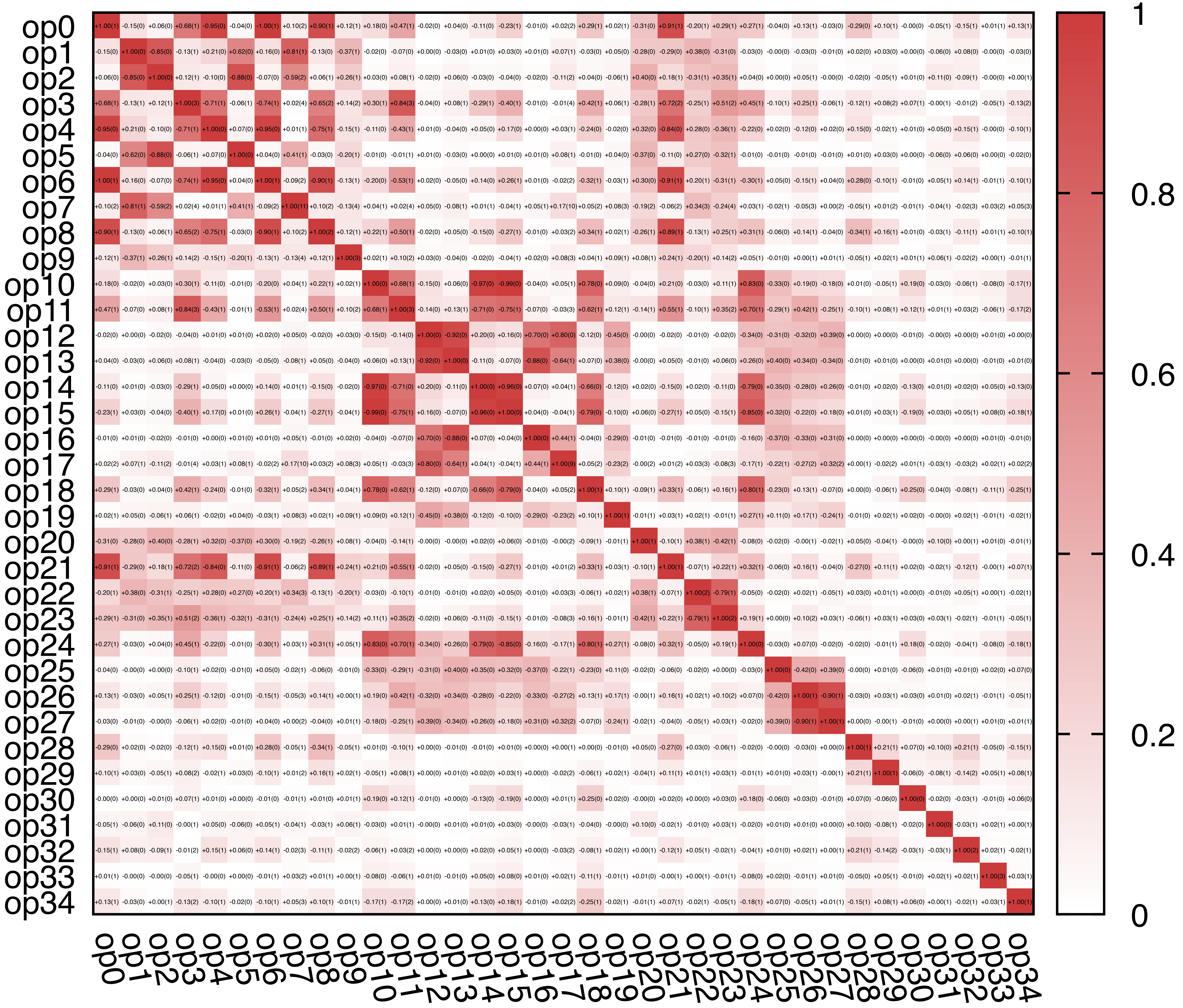
Many ops. for a good GEVP

Distillation → 0905.2160

Time src avg. correlations

Some highly correlated

More than a few relevant ops.



$I = 0\ \pi\pi$

Many fits for a good E_n

Many fits for diff

t_0

t_{min}, t_{max}

N_{exp}**(1-2)**

Model averaging technique

2008.01069

2208.13755