Timelike pion form factor from lattice QCD

HADRON 2023

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June - 9 - 2023

In collaboration with J. Dudek

hadspec

WILLIAM & MARY
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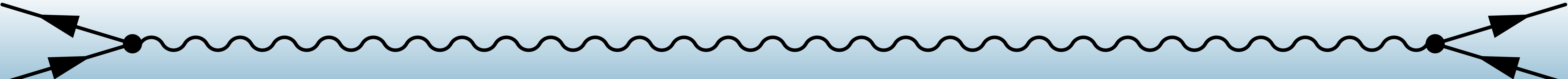
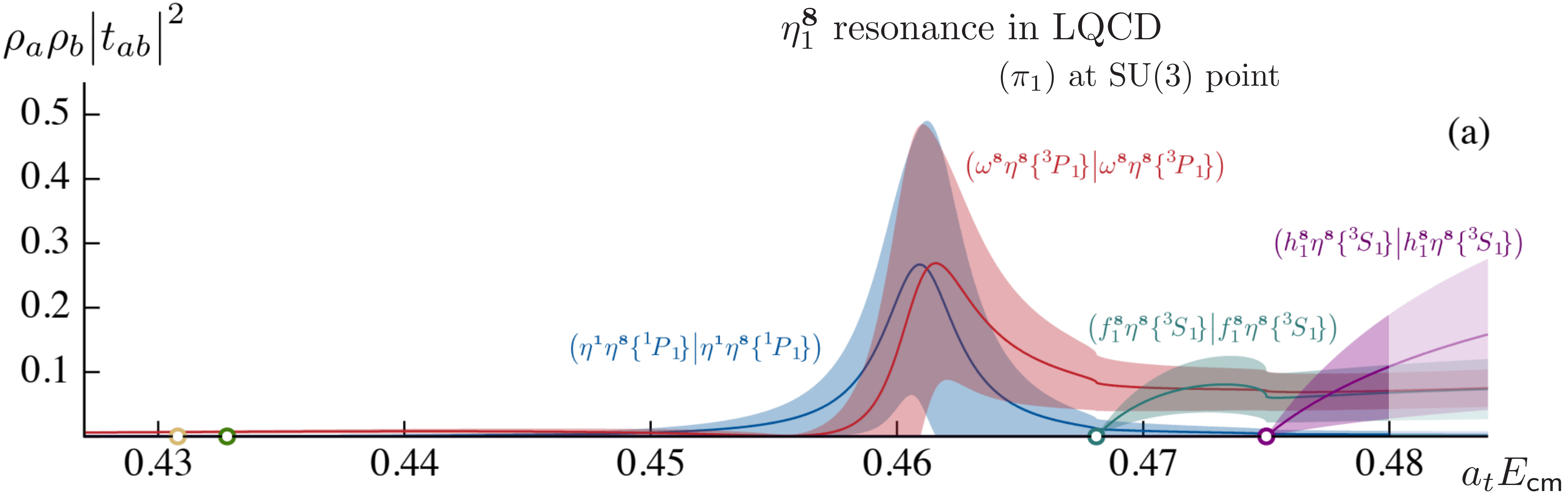
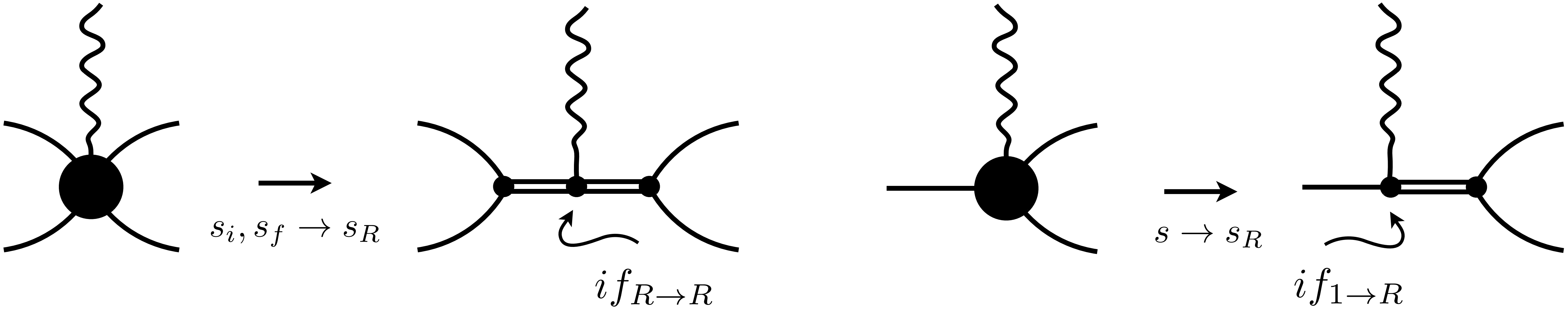


Resonance structure and production

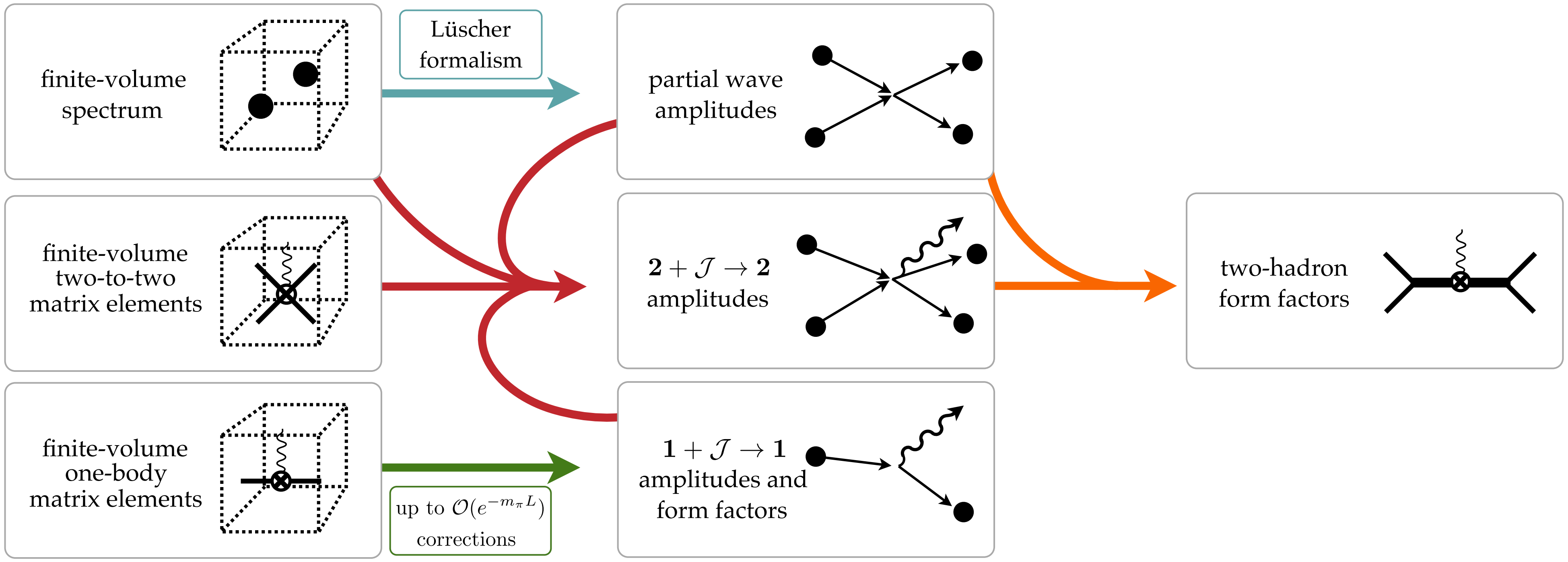


Diagonal elements of the scattering amplitude from [2009.10034 Woss et al.]



[1812.10504 FO et al.]

Finite-volume roadmap



finite-volume spectrum

Lüscher formalism

partial wave amplitudes

finite-volume two-to-two matrix elements

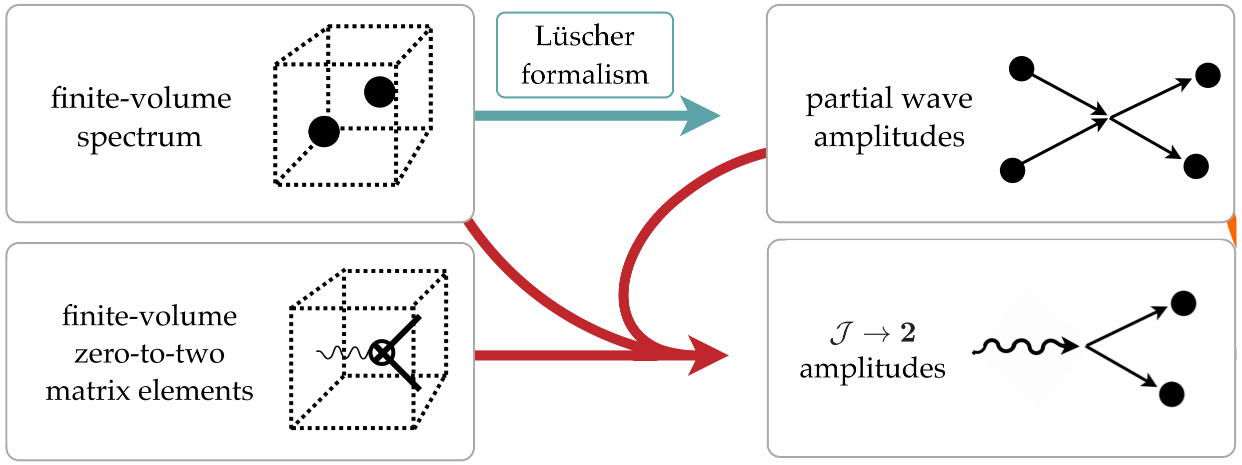
$\mathbf{2} + \mathcal{J} \to \mathbf{2}$ amplitudes

two-hadron form factors

finite-volume one-body matrix elements

up to $\mathcal{O}(e^{-m_\pi L})$ corrections

$\mathbf{1} + \mathcal{J} \to \mathbf{1}$ amplitudes and form factors



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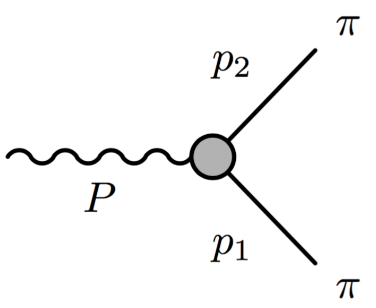

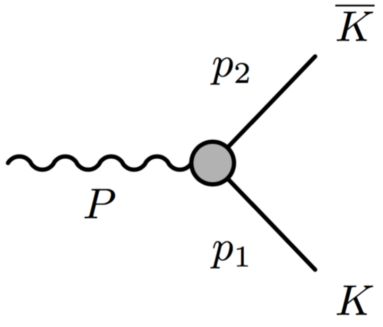

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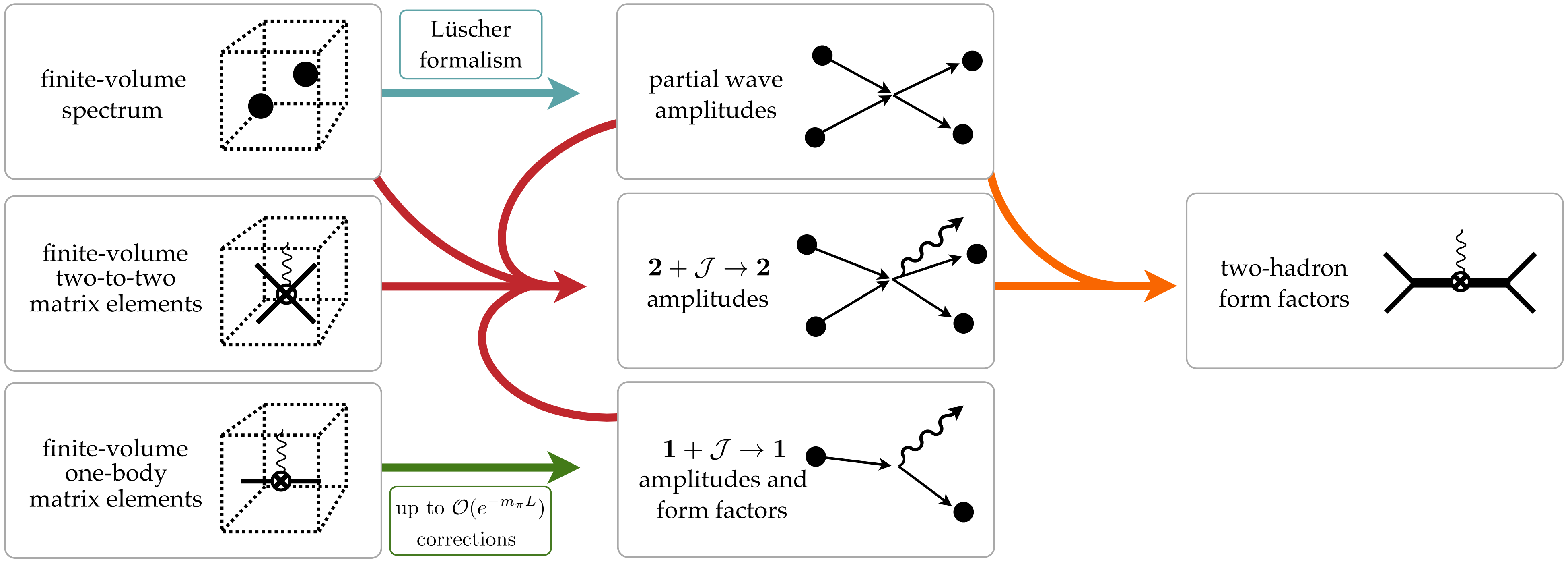

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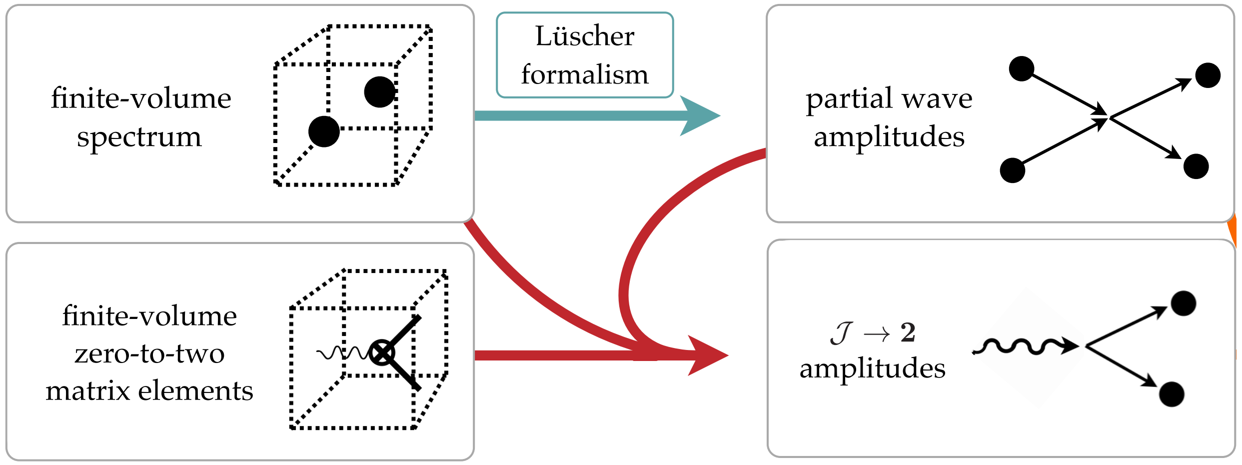
This talk:



finite-volume spectrum

finite-volume zero-to-two matrix elements

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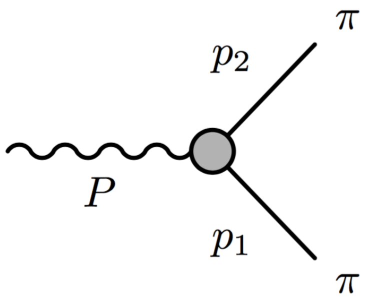
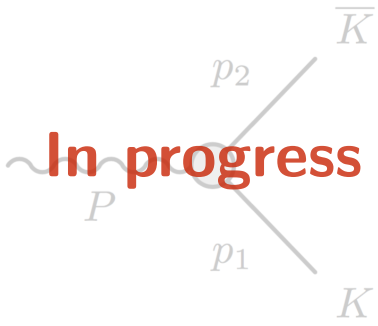
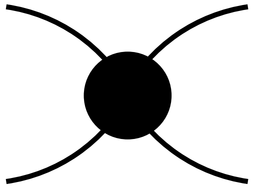
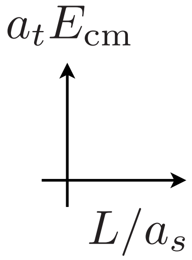


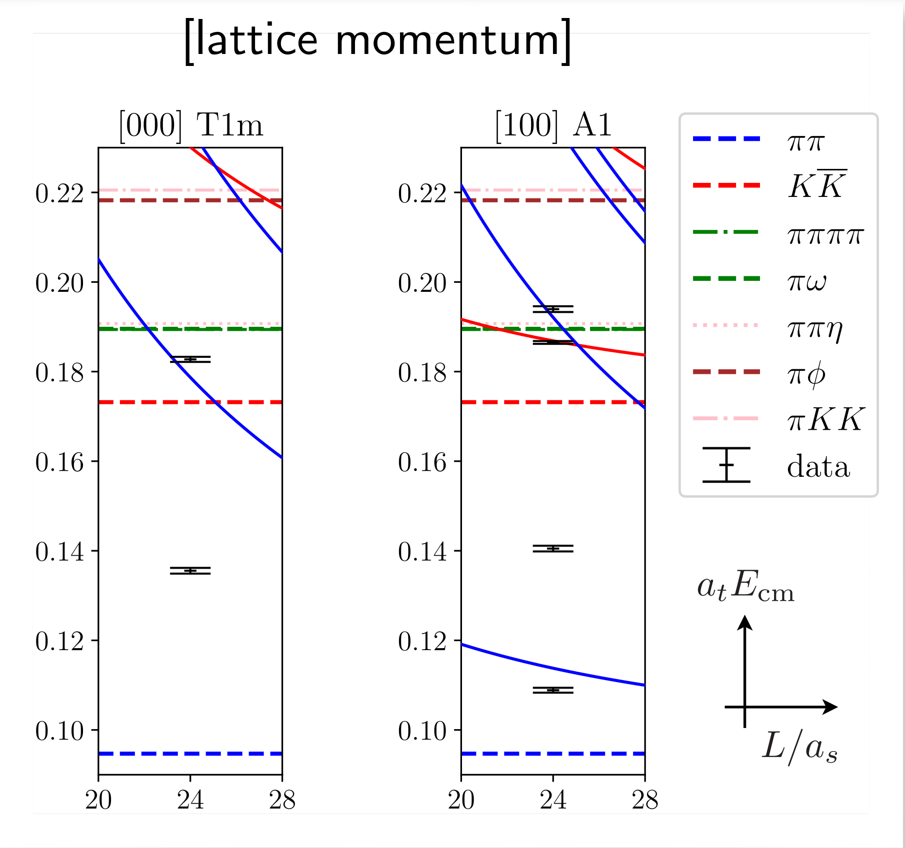


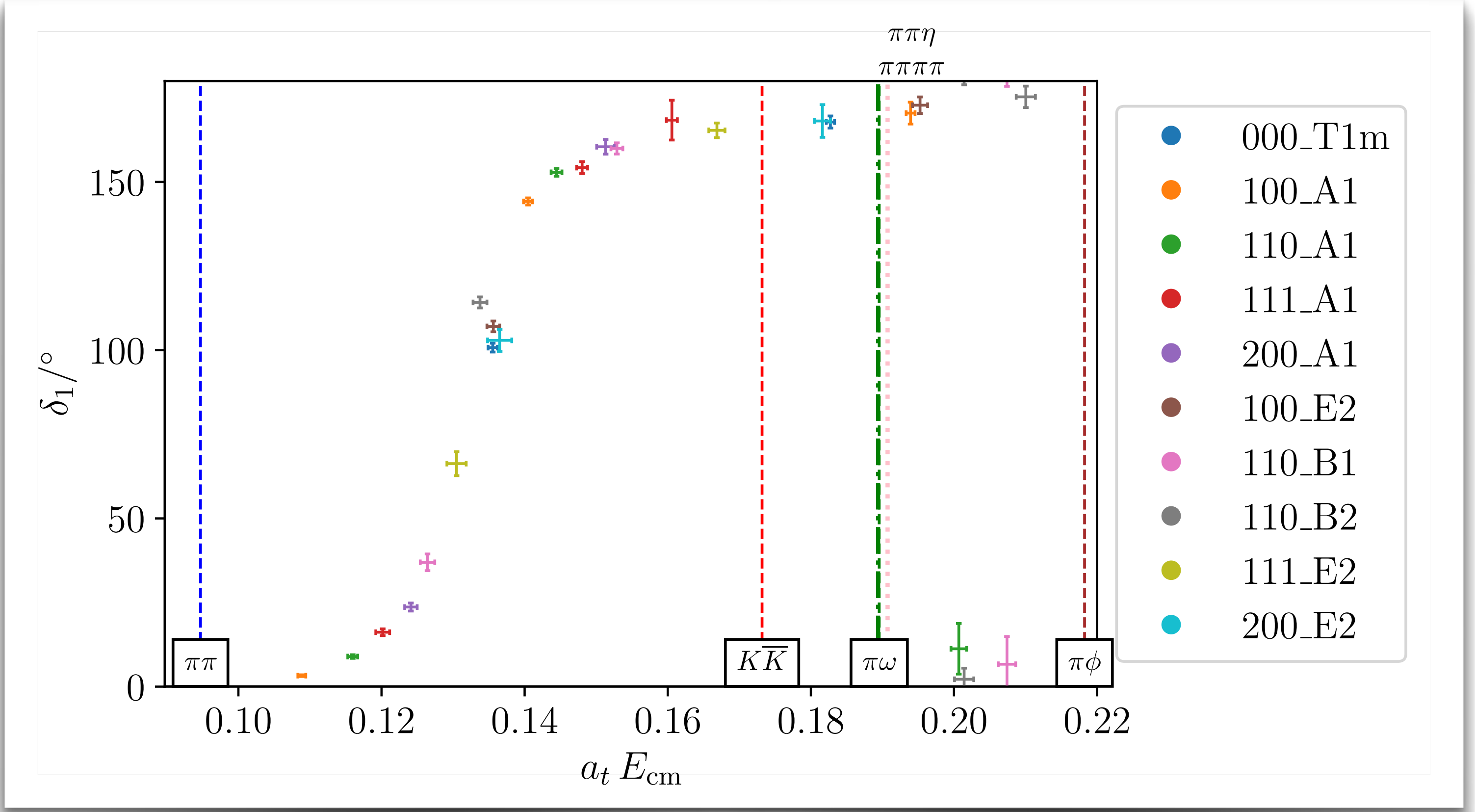
In progress



Scattering in the finite volume







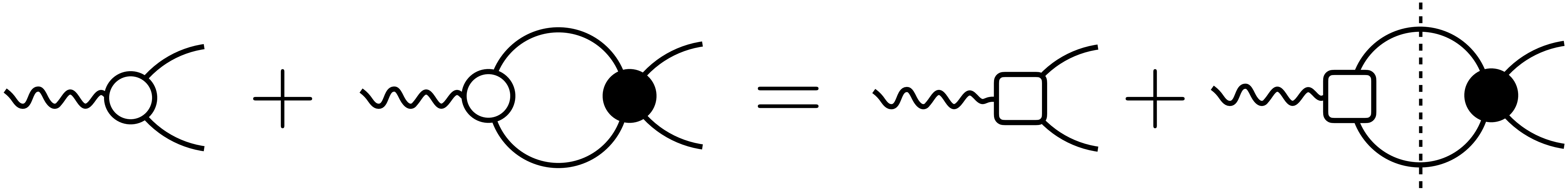
$$\det\left(F^{-1}(E_n, L) + \mathcal{M}(E_n)\right) = 0$$

Known geometric function

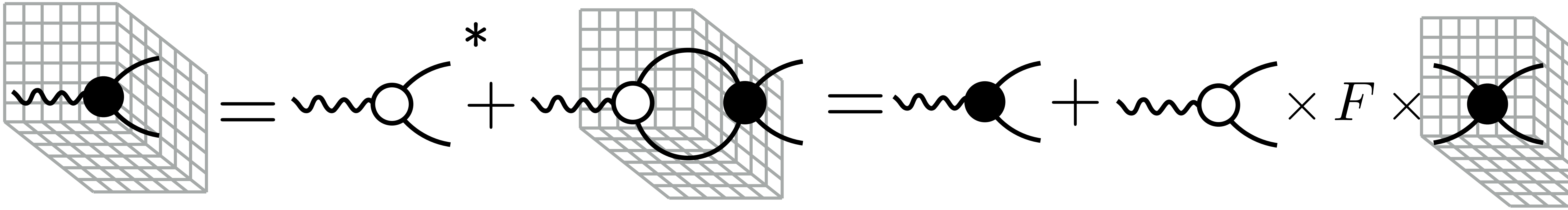
Finite-volume correlation function poles in momentum space

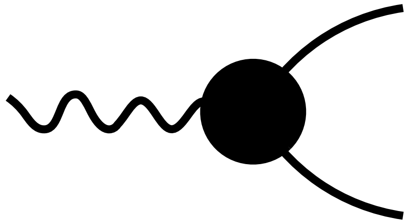
Finite volume corrections

Infinite volume: Watson's theorem. $\mathcal{H}(s) = \mathcal{A}_{02}(s)\,\mathcal{M}(s)$

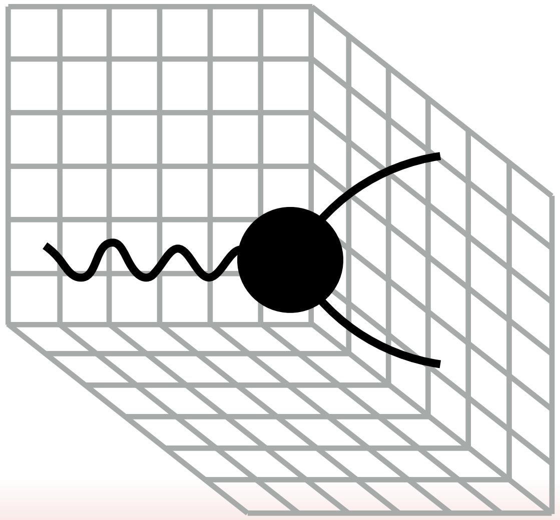


Finite volume: Lellouch-Lüscher factor.





$$\times \lim_{E\to E_n} \frac{E - E_n}{F^{-1} + \mathcal{M}} =$$



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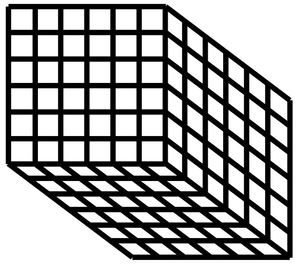
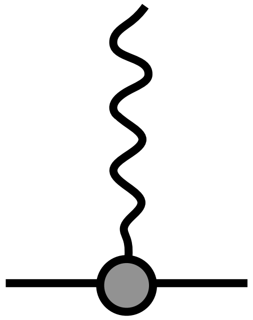
*Up to exponentially small corrections

Scattering and pair-production $J^P(I^G)=1^-(1^+)$

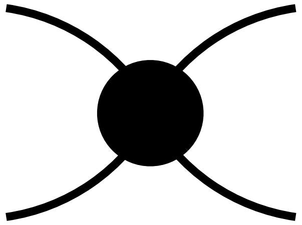


$$L = 2.7 \text{ fm}$$

	$a_t m$	m/MeV
π	0.0474	284
K	0.0866	519



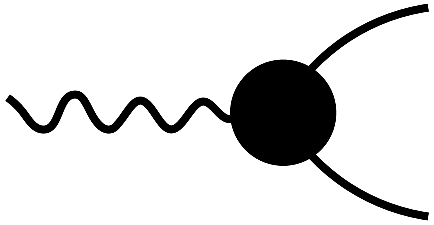
For renormalization of the current



$$C_{ab}(t-t_s) = \left\langle O_a(t) O_b^\dagger(t_s) \right\rangle$$

- Two-meson operators
- $q\bar{q}$ operators

1. GEVP $\quad C_{ab}(t) v_b^n = C_{ab}(t_0) v_b^n \lambda_n(t-t_0)$
2. Fit eigenvalues $\quad \lambda_n(t-t_0) = e^{-E_n(t-t_0)}$

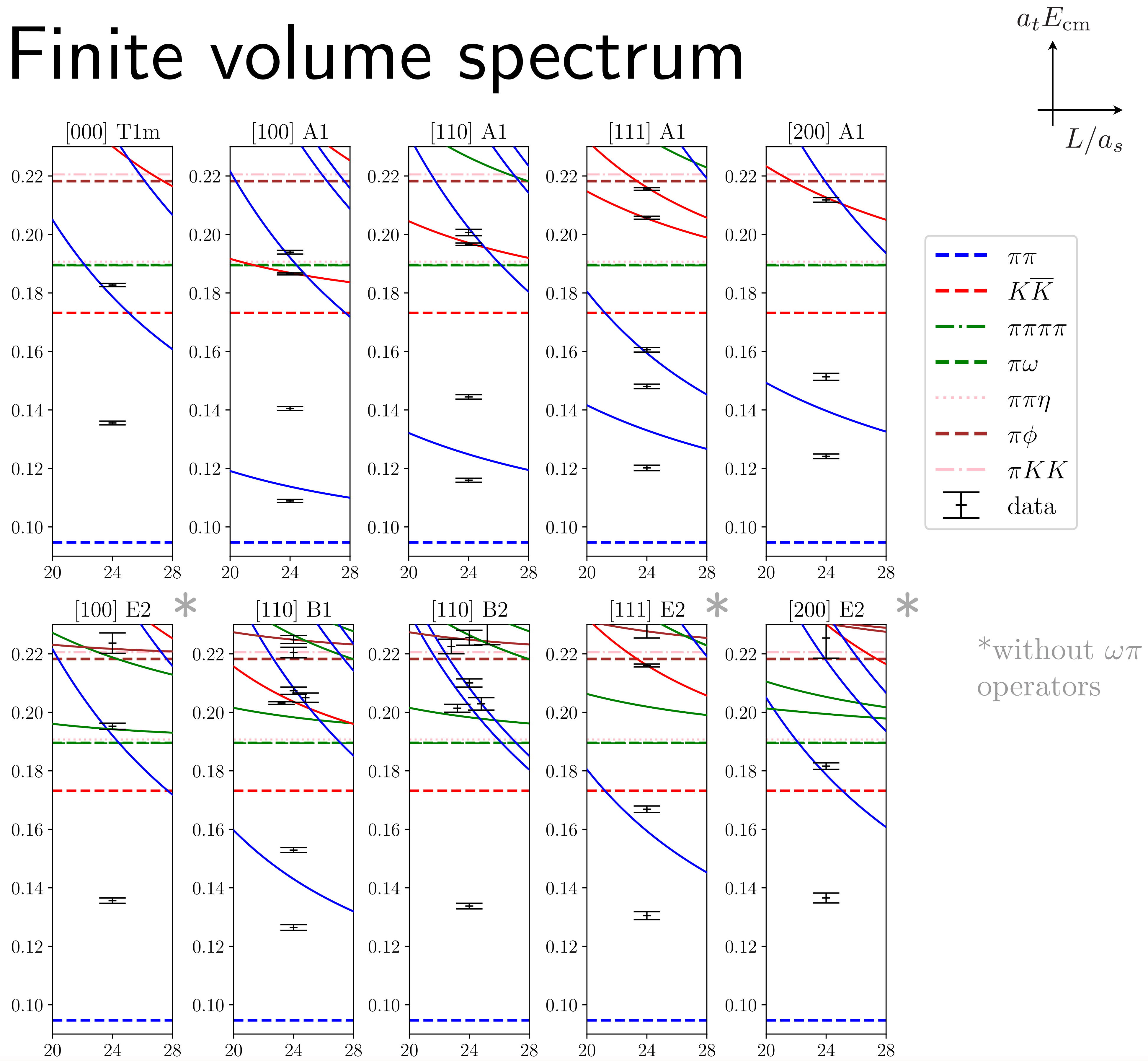


$$\left\langle \mathcal{J}(t) \Omega_n^\dagger(0) \right\rangle \propto \langle 0 | \mathcal{J}(0) | n \rangle$$

- Optimized operators

1. Extract invariant $\quad \mathcal{H}_L^\mu = K^\mu \mathcal{F}_L$
2. FV correction $\quad \mu_0'^\star = \dfrac{\partial}{\partial E^\star}(\mathcal{M}^{-1} + F)$

Finite volume spectrum



*without $\omega\pi$ operators

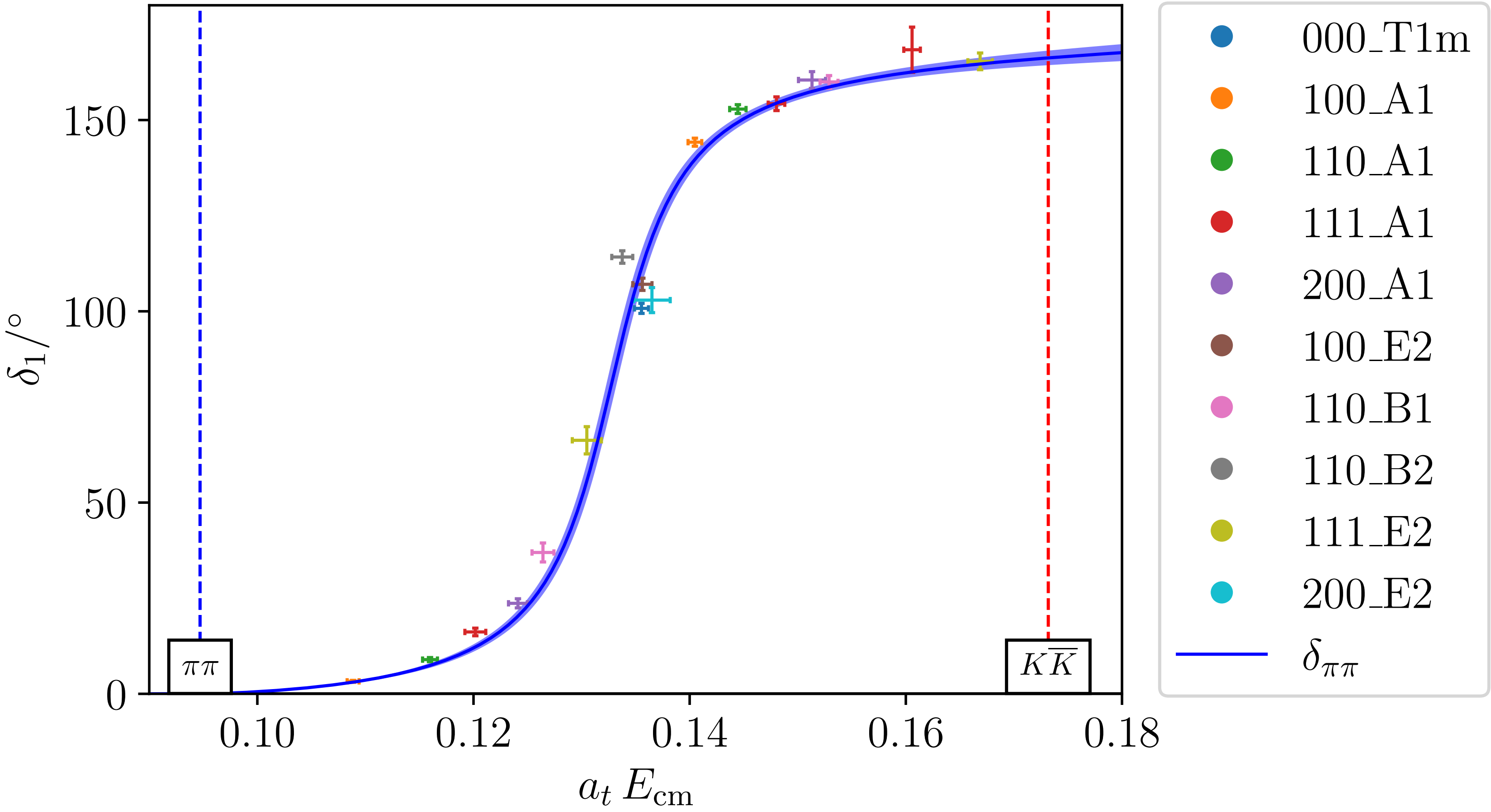
- 17 elastic levels
- 25 $\pi\pi$-like levels
- 7 KK-like levels

Elastic scattering

$$\mathcal{M}_\ell = \frac{E_{\mathrm{cm}}}{2q^\star} e^{i\delta_\ell} \sin\delta_\ell$$

$$\mathcal{M}_\ell^{-1} = \frac{1}{(2q^\star)^\ell} K_\ell^{-1} \frac{1}{(2q^\star)^\ell} - i\rho_{\mathrm{CM}}$$

$$K_1 = \frac{g^2}{-s+m^2} + \gamma$$

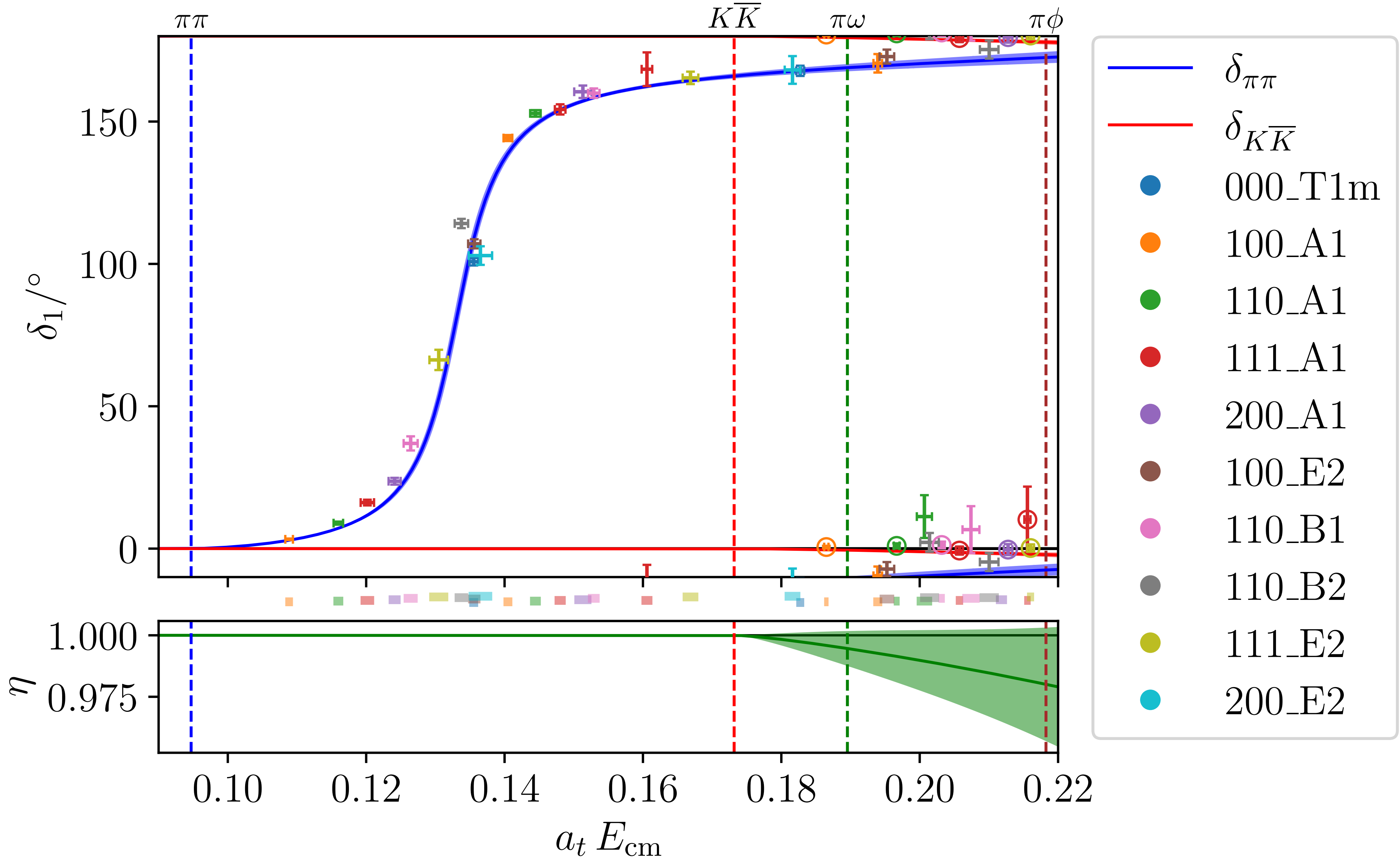


Coupled channel fit

$$\mathcal{M}_{\ell,aa} = \frac{\eta e^{2i\delta_{\ell,a}} - 1}{2i\rho_a}$$

$$K_{ab} = \frac{g_a g_b}{-s + m_r^2} + \gamma_{ab}$$

$$\mathcal{M}_{\ell,ab} = \frac{\sqrt{1-\eta^2} e^{i(\delta_{\ell,a}+\delta_{\ell,b})}}{2\sqrt{\rho_a \rho_b}}$$

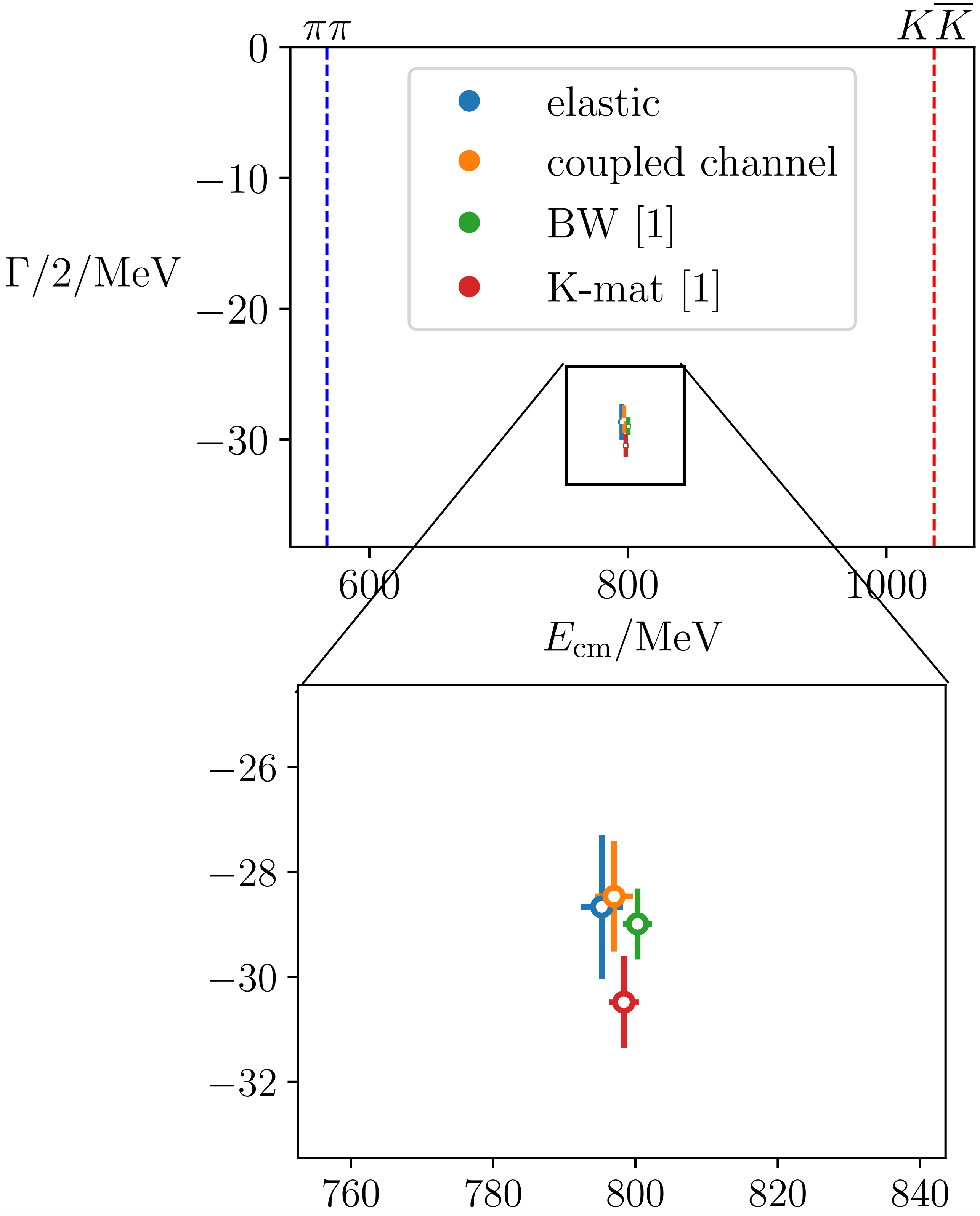


Rho resonance

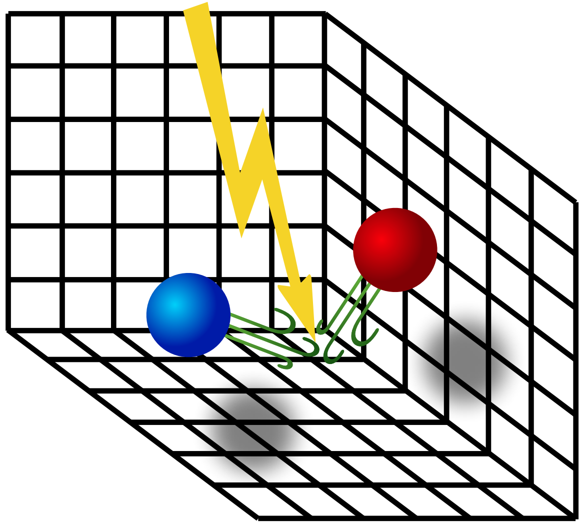
$$\mathcal{M}(s) \sim \frac{g_R^2}{(m_\rho - i\Gamma_\rho/2)^2 - s}$$

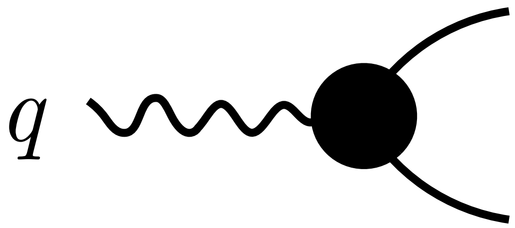
$$g_R = 254 \pm 5 \text{ MeV}$$

$$\mathrm{Re}(m_\rho) = 797 \pm 2.6 \text{ MeV}$$
$$\Gamma_\rho/2 = 28.5 \pm 1.0 \text{ MeV}$$

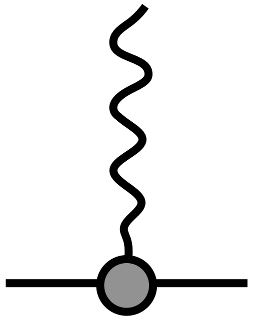


Pair production





$q^2 < 0$



$$\mathcal{H}^\mu_L = K^\mu \mathcal{F}_L$$

$$\mathcal{H}^\mu(q) = K^\mu f(q^2)$$

Finite-volume matrix elements

$$f = \boxed{\frac{\mathcal{M}}{q^{\star 2}} \frac{1}{\tilde{r}_n}} \mathcal{F}_L$$

Lellouch-Lüscher factor

$q^\star$

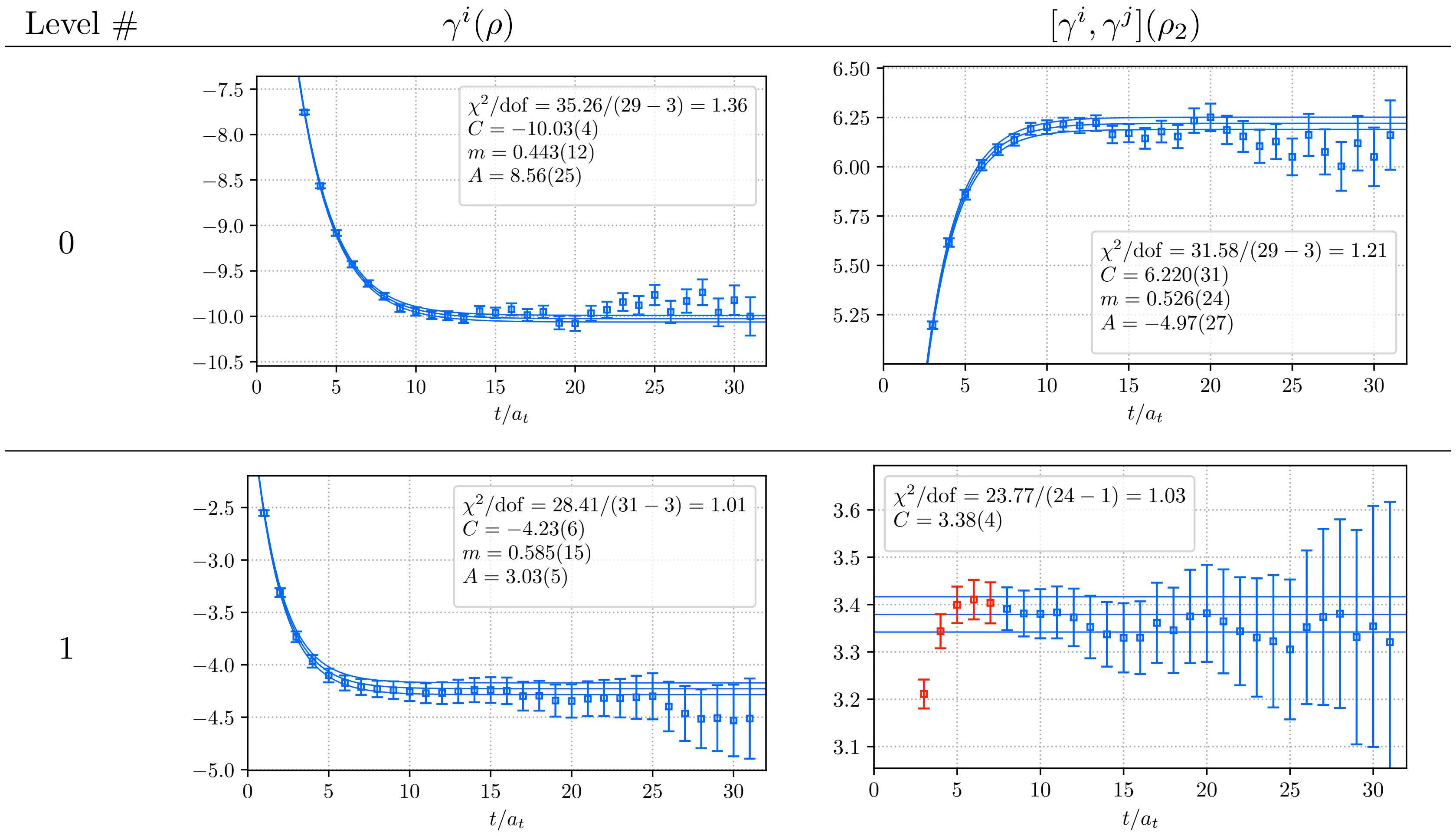
Relative momentum
in cm frame

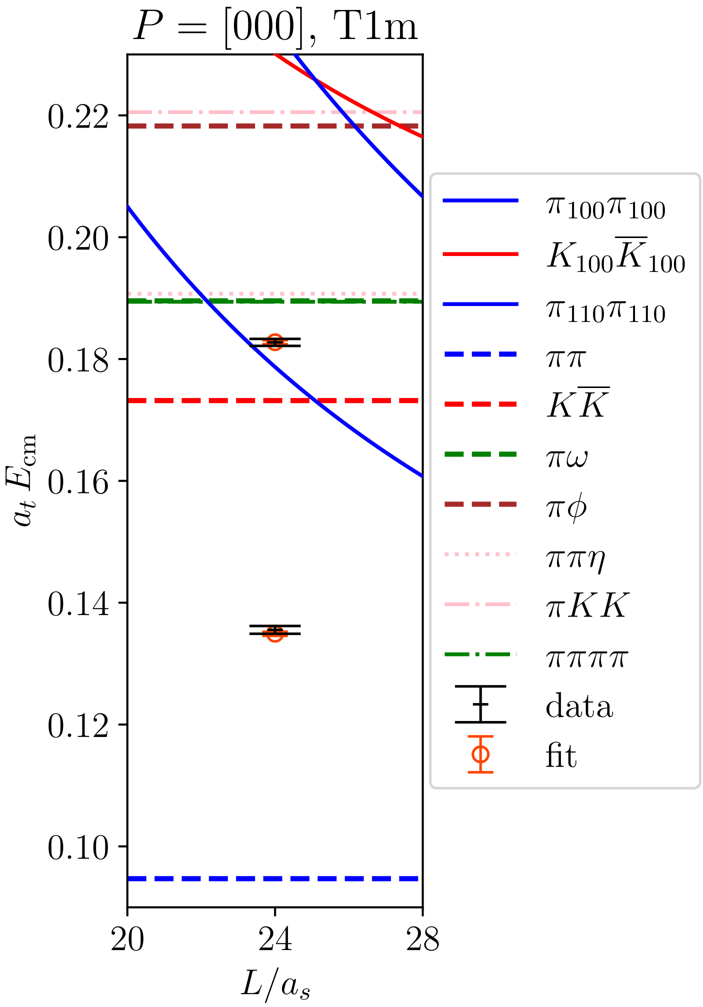
Matrix elements

$$\frac{\left\langle \mathcal{J}(t)\Omega_n^\dagger(0)\right\rangle}{\left\langle \Omega_n(t)\Omega_n^\dagger(0)\right\rangle} = \langle 0|\mathcal{J}(0)|n\rangle\, Z_n^{-1/2} + \mathcal{O}(e^{-(E_N-E_n)t})$$

Optimized operators

Fit function: $C + Ae^{-mt}$





$$\langle 0|\mathcal{J}^{\rho}_{\mathrm{impro}}|\mathfrak{n}\rangle = \langle 0|\rho|\mathfrak{n}\rangle - \frac{1}{4}(1-\xi)a_t E_n \,\langle 0|\rho_2|\mathfrak{n}\rangle$$

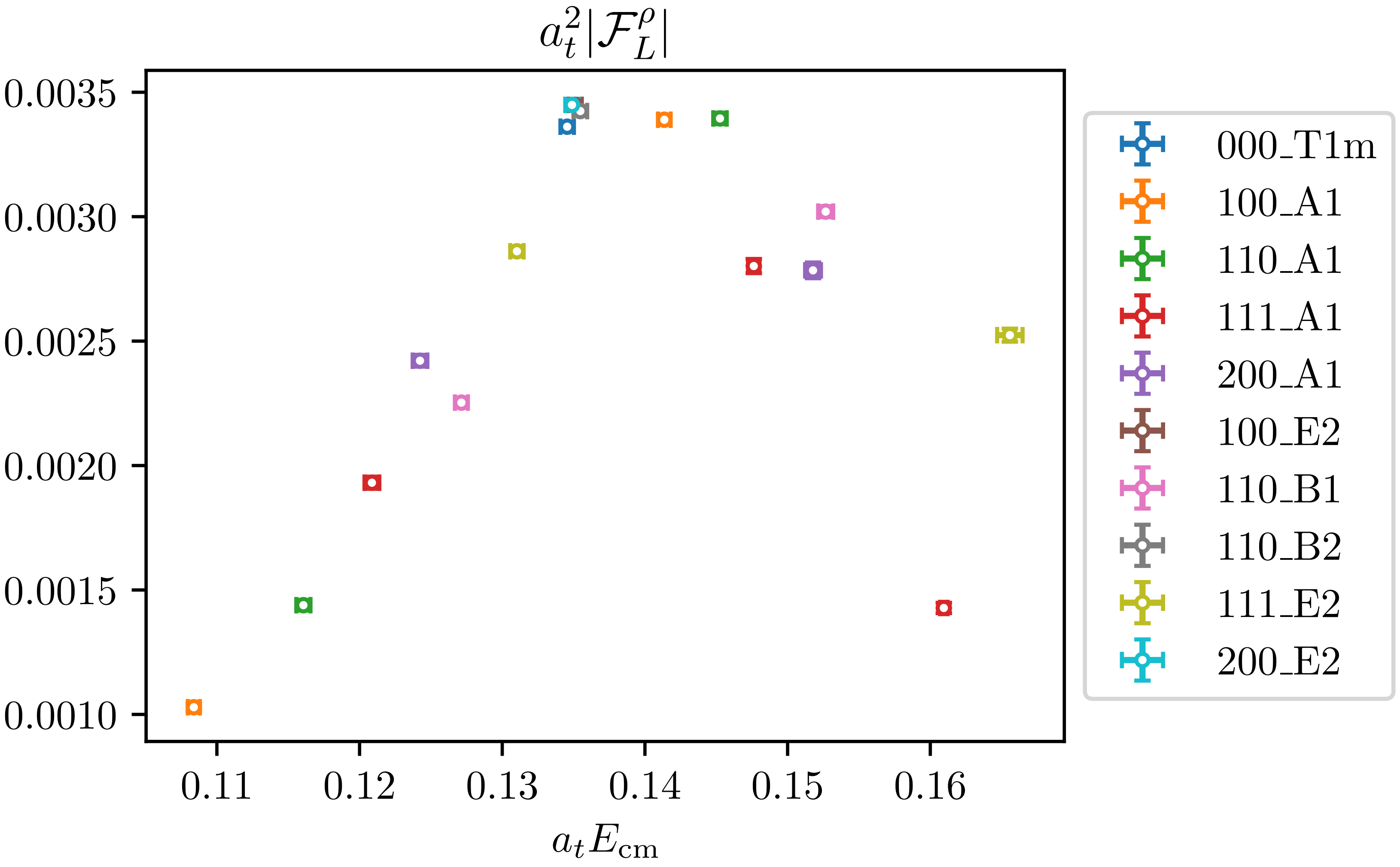
anisotropy

Invariant matrix element

$$\mathcal{H}^{\mu}_{L} = K^{\mu}\mathcal{F}_{L}$$

$$\langle 0|\mathcal{J}^{\mu}(0)|n, P, 1\lambda\rangle = \frac{2}{\sqrt{3}}\epsilon^{\mu}(P,\lambda)\mathcal{F}_{L}(P^{2})$$

$$\mathcal{F}_{L}(P^{2}) = \sqrt{\frac{2E_{n}}{L^{3}}}\frac{Z_{n}^{1/2}Z_{V}C_{n}^{\mathbf{d},\Lambda,\rho}}{K_{n}(\vec{d},\Lambda,J)}$$



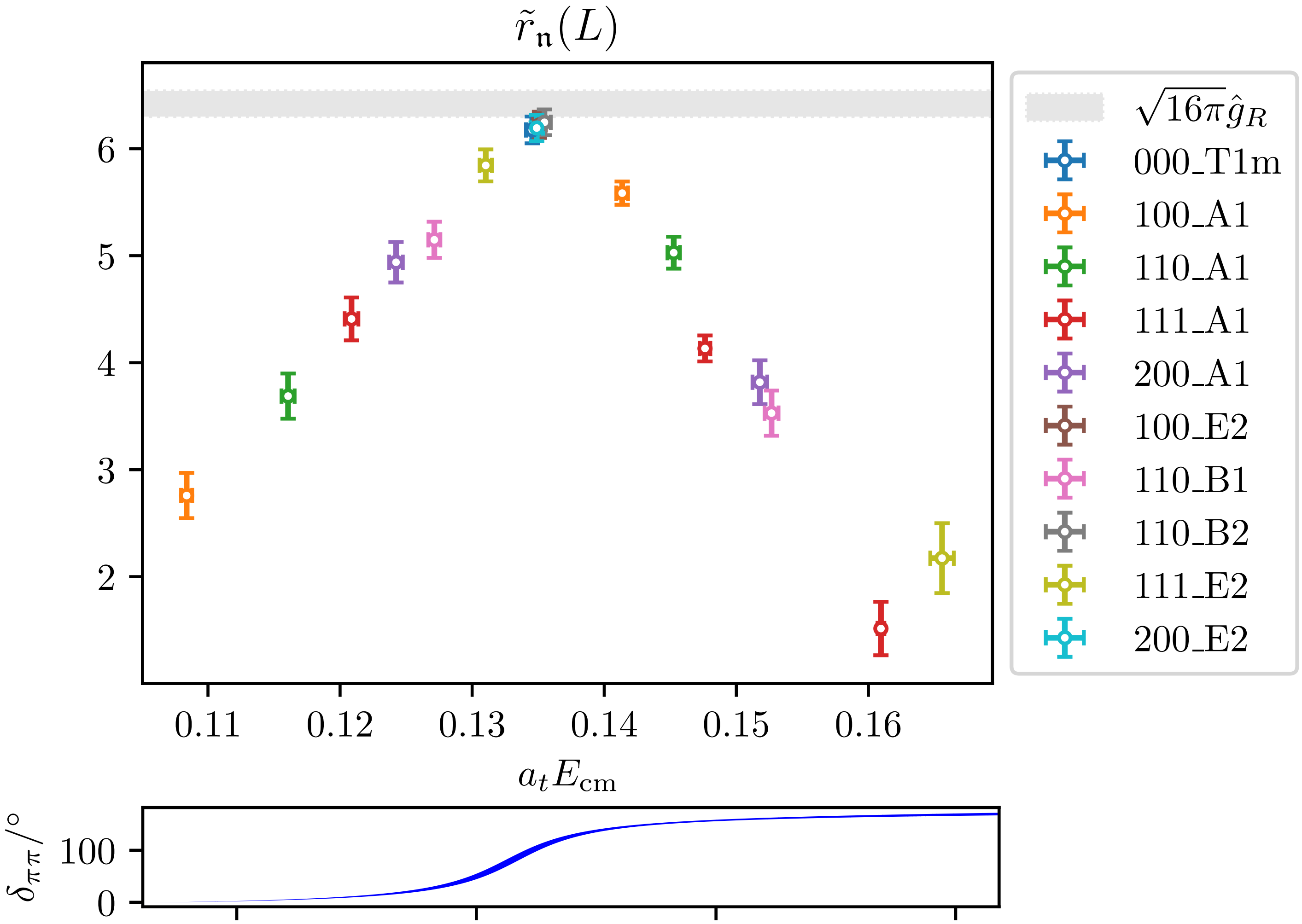
Finite volume correction

- Lellouch-Lüscher factor

$$\mu_0'^{\star} = \frac{\partial}{\partial E^\star}\left(\mathcal{M}^{-1} + F\right)$$

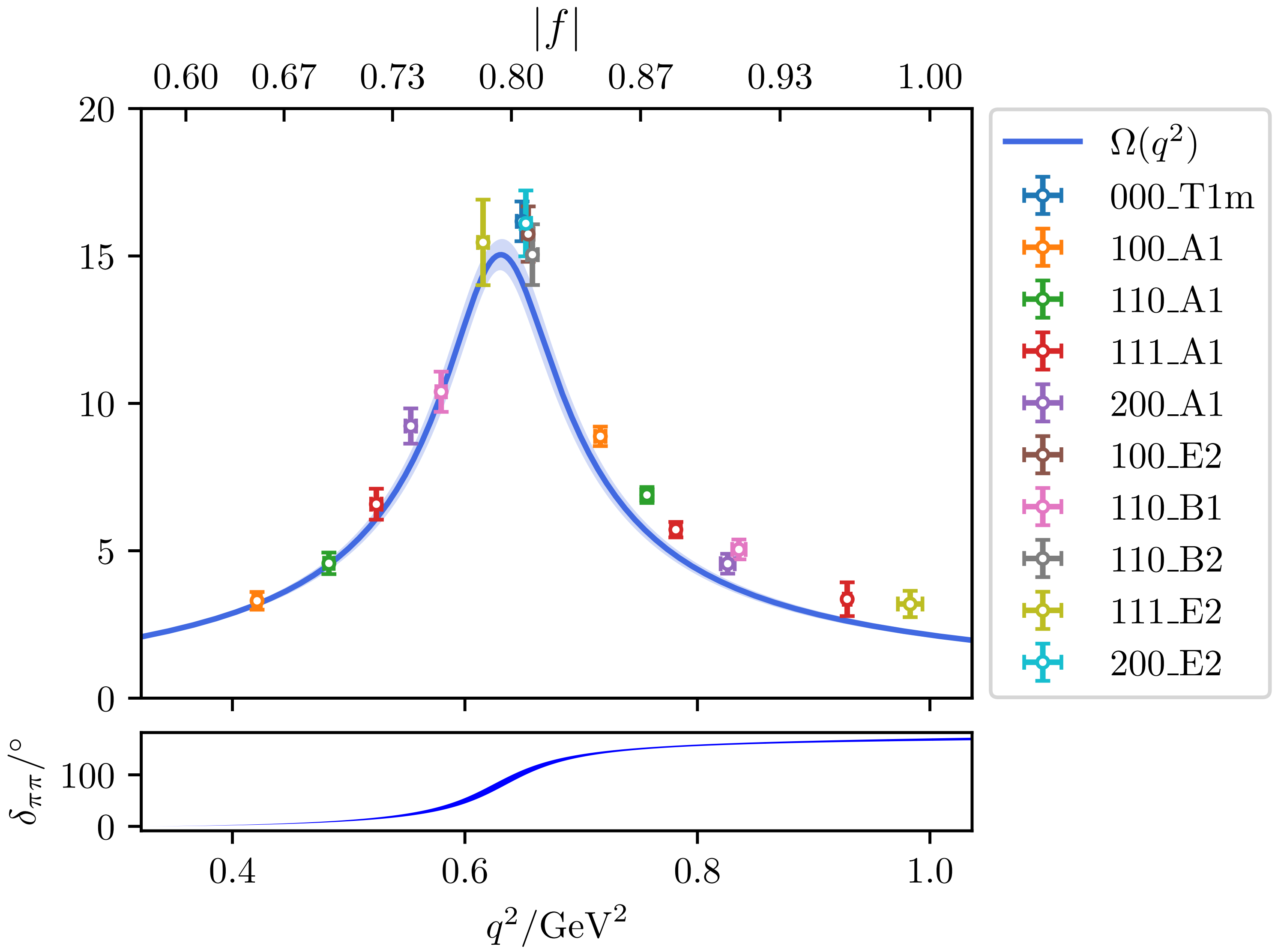
$$\tilde{r}_{\mathfrak{n}} = \frac{1}{q^\star}\sqrt{\frac{-2E_n^\star}{\mu_0'^\star}} \sim \sqrt{16\pi}g_R/q^\star$$

In the narrow width limit



Timelike form factor

$$f = \frac{\mathcal{M}}{q^{\star 2}} \frac{1}{\tilde{r}_n} \mathcal{F}_L$$



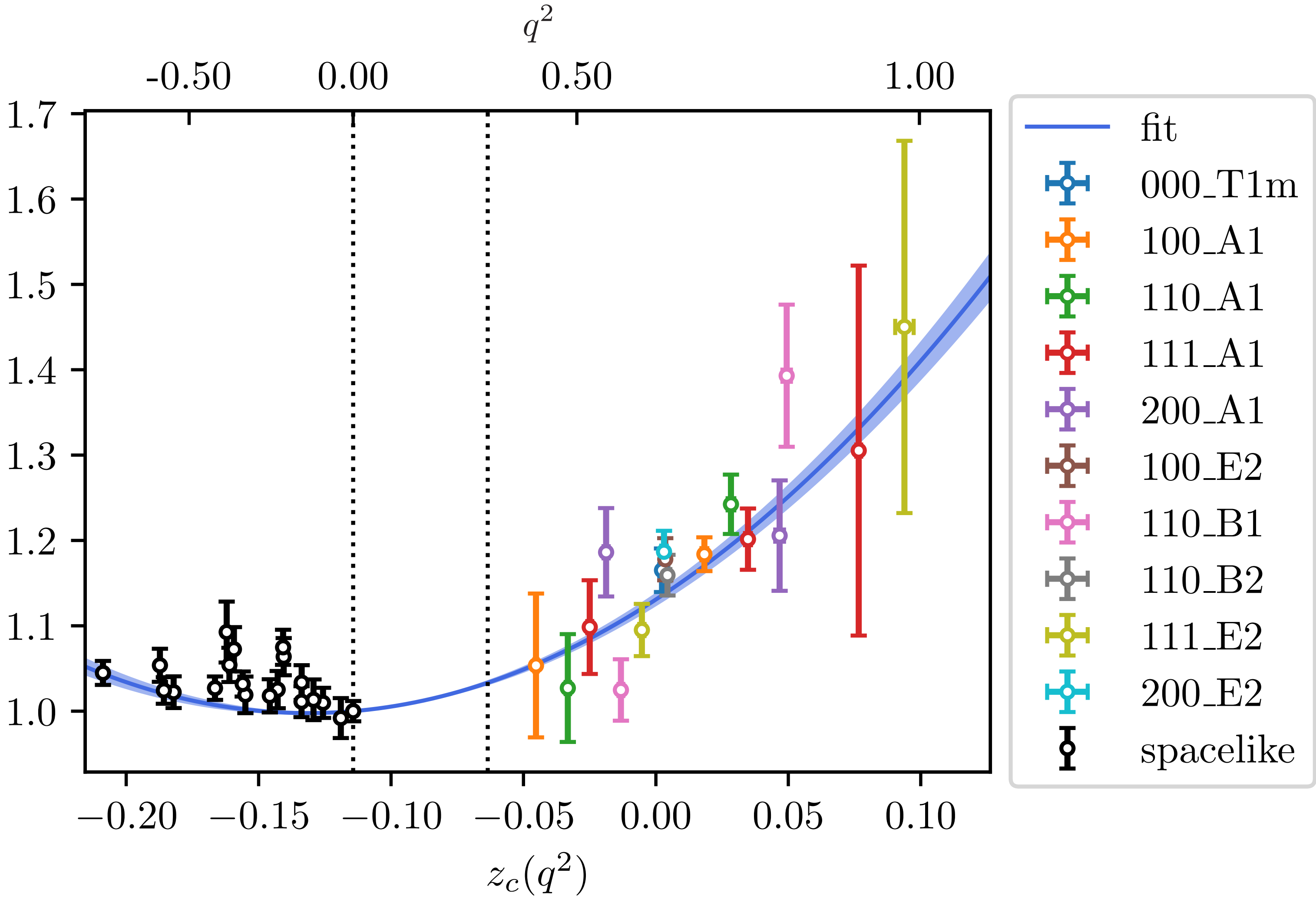
Watson's theorem:

$$f = \Omega \times \mathcal{F}$$

$$\Omega(s) = \exp\left(\frac{s}{\pi} \int_{s_{\mathrm{thr}}}^{\infty} \mathrm{d}s' \frac{\delta(s')}{s'(s'-s)}\right)$$

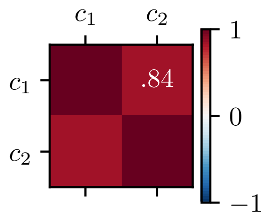
Smooth function fit

$$\mathcal{F} = f/\Omega$$



$$\chi^2/n_{\rm dof} = 87.7/(37-2) = 2.51$$

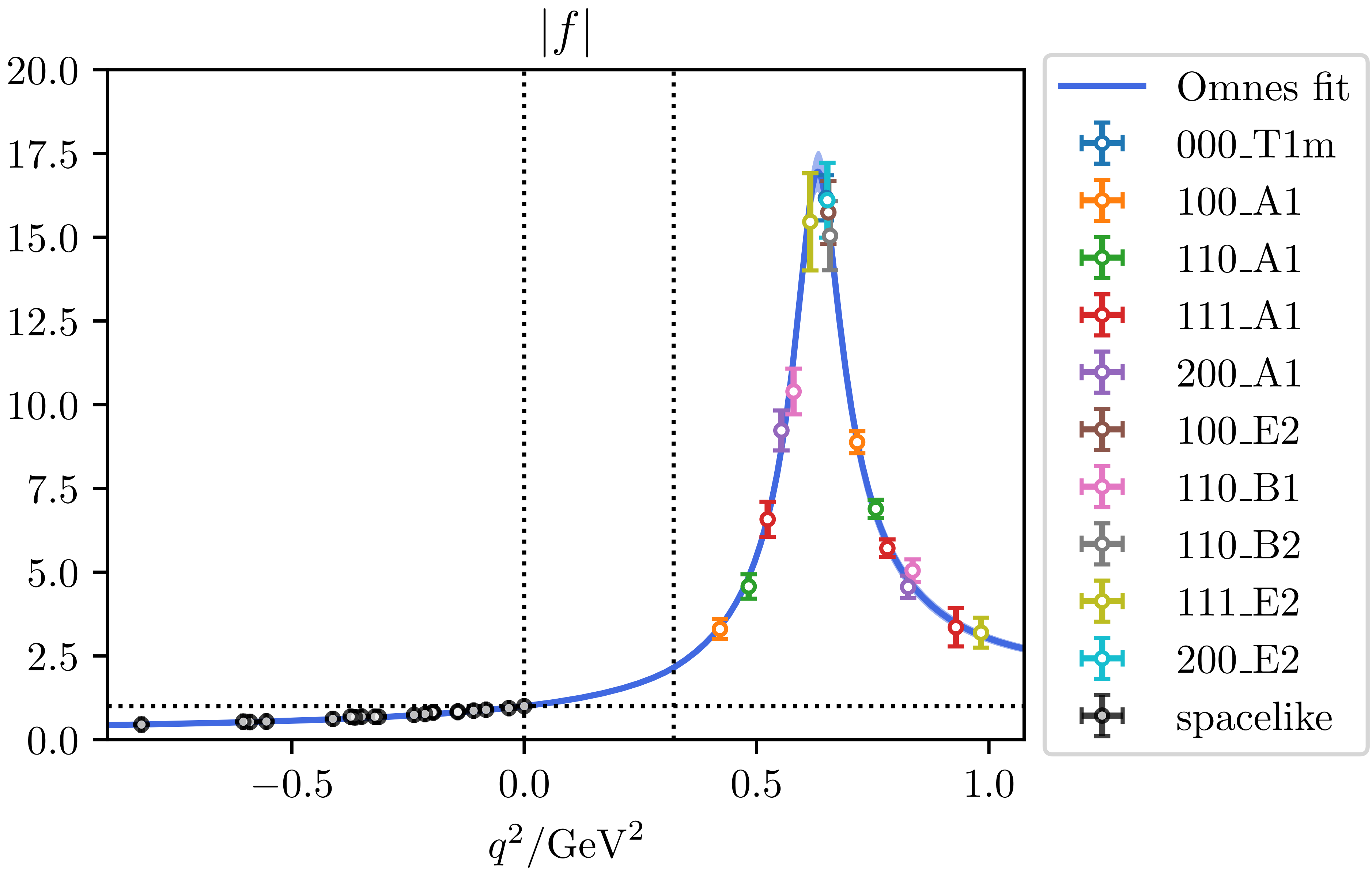
$$Q \text{ [fixed]} = 1(0)$$
$$c_1 = 2.03(11)$$
$$c_2 = 7.7(5)$$



$$\mathcal{F}(q^2) = Q + \sum_{n=1}^{N} c_n \left(z_c(q^2) - z_c(0)\right)^n$$

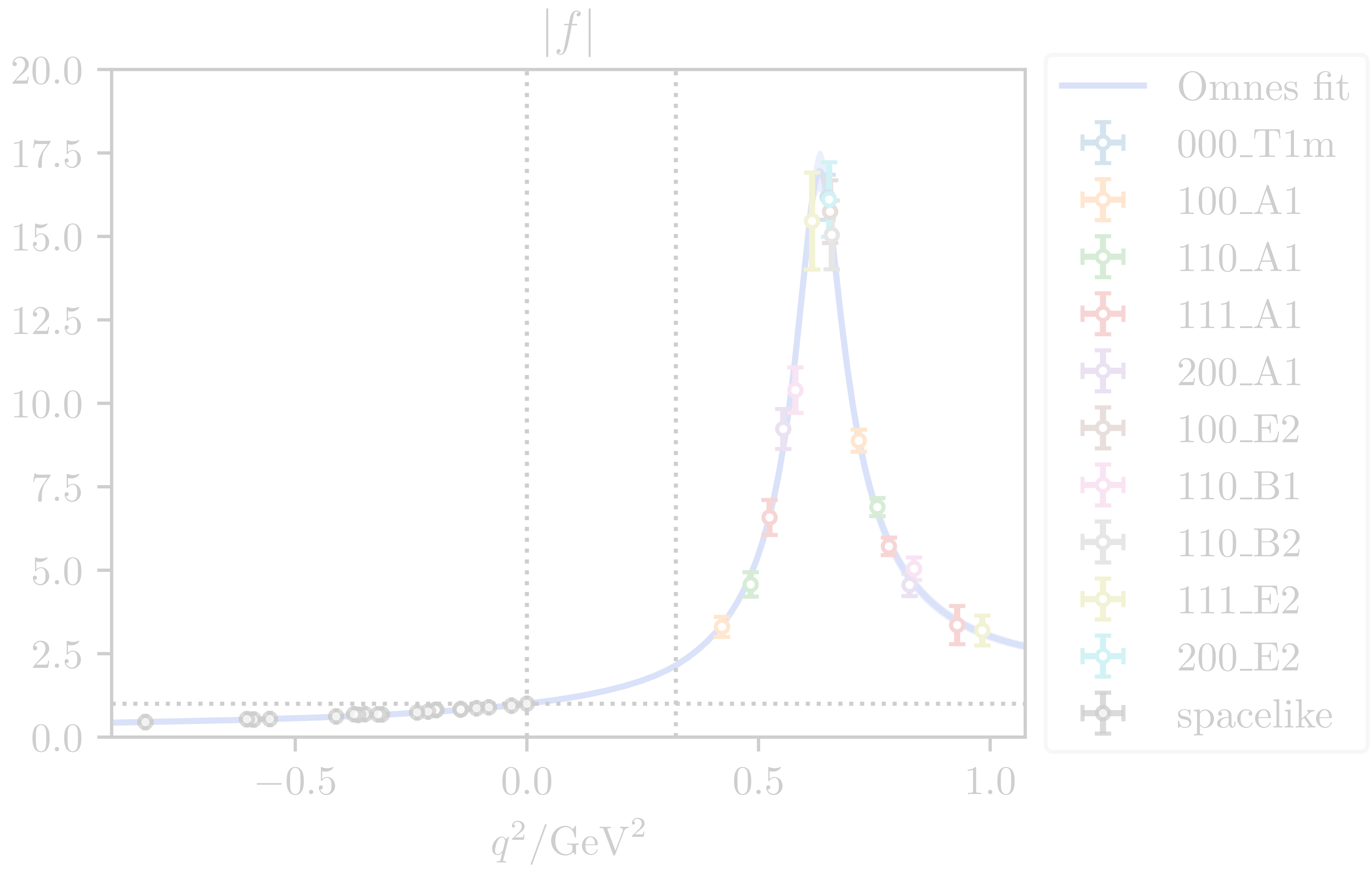
$$z_c(q^2) = \frac{\sqrt{s_c - s_{in}} - \sqrt{s_c - q^2}}{\sqrt{s_c - s_{in}} + \sqrt{s_c - q^2}}$$

Form factor fit: spacelike and timelike region



$$\sqrt{\langle r_\pi^2 \rangle} = 0.616(7)\ \text{fm}$$

Form factor fit: spacelike and timelike region



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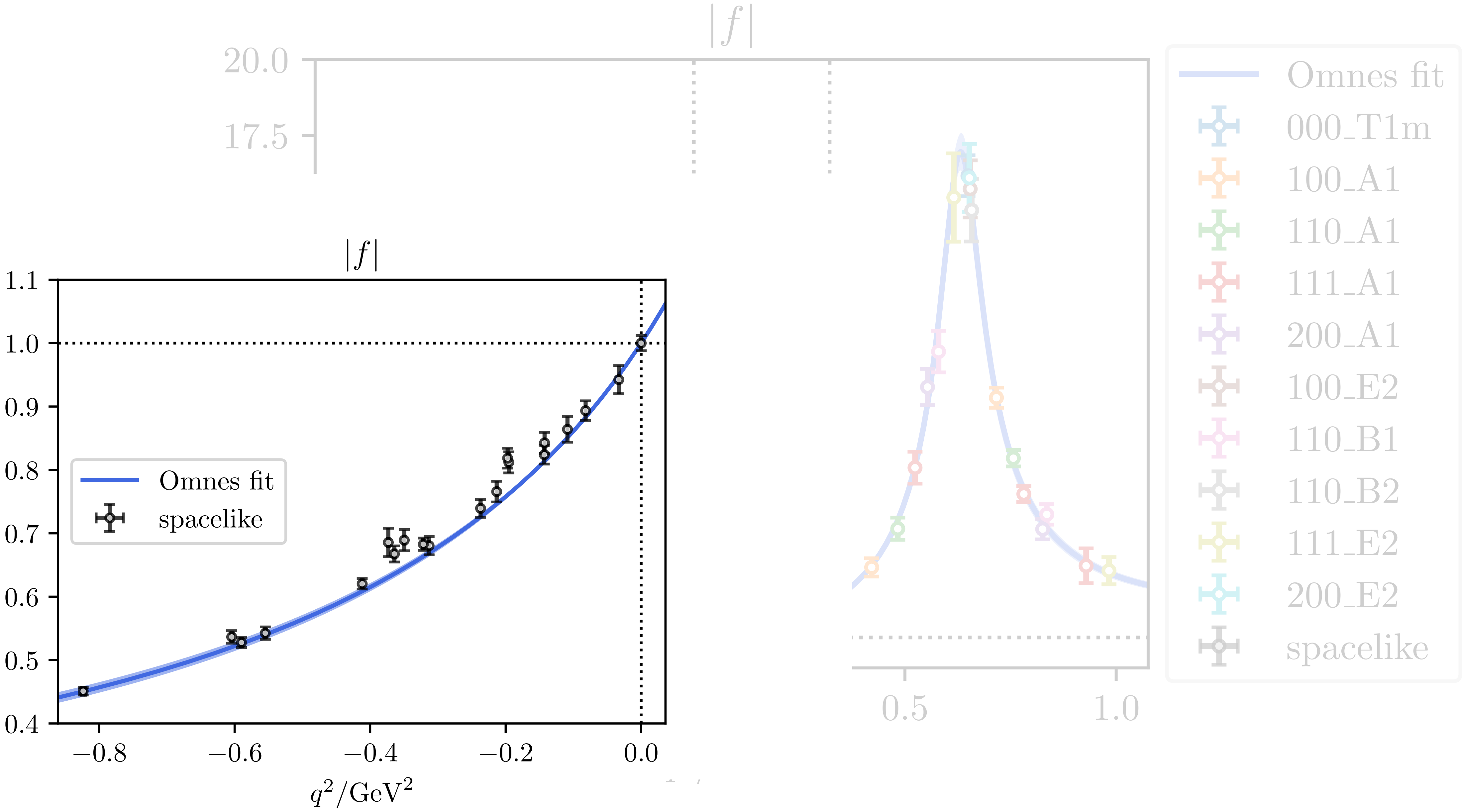

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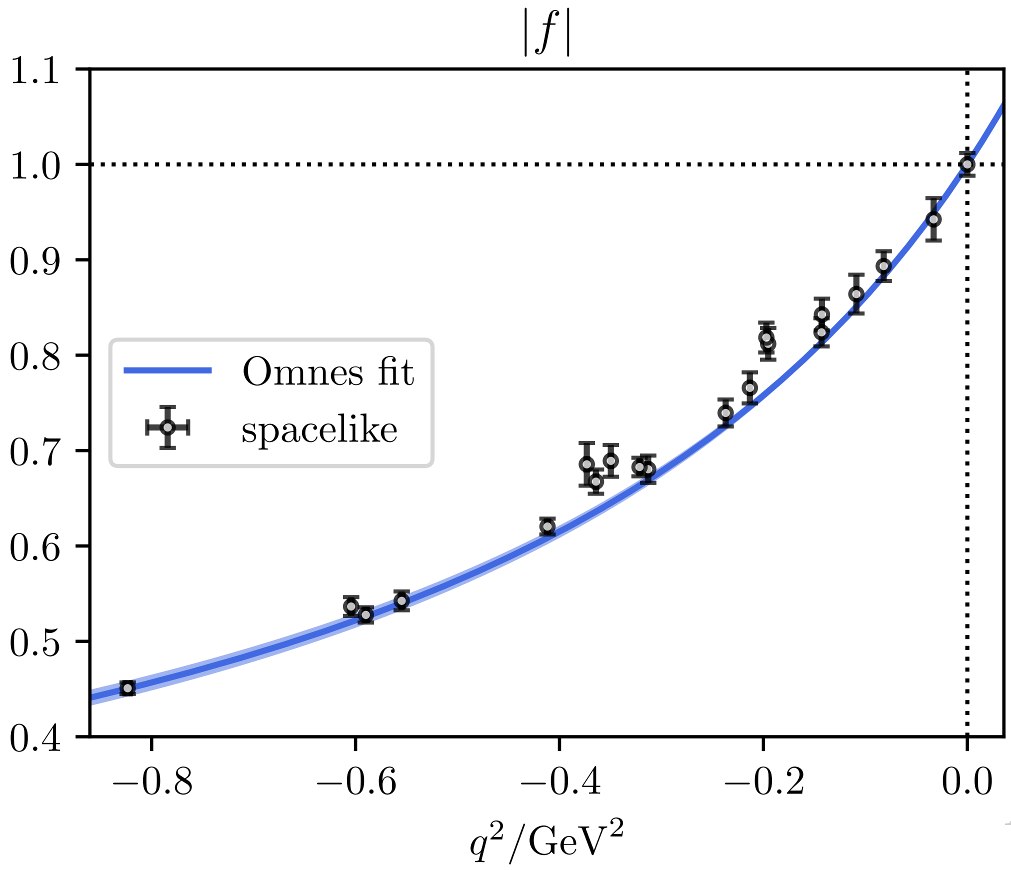
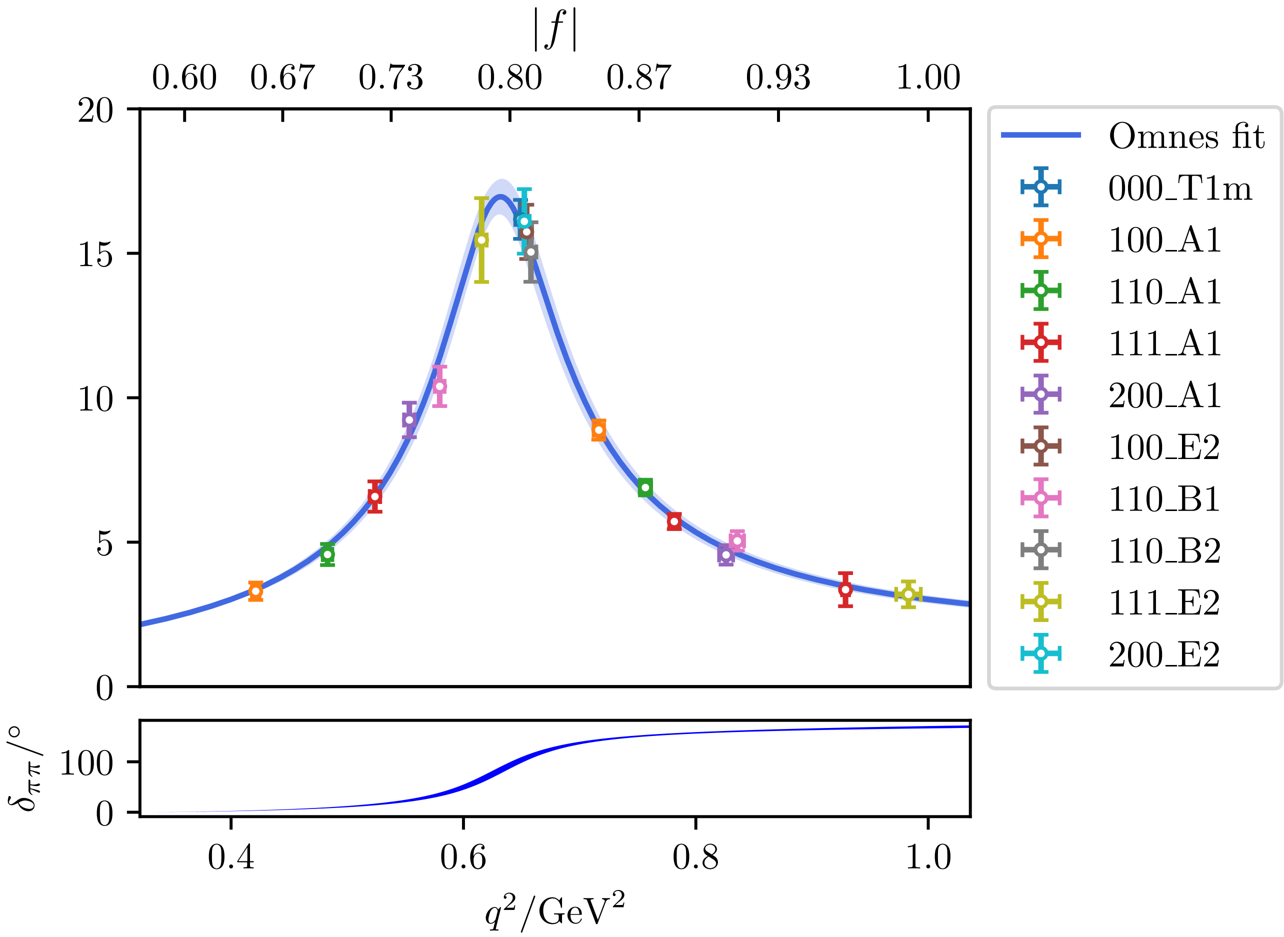
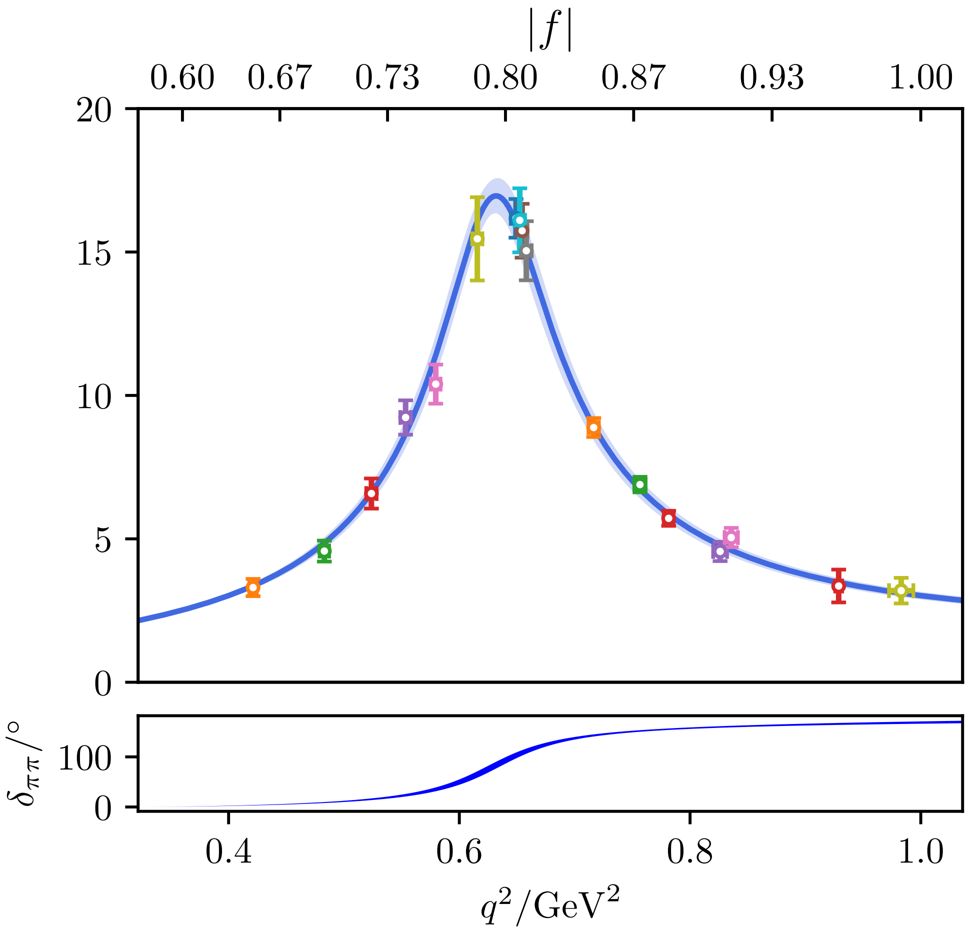


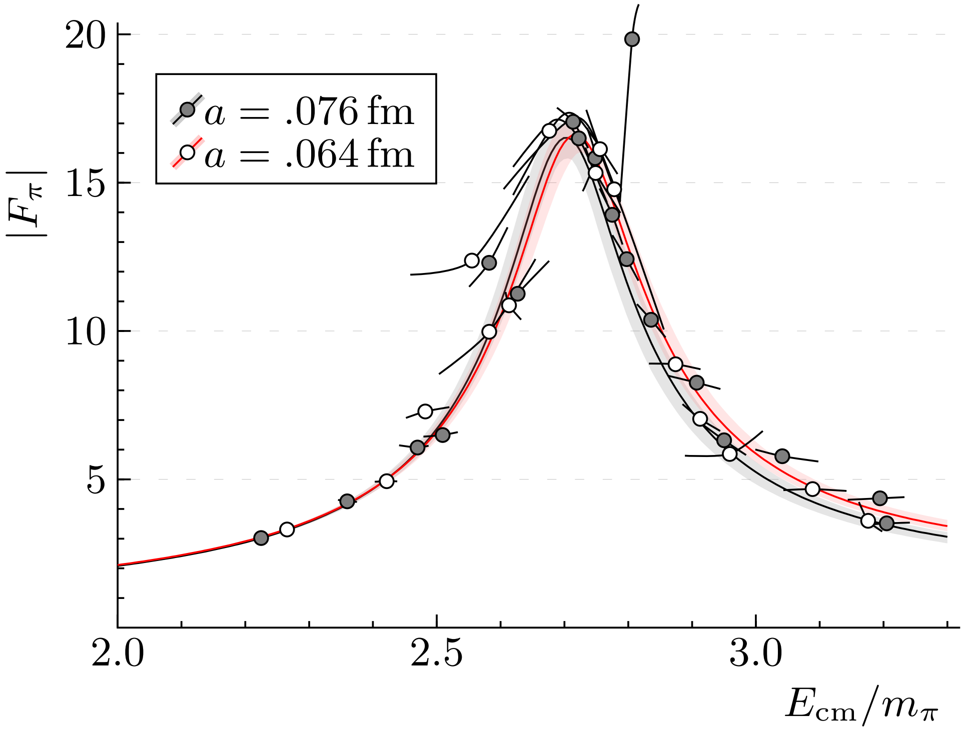


$$\sqrt{\langle r_\pi^2 \rangle} = 0.616(7)\ \mathrm{fm}$$

Consistency with existing results

$$m_\pi \approx 280 \text{ MeV}$$



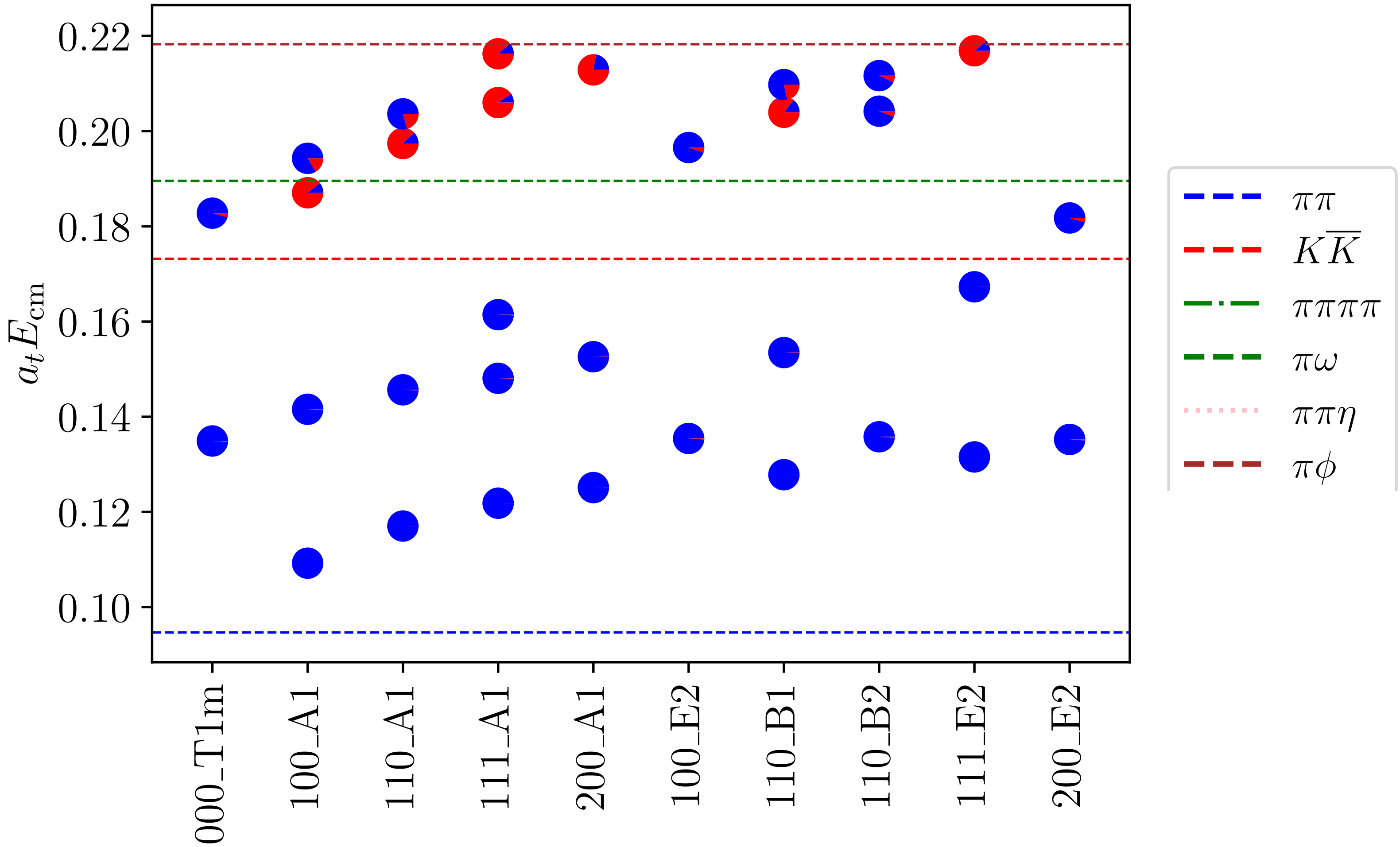


[arXiv:1808.05007] Andersen, et al.

Coupled channel Finite Volume correction

$$\lambda_0'^{\star} w_0 w_0^{\intercal} = \frac{\partial}{\partial E^{\star}}\left(\mathcal{M} + F^{-1}\right)$$

$$\sum_a w_{0,a}^2 = 1$$



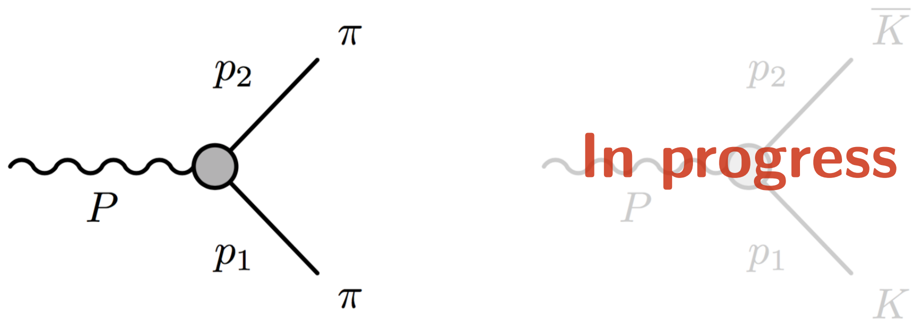
$$\mathcal{F}_L \propto w_{0,a} f_a$$

Summary and outlook

- Isovector *p*-wave in elastic and coupled channel regions
  - Finite-volume spectrum
  - Scattering amplitude

- Pair production amplitude
  - Zero-to-two finite volume matrix elements
  - Lellouch-Lüscher factor
  - Form factor fit across spacelike and timelike region

- Future work to extend the analysis to the coupled channel region.



Back up slides

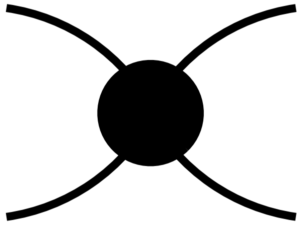
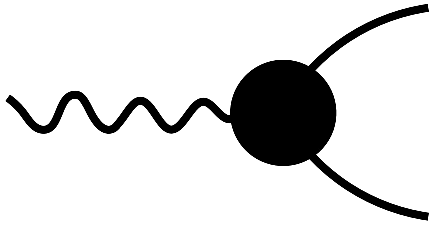
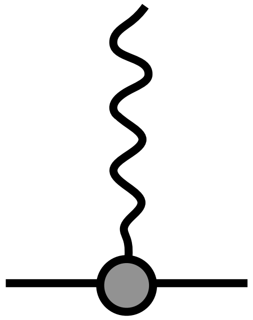
Operator basis

	$a_t E_{\rm th}$	$E_{\rm th}/{\rm MeV}$
$\pi\pi$	0.0947	567
$K\overline{K}$	0.1732	1037
$\pi\pi\pi\pi$	0.1894	1134
$\pi\omega$	0.1896	1135
$\pi\pi\eta$	0.1908	1142
$\pi\phi$	0.2183	1307
$\pi K\overline{K}$	0.2206	1320
$\pi\eta\,\omega$	0.2856	1710

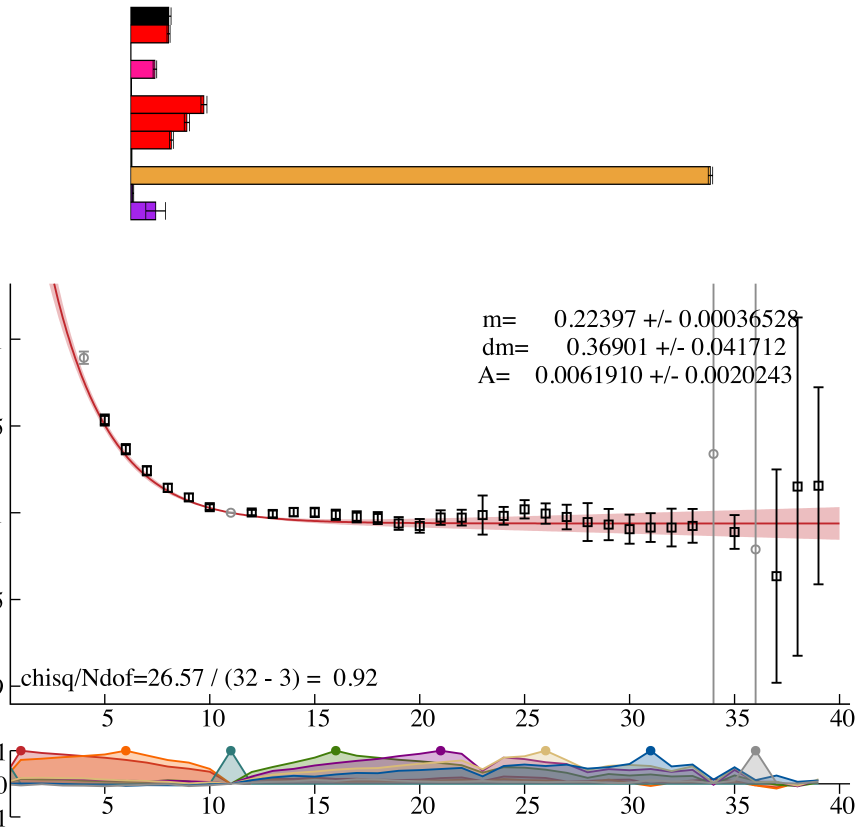
[000] T_1^-	[100] A_1	[110] A_1	[111] A_1	[200] A_1
$11 \times \bar{\psi}\mathbf{\Gamma}\psi$	$8 \times \bar{\psi}\mathbf{\Gamma}\psi$	$9 \times \bar{\psi}\mathbf{\Gamma}\psi$	$10 \times \bar{\psi}\mathbf{\Gamma}\psi$	$11 \times \bar{\psi}\mathbf{\Gamma}\psi$
$\pi_{[100]}\pi_{[100]}$	$\pi_{[100]}\pi_{[000]}$	$\pi_{[110]}\pi_{[000]}$	$\pi_{[111]}\pi_{[000]}$	$\pi_{[200]}\pi_{[000]}$
	$K_{[100]}\overline{K}_{[000]}$	$K_{[110]}\overline{K}_{[000]}$	$\pi_{[110]}\pi_{[100]}$	$K_{[200]}\overline{K}_{[000]}$
	$\pi_{[110]}\pi_{[100]}$	$\pi_{[111]}\pi_{[100]}$	$K_{[111]}\overline{K}_{[000]}$	
			$K_{[110]}\overline{K}_{[100]}$	

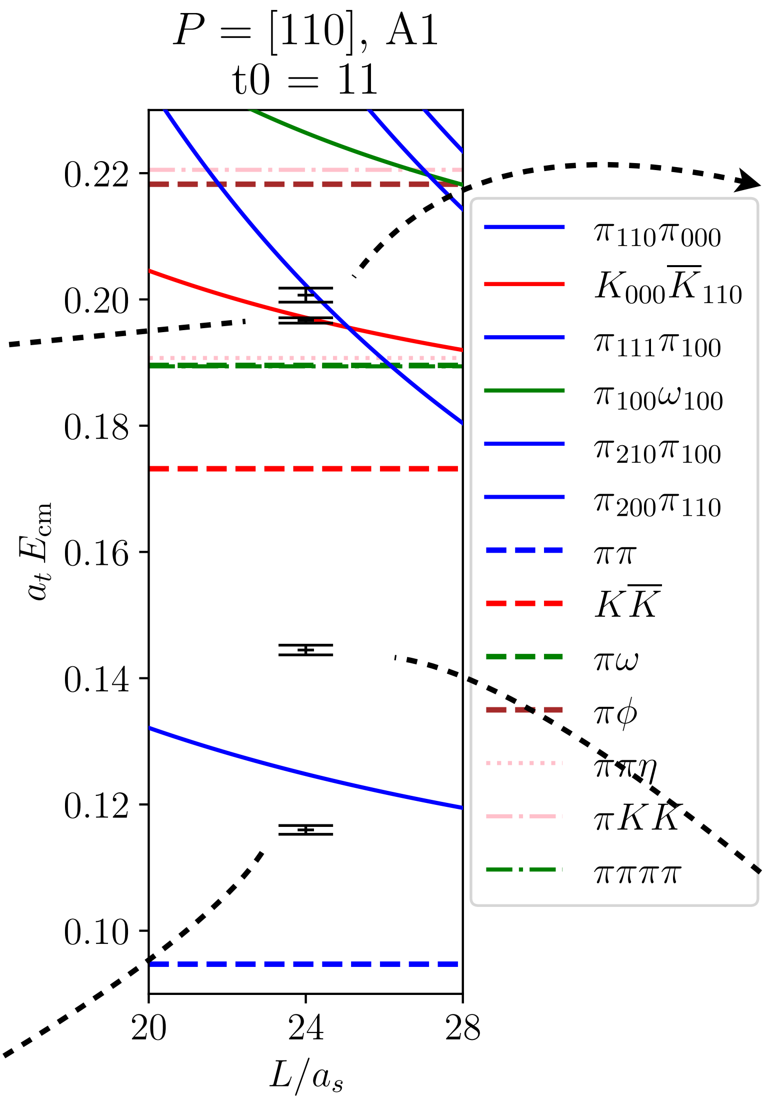
[100] E_2	[110] B_1	[110] B_2	[111] E_2	[200] E_2
$17 \times \bar{\psi}\mathbf{\Gamma}\psi$	$12 \times \bar{\psi}\mathbf{\Gamma}\psi$	$15 \times \bar{\psi}\mathbf{\Gamma}\psi$	$12 \times \bar{\psi}\mathbf{\Gamma}\psi$	$15 \times \bar{\psi}\mathbf{\Gamma}\psi$
$\pi_{[110]}\pi_{[100]}$	$\pi_{[100]}\pi_{[100]}$	$\omega_{[110]}\pi_{[000]}$	$\pi_{[110]}\pi_{[100]}$	$\pi_{[110]}\pi_{[110]}$
	$\omega_{[110]}\pi_{[000]}$	$\pi_{[111]}\pi_{[100]}$	$K_{[110]}\overline{K}_{[100]}$	
	$K_{[100]}\overline{K}_{[100]}$	$\pi_{[110]}\pi_{[110]}$		
	$\pi_{[110]}\pi_{[110]}$	$\phi_{[110]}\pi_{[000]}$		
	$\phi_{[110]}\pi_{[000]}$			
	$\omega_{[100]}\pi_{[100]}$			

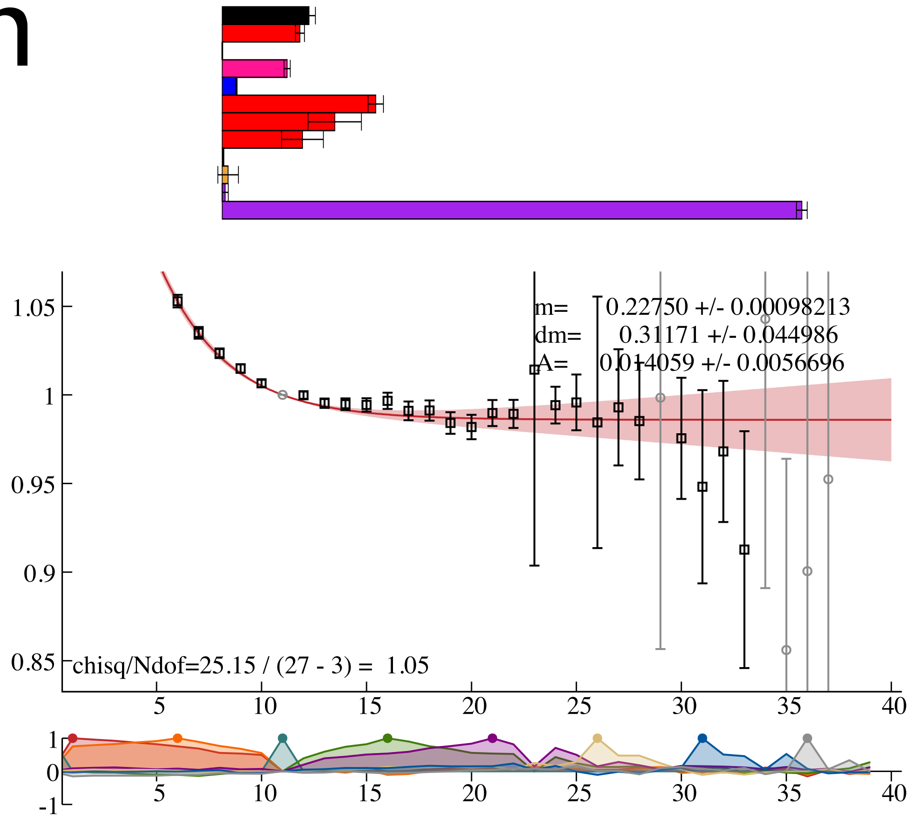
MC statistics

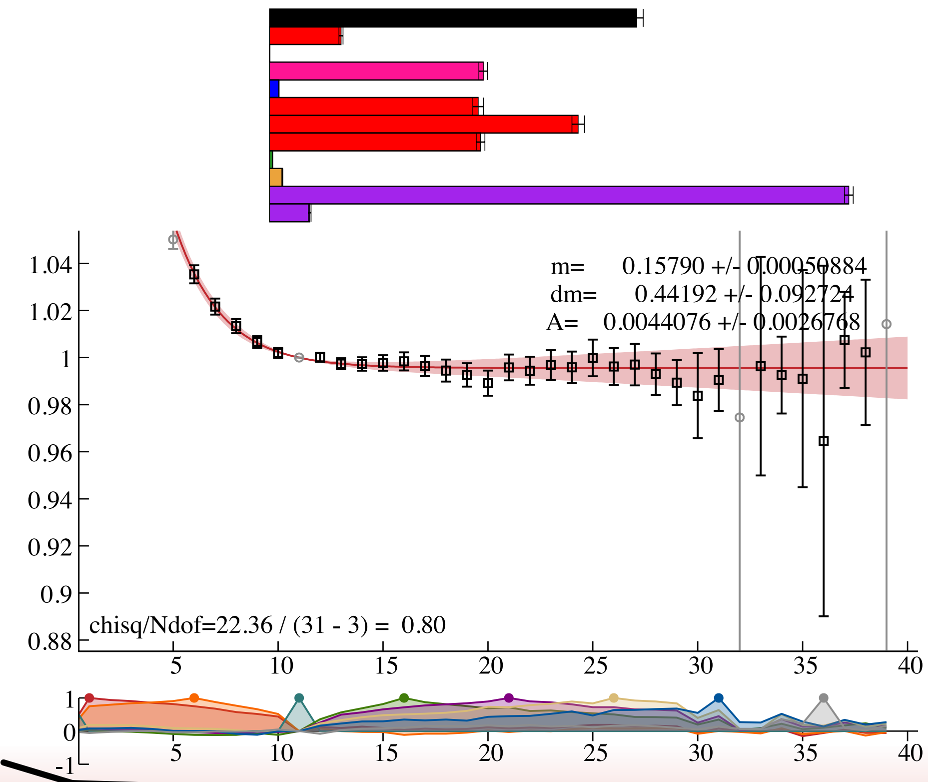
			 For renormalization of the current
MC configurations	400	348	348
# time sources	4	1	1
Correlation timeslice extent	40	32	32

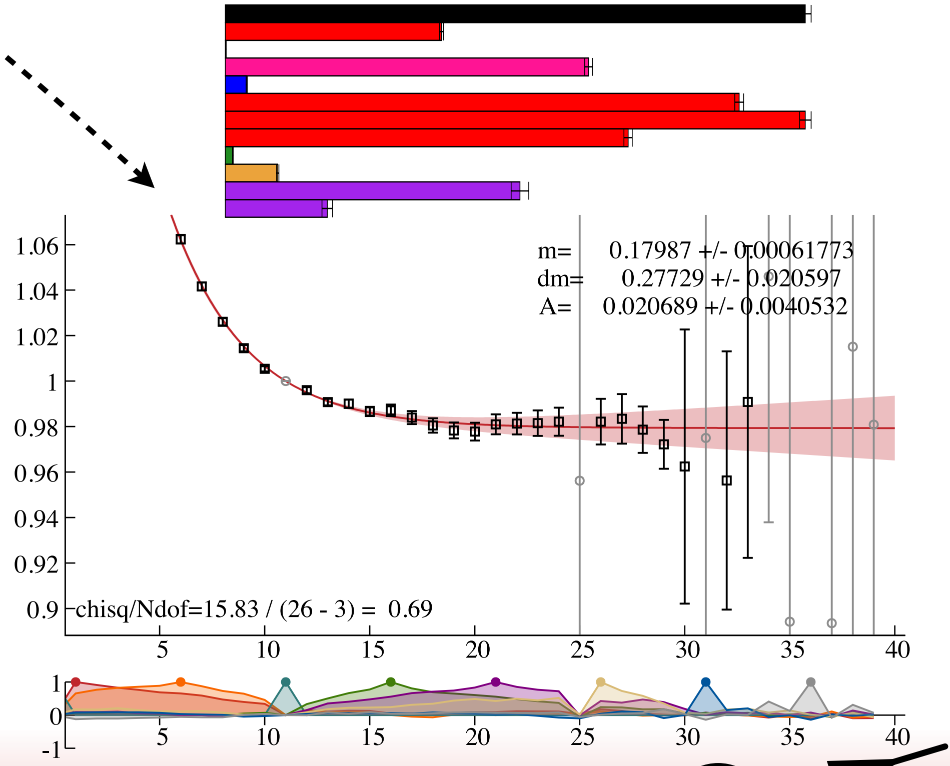
Typical spectrum











Multimeson $\pi\pi$ $K\overline{K}$

qbarq ρ ρ_2 ρ_3 b_0

hybrids

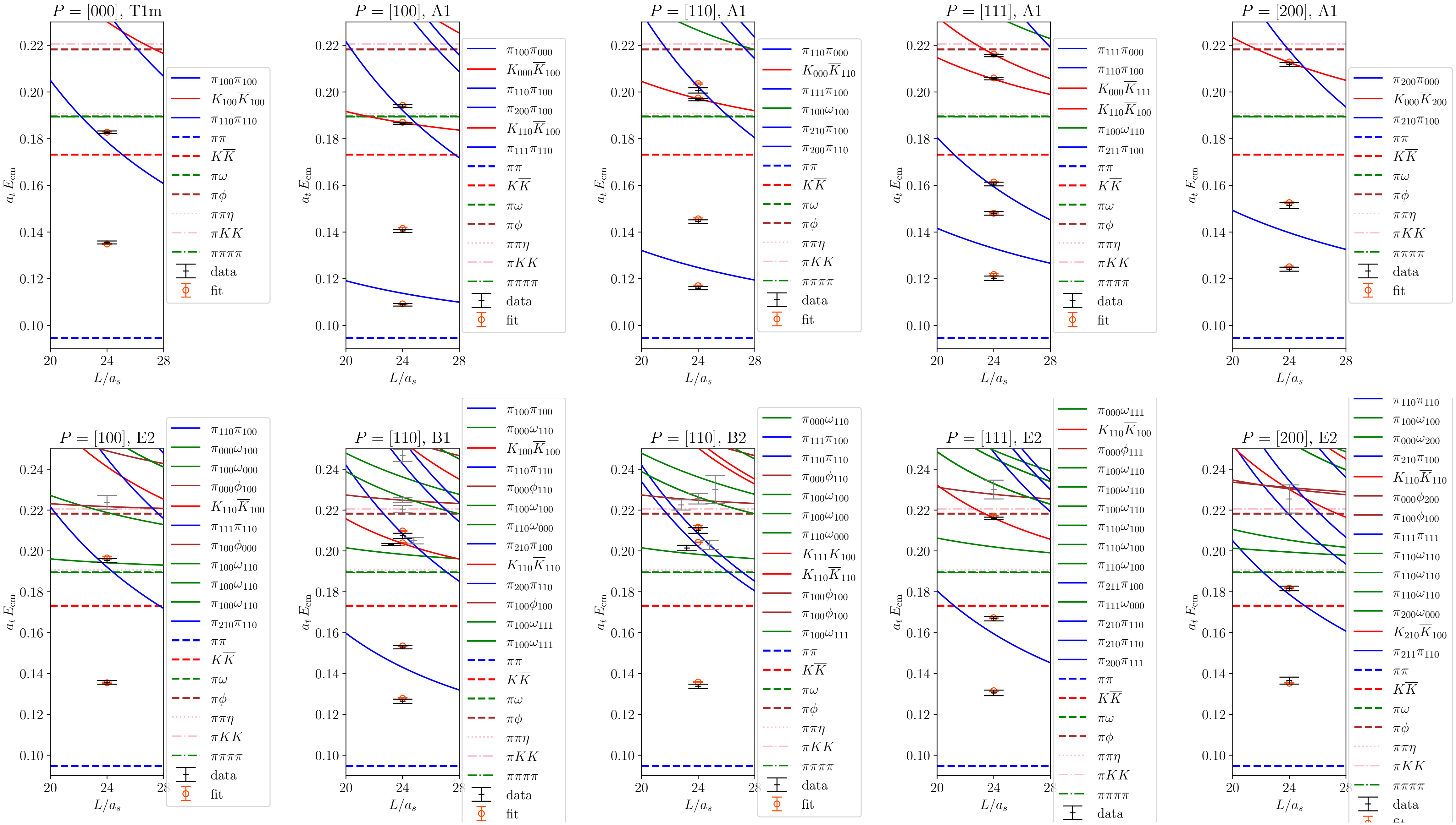
$$\left(\gamma^0\gamma^5 \times D^{[2]}_{J=1}\right)^{J=1}$$

Coupled channels fit: $\pi\pi$ $K\overline{K}$

$$\mathcal{M}^{-1}_{\ell,ab} = \frac{1}{(2q^\star_a)^\ell} K^{-1}_{\ell,ab} \frac{1}{(2q^\star_b)^\ell} - i\rho_{\mathrm{CM},ab}$$

$$K_{ab} = \frac{g_a g_b}{-s+m_r^2} + \gamma_{ab}$$

$$\chi^2/\mathrm{dof} = 28.70/(32-6) = 1.10$$

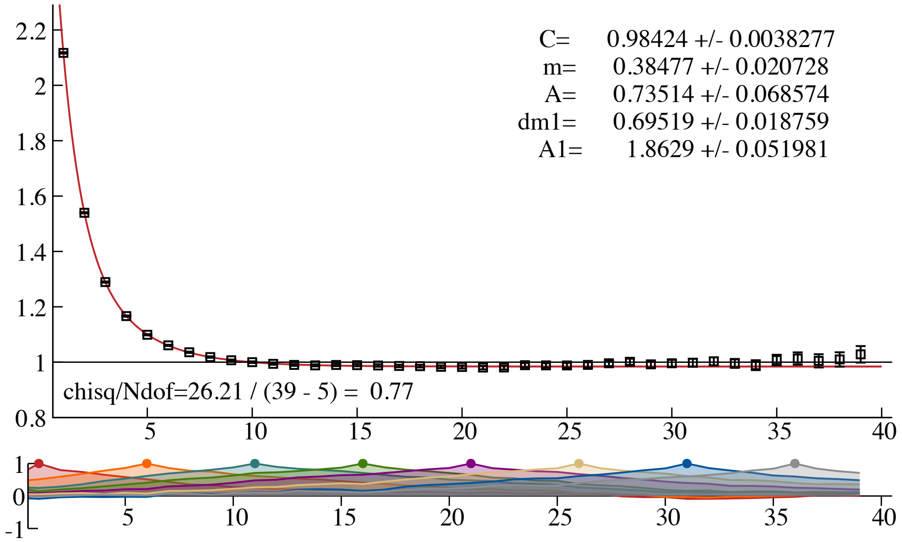


Optimized operators

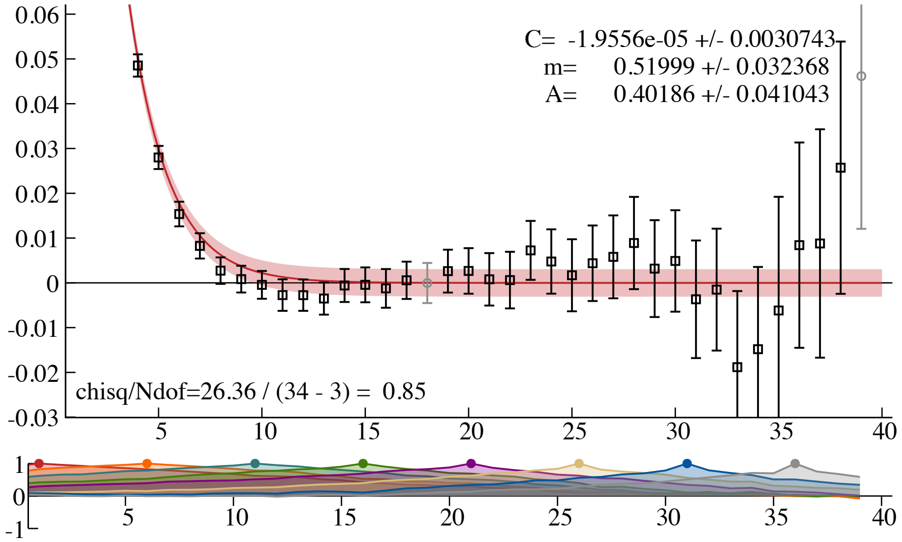
[000] T1m irrep

$$e^{\sqrt{E_n E_m}t}\,\langle 0|\Omega_n(t)\Omega_m^\dagger(0)|0\rangle = \delta_{n,m} + \mathcal{O}(e^{-(E_N-\sqrt{E_n E_m})t}$$

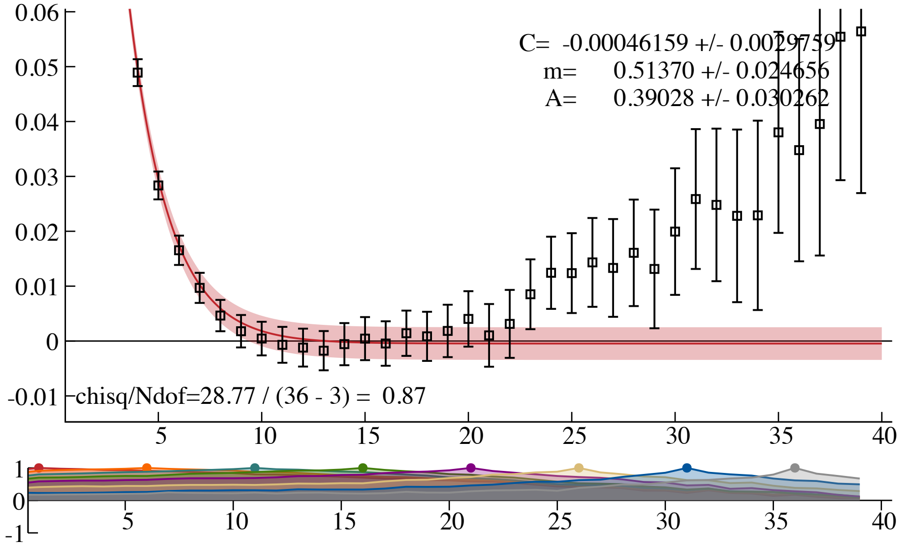
C00



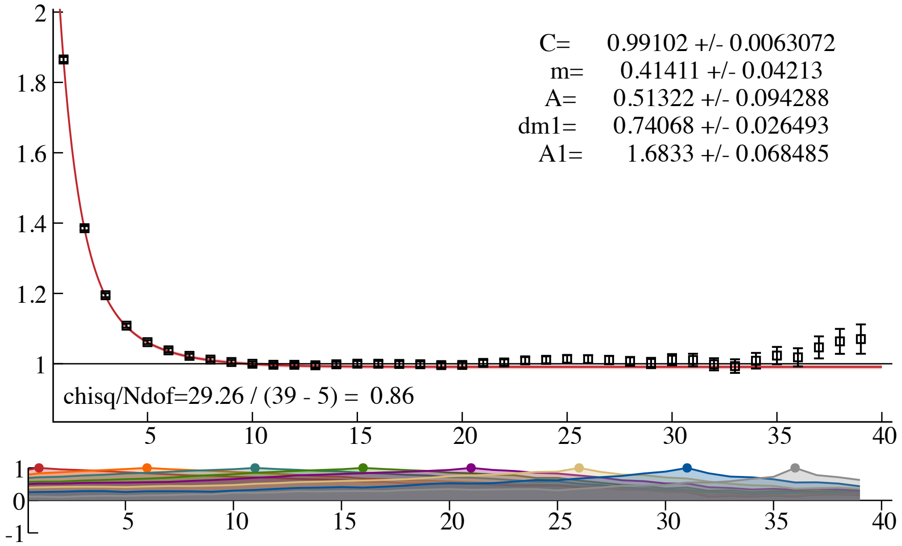
C01



C10



C11



Typical three point fit

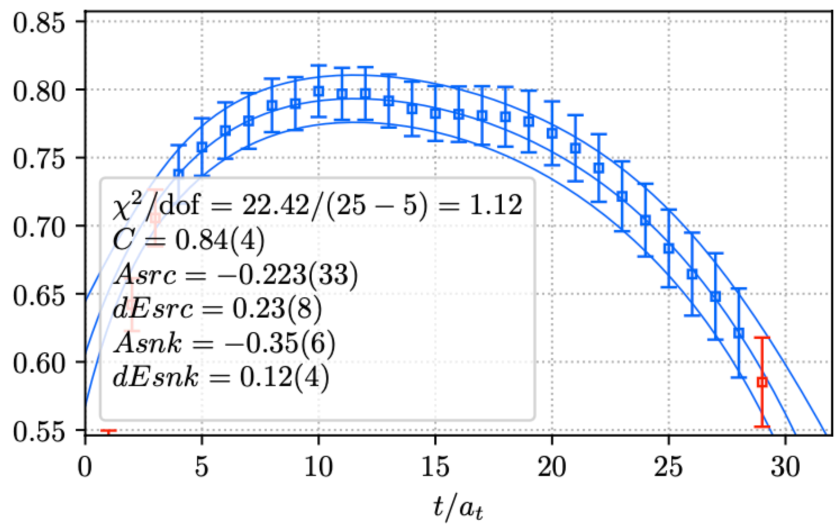
$$\left\langle \pi_{[1-10]} \middle| \mathcal{J}^{\rho}_{\text{impro}} \middle| \pi_{[200]} \right\rangle$$

$$\frac{\langle \pi(\tau,\vec{p_f})\mathcal{J}(t)\pi^{\dagger}(0,\vec{p_i})\rangle_{\text{rel}}}{\langle \pi(\tau-t,\vec{p_f})\pi^{\dagger}(0,\vec{p_f})\rangle\,\langle \pi(t,\vec{p_i})\pi^{\dagger}(0,\vec{p_i})\rangle} = Z_f^{-1/2} Z_i^{-1/2}\,\langle \pi(\vec{p_f})|\mathcal{J}(0)|\pi(\vec{p_i})\rangle + \dots$$

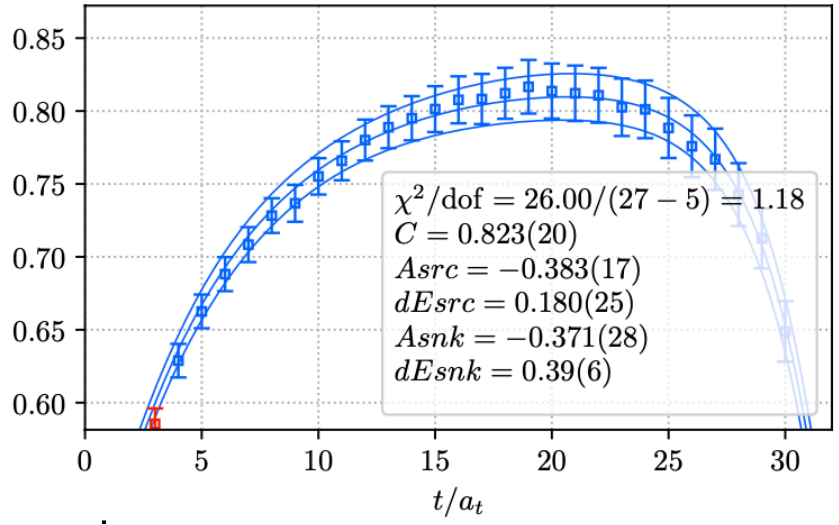
$$C + A_{src}e^{-dE_{src}t} + A_{snk}e^{-dE_{snk}(\tau-t)}$$

Current irrep

A1



B1



B2 kinematic factor is zero.

Kinematic factor

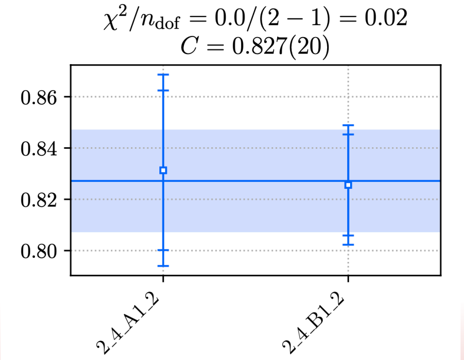
$$\langle \pi(p_1)|\mathcal{J}^i(\vec{q})|\pi(p_2)\rangle = (p_1+p_2)^i f(Q^2)\,,$$

$$\mathcal{S}^{\Lambda,\mu}_{m}\varepsilon^{m}_{i}(\vec{q})\,\langle \pi(p_1)|\mathcal{J}^i(\vec{q})|\pi(p)\rangle = \langle \pi(p_1)|\mathcal{J}^{\Lambda,\mu}(\vec{q})|\pi(p_2)\rangle\ ,$$

Improvement

$$\langle \mathfrak{m}|\mathcal{J}^{\rho}_{\text{impro}}|\mathfrak{n}\rangle = \langle \mathfrak{m}|\rho|\mathfrak{n}\rangle + \frac{1}{4}(1-\xi)a_t(E_m - E_n)\,\langle \mathfrak{m}|\rho_2|\mathfrak{n}\rangle$$

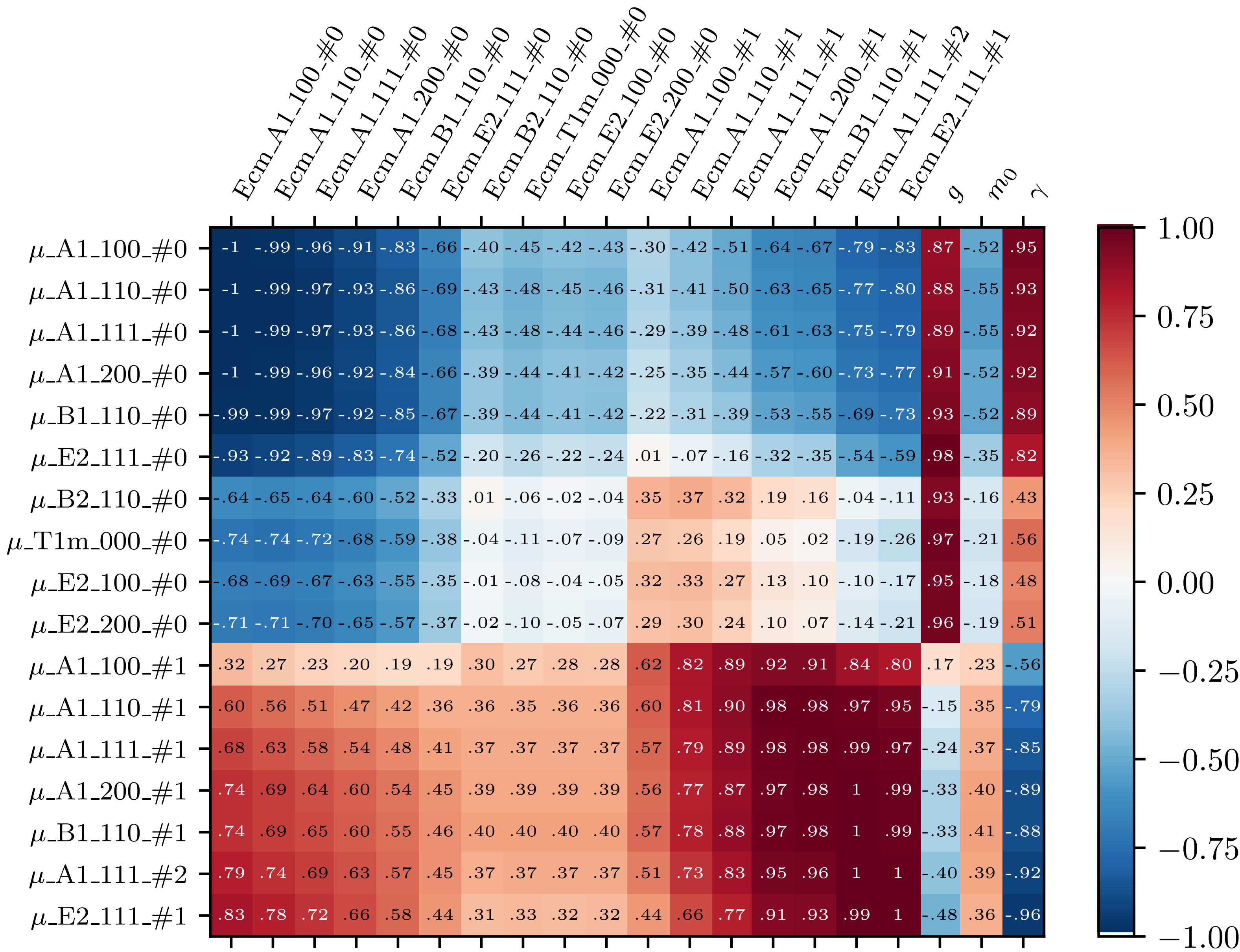
Average over irreps



Correlation between slope and energies

- Slopes away from the resonance are highly correlated to the Lüscher energy.

- The slopes close to the resonance have high correlation to *g*.



	Ecm_A1_100_#0	Ecm_A1_110_#0	Ecm_A1_111_#0	Ecm_A1_200_#0	Ecm_B1_110_#0	Ecm_E2_111_#0	Ecm_B2_110_#0	Ecm_T1m_000_#0	Ecm_E2_100_#0	Ecm_E2_200_#0	Ecm_A1_100_#1	Ecm_A1_110_#1	Ecm_A1_111_#1	Ecm_A1_200_#1	Ecm_B1_110_#1	Ecm_A1_111_#2	Ecm_E2_111_#1	g	m_0	γ
μ_A1_100_#0	-1	-.99	-.96	-.91	-.83	-.66	-.40	-.45	-.42	-.43	-.30	-.42	-.51	-.64	-.67	-.79	-.83	.87	-.52	.95
μ_A1_110_#0	-1	-.99	-.97	-.93	-.86	-.69	-.43	-.48	-.45	-.46	-.31	-.41	-.50	-.63	-.65	-.77	-.80	.88	-.55	.93
μ_A1_111_#0	-1	-.99	-.97	-.93	-.86	-.68	-.43	-.48	-.44	-.46	-.29	-.39	-.48	-.61	-.63	-.75	-.79	.89	-.55	.92
μ_A1_200_#0	-1	-.99	-.96	-.92	-.84	-.66	-.39	-.44	-.41	-.42	-.25	-.35	-.44	-.57	-.60	-.73	-.77	.91	-.52	.92
μ_B1_110_#0	-.99	-.99	-.97	-.92	-.85	-.67	-.39	-.44	-.41	-.42	-.22	-.31	-.39	-.53	-.55	-.69	-.73	.93	-.52	.89
μ_E2_111_#0	-.93	-.92	-.89	-.83	-.74	-.52	-.20	-.26	-.22	-.24	.01	-.07	-.16	-.32	-.35	-.54	-.59	.98	-.35	.82
μ_B2_110_#0	-.64	-.65	-.64	-.60	-.52	-.33	.01	-.06	-.02	-.04	.35	.37	.32	.19	.16	-.04	-.11	.93	-.16	.43
μ_T1m_000_#0	-.74	-.74	-.72	-.68	-.59	-.38	-.04	-.11	-.07	-.09	.27	.26	.19	.05	.02	-.19	-.26	.97	-.21	.56
μ_E2_100_#0	-.68	-.69	-.67	-.63	-.55	-.35	-.01	-.08	-.04	-.05	.32	.33	.27	.13	.10	-.10	-.17	.95	-.18	.48
μ_E2_200_#0	-.71	-.71	-.70	-.65	-.57	-.37	-.02	-.10	-.05	-.07	.29	.30	.24	.10	.07	-.14	-.21	.96	-.19	.51
μ_A1_100_#1	.32	.27	.23	.20	.19	.19	.30	.27	.28	.28	.62	.82	.89	.92	.91	.84	.80	.17	.23	-.56
μ_A1_110_#1	.60	.56	.51	.47	.42	.36	.36	.35	.36	.36	.60	.81	.90	.98	.98	.97	.95	-.15	.35	-.79
μ_A1_111_#1	.68	.63	.58	.54	.48	.41	.37	.37	.37	.37	.57	.79	.89	.98	.98	.99	.97	-.24	.37	-.85
μ_A1_200_#1	.74	.69	.64	.60	.54	.45	.39	.39	.39	.39	.56	.77	.87	.97	.98	1	.99	-.33	.40	-.89
μ_B1_110_#1	.74	.69	.65	.60	.55	.46	.40	.40	.40	.40	.57	.78	.88	.97	.98	1	.99	-.33	.41	-.88
μ_A1_111_#2	.79	.74	.69	.63	.57	.45	.37	.37	.37	.37	.51	.73	.83	.95	.96	1	1	-.40	.39	-.92
μ_E2_111_#1	.83	.78	.72	.66	.58	.44	.31	.33	.32	.32	.44	.66	.77	.91	.93	.99	1	-.48	.36	-.96

Correlation matrix between slopes, Lüscher energies and scattering parameters