Neutrino Physics and Astrophysics: Tutorial

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Contents

1 Exercises

1.1 Majorana mass matrix and $\mu - \tau$ symmetry

1.1.1 Introduction

Let us remind that a Majorana mass term for a single field can be written as

$$\begin{aligned} m\overline{\nu_R}\nu_L =& m(\mathcal{C}\overline{\nu_L}^T)^\dagger\gamma^0\nu_L = m[\mathcal{C}(\nu_L^\dagger\gamma^0)^T]^\dagger\gamma^0\nu_L = m(\mathcal{C}\gamma^{0T}\nu_L^*)^\dagger\gamma^0\nu_L = \\ =& m\nu_L^T\gamma^{0*}\mathcal{C}^\dagger\gamma^0\nu_L = m\nu_L^T\gamma^{0T}\mathcal{C}^\dagger\gamma^0\nu_L = -m\nu_L^T C^\dagger\gamma^0\gamma^0\nu_L = -m\nu_L^T C^\dagger\nu_L \end{aligned} \tag{1}$$

where we used $\nu_R = \mathcal{C}\overline{\nu_L}^T$, $(\gamma^0)^\dagger = \gamma^0$, $C\gamma^{\mu T}\mathcal{C}^{-1} = -\gamma^\mu$ and $\mathcal{C}^{-1} = \mathcal{C}^\dagger = \mathcal{C}^T = -\mathcal{C}$. For more than one field the mass term can be written as $\nu_L^T\mathcal{C}^\dagger M\nu_L$, where M is a $N \times N$ mass matrix with N being the number of fields involved.

1.1.2 Questions

1. Prove that the Majorana mass matrix is symmetric. Use the fact that mass term is a number and thus it is equal to its tranpose, then use the properties of the charge conjugation operator $\mathcal{C}$.

2. If we further assume that the mass matrix is symmetric under the transformation $\nu_\mu \longleftrightarrow -\nu_\tau$ can we make a precise prediction for θ_{13} and θ_{23}? Just diagonalize the mass matrix, even using the help of a software, and arrange the eigenvectors so that the one that is constant (independent from the elements of the mass matrix) is the rightmost one in the diagonalization matrix. In principle you can use a python code I prepared for performing symbolic diagonalization. To use it you have to go at the following link `https://www.kaggle.com/francescocapozzi/mu-tau-symmetry/edit` and sign in with francesco.capozzi@univaq.it (username) and ggi2023 (password). This is an example of flavor symmetry which is introduced to explain the structure of the PMNS matrix.

1.1.3 Solutions

1. $\nu_L^T\mathcal{C}^\dagger M\nu_L = (\nu_L^T\mathcal{C}^\dagger M\nu_L)^T = -\nu_L^T(\mathcal{C}^\dagger)^T M^T\nu_L = -\nu_L^T(\mathcal{C}^T)^\dagger M^T\nu_L = \nu_L^T\mathcal{C}^\dagger M^T\nu_L$, where in the first step we have used the fact that the mass term is equal to its transpose, then we have changed sign because of the anticommutation of fermion fields. We have also used $\mathcal{C}^T = -\mathcal{C}$. The previous equation implies $M = M^T$.

2. The code with the diagonalization of M that is invariant under $\nu_\mu \leftrightarrow -\nu_\tau$ can be found at the following link `https://www.kaggle.com/code/`

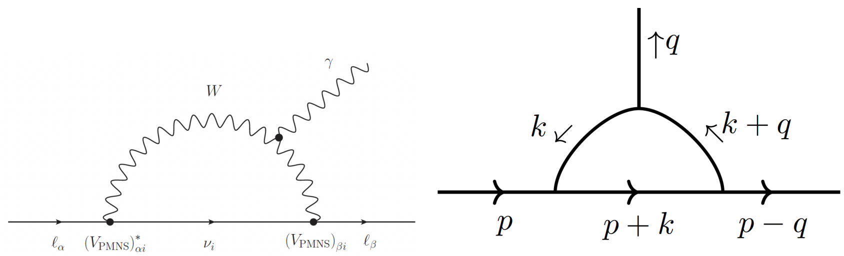

Figure 1: Feynman diagram of $l_\alpha \to l_\beta \gamma$.

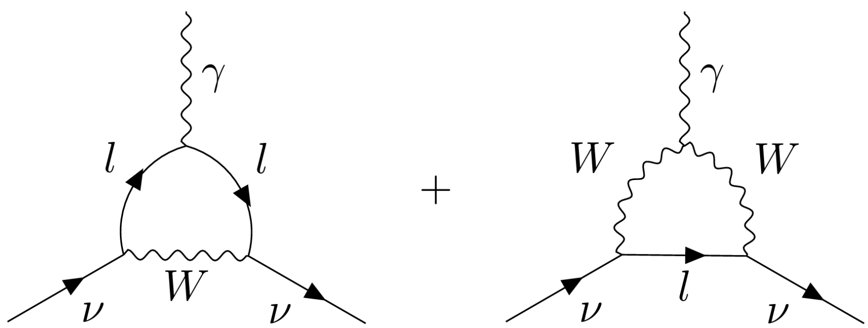

Figure 2: Feynman diagrams contributing to the magnetic moment.

`francescocapozzi/mu-tau-symmetry-solution/edit`. We choose the eigenvector $(0,1,1)^T$ as the last column of the diagonalization matrix, which is equivalent to $(U_{e3}, U_{\mu 3}, U_{\tau 3})^T$. This implies $\theta_{13} = 0$ and $\theta_{23} = \pi/4$.

1.2 Lepton flavor violation: $\mu \to e\gamma$

1.2.1 Question

A typical signature of neutrino mixing is lepton flavor violation, such as the process $\mu \to e\gamma$. However, the rate of this process is so suppressed to be basically unobservable in the Standard Model. Estimate the suppression factor, without doing a full calculation of the loop integral. Figure 1 shows the corresponding Feyman diagram.

(Hint: consider just the product of the propagators in the loop. Remember that the propagator for a fermion with mass m and momentum p is $\frac{\not{p}+m}{p^2-m^2}$, whereas the one for a spin-1 boson is $\frac{g_{\mu\nu}-p_\mu p_\nu/m^2}{p^2-m^2}$. Assume the W mass is much larger than the momentum that runs in the loop.)

1.2.2 Solution

The loop in the Feynman diagram in Fig. 1 contains a sum over neutrino propagators for the mass eigenstate and the product $U^*_{\mu i}U_{ei}$. Therefore the amplitude for the decay is proportional to

$$
\begin{aligned}
\mathcal{M} \propto & \sum_{i=1}^{3} \frac{U^*_{\mu i}U_{ei}(\not{p}+\not{k}+m_i)}{(p+k)^2-m_i^2} \simeq \sum_{i=1}^{3} U^*_{\mu i}U_{ei}(\not{p}+\not{k})\frac{1}{(p+k)^2}\frac{1}{1-\left[\frac{m_i}{p+k}\right]^2} \\
\simeq & \sum_{i=1}^{3} U^*_{\mu i}U_{ei}\frac{\not{p}+\not{k}}{(p+k)^2}\left[1+\frac{m_i^2}{(p+k)^2}+\dots\right] \\
= & (\not{p}+\not{k})\sum_{i=1}^{3} U^*_{\mu i}U_{ei}\left[\frac{1}{(p+k)^2}+\frac{m_i^2}{[(p+k)^2]^2}+\dots\right] = (\not{p}+\not{k})\sum_{i=1}^{3} U^*_{\mu i}U_{ei}\frac{m_i^2}{[(p+k)^2]^2}+\dots
\end{aligned}
\quad , \tag{2}
$$

where we have approximated the expression taking advantage of the smallness of the neutrino masses. In the last step we used the relation $\sum_i U^*_{\mu i}U_{ei} = (UU^\dagger)_{\mu e} = \mathbb{1}_{\mu e} = 0$ (GIM mechanism). The other propagators in the loop are the ones of the W boson and each of them can be approximated by $\frac{1}{M_W^2}$. Therefore, the ratio $\frac{m_i^2}{M_W^2}$ can be pulled out of the loop integral, whereas the other $\frac{1}{M_W^2}$ factor can be used to introduce the fermi constant G_F. The amplitude of the process can be written as

$$
\mathcal{M} \propto e\, G_F \sum_{i=1}^{3} U^*_{\mu i}U_{ei}\frac{m_i^2}{M_W^2}\,, \tag{3}
$$

where we have introduced also the muon charge e to take into account the only electromagnetic vertex. The decay rate is proportional to the squared amplitude and, apart from adimensional factors, it must contain an energy scale to the fifth power in order to give the correct dimensions. Since we are dealing with the decay of the muon, such an energy scale is given by the muon mass m_μ. Therefore the decay rate is given by

$$
\Gamma(\mu\to e\gamma) \propto \alpha G_F^2 m_\mu^5 \left(\sum_{i=1}^{3} U^*_{\mu i}U_{ei}\frac{m_i^2}{M_W^2}\right)^2 . \tag{4}
$$

This means that $\Gamma(\mu\to e\gamma)$ is proportional to $\left(\frac{m_i^2}{M_W^2}\right)^4$. Such a ratio is of the order of 10^{-48} assuming $m_i < 0.1$ eV. With such a strong suppression the rate is basically unobservable.

1.3 Neutrino magnetic moment

1.3.1 Question

Another consequence of neutrino masses is neutrinos having electromagnetic properties. In particular they can have a magnetic moment. The main contribution to the magnetic moment comes from the Feyman diagrams in Fig. 2. Estimate the value of the magnetic moment and compare it with the Bohr magneton $\mu_B = e/(2m_e)$. Just use dimensional analysis and remember that a magnetic moment induces a chirality flip between initial and final states. Concerning phenomenology, what are the consequences of neutrinos having a magnetic moment?

1.3.2 Solution

The effective lagrangian for magnetic moment interactions is

$$\mathcal{L}_{\text{eff}} \propto F_{\mu\nu}\bar{\psi}\sigma^{\mu\nu}\psi \,. \tag{5}$$

We can rewrite the effective lagrangian in terms of left and right-handed parts of the fields

$$\mathcal{L}_{\text{eff}} \propto F_{\mu\nu}(\overline{\psi_L} + \overline{\psi_R})\sigma^{\mu\nu}(\psi_L + \psi_R) = F_{\mu\nu}\left(\bar{\psi}P_R + \bar{\psi}P_L\right)\sigma^{\mu\nu}(P_L\psi + P_R\psi)\,. \tag{6}$$

Since γ^5 anticommutes with γ^μ, both P_L and P_R commute with $\sigma^{\mu\nu}$. Then, using the fact that $P_R P_L = P_L P_R = 0$ we have

$$\mathcal{L}_{\text{eff}} \propto F_{\mu\nu}\overline{\psi_L}\sigma^{\mu\nu}\psi_R + F_{\mu\nu}\overline{\psi_R}\sigma^{\mu\nu}\psi_L \,. \tag{7}$$

Equation 7 shows that magnetic moment interactions induces a chirality flip. Thus, to have neutrino magnetic moments one needs to introduce right handed neutrinos, which is why this can happen with massive neutrinos. These right handed neutrinos can be the singlet fields appearing in a Dirac mass term. They can be also heavier singlets which also have a Majorana mass term. The right handed part can also comes from ν_L^C, which is part of the Majorana field. The Feynman diagrams that contribute to the neutrino magnetic moment in the Standard Model are given in Fig. 2. These diagrams involve electroweak vertices, which only occur for left-chiral neutrinos. Thus, the chirality flip occurs on the incoming or outgoing neutrino with a mass insertion, from $m\overline{\psi_R}\psi_L + h.c.$ Therefore, one has $\mu_\nu \propto m_\nu$. Since in these diagrams there is one electromagnetic vertex and two weak ones, we obtain

$$\mu_\nu \simeq \frac{eG_F m_\nu}{10\pi^2} = \frac{2m_e\mu_B G_F m_\nu}{10\pi^2} \simeq 10^{-19}\mu_B\left(\frac{m_\nu}{1\,\text{eV}}\right)\,, \tag{8}$$

where $\mu_B = e/(2m_e)$ is the Bohr magneton and we introduced a factor $10\pi^2$ that represents the suppression from the loop calculation. Thus, the neutrino magnetic moment in the Standard Model is suppressed by the smallness of neutrino masses.

There are multiple consequences of the neutrino magnetic moment. These possibilities can be understood by looking at the Feyman diagrams contributing to it.

- There can be neutrino decay into a photon and a lighter neutrino.

- If neutrinos are Dirac particles, there can be spin-flavor conversion of a left-handed neutrino into a right-handed one in an external magnetic field. If they are Majorana particles spin-flavor conversion can convert a left-handed neutrino into a right-handend antineutrino. Both these possibilities have consequences in the context of solar and supernova neutrinos, where magnetic fields are known to be present, but their size is uncertain

- There can be plasmon decay into two neutrinos. This process is an important source of cooling for stars. For example it represents the dominant cooling in the degenerate core of a red-giant star.

- There can be a contribution to neutrino elastic scattering with electrons or to neutrino coherent scattering with nuclei.

1.4 Flavor ratio of high energy astrophysical neutrinos

1.4.1 Question

High energy astrophysical neutrinos are produced in a generic source S with a flavor composition $f_{\nu_\alpha,S}$, where $\nu_\alpha = \nu_e, \nu_\mu, \nu_\tau$ and $\sum_\alpha f_{\nu_\alpha,S} = 1$. The flavor composition on Earth $f_{\nu_\alpha,\oplus}$ is affected by neutrino oscillations. Assuming we have no information at all on $f_{\nu_\alpha,S}$ and taking into account current uncertainties on the mixing parameters (at 3σ), find out what are all possible values of $f_{\nu_\alpha,\oplus}$. To help you, I have prepared a python code. To use it you have to go at the following link `https://www.kaggle.com/code/francescocapozzi/flavor-ratio-of-high-energy-neutrinos/edit` and sign in with francesco.capozzi@univaq.it (username) and ggi2023 (password). Before running the code you need to write a proper algorithm for the function "neutrino_oscillations". This function takes as input an array with 3 entries "f_production" ($f_{\nu_\alpha,S}$) and the 3×3 mixing matrix "U_PMNS". The function should return $f_{\nu_e,\oplus}$, $f_{\nu_\mu,\oplus}$, $f_{\nu_\tau,\oplus}$. For calculating the oscillation probability assume that the propagation distance is large enough to induce decoherence between mass eigenstates.

1.4.2 Solution

The equation connecting $f_{\nu_\alpha,S}$ to $f_{\nu_\alpha,\oplus}$ is

$$f_{\nu_\alpha,\oplus} = \sum_\beta f_{\nu_\beta,S} P(\nu_\beta \to \nu_\alpha)\,, \tag{9}$$

and the oscillation probability is given by

$$P(\nu_\beta \to \nu_\alpha) = \sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2\,. \tag{10}$$

The final version of the code can be found at the following link `https://www.kaggle.com/code/francescocapozzi/flavor-ratio-of-high-energy-neutrinos-solution/edit`.

1.5 Solar neutrinos oscillation probability

1.5.1 Questions

1. Let us assume that the propagation of solar neutrinos is adiabatic, and that there are only two flavours ν_e, ν_x and two mass eigenstate ν_1, ν_2. Let us also assume that the distance between the Sun and the Earth is large enough to make induce decoherence among mass eigenstates. Derive the survival probability of ν_e as a function of the mixing angle θ_{12} in the limits $V_{CC} \gg \frac{\Delta m^2_{21} \cos 2\theta_{12}}{2E}$ and $V_{CC} \ll \frac{\Delta m^2_{21} \cos 2\theta_{12}}{2E}$.

2. How does the expression of $P(\nu_e \to \nu_e)$ change in 3 flavours, assuming matter effects are negligible in the 13 sector?

3. The Sun converts protons into He through the reaction $4p + 2e^- \to {}^4\mathrm{He} + 2\nu_e$. Knowing that the solar luminosity is $L_\odot = 2.4 \times 10^{45}$ eV s^{-1}, estimate the solar neutrino flux on Earth. Basically all this flux goes into the so called pp neutrinos, which are the solar neutrinos with the lowest energy (up to $\sim$ 0.4 MeV). If the measured flux on Earth of pp neutrinos is 3.3×10^{10} cm^{-2} s^{-1}, independently from the neutrino energy, what is the value of $\sin^2\theta_{12}$?

1.5.2 Solutions

1. The survival probability of ν_e in two flavors and assuming adiabatic propagation in matter is given by

$$P_{\nu_e \to \nu_e} = \frac{1}{2} + \frac{1}{2}\cos 2\theta_M^{(i)} \cos 2\theta_M^{(f)} + \frac{1}{2}\sin 2\theta_M^{(i)} \sin 2\theta_M^{(f)} \cos\left(\int_0^x dx' \frac{\Delta m^2_M(x')}{2E}\right), \tag{11}$$

where $\theta_M^{(f)}$ and $\theta_M^{(i)}$ are the mixing angles at detection and at production. Since $\frac{\Delta m^2_{21} d_\odot}{2E} \gg 1$, the term with the integral averages to 0. Detection can be assumed to be always in vacuum, thus $\theta_M^{(f)} = \theta_{12}$. Concerning $\theta_M^{(i)}$ we have to consider two regimes.

   - When $V_{CC} \ll \frac{\Delta m^2_{21} \cos 2\theta_{12}}{2E}$ we have that matter effects are negligible, which implies $\theta_M^{(i)} = \theta_{12}$. In this case the survival probability becomes $P_{\nu_e \to \nu_e} \simeq \frac{1}{2}(1 + \cos^2 2\theta_{12}) = 1 - \frac{1}{2}\sin^2 2\theta_{12}$.
   - When $V_{CC} \gg \frac{\Delta m^2_{21} \cos 2\theta_{12}}{2E}$ matter effects are dominating and $\theta_M^{(i)} = \pi/2$, which implies $P_{\nu_e \to \nu_e} \simeq \frac{1}{2}(1 - \cos 2\theta_{12}) = \sin^2\theta_{12}$.

2. In three flavours the expression for the survival probability is given by

$$P_{\nu_e \to \nu_e} \simeq \sum_{i=1}^{3} |U_{ei}^{(i)}|^2 |U_{ei}^{(f)}|^2, \tag{12}$$

where $U_{ei}^{(i)}$ and $U_{ei}^{(f)}$ are the elements of the mixing matrix at production and detection, respectively. Since detection can be assumed to be alway in vacuum we have that

$$U_{ei}^{(f)} = U_{ei} = (\cos\theta_{13}\cos\theta_{12}, \cos\theta_{13}\sin\theta_{12}, \sin\theta_{13}e^{-i\delta})\,. \tag{13}$$

- When $V_{CC} \ll \frac{\Delta m_{21}^2 \cos 2\theta_{12}}{2E}$, the values of $U_{ei}^{(i)}$ are equal to the ones in vacuum, thus

$$\begin{aligned} P_{\nu_e\to\nu_e} \simeq \sum_{i=1}^{3} |U_{ei}|^4 &= \cos^4\theta_{13}\cos^4\theta_{12} + \cos^4\theta_{13}\sin^4\theta_{12} + \sin^4\theta_{13} = \\ &= \cos^4\theta_{13}\left(1 - \frac{1}{2}\sin^2 2\theta_{12}\right) + \sin^4\theta_{13}\,. \end{aligned} \tag{14}$$

- When $V_{CC} \ll \frac{\Delta m_{21}^2 \cos 2\theta_{12}}{2E}$ we have that $\theta_{12,M}^{(i)} = \pi/2$, thus

$$P_{\nu_e\to\nu_e} \simeq \sum_{i=1}^{3} |U_{ei}|^2 |U_{ei}(\theta_{12} = \pi/2)|^2 = \cos^4\theta_{13}\sin^2\theta_{12} + \sin^4\theta_{13}\,. \tag{15}$$

3. Assuming that all the nuclear energy produced in the Sun (the Q-value of the pp chain) is then emitted from the surface, the flux of neutrinos at Earth is given by

$$\Phi_\nu^\odot = 2\frac{L_\odot}{Q_{pp}4\pi d_\odot^2} = 6\times 10^{10}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\,, \tag{16}$$

where Q_{pp} is the Q-value of the pp chain, $Q_{pp} = 4m_p + 2m_e - m(^4\mathrm{He}) \simeq$ 26.73 MeV, and $d_\odot = 1.5\times 10^{13}$ cm is the distance of the Sun from the Earth. Because $\Phi_\nu^\odot = \Phi_\nu^{\mathrm{pp},\odot}$ and the measured flux on Earth is $\Phi_\nu^{\mathrm{pp},\oplus} = 3.3\times 10^{10}$ cm^{-2} s^{-1}, we have that

$$P(\nu_e\to\nu_e) = \Phi_\nu^{\mathrm{pp},\oplus}/\Phi_\nu^{\mathrm{pp},\odot} = 0.55 = 1 - \frac{1}{2}\sin^2 2\theta_{12}\,. \tag{17}$$

The equation above implies that $\sin^2\theta_{12} \simeq 0.34$. Note that we have used $P(\nu_e\to\nu_e) = 1-\frac{1}{2}\sin^2 2\theta_{12}$ because pp neutrinos have the lowest energies among solar neutrinos, so they are less affected by matter effects, and the measured flux is independent from the neutrino energy.

1.6 Supernova neutrino flux, distance to SN1987a and upper limit on the neutrino mass

1.6.1 Questions

A core-collapse supernova is the final stage of the life of a massive star and it consists in a collapse to a neutron star ($R_{\mathrm{NS}} \sim 20$ km, $M_{\mathrm{NS}} \sim 1.5M_\odot$) and a violent emission of the outer layers through an explosion driven by a shock wave. In this process a large number of neutrinos of all flavors is produced.

1. Estimate the flux of supernova neutrinos as a function of distance (in kiloparsec), assuming that all the gravitational energies released in the explosion is converted into neutrinos with an energy of 10 MeV.

2. The only core-collapse supernova observed in neutrinos occurred in 1987. The number of $\bar{\nu}_e$ detected by the Kamiokande detector (2.1 kt of water) through inverse beta decay ($\bar{\nu}_e + p \to e^+ + n$) was ~ 10. Estimate the distance of this supernova.

3. The highest and lowest energy events detected from SN1987a by Kamiokande had energies equal to $E_1 \simeq 30$ MeV and $E_2 \simeq 10$ MeV, respectively. Moreover all neutrinos were detected in a time window of ~ 10 seconds. Since they are massive particles, neutrinos with different energies have a different propagation time between the supernova and the detector. The time delay of course depends on the neutrino mass. Working in the approximation $m_\nu^2/E_\nu^2 \ll 1$ (ultra-relativistic limit), derive an upper bound on the neutrino mass.

4. Supernova neutrinos are produced in the core of the exploding star where the density can get as high as $\rho \sim 10^{14}$ gr/cm^3. What is the mean free path of neutrinos in such an environment? The main scattering process here is the charged current weak interaction with nucleons, which are all free.

5. How many events we would have seen for the supernova of 1987 if there was a detector with 10 ton of Xenon, capable of detecting neutrinos through coherent scattering on ^{132}Xe?

[Useful quantities and conversion factors: $G = 6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, $M_\odot = 2 \times 10^{30}$ kg, 1 kpc $= 3.086 \times 10^{21}$ cm, 1 erg $= 6.242 \times 10^5$ MeV, 1 J $= 6.242 \times 10^{12}$ MeV, $\sigma(\bar{\nu}_e p \to e^+ n) \simeq 10^{-43}\,\text{cm}^2\, E_\nu^2/\text{MeV}^2$]

1.6.2 Solutions

1. The gravitational energy released in a core-collapse explosion is given by

$$\begin{aligned} E_{\text{grav}} \simeq & - GM_C^2 \left(\frac{1}{R_C} - \frac{1}{R_{NS}} \right) \simeq \frac{GM_C^2}{R_{NC}} \simeq \frac{6.67 \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2} 2.25 M_\odot^2}{2 \times 10^4\,\text{m}} \simeq \\ & \simeq 7.5 \times 10^{-15} \times (2 \times 10^{30})^2\,\text{m}^2\,\text{kg}\,\text{s}^{-2} = 3 \times 10^{46}\,\text{J} \simeq 3 \times 10^{53}\,\text{erg}\,, \end{aligned} \tag{18}$$

where M_C is the mass of the core which becomes a neutron star, which we assume to be $M_C = 1.5 M_\odot = 3 \times 10^{30}$ kg, $G = 6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2} is the Newton constant, $R_{NC} = 20$ km is the radius of the final neutron star, $R_C = 10^4$ km is the radius of the initial core at the beginning of the collapse. Assuming the average energy of neutrinos is $\langle E_\nu \rangle = 10$ MeV, and that all the gravitational energy is converted into neutrinos, their flux at a distance d is given by

$$\begin{aligned}\Phi_\nu =& \frac{E_{\rm grav}}{\langle E_\nu \rangle 4\pi d^2} \simeq \frac{3\times 10^{53}\,{\rm erg}}{10\,{\rm MeV}} \frac{1}{4\times 3.14\times 10^2\,{\rm kpc}^2 \left(\frac{d}{10\,{\rm kpc}}\right)^2} \simeq \\ \simeq& \frac{3\times 624151\times 10^{53}\,{\rm MeV}}{10\,{\rm MeV}} \frac{1}{4\times 3.14\times 10^2\times 9.5234\times 10^{42}\,{\rm cm}^2 \left(\frac{d}{10\,{\rm kpc}}\right)^2} \simeq \\ \simeq& 1.57\times 10^{12}\left(\frac{d}{10\,{\rm kpc}}\right)^{-2}\,{\rm cm}^{-2}\,,\end{aligned} \tag{19}$$

where we used the conversions factor $1\,{\rm kpc} = 3.086\times 10^{21}$ cm and $1\,{\rm erg} = 624151\,{\rm MeV}$.

2. The number of $\bar\nu_e$ detected by Kamiokande on the 23rd of February 1987 was ~ 10. Such a number can be obtained from the following formula

$$N_{\rm events} = \frac{\Phi_\nu}{6}\sigma_{\rm IBD} N_{\rm p}\,, \tag{20}$$

where Φ_ν is the total flux of supernova neutrinos at a distance d, $N_{\rm p} \simeq \frac{2}{18}\frac{2.1\times 10^6\,{\rm kg}}{m_p} \simeq \frac{2}{18}\frac{2.1\times 10^6\,{\rm kg}}{1.6\times 10^{-27}\,{\rm kg}} \simeq 1.4\times 10^{32}$ is the number of free protons in the detector and $\sigma_{\rm IBD}$ is the cross section for the inverse beta decay ($\bar\nu_e + p \to n + e^+$), the detection channel used in Kamiokande, which is given by

$$\sigma_{\rm IBD} \simeq 10^{-43}\frac{E_\nu^2}{{\rm MeV}^2}\,{\rm cm}^2\,. \tag{21}$$

We have divided by 6 the total flux because Kamiokande is only sensitive to $\bar\nu_e$ and we are assuming that all neutrino species are emitted with the same flux. Imposing $N_{\rm events} = 10$ and assuming $E_\nu = \langle E_\nu \rangle = 10$ MeV we can estimate the distance of SN1987a

$$\begin{aligned} d \simeq& \sqrt{\frac{1.57\times 10^{12}\times 10^2}{6N_{\rm events}} 10^{-43}\frac{\langle E_\nu\rangle^2}{{\rm MeV}^2}\times 1.4\times 10^{32}\,{\rm kpc}} \simeq \\ \simeq& \sqrt{\frac{1.57\times 10^{12}\times 10^2}{6\times 10} 10^{-43}\times 10^2\times 1.4\times 10^{32}\,{\rm kpc}} \simeq 60\,{\rm kpc}\,.\end{aligned} \tag{22}$$

The measured distance is actually $d \simeq 51$ kpc, which is close to our estimate.

3. Relativistic particles have a velocity that can be approximated by

$$v = \frac{p}{E} = \sqrt{1-\frac{m^2}{2E^2}} \simeq 1 - \frac{m^2}{2E^2}\,, \tag{23}$$

where we are using natural units. Therefore, massive neutrinos travelling for a distance D will have a delay with respect to massless particles given by

$$\Delta t = \frac{D}{v} - D \simeq D\left(1+\frac{m_\nu^2}{2E^2}\right) - D \simeq \frac{m_\nu^2}{2E^2}D = 2.57\left(\frac{m_\nu}{\mathrm{eV}}\right)^2\left(\frac{E}{\mathrm{MeV}}\right)^{-2}\frac{D}{50\,\mathrm{kpc}}\,\mathrm{s}\,. \tag{24}$$

Neutrinos which are produced at the same time with different energies will reach a detector with the following arrival time difference

$$\Delta T(E_1,E_2) = \frac{D}{v_1} - \frac{D}{v_2} \simeq D\left(1+\frac{m_\nu^2}{2E_1^2}\right) - D\left(1+\frac{m_\nu^2}{2E_2^2}\right) = \frac{Dm_\nu^2(E_2^2-E_1^2)}{2E_1^2E_2^2}\,. \tag{25}$$

The highest and lowest energy events detected from SN1987a by Kamiokande had $E_1 \simeq 10$ MeV and $E_2 \simeq 30$ MeV. Since the total time interval in which the ~ 10 events were detected was of $\Delta T_0 \simeq 12$ s, we can get a conservative bound on the neutrino mass by imposing $\Delta T(E_1 = 10\,\mathrm{MeV}, E_2 = 30\,\mathrm{MeV}) < \Delta T_0$. We obtain

$$\begin{aligned} m_\nu &< \sqrt{\left[\left(\frac{10\,\mathrm{MeV}}{E_1}\right)^2 - \left(\frac{10\,\mathrm{MeV}}{E_2}\right)^2\right]^{-1}\frac{\Delta T_0}{10\,\mathrm{s}}\frac{10\,\mathrm{kpc}}{D}}\,45\,\mathrm{eV} \quad \Longrightarrow \\ m_\nu &< 20\,\mathrm{eV}\,. \end{aligned} \tag{26}$$

4. The number density of nucleons is given by

$$N_{\mathrm{nucleons}} = \frac{\rho}{m_p} \simeq \frac{10^{14}\,\mathrm{gr/cm^3}}{m_p} \simeq \frac{10^{14}\,\mathrm{gr/cm^3}}{1.6\times 10^{-24}\,\mathrm{g}} \simeq 10^{38}\,\mathrm{cm^{-3}}\,, \tag{27}$$

where m_p is the proton mass. The cross section is $\sigma_{\mathrm{IBD}} \simeq 10^{-43}\,\mathrm{cm}^2 E_\nu^2/\mathrm{MeV}^2$. Assuming $E_\nu = \langle E_\nu\rangle = 10$ MeV, the mean fee path is

$$l_{\mathrm{mfp}} = \frac{1}{N_{\mathrm{nucleons}}\sigma_{\mathrm{IBD}}} \simeq \frac{\mathrm{cm}}{10^{38}\times 10^{-43}\times 100} \simeq 10^3\,\mathrm{cm} = 10\,\mathrm{m}\,, \tag{28}$$

which means neutrinos are trapped.

5. The number of events detected by a detector of Xenon would be

$$N_{\mathrm{events}} = \Phi_\nu \sigma_{\mathrm{coh}} N_{\mathrm{Xe}}\,, \tag{29}$$

where Φ_ν is the total flux of supernova neutrinos at a distance d (including all flavors), $N_{\mathrm{Xe}} \simeq 10^4\,\mathrm{kg}/(132 m_p) = 4.73\times 10^{28}$ is the number of Xenon nuclei in the detector and $\sigma_{\mathrm{coh}} = 10^{-43}\,\mathrm{cm}^2\,A^2E_\nu^2/\mathrm{MeV}^2$ is the cross section for the coherent scattering on nuclei with $A = 132$ being the mass number of Xenon. Using $d = 50$ kpc we get $N_{\mathrm{events}} \simeq 520$.

1.7 Neutrino interactions at low and high energy

1.7.1 Questions

1. At what energy can a neutrino produce a W boson on-shell and what is the Feyman diagram for this process?

2. Estimate the cross section for such a process assuming it behaves like a Breit-Wigner resonance.

1.7.2 Solutions

1. The Glashow resonance occurs when a $\bar{\nu}_e$ has enough energy to produce a W^- on-shell when scattering on an electron at rest. The neutrino energy required for such a process can be derived from the conservation law of momentum $(p_\nu + p_e)^2 = p_W^2 \implies m_\nu^2 + m_e^2 + 2p_\nu \cdot p_e = M_W^2 \implies 2p_\nu \cdot p_e \simeq M_W^2 \implies 2m_e E_\nu \simeq M_W^2 \implies E_\nu \simeq \frac{M_W^2}{2m_e} \simeq 6.3\,\mathrm{PeV}$

2. The cross section at the Glashow resonance can be obtained by approximating it with a Breit-Wigner formula

$$\sigma_{\mathrm{Gl}}(E) = \frac{2J+1}{2j+1}\frac{4\pi}{p_{\mathrm{cm}}^2}\frac{\Gamma_W^2/4}{(E_{\mathrm{cm}} - M_W^2)^2 + \Gamma_W^2/4}\mathrm{Br}(W \to \bar{\nu}_e e^-)\mathrm{Br}(W \to \mathrm{hadrons})\,, \tag{30}$$

where $\Gamma_W \simeq 2.1$ GeV is the total width of the W boson, $\mathrm{Br}(W \to \bar{\nu}_e e^-) \simeq 0.11$ $\mathrm{Br}(W \to \mathrm{hadrons}) \simeq 0.67$ are the branching ratios to the lepton and hadron channel, respectively, E_{cm} and p_{cm} are the initial energy and the momentum in center of mass, $J = 1$ and $j = 1/2$ are the spins of the W boson and the electron, respectively. At the resonance the cross section is equal to $\sigma_{\mathrm{Gl}}(M_W) \simeq 10^{-31}\,\mathrm{cm}^2$.

1.8 Reactor neutrino flux and interactions

1.8.1 Questions

1. Consider a research nuclear reactor with a thermal power of $P = 1$ GW which mainly has U_{235} (fission reaction ${}^{235}_{92}\mathrm{U} + {}^{1}_{0}n \to {}^{144}_{55}\mathrm{Cs} + {}^{90}_{37}\mathrm{Rb} + 2{}^{1}_{0}n$). If an average of 6 $\bar{\nu}_e$ are emitted from each fission, what would be the neutrino flux at a location 1 km away from the core?

2. For energies typical of reactor neutrinos (~ 3 MeV) and for a propagation distance of 1 km the following hierarchy of oscillation frequencies hold $\frac{\Delta m_{31}^2 L}{4E} \simeq \frac{\Delta m_{32}^2 L}{4E} \sim O(1)$ and $\frac{\Delta m_{21}^2 L}{4E} \ll 1$, where $\Delta m_{31}^2 \simeq \Delta m_{32}^2 \simeq 2.5 \times 10^{-3}$ eV^2 and $\Delta m_{21}^2 \simeq 7 \times 10^{-5}$ eV^2. Derive an approximation for the survival probability of $\bar{\nu}_e$ $P(\bar{\nu}_e \to \bar{\nu}_e)$.

3. We plan to detect these reactor neutrinos through inverse beta decay. How much mass of the liquid scintillator $C_{18}\mathrm{H}_{30}$ would we need to have an event rate of about 100 events/day. What is the required mass in case

one uses coherent scattering on CsI instead of inverse beta decay? In both cases assume all neutrinos have an energy $E_\nu = 3$ MeV.

[Useful quantities and conversion factors: 1 J $= 6.242 \times 10^{12}$ MeV, $\sigma(\bar{\nu}_e p \to e^+ n) \simeq 10^{-43}\,\mathrm{cm}^2\, E_\nu^2/\mathrm{MeV}^2$]

1.8.2 Solutions

1. The masses of the nuclei involved in the fission reaction are

   - $m(^{235}\mathrm{U}) = 235.044\,u$,
   - $m(^{144}\mathrm{Cs}) = 143.932\,u$,
   - $m(^{90}\mathrm{Rb}) = 89.915\,u$,
   - $m(^1_0 n) = 1.009\,u$,

   where $1u = 931.5$ MeV. Furthermore $1\,\mathrm{MW} \simeq 6.25 \times 10^{18}$ MeV/s. The energy released in one fission reaction is given by

   $$\begin{aligned}\Delta E =& m(^{235}\mathrm{U}) + m(^1_0 n) - m(^{144}\mathrm{Cs}) - m(^{90}\mathrm{Rb}) - 2m(^1_0 n) = 0.188\,u = \\ =& 0.188 \times 931.5\,\mathrm{MeV} = 175.1\,\mathrm{MeV}\,.\end{aligned} \tag{31}$$

   The neutrino flux is given by

   $$\begin{aligned}\phi_{\bar{\nu}_e} =& \frac{\text{Number of neutrinos}}{\text{area} \times \text{time}} = \frac{6N_{\text{fissions}}}{4\pi L^2 \Delta t} = \frac{6P}{4\pi L^2 \Delta E} \simeq \\ \simeq& 1.5 \times 10^9\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1} \left(\frac{P}{1\,\mathrm{GW}}\right)\left(\frac{1\,\mathrm{km}}{L}\right)^2 \left(\frac{200\,\mathrm{MeV}}{\Delta E}\right) = \\ \simeq& 1.7 \times 10^9\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\,.\end{aligned} \tag{32}$$

2. We start from

   $$P(\bar{\nu}_\alpha \to \bar{\nu}_\alpha) = 1 - 4\sum_{i>j} \mathrm{Re}[J^{ij}_{\alpha\alpha}] \sin^2\left(\frac{\Delta m^2_{ij} L}{4E}\right)\,, \tag{33}$$

   where $J^{ij}_{\alpha\beta} = U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}$.

   We can use the following approximations: $\Delta m^2_{31} \simeq \Delta m^2_{32} \simeq \Delta m^2$ and $\Delta m^2_{21} \simeq 0$. The oscillation probability becomes

   $$\begin{aligned}P(\bar{\nu}_e \to \bar{\nu}_e) \simeq& 1 - 4\mathrm{Re}[J^{13}_{ee} + J^{23}_{ee}] \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \\ =& 1 - 4|U_{e3}|^2(1 - |U_{e3}|^2) \sin^2\left(\frac{\Delta m^2 L}{4E}\right) = \\ =& 1 - \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 L}{4E}\right)\,,\end{aligned} \tag{34}$$

   where we used the unitarity condition $|U_{e1}|^2 + |U_{e2}|^2 = 1 - |U_{e3}|^2$.

3. The cross section for inverse beta decay is approximately given by

$$\sigma_{\rm IBD} \simeq 10^{-43} \frac{E_\nu^2}{\rm MeV^2}\, {\rm cm}^2\,. \tag{35}$$

The number of events per unit time is given by

$$\frac{N_{\rm evts}}{\Delta t} = \phi_\nu \sigma_{\rm IBD} N_p^{\rm free} P(\bar\nu_e \to \bar\nu_e)\,, \tag{36}$$

where $N_p^{\rm free}$ is the number of free protons in the detector. For $C_{18}{\rm H}_{30}$ this number is given by $N_p^{\rm free} = \frac{M}{m_p}\frac{30}{18\times 12+30}$, where M is the mass of the scintillator and m_p is the proton mass. One can show that in this case $P(\bar\nu_e \to \bar\nu_e) \simeq 0.95$. Imposing $\frac{N_{\rm evts}}{\Delta t} = 100$ events/day, we can calculate M by

$$M = \frac{N_{\rm evts}}{\Delta t} \frac{m_p(18\times 12+30)}{30\, \sigma_{\rm IBD}\, \phi_{\bar\nu_e} P(\bar\nu_e\to\bar\nu_e)} \simeq 9.4\times 10^3\,{\rm kg}\,. \tag{37}$$

Now let us consider the case of coherent scattering on CsI. The cross section for $E_\nu = 3$ MeV is

$$\sigma_{{\rm CE}\nu{\rm NS}} \simeq 10^{-43} A^2 \frac{E_\nu^2}{\rm MeV^2}\,{\rm cm}^2\,. \tag{38}$$

Therefore

$$M_{{\rm CE}\nu{\rm NS}} = \frac{N_{\rm evts}}{\Delta t}\frac{130 m_p}{\sigma_{{\rm CE}\nu{\rm NS}}\, \phi_{\bar\nu_e} P(\bar\nu_e\to\bar\nu_e)} \simeq 9\,{\rm kg}\,. \tag{39}$$

1.9 Neutrino cosmology

1.9.1 Introduction

A brief recap of cosmology. In the relativistic limit ($T_\chi \gg m_\chi$ and $T_\chi \gg \mu_\chi$) the number, energy and pressure densities of relativistic particles in thermal equilibrium are given by

$$n_\chi \simeq \begin{cases} \dfrac{\zeta(3)}{\pi^2} g_\chi T_\chi^3 & (\chi = \text{boson}) \\ \dfrac{3}{4}\dfrac{\zeta(3)}{\pi^2} g_\chi T_\chi^3 & (\chi = \text{fermion}) \end{cases} \tag{40}$$

$$\rho_\chi \simeq \begin{cases} \dfrac{\pi^2}{30} g_\chi T_\chi^4 & (\chi = \text{boson}) \\ \dfrac{7}{8}\dfrac{\pi^2}{30} g_\chi T_\chi^4 & (\chi = \text{fermion}) \end{cases} \tag{41}$$

$$p_\chi \simeq \frac{1}{3}\rho_\chi\,. \tag{42}$$

In the non-relativistic limit the following relations hold

$$\begin{aligned}
n_\chi \simeq & g_\chi \left(\frac{m_\chi T_\chi}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_\chi - m_\chi}{T_\chi}\right) , \\
\rho_\chi \simeq & m_\chi n_\chi \left(1 + \frac{3T_\chi}{2m_\chi}\right) , \\
p_\chi \simeq & n_\chi T_\chi \ll \rho_\chi .
\end{aligned} \tag{43}$$

Let us now review the role of the difference between the number densities of neutrinos and antineutrinos, called neutrino asymmetry,

$$\eta_{\nu_\alpha} = \frac{n_{\nu_\alpha} - n_{\bar{\nu}_\alpha}}{n_\gamma} = \frac{\pi^2}{12\zeta(3)} \left(\frac{T_\nu}{T_\gamma}\right)^3 \left[\xi_{\nu_\alpha} + \frac{\xi_{\nu_\alpha}^3}{\pi^2}\right] , \tag{44}$$

where $\xi_{\nu_\alpha} = \mu_{\nu_\alpha}/T_\nu$. This quantity is assumed to be extremely small, i.e. of the same order of baryon or charged lepton asymmetries. However, the constraints on such asymmetries are actually relatively weak. The energy density of neutrinos and antineutrinos in tems of ξ is given by

$$\rho_{\nu_\alpha} + \rho_{\bar{\nu}_\alpha} = \frac{7}{8}\frac{\pi^2}{15} T_\nu^4 \left[1 + \frac{30}{7}\left(\frac{\xi_{\nu_\alpha}}{\pi}\right)^2 + \frac{15}{7}\left(\frac{\xi_{\nu_\alpha}}{\pi}\right)^4\right] . \tag{45}$$

Neutrinos decoupled from the primordial plasma when the temperature was around 1 MeV and they were relativstic at that time. Since then their energy distribution remained the relativistic one with a temperature that decreased as a^{-1}, where a is the scale factor. Their number density decreased as a^{-3}. The temperature of neutrinos today is very simlar to the one of CMB photons (~ 2 K$\simeq 10^{-4}$ eV).

1.9.2 Questions

1. Are neutrinos relativistic today?
2. What is the number density of neutrinos today?
3. What is their energy density today? Can we derive a limit on the neutrino mass? [Hint: compare the energy density of neutrinos today with the critical energy density of the universe today $\rho_c^0 = 5.5 \times 10^3$ eV cm^{-3}]
4. Derive a limit on the neutrino-antineutrino asymmetry parameter ξ_{ν_α}. [Hint: Use the same approach employed for the constraint on the neutrino mass of the previous question]

1.9.3 Solutions

1. The smallest squared mass difference meaasured by oscillation experiments is $\Delta m^2_{21} \simeq 7 \times 10^{-5}$ eV^2. The temperature of neutrinos today is similar

to the one of CMB, so $T_\nu^0 \simeq 2\,\mathrm{K} \simeq 1.7\times10^{-4}$ eV. Since $T_\nu^0 < \sqrt{\Delta m_{21}^2} < \sqrt{\Delta m_{31}^2}$, at least two of the three mass eigenstates are non-relativistic today. The redshift for which such neutrinos became non-relativistic can be obtained imposing that $T_\nu = m_i/3$, which implies $T_\nu^0(1+z) \simeq T_\nu^0 z = \frac{m_i}{3} \implies z_{\nu_i-\mathrm{nr}} \simeq \frac{m_i}{3T_0^\nu} = 2\times10^3\left(\frac{m_i}{\mathrm{eV}}\right)$.

2. Since neutrinos decoupled when they were relativistic, they still have a relativistic energy distribution. Therefore, for calculating the number density we have to use the limit of relativistic neutrinos. Using $T_\nu^0 = 1.9$ K, we obtain for a single specie of neutrinos

$$n_\nu^0 = \frac{3}{4}\frac{\zeta(3)}{\pi^2} g_\nu (T_\nu^0)^3 \simeq 56\,\mathrm{cm}^{-3}\,, \tag{46}$$

where $g_\nu = 1$.

3. Concerning the energy density, the one in the non-relativistic case dominates. Assuming all neutrinos are non-relativistic the energy density is given by $\rho_\nu^0 = \sum_i m_{\nu i}(n_\nu^0 + n_{\bar\nu}^0)$. Let us divide this quantity by the critical energy density ρ_c^0:

$$\Omega_\nu^{\mathrm{nr}} = \frac{\sum_i m_{\nu i}(n_\nu^0 + n_{\bar\nu}^0)}{\rho_c^0} \simeq \frac{2\times 56\,\mathrm{cm}^{-3}}{10^4\,h^2\,\mathrm{eV\,cm}^{-3}}\sum_i m_i \simeq \frac{\sum_i m_i}{90\,\mathrm{eV}\,h^2}\,, \tag{47}$$

where we assumed $n_\nu^0 = n_{\bar\nu}^0 = 56$ cm^{-3}. Since current data shows that the total energy density of the universe is very similar to the ρ_c^0, imposing $\rho_\nu^0 < \rho_c^0$ or $\Omega_\nu < 1$ gives a conservative upper bound on neutrino masses: $\sum_i m_{\nu i} \leqslant 90\,\mathrm{eV}\,h^2$.

4. When neutrinos were relativistic, the contribution to their energy density coming from an asymmetry is given by

$$\rho_{\nu_\alpha} + \rho_{\bar\nu_\alpha} = \frac{7}{8}\frac{\pi^2}{15}T_\nu^4\left[1 + \frac{30}{7}\left(\frac{\xi_{\nu_\alpha}}{\pi}\right)^2 + \frac{15}{7}\left(\frac{\xi_{\nu_\alpha}}{\pi}\right)^4\right]. \tag{48}$$

The value ξ_{ν_α} remained constant after neutrino decoupling, thus, even if (some) neutrinos became non-relativistic, the contribution to the energy density from asymmetries is given by the equation before also today. The asymmetry cannot be too large, otherwise the extra contribution to the energy density of the universe would be greater than observed values. Taking $\mu_{\nu_\alpha} \gg T_\nu$ (which means only the $\xi_{\nu_\alpha}^4$ survives in the square brackets) and imposing that $\sum_\alpha \rho_{\nu_\alpha} + \rho_{\bar\nu_\alpha} \lesssim \rho_c^0$ we obtain

$$\sum_{\alpha=e,\mu,\tau} (\mu_{\nu_\alpha}^0)^4 \lesssim 8\pi^2\rho_c^0 \simeq (7.6\times10^{-3}\,\mathrm{eV})^4\,. \tag{49}$$

This bound can be converted into an equivalent one for ξ_{ν_α} by using the value of $T_\nu^0 = 1.9$ K$= 1.7\times10^{-4}$ eV

$$\sum_{\alpha=e,\mu,\tau} (\xi_{\nu_\alpha})^4 \lesssim (44)^4\,. \tag{50}$$

In principle, the asymmetries of the three flavor neutrinos could be very different, but neutrino oscillations equilibrate the distribution functions of the three flavors before neutrino decoupling. Therefore, the three flavor asymmetries $\xi_{\nu_e}, \xi_{\nu_\mu}$ and ξ_{ν_τ} are almost equal after decoupling and the bound becomes

$$-33 \lesssim \xi_{\nu_e} \simeq \xi_{\nu_\mu} \simeq \xi_{\nu_\tau} \lesssim 33\,. \tag{51}$$

1.10 Neutrino oscillations with polarization vectors and self-interactions

1.10.1 Questions

I have prepared a code to be used to solve the system of differential equations describing a system of interacting neutrinos in terms of polarization vectors. To use it you have to go at the following link `https://www.kaggle.com/code/francescocapozzi/oscillation-probability-with-polarization-vector/edit` and sign in with francesco.capozzi@univaq.it as username and ggi2023 as password. The code is written in python. Just modify the parameters at the beginning of the script.

1. First consider the case without antineutrinos and see how the evolution of the polarization vector changes with μ. Start with $\mu = 0$ and then increase it until it becomes much larger than ω. What happens when $\mu \gg \omega$?

2. Now include also antineutrinos. In particular you can modify the ratio between the number of antineutrinos and the one of neutrinos by changing the parameter "antinu_nu_ratio" between 0 and 1. Repeat the same process used before by varying μ and use both normal and inverted ordering. What happens when $\mu \gg \omega$ in normal ordering? And in inverted ordering? What is the role of the size of mixing angle θ?

1.11 Waxman-Bahcall bound

1.11.1 Question

From observations of ultra high energy cosmic rays (mainly protons), one can derive that their energy production rate in the energy range $[10^{19}, 10^{21}]$ eV is

$$E_p^2 \frac{d\dot{N}_p}{dE_p} \simeq 5 \times 10^{44}\, \mathrm{erg\, Mpc^{-3}\, yr^{-1}}\,. \tag{52}$$

The main interaction channel of such high energy protons is with CMB photons and photons produced in the same source

$$p + \gamma \to p/n + \pi^0/\pi^+\,. \tag{53}$$

So a flux of high energy neutrinos is expected. From this, one can derive an upper bound on the energy flux of muon neutrinos today.

[Hint. The probability for π^0 production and for π^+ production is roughly the same. In π^+ decay chain roughly half of the pion's energy goes to muon neutrinos. The units of the energy flux should be in GeV cm^{-2} s^{-1} sr-1, where sr stands for steradians.]

1.11.2 Solution

The total energy density of cosmic rays in the history of the Universe is given by $H_0^{-1} E_p^2 \frac{d\dot{N}_p}{dE_p}$, where H_0 is the hubble parameter, whose inverse gives the age of the Universe $H_0^{-1} \simeq 10^{10}$ years. To transform this quantity to the energy flux of neutrinos per unit solid angle we need to multiply by a factor $c/(4\pi)$, where c is the speed of light. Finally there should be an extra factor 1/4, which takes into account that only the probability of producing a π^+ in $p\gamma$ interactions is 0.5 and and that only half of the π^+ energy goes into neutrino. Thus:

$$E_\nu^2 \Phi_{\nu_\mu} \lesssim \frac{1}{4} \frac{c}{4\pi} H_0^{-1} \frac{d\dot{N}_p}{dE_p} \simeq 1.5 \times 10^8 \,\mathrm{GeV\,cm^{-2}\,s^{-1}\,sr^{-1}}\,. \tag{54}$$