GGI School on Theoretical Aspects of Astroparticle Physics, Cosmology, and Gravitation 2023

Particle dark matter

Solution for exercise sheet 2

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Problem 1: Including spin-statistical factors for 2-to-2 processes

Consider the collision operator for the number density from a process $12 \to 34$

$$\frac{C_{n,12\to 34}}{\kappa_{12\to 34}} = \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2} \frac{\mathrm{d}^3 p_3}{(2\pi)^3 2E_3} \frac{\mathrm{d}^3 p_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta(\underline{p}_1 + \underline{p}_2 - \underline{p}_3 - \underline{p}_4) |\mathcal{M}_{12\to 34}|^2 \times f_1 f_2 (1 \pm f_3)(1 \pm f_4) \,. \tag{1}$$

The integrations here are over the whole $\mathbb{R}^3$ for all momenta; possible overcounting of physically identical regions is compensated with the symmetry factor $\kappa_{12\to 34}$. We aim to derive the most simplified expression for a general matrix element $|\mathcal{M}_{12\to 34}|^2$ and keeping the spin-statistical factors $1 \pm f_{3,4}$. Rotational invariance can be used to go to a coordinate system where

$$\mathbf{p}_1 = p_1(0,0,1)^T \ , \ \ \mathbf{p}_2 = p_2(\sin\beta, 0, \cos\beta)^T \ , \ \ \mathbf{p}_3 = p_3(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T \ . \tag{2}$$

1. We find

$$\frac{C_{n,12\to 34}}{\kappa_{12\to 34}} = 2(2\pi)^2 \int_0^\infty \frac{\mathrm{d}p_1\, p_1^2}{(2\pi)^3 2E_1} \int_{-1}^1 \mathrm{d}\cos\beta \int_0^\infty \frac{\mathrm{d}p_2\, p_2^2}{(2\pi)^3 2E_2} \int_0^{2\pi} \mathrm{d}\phi \int_{-1}^1 \mathrm{d}\cos\theta \times \int_0^\infty \frac{\mathrm{d}p_3\, p_3^2}{(2\pi)^3 2E_3} \int \frac{\mathrm{d}^3 p_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta(\underline{p}_1 + \underline{p}_2 - \underline{p}_3 - \underline{p}_4) |\mathcal{M}_{12\to 34}|^2 \times f_1 f_2 (1 \pm f_3)(1 \pm f_4) \,. \tag{3}$$

2. $|\mathcal{M}_{12\to 34}|^2$ can be written in a form such that it only on s and t (using $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$).[1] In the above coordinate system, we have

$$s = (\underline{p}_1 + \underline{p}_2)^2 = m_1^2 + m_2^2 + 2E_1E_2 - 2p_1p_2\cos\beta \ , \tag{4}$$

$$t = (\underline{p}_1 - \underline{p}_3)^2 = m_1^2 + m_3^2 - 2E_1E_3 + 2p_1p_3\cos\theta \ . \tag{5}$$

3. Integration over $\mathbf{p}_4$ can be performed trivially with the help of the spatial part of the δ-distribution. Then,

$$E_4 = \sqrt{m_4^2 + (\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3)^2} = \sqrt{m_4^2 + p_1^2 + p_2^2 + p_3^2 + 2p_1p_2\cos\beta - 2p_1p_3\cos\theta - 2p_2p_3(\sin\beta\sin\theta\cos\phi + \cos\beta\cos\theta)} \ , \tag{6}$$

which needs to be entered into the remaining $\delta(E_1 + E_2 - E_3 - E_4)$. At this point, it is useful to note that the entire integrand only depends on ϕ through $\cos\phi$ and we can therefore multiply by 2 and restrict the integration in ϕ to the interval $[0, \pi]$. Reminding

[1] Note that strictly speaking, the relation between s, t, and u becomes enforced by the four-momentum conserving δ-distribution.

ourselves that $\delta(g(x)) = \delta(x - x_0)/|g'(x_0)|$, where x_0 is a zero of $g(x)$ (in this case the only one), we then find

$$\delta(E_1 + E_2 - E_3 - E_4) = \frac{E_1 + E_2 - E_3}{2p_2p_3 \sin\beta \sin\theta}\delta(\cos\phi - c_\phi) \; , \tag{7}$$

where

$$c_\phi = \frac{1}{2p_2p_3 \sin\beta \sin\theta}[m_1^2 + m_2^2 + m_3^2 - m_4^2 + 2(E_1E_2 - E_1E_3 - E_2E_3) - 2p_1p_2 \cos\beta \\ + 2p_1p_3 \cos\theta + 2p_2p_3 \cos\beta \cos\theta] \; . \tag{8}$$

For later reference, we define

$$Q = m_1^2 + m_2^2 + m_3^2 - m_4^2 \; , \tag{9}$$

$$\gamma = E_1E_2 - E_1E_3 - E_2E_3 \; . \tag{10}$$

4. We start by deriving the restricted integration region. As hinted, we have

$$c_\phi^2 \leq 1 \tag{11}$$

$$(Q + 2\gamma - 2p_1p_2 \cos\beta + 2p_1p_3 \cos\theta + 2p_2p_3 \cos\beta \cos\theta)^2 \leq 4p_2^2p_3^2(1 - \cos^2\beta)(1 - \cos^2\theta) \tag{12}$$

This is a quadratic inequality. After some simplification and rewriting terms (Mathematica is of course highly helpful here), one arrives at the expression given in the hint

$$0 \geq \cos^2\theta + \frac{b}{a}\cos\theta + \frac{c}{a} \; , \tag{13}$$

$$\begin{aligned} a &= -4p_2^2p_3^2(1 - \cos^2\beta) - (2p_1p_3 + 2p_2p_3 \cos\beta)^2 \\ &= -4p_2^2p_3^2 - 4p_1^2p_3^2 - 8p_1p_2p_3^2 \cos\beta && (14) \\ &= -4p_3^2[(E_1 + E_2)^2 - s] \; , && (15) \\ b &= -4(p_1p_3 + p_2p_3 \cos\beta)(Q + 2\gamma - 2p_1p_2 \cos\beta) && (16) \\ &= -4p_3\left[p_1 + \frac{m_1^2 + m_2^2 + 2E_1E_2 - s}{2p_1}\right][s + m_3^2 - m_4^2 - 2E_3(E_1 + E_2)] && (17) \\ &= \frac{2p_3}{p_1}[s - 2E_1(E_1 + E_2) + m_1^2 - m_2^2][s - 2E_3(E_1 + E_2) + m_3^2 - m_4^2] \; , && (18) \\ c &= -(Q + 2\gamma - 2p_1p_2 \cos\beta)^2 + 4p_2^2p_3^2(1 - \cos^2\beta) && (19) \\ &= -[2E_3(E_1 + E_2) - m_3^2 + m_4^2 - s]^2 - p_3^2\left[4p_2^2 - \left(\frac{m_1^2 + m_2^2 + 2E_1E_2 - s}{p_1}\right)^2\right] && (20) \\ &= -[2E_3(E_1 + E_2) - m_3^2 + m_4^2 - s]^2 && (21) \\ &\quad - \frac{p_3^2}{p_1^2}[4p_2^2p_1^2 - (m_1^2 + m_2^2 + 2E_1E_2)^2 + 2(m_1^2 + m_2^2 + 2E_1E_2)s - s^2] && (22) \\ &= -[2E_3(E_1 + E_2) - m_3^2 + m_4^2 - s]^2 - \frac{p_3^2}{p_1^2}(s - s_{12,-})(s - s_{12,+}) \; , && (23) \\ s_{12/34,\pm} &= m_{1/3}^2 + m_{2/4}^2 + 2E_{1/3}E_{2/4} \pm 2p_{1/3}p_{2/4} \; . && (24) \end{aligned}$$

Note that as $s = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 \leq (E_1 + E_2)^2$ one has $a \leq 0$. The inequality (13) leads to

$$c_{\theta,+} \leq \cos\theta \leq c_{\theta,-} \; , c_{\theta,\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \; , \tag{25}$$

which shows that the kinematics can only be fulfilled if

$$b^2 - 4ac = 4(p_3^2/p_1^2)(s - s_{12,-})(s - s_{12,+})(s - s_{34,-})(s - s_{34,+}) \geq 0 \,. \tag{26}$$

Defining $R_\theta = \{-1 \leq \cos\theta \leq 1 \mid c_{\theta,+} \leq \cos\theta \leq c_{\theta,-}\}$, the integration over ϕ gives

$$\begin{aligned}\frac{C_{n,12\to34}}{\kappa_{12\to34}} &= 2(2\pi)^3 \int_0^\infty \frac{\mathrm{d}p_1\, p_1^2}{(2\pi)^3 2E_1} \int_{-1}^1 \mathrm{d}\cos\beta \int_0^\infty \frac{\mathrm{d}p_2\, p_2^2}{(2\pi)^3 2E_2} \int_0^\infty \frac{\mathrm{d}p_3\, p_3^2}{(2\pi)^3 2E_3} \theta(b^2 - 4ac) \\ &\times f_1 f_2 (1 \pm f_3)(1 \pm f_4) \int_{R_\theta} \frac{\mathrm{d}\cos\theta}{\sqrt{a\cos^2\theta + b\cos\theta + c}} |\mathcal{M}_{12\to34}|^2 \,, \end{aligned}\tag{27}$$

where we used that $\mathrm{d}\phi = -\mathrm{d}\cos\phi/\sqrt{1-\cos^2\phi}$ ($\phi \in [0,\pi]$, $\cos\phi \in [-1,1]$, the "$-$"-sign switches the integration direction) and

$$2p_2 p_3 \sin\beta \sin\theta \sqrt{1 - c_\phi^2} = \sqrt{a\cos^2\theta + b\cos\theta + c} \,. \tag{28}$$

5. For the variable transformation, we first note that ($i = 1, 2, 3$)

$$\mathrm{d}s = -2p_1 p_2 \mathrm{d}\cos\beta \,, \tag{29}$$

$$E_i \mathrm{d}E_i = p_i \mathrm{d}p_i \,. \tag{30}$$

From $-1 \leq \cos\beta \leq 1$ and after using the "$-$"-sign to switch integration directions, we have $s_{12,-} \leq s \leq s_{12,+}$. Due to the θ-function and with Eq. (26), the integration over s becomes restricted to the region $R_s = \{s \in \mathbb{R} \mid \max(s_{12,-}, s_{34,-}) \leq s \leq \min(s_{12,+}, s_{34,+})\}$ and we find

$$\begin{aligned} C_{n,12\to34} &= \frac{\kappa_{12\to34}}{4(2\pi)^6} \int_{m_1}^\infty \mathrm{d}E_1 \int_{m_2}^\infty \mathrm{d}E_2 \int_{m_3}^\infty \mathrm{d}E_3\, p_3 f_1 f_2 (1 \pm f_3)(1 \pm f_4) \int_{R_s} \mathrm{d}s \\ &\times \int_{R_\theta} \frac{\mathrm{d}\cos\theta}{\sqrt{a\cos^2\theta + b\cos\theta + c}} |\mathcal{M}_{12\to34}|^2 \,, \end{aligned}\tag{31}$$

It is already obvious that not for all E_1, E_2, and E_3, the region R_s is not empty, e.g. $E_2 \geq m_3 + m_4 - E_1$ and $E_3 \leq E_1 + E_2 - m_4$ must hold for sure.[2] This motivates switching integration order to s, E_1, E_2, E_3, and $\cos\theta$ (from left to right, last to first integral). To perform these switches, we need to find the corresponding regions

$$s_{12,-} \leq s \leq s_{12,+} \tag{32}$$

$$\Rightarrow s \geq (m_1 + m_2)^2 \quad \text{and}$$

$$(s - m_1^2 - m_2^2 - 2E_1E_2)^2 \leq 4(E_1^2 - m_1^2)(E_2^2 - m_2^2) \tag{33}$$

$$(s - m_1^2 - m_2^2)^2 - 4(s - m_1^2 - m_2^2)E_1E_2 + 4E_1^2E_2^2 \leq 4(E_1^2 - m_1^2)(E_2^2 - m_2^2) \tag{34}$$

$$(s - m_1^2 - m_2^2)^2 + 4m_2^2(E_1^2 - m_1^2) - 4(s - m_1^2 - m_2^2)E_1E_2 + 4E_2^2 m_1^2 \leq 0 \tag{35}$$

$$E_{2,-} \leq E_2 \leq E_{2,+} \tag{36}$$

$$E_{2,\pm} = \frac{1}{2m_1^2}\left(E_1[s - m_1^2 - m_2^2] \pm \sqrt{E_1^2[s - m_1^2 - m_2^2]^2 - m_1^2(s - m_1^2 - m_2^2)^2 - 4m_1^2m_2^2(E_1^2 - m_1^2)}\right) \tag{37}$$

$$= \frac{1}{2m_1^2}\left(E_1[s - m_1^2 - m_2^2] \pm p_1\sqrt{s^2 + (m_1^2 - m_2^2)^2 - 2s(m_1^2 + m_2^2)}\right) \tag{38}$$

$$R_2 = \{E_2 \geq m_2 \mid E_{2,-} \leq E_2 \leq E_{2,+}\} \,, \tag{39}$$

[2]Similarly, R_θ is not always non-empty, but the expressions here would become very complicated and it is more useful to deal with this when performing numerical integrations in the end.

and

$$
\begin{aligned}
& s_{34,-} \leq s \leq s_{34,+} && (40)\\
& \Rightarrow s \geq (m_3+m_4)^2 \quad \text{and} \\
& [s - m_3^2 - m_4^2 - 2E_3(E_1+E_2-E_3)]^2 \leq 4(E_3^2 - m_3^2)[(E_1+E_2-E_3)^2 - m_4^2] && (41)\\
& (s-m_3^2-m_4^2)^2 - 4(s-m_3^2-m_4^2)E_3(E_1+E_2-E_3) + 4E_3^2(E_1+E_2-E_3)^2 \leq \\
& 4E_3^2(E_1+E_2-E_3)^2 - 4E_3^2(m_3^2+m_4^2) + 8E_3m_3^2(E_1+E_2) - 4m_3^2[(E_1+E_2)^2 - m_4^2] && (42)\\
& (s-m_3^2-m_4^2)^2 + 4m_3^2[(E_1+E_2)^2 - m_4^2] - 4(s+m_3^2-m_4^2)(E_1+E_2)E_3 + 4sE_3^2 \leq 0 \\
& && (43)\\
& E_{3,-} \leq E_3 \leq E_{3,+} && (44)\\
& E_{3,\pm} = \frac{1}{2s}\Bigg([E_1+E_2][s+m_3^2-m_4^2] \\
& \qquad \pm \sqrt{(s+m_3^2-m_4^2)^2(E_1+E_2)^2 - s(s-m_3^2-m_4^2)^2 - 4sm_3^2[(E_1+E_2)^2-m_4^2]}\Bigg) \\
& \quad = \frac{1}{2s}\left([E_1+E_2][s+m_3^2-m_4^2] \pm \sqrt{[(E_1+E_2)^2 - s][s^2 + (m_3^2-m_4^2)^2 - 2s(m_3^2+m_4^2)]}\right) \\
& && (45)\\
& R_3 = \{E_3 \geq m_3 \mid E_{3,-} \leq E_3 \leq E_{3,+}\} \,. && (46)
\end{aligned}
$$

This gives

$$
\begin{aligned}
C_{n,12\to34} = \frac{\kappa_{12\to34}}{4(2\pi)^6} \int_{s_{\min}}^{\infty} \mathrm{d}s \int_{m_1}^{\infty} \mathrm{d}E_1 \int_{R_2} \mathrm{d}E_2 \int_{R_3}^{\infty} \mathrm{d}E_3\, p_3 f_1 f_2 (1 \pm f_3)(1 \pm f_4) \\
\times \int_{R_\theta} \frac{\mathrm{d}\cos\theta}{\sqrt{a\cos^2\theta + b\cos\theta + c}} |\mathcal{M}_{12\to34}|^2 \,, \qquad (47)
\end{aligned}
$$

where $s_{\min} = \max([m_1+m_2]^2, [m_3+m_4]^2)$.

In some cases the integration over $\cos\theta$ can be performed analytically after a variable transformation to $x = -2\arcsin(\sqrt{(c_{\theta,-} - \cos\theta)/(c_{\theta,-} - c_{\theta,+})})$. We conclude with some remarks on possible further applications. The operator $C_{n,34\to12}$ easily follows by replacing

$$
f_1 f_2 (1 \pm f_3)(1 \pm f_4) \to f_3 f_4 (1 \pm f_1)(1 \pm f_2) \,, \qquad (48)
$$

$$
|\mathcal{M}_{12\to34}|^2 \to |\mathcal{M}_{34\to12}|^2 \,. \qquad (49)
$$

Similarly, collision operators for the Boltzmann equation for the energy density of a particle j with known phase-space distribution function

$$
\dot{\rho}_j + 3H(\rho_j + P_j) = C_{\rho,j} \qquad (50)
$$

can be obtained by including a factor E_j in the integrand. Another possible application of Eq. (47) is freeze-in. This is obvious when having the particle 1 freezing in and considering $C_{n,34\to12}$ for $f_1 \ll 1$, where $1 \pm f_1 \simeq 1$. Therefore, the integration over E_1 can be removed and the collision operator for the Boltzmann equation of the phase-space density can be recovered *including* the spin-statistical factor $1 \pm f_2$, which can potentially be relevant if particle 2 has a large occupation number.

Problem 2: Relic density calculations with DarkSUSY

You should all have a running version of DarkSUSY *at this point. In case of questions about how to use the code, remember to take a look at the tutorial at* ***https://darksusy.hepforge.org/tutorials/TOOLS_2021/DarkSUSY_getting_started.pdf*** *if you haven't done so yet.*

For concreteness, we will consider a simple toy model with DM composed of Dirac fermions ψ that couples to the standard model only via the 5D operator

$$\mathcal{L} \supset \Lambda^{-1}\bar{\psi}\psi\,|H|^2 \;, \tag{51}$$

where H is the SM Higgs doublet and Λ is a parameter with mass dimension one.

1. The relevant part of the Lagrangian in the scalar singlet model is $\mathcal{L} \supset (\lambda/2)S^2|H|^2$. Clearly, this has the same types of diagrams as the model of fermionic DM. To simplify the discussion, we look at the Lagrangians after electroweak symmetry breaking. Most of the annihilation channels (e.g. l^+l^-, $\bar{q}q$, $\gamma\gamma$, ...) are mediated by a Higgs in the s-channel. For these, it is enough to replace the factor corresponding to the three-point vertex SSh from the initial state in the annihilation cross-section with the expression arising for fermionic DM. The scalar singlet model has the three-point vertex $\mathcal{L} \supset (\lambda/2)v_0S^2h$ with the vev v_0 and the Higgs field h, which leads to a matrix element for the (inverse) decay

   $$|\mathcal{M}_{SS\to h}|^2 = |\mathcal{M}_{h\to SS}|^2 = \lambda^2 v_0^2 \;. \tag{52}$$

   From Eq. (51) we find for the model of fermionic DM

   $$|\mathcal{M}_{\bar{\psi}\psi\to h}|^2 = |\mathcal{M}_{h\to\bar{\psi}\psi}|^2 = \frac{v_0^2}{\Lambda^2}\mathrm{Tr}[(\not{p}_{\bar{\psi}} - m_\psi)(\not{p}_{\psi} + m_\psi)] = \frac{2v_0^2}{\Lambda^2}(s - 4m_\psi^2) \tag{53}$$

   with the four-momenta $\underline{p}_{\bar{\psi}}$ and $\underline{p}_{\psi}$ of $\bar{\psi}$ and ψ and the Mandelstam variable $s = (\underline{p}_{\bar{\psi}} + \underline{p}_{\psi})^2$. Note however that it is the *spin-averaged* annihilation rate, and hence matrix elements, that enter in the Boltzmann equation for the number density. We thus need to further divide by $g_{\bar{\psi}}g_\psi = 4$. In summary, we need to make the replacement

   $$\lambda^2 \to (s - 4m_\psi^2)/(2\Lambda^2) \tag{54}$$

   in Eq. (D.3) in the DarkSUSY paper.

   For the annihilations into two Higgs bosons more diagrams are relevant. For the scalar singlet model these are Higgs-mediated s-channel, S-mediated s-, t-, and u-channel as well as contact terms from the four-point vertex S^2h^2. In the model of fermionic DM, there are the same types of diagrams present. However, adapting the annihilation cross-section in this case clearly requires more than an overall replacement in the prefactor. To keep things simple in this exercise, we just *assume* that there is the same overall modification as for the other annihilations.

2. Setup a folder exercise with a subfolder replacables somewhere *outside* the folder with the DarkSUSY release and copy examples/aux/oh2_ScalarSinglet.f from the DarkSUSY folder into exercise/. You also need to copy examples/aux/makefile to your new folder exercise/, such that you can compile by simply typing

   ```
   make oh2_ScalarSinglet
   ```

   We start by discussing the replacable function needed for the annihilation rate. Copy src_models/silveira_zee/an/dssigmavpartial.f from the DarkSUSY folder into exercise/replacables/. In this copied function you need to change line 171 to

   ```
   100     dssigmavpartial = gev2cm3s*kin*sv*0.5d0*(s-4.*mx2)
   ```

NB: in order for the compiler to find functions in this replacables/ folder, you need to add it to the oh2_ScalarSinglet block in the makefile; compare, e.g., the oh2_generic_wimp_threshold with the oh2_generic_wimp block for how exactly to do this (or re-check the tutorial).

To make the rest of the replacement, i.e. going from λ to $1/\Lambda$, you need to adapt the file oh2_ScalarSinglet.f in your folder. It is convenient to define a variable inputLambda instead of inputlambda and initialize the models with $1/\Lambda$

```
call dsgivemodel_silveira_zee(1.d0/inputLambda,inputmass)
```

in the two occurences where this turns up. Note that you should also change the starting value for inputLambda whenever it is initialized; values around inputmass are reasonable. Furthermore, the scaling of the relic density with respect to Λ is inverted compared to before, so in line 145 you should change to a division by stepsize

```
inputLambda = inputLambda/stepsize
```

According to Eq. (51), the DM particle annihilates with its anti-particle. Unlike the scalar singlet model, DM is therefore not self-conjugate anymore and we need to also adapt the code for this. To do so, copy src_models/silveira_zee/rd/dsrdparticles.f from the DarkSUSY folder to exercise/replacables/ and change line 65 to

```
selfcon = 2
```

Note that in principle, the DM fermion and anti-fermion have two internal degrees of freedom now. For this, copy src_models/silveira_zee/ini/dsinit_module.f from the DarkSUSY folder to exercise/replacables/ and change line 46 to

```
kdof(kdm) = 2
```

This only has a logarithmic impact on the relic density by slightly changing the temperature of freeze-out, see e.g. Kolb & Turner, similar to the change *differing* from the factor of 2 when going from a self-conjugate particle to a non-self-conjugate particle.

Before comparing the values of λ and Λ required to obtain the observed DM relic density, we should first understand what to expect. In the scalar singlet model, the annihilations in terms of a partial-wave decomposition are dominated by the s-wave part,[3] i.e. there is no velocity suppression and in the non-relativistic limit

$$(\sigma v)_{\text{scalar}} = \lambda^2 a_{\text{scalar}} \simeq \text{const} \, . \tag{55}$$

for some a. You can see this by considering the quantum numbers of the initial and final states, cf. e.g. `https://arxiv.org/abs/1305.1611`. This is no longer the case for the model of fermionic DM. Explicitly, the overall modification of σv introduces a factor $(s - 4m_\psi^2)$, which becomes proportional to the Møller velocity squared v^2 in the non-relativistic limit. This directly makes it obvious that the annihilations are now p-wave. We can write for the cross-section of fermionic DM in the non-relativistic limit

$$\sigma_{\text{fermion}} = \left(\frac{\sigma v}{v}\right)_{\text{fermion}} = \frac{a_{\text{scalar}}}{2\Lambda^2}\frac{s-4m_\psi^2}{v} = \frac{a_{\text{scalar}}}{2\Lambda^2}\frac{2E_\psi E_{\bar\psi}}{\sqrt{s}}\sqrt{s-4m_\psi^2} \simeq \frac{a_{\text{scalar}}}{\Lambda^2} m_\psi^2 \sqrt{\tilde s - 1} \, , \tag{56}$$

where $\tilde s = s/(4m_\psi^2)$ and we introduced an additional factor of 2 compared to the scalar singlet case to account for the particle not being self-conjugate. Inserting into the expression

[3] s here referring to the partial wave, *not* the Mandelstam variable.

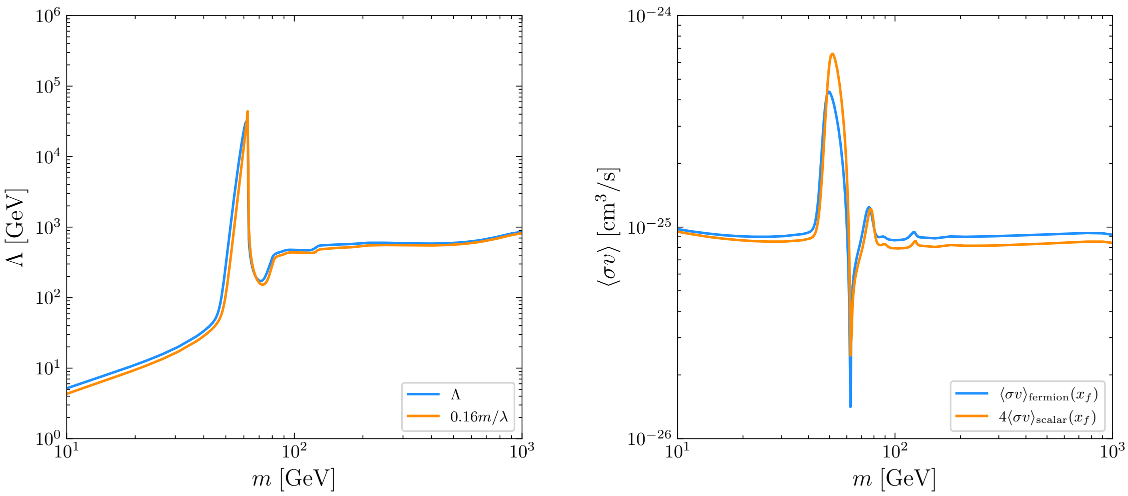

Figure 1: *Left:* Value of Λ required to obtain the observed DM relic abundance as a function of DM mass m (blue) as well as the expected value by comparison to the scalar singlet model (orange). *Right:* $\langle\sigma v\rangle$ at freeze-out as a function of the DM mass m when the observed DM relic abundance is obtained for the model of fermionic DM (blue) and the scalar singlet model (orange).

for the thermal average from Gondolo-Gelmini gives

$$\begin{aligned}\langle\sigma v\rangle_{\rm fermion} &= \frac{4x}{K_2^2(x)}\int_1^\infty \mathrm{d}\tilde{s}\,(\tilde{s}-1)\sqrt{\tilde{s}}K_1(2\sqrt{\tilde{s}}x)\sigma_{\rm fermion} \\ &\simeq \frac{4x}{K_2^2(x)}\int_1^\infty \mathrm{d}\tilde{s}\,(\tilde{s}-1)\sqrt{\tilde{s}}K_1(2\sqrt{\tilde{s}}x)\frac{a}{\Lambda^2}m_\psi^2\sqrt{\tilde{s}-1} \qquad (57) \\ &\simeq \frac{3a_{\rm scalar}m_\psi^2}{x\Lambda^2}\,. \qquad (58)\end{aligned}$$

This is to be compared at freeze-out $x = x_f \sim 20$ with

$$4\langle\sigma v\rangle_{\rm scalar} = 4\lambda^2 a_{\rm scalar}\,, \qquad (59)$$

where there is a factor of 2 due to the fermionic DM not being self-conjugate and *another* factor of 2 due to the annihilations being p-wave suppressed. The latter is discussed e.g. in the book by Kolb & Turner, and we show the calculation in the appendix at the end of these solutions.

Coming back to the comparison of the model for fermionic DM with the scalar singlet model, we would expect that to obtain the observed DM relic abundance

$$\Lambda \sim \frac{m_\psi}{\lambda}\sqrt{\frac{3}{120}} \sim 0.16\frac{m_\psi}{\lambda}\,, \qquad (60)$$

where λ is the value for the scalar singlet to obtain the observed DM relic abundance. We show this in Fig. 1 and find good agreement. Around $m \sim m_h/2$ the impact of the s-channel resonance of the SM Higgs is clearly visible in both models.

As a final remark for this part of the exercise, note that including the factors of 2 for going from self-conjugate to non-self-conjugate DM and s-wave to p-wave annihilations are not exact as there are also corrections to the freeze-out temperature.

3. To find the value of $\langle\sigma v\rangle$ at freeze-out, we can use that DarkSUSY already computes x_f when performing the calculation for the relic density, cf.

```
oh2=dsrdomega(0,20,xf,ierr,iwarn,nfc)
```

Consequently, we only have to find the function that computes $\langle\sigma v\rangle$. This can e.g. be done by going through examples/aux/ScalarSinglet_thermal_averages.f in your DarkSUSY folder, where this is done in line 80 with the function dsrdthav. This needs to be called with the value of x, where $\langle\sigma v\rangle$ should be computed, and the invariant annihilation rate. The latter is accessible by the function dsanwx after a DarkSUSY model has been initialized. Consequently, calling

```
dsrdthav(xf,dsanwx)
```

gives $\langle\sigma v\rangle$ at x_f in units GeV^{-2}. For this call to work you should also tell the compiler about these functions by changing line 24 to

```
real*8 dsrdomega, dskdmcut, dskdtkd, dsmwimp, dsrdthav, dsanwx
external dsanwx
```

Note that conversion from GeV^{-2} to cm^3/s can be done with the constant gev2cm3s from src/include/dsmpconst.f, which first needs to be included

```
include 'dsmpconst.h'
```

Having done these modifications to the code for fermionic DM as well as a copy of the code for the scalar singlet model, we can plot the value of $\langle\sigma v\rangle$ at freeze-out. This is done in Fig. 1. We notice that the factor of 4 in Eq. (59) is a good approximation, but not exact due to corrections on x_f. Furthermore, around the resonance $m \sim m_h/2$, where an expansion $\langle\sigma v\rangle \simeq a + 6b/x$ is no good approximation anymore and one needs to solve the Boltzmann equation numerically. Note that this would generally also require going to a coupled system as kinetic equilibrium is no longer guaranteed during freeze-out.

Appendix: Analytical approximation for freeze-out including velocity-suppressed terms

In this appendix, we perform an analytical treatment for DM freeze-out to show that $\langle\sigma v\rangle$ at freeze-out for p-wave annihilations needs to be around twice as large as for s-wave annihilations. The treatment follows along the corresponding section of the book by Kolb & Turner. First going to the Boltzmann equation for the yield of DM $Y = n/s_{\mathrm{tot}}$, where s_{tot} is the total entropy density (including all SM particles) satisfying $\dot{s}_{\mathrm{tot}} + 3Hs_{\mathrm{tot}} = 0$, gives

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \frac{\dot{n}}{s_{\mathrm{tot}}} - \frac{n\dot{s}_{\mathrm{tot}}}{s_{\mathrm{tot}}^2} = s_{\mathrm{tot}}\langle\sigma v\rangle[Y_{\mathrm{eq}}^2 - Y^2] \;, \tag{61}$$

where $Y_{\mathrm{eq}} = n_{\mathrm{eq}}/s_{\mathrm{tot}}$. This can be rewritten in terms of $x = m/T$ using $\mathrm{d}x = -(x^2/m)\mathrm{d}T \simeq (H(m)/x)\mathrm{d}t$ with the Hubble rate evaluated at the DM mass m such that

$$\frac{\mathrm{d}Y}{\mathrm{d}x} \simeq -\frac{x s_{\mathrm{tot}}\langle\sigma v\rangle}{H(m)}[Y_{\mathrm{eq}}^2 - Y^2] \tag{62}$$

$$\simeq -\alpha x^{-2}\langle\sigma v\rangle[Y_{\mathrm{eq}}^2 - Y^2] \;, \tag{63}$$

where $\alpha = 2\pi^2 g_{*,s} m^3/(45Hm)$ with g_{*s} the effective number of degrees of freedom for the entropy density s_{tot}. Now, we introduce $\Delta = Y - Y_{\mathrm{eq}}$ such that

$$\frac{\mathrm{d}\Delta}{\mathrm{d}x} \simeq -\frac{\mathrm{d}Y_{\mathrm{eq}}}{\mathrm{d}x} - \alpha x^{-2}\langle\sigma v\rangle\Delta(\Delta + 2Y_{\mathrm{eq}}) \;. \tag{64}$$

Before freeze-out at x_f, i.e. for $x \ll x_f$, $Y \simeq Y_{\rm eq}$ such that $\mathrm{d}\Delta/\mathrm{d}x \simeq 0$, $\Delta \ll Y_{\rm eq}$, and therefore

$$\Delta \simeq -\frac{x^2}{2\alpha\langle\sigma v\rangle Y_{\rm eq}}\frac{\mathrm{d}Y_{\rm eq}}{\mathrm{d}x} \simeq \frac{x^2}{2\alpha\langle\sigma v\rangle}\ , \tag{65}$$

where we used the fact that still $x \gg 1$, $Y_{\rm eq} \propto x^{3/2}\mathrm{e}^{-x}$ and therefore $\mathrm{d}Y_{\rm eq}/\mathrm{d}x \simeq -Y_{\rm eq}$. After freeze-out for $x > x_f$ we can approximate $\Delta \simeq Y \gg Y_{\rm eq}$ and hence

$$\frac{\mathrm{d}\Delta}{\mathrm{d}x} \simeq -\alpha x^{-2}\langle\sigma v\rangle\Delta^2\ . \tag{66}$$

Expanding in powers of x, one has[4]

$$\sigma v \simeq a + bv^2 + \dots\ , \tag{67}$$
$$\langle\sigma v\rangle \simeq a + 6b/x + \dots\ , \tag{68}$$

where s-wave annihilations have non-vanishing contribution from the first term, whereas p-wave annihilations have $a = 0$. Inserting into Eq. (66) yields for the final $\Delta_\infty = \Delta(x \to \infty)$

$$\int_{\Delta(x_f)}^{\Delta_\infty}\frac{\mathrm{d}\Delta}{\Delta^2} \simeq -\alpha\int_{x_f}^{\infty}\frac{\mathrm{d}x}{x^2}\left(a + \frac{6b}{x}\right) \tag{69}$$

$$\frac{1}{\Delta(x_f)} - \frac{1}{\Delta_\infty} \simeq -\frac{\alpha}{x_f}\left(a + \frac{3b}{x_f}\right) \tag{70}$$

$$Y_\infty \simeq \Delta_\infty \simeq \left[\frac{2\alpha}{x_f^2}\left(a + \frac{6b}{x_f}\right) + \frac{\alpha}{x_f}\left(a + \frac{3b}{x_f}\right)\right]^{-1}\ . \tag{71}$$

Given that typically $x_f \sim 20$ and either $a \gg b/x_f$ or $a \ll b/x_f$, we can approximate

$$Y_\infty \simeq \Delta_\infty \simeq \frac{x_f}{\alpha(a + 3b/x_f)}\ . \tag{72}$$

Since we moved from a prefactor of 6 for b in Eq. (68) to 3 in Eq. (72), we find the expected factor of 2 that $\langle\sigma v\rangle$ at freeze-out needs to be increased for p-wave annihilations compared to s-wave annihilations.

[4]The factor of 6 in front of the second term of $\langle\sigma v\rangle$ appears due to the thermal average of v, cf. Eq. (58)